Principle of Mathematical Induction

- **Principle of Mathematical Induction**
  - There are some mathematical statements or results that are formulated in terms of *n*, where *n* is a positive integer. To prove such statements, the well-suited principle that is used, based on the specific technique, is known as the principle of mathematical induction.
  - To prove a given statement in terms of *n*, firstly, we assume the statement as P (*n*).

Thereafter, we examine the correctness of the statement for $n = 1$, i.e., P (1) is true.
Then, assuming that the statement is true for $n = k$, where *k* is a positive integer, we prove that the statement is true for $n = k + 1$, i.e., truth of P (*k*) implies the truth of P ($k + 1$). Then, we say P (*n*) is true for all natural numbers *n*.

Example: For all $n \in \mathbb{N}$, prove that

$$\frac{4}{3} + \left(\frac{4}{3}\right)^2 + \ldots\ldots + \left(\frac{4}{3}\right)^n = 4\left[\left(\frac{4}{3}\right)^n - 1\right]$$

Solution:
Let the given statement be P (*n*), i.e.,

$$P(n): \frac{4}{3} + \left(\frac{4}{3}\right)^2 + \ldots\ldots + \left(\frac{4}{3}\right)^n = 4\left[\left(\frac{4}{3}\right)^n - 1\right]$$

For $n = 1, P(n): \frac{4}{3} = 4\left[\frac{4}{3} - 1\right] = 4 \times \frac{1}{3} = \frac{4}{3}$, which is true.
Now, assume that P(*x*) is true for some positive integer *k*. This means

$$\frac{4}{3} + \left(\frac{4}{3}\right)^2 + \ldots\ldots + \left(\frac{4}{3}\right)^k = 4\left[\left(\frac{4}{3}\right)^k - 1\right] \qquad -(1)$$

We shall now prove that P ($k + 1$) is also true.

Now, we have

$$\left[\frac{4}{3}+\left(\frac{4}{3}\right)^{2}+\ldots\ldots+\left(\frac{4}{3}\right)^{k}\right]+\left(\frac{4}{3}\right)^{k+1}$$

$$=4\left[\left(\frac{4}{3}\right)^{k}-1\right]+\left(\frac{4}{3}\right)^{k+1}$$

$$=4\left(\frac{4}{3}\right)^{k}-4+\left(\frac{4}{3}\right)^{k}\times\frac{4}{3}$$

$$=\left(\frac{4}{3}\right)^{k}\times\left[4+\frac{4}{3}\right]-4$$

$$=\left(\frac{4}{3}\right)^{k}\times\frac{16}{3}-4$$

$$=\left(\frac{4}{3}\right)^{k}\times\frac{4}{3}\times 4-4$$

$$=\left(\frac{4}{3}\right)^{k+1}\times 4-4=4\left[\left(\frac{4}{3}\right)^{k+1}-1\right]$$

Thus, P $(k+1)$ is true whenever P (k) is true. Hence, from the principle of mathematical induction, the statement P (n) is true for all natural numbers n.