Chapter 8

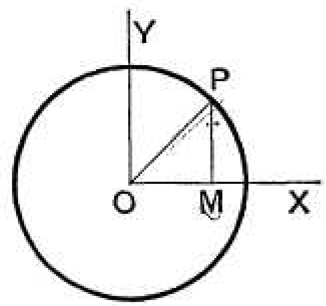
THE CIRCLE

138. Def. A circle is the locus of a point which moves so that its distance from a fixed point, called the centre, is equal to a given distance. The given distance is called the radius of the circle.

139. *To find the equation to a circle, the axes of coordinates being two straight lines through its centre at right angles.*

Let O be the centre of the circle and let a be its radius.

Let OX and OY be the axes of coordinates.



Let P be any point on the circumference of the circle, and let its coordinates be x and y.

Draw PM perpendicular to OX and join OP.

Then (Euc. I. 47)

$$OM^2 + MP^2 = a^2,$$

i.e.

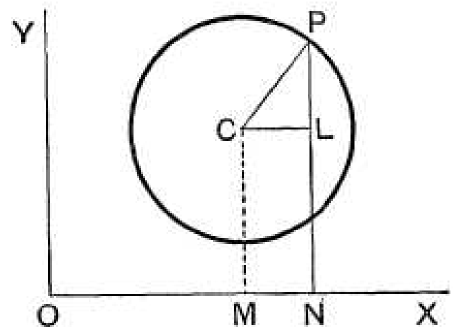
$$\mathbf{x^2 + y^2 = a^2}.$$

This being the relation which holds between the coordinates of any point on the circumference is, by Art. 42, the required equation.

140. *To find the equation to a circle referred to any rectangular axes.*

Let OX and OY be the two rectangular axes.

Let C be the centre of the circle and a its radius.



Take any point P on the circumference and draw perpendiculars CM and PN upon OX; let P be the point (x, y).

Draw CL perpendicular to NP.

Let the coordinates of C be h and k; these are supposed to be known.

We have $CL = MN = ON - OM = x - h$,

and $LP = NP - NL = NP - MC = y - k$.

Hence, since $CL^2 + LP^2 = CP^2$,

we have $$\mathbf{(x - h)^2 + (y - k)^2 = a^2} \quad \text{..............(1).}$$

This is the required equation.

Ex. The equation to the circle, whose centre is the point $(-3, 4)$ and whose radius is 7, is

$$(x+3)^2 + (y-4)^2 = 7^2,$$

i.e. $$x^2 + y^2 + 6x - 8y = 24.$$

141. Some particular cases of the preceding article may be noticed:

(α) Let the origin O be on the circle so that, in this case,

$$OM^2 + MC^2 = a^2,$$

i.e. $$h^2 + k^2 = a^2.$$

The equation (1) then becomes

$$(x-h)^2 + (y-k)^2 = h^2 + k^2,$$

i.e. $$x^2 + y^2 - 2hx - 2ky = 0.$$

(β) Let the origin be not on the curve, but let the centre lie on the axis of x. In this case $k = 0$, and the equation becomes

$$(x-h)^2 + y^2 = a^2.$$

(γ) Let the origin be on the curve and let the axis of x be a diameter. We now have $k = 0$ and $a = h$, so that the equation becomes

$$x^2 + y^2 - 2hx = 0.$$

(δ) By taking O at C, and thus making both h and k zero, we have the case of Art. 139.

(ϵ) The circle will touch the axis of x if MC be equal to the radius, *i.e.* if $k=a$.

The equation to a circle touching the axis of x is therefore

$$x^2+y^2-2hx-2ky+h^2=0.$$

Similarly, one touching the axis of y is

$$x^2+y^2-2hx-2ky+k^2=0.$$

142. *To prove that the equation*

$$x^2+y^2+2gx+2fy+c=0\ldots\ldots\ldots\ldots\ldots(1),$$

always represents a circle for all values of g, f, and c, and to find its centre and radius. [The axes are assumed to be rectangular.]

This equation may be written

$$(x^2+2gx+g^2)+(y^2+2fy+f^2)=g^2+f^2-c,$$

i.e. $$(x+g)^2+(y+f)^2=\{\sqrt{g^2+f^2-c}\}^2.$$

Comparing this with the equation (1) of Art. 140, we see that the equations are the same if

$$h=-g,\quad k=-f,\ \text{and}\ a=\sqrt{g^2+f^2-c}.$$

Hence (1) represents a circle whose centre is the point $(-g, -f)$, and whose radius is $\sqrt{g^2+f^2-c}$.

If $g^2+f^2>c$, the radius of this circle is real.

If $g^2+f^2=c$, the radius vanishes, *i.e.* the circle becomes a point coinciding with the point $(-g, -f)$. Such a circle is called a point-circle.

If $g^2+f^2<c$, the radius of the circle is imaginary. In this case the equation does not represent any real geometrical locus. It is better not to say that the circle does not exist, but to say that it is a circle with a real centre and an imaginary radius.

Ex. 1. The equation $x^2+y^2+4x-6y=0$ can be written in the form

$$(x+2)^2+(y-3)^2=13=(\sqrt{13})^2,$$

and therefore represents a circle whose centre is the point $(-2, 3)$ and whose radius is $\sqrt{13}$.

Ex. 2. The equation $45x^2+45y^2-60x+36y+19=0$ is equivalent to

$$x^2+y^2-\frac{4}{3}x+\frac{4}{5}y=-\frac{19}{45},$$

i. e.

$$\left(x-\frac{2}{3}\right)^2+\left(y+\frac{2}{5}\right)^2=\frac{4}{9}+\frac{4}{25}-\frac{19}{45}=\frac{41}{225},$$

and therefore represents a circle whose centre is the point $\left(\frac{2}{3}, -\frac{2}{5}\right)$ and whose radius is $\frac{\sqrt{41}}{15}$.

143. *Condition that the general equation of the second degree may represent a circle.*

The equation (1) of the preceding article, multiplied by any arbitrary constant, is a particular case of the general equation of the second degree (Art. 114) in which there is no term containing xy and in which the coefficients of x^2 and y^2 are equal.

The general equation of the second degree in rectangular coordinates therefore represents a circle **if the coefficients of x^2 and y^2 be the same and if the coefficient of xy be zero.**

144. The equation (1) of Art. 142 is called the **general equation of a circle,** since it can, by a proper choice of g, f, and c, be made to represent *any* circle.

The three constants g, f, and c in the general equation correspond to the geometrical fact that a circle can be found to satisfy three independent geometrical conditions and no more. Thus a circle is determined when three points on it are given, or when it is required to touch three straight lines.

145. *To find the equation to the circle which is described on the line joining the points (x_1, y_1) and (x_2, y_2) as diameter.*

Let A be the point (x_1, y_1) and B be the point (x_2, y_2), and let the coordinates of any point P on the circle be h and k.

The equation to AP is (Art. 62)

$$y-y_1=\frac{k-y_1}{h-x_1}(x-x_1) \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

and the equation to BP is

$$y-y_2=\frac{k-y_2}{h-x_2}(x-x_2) \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

But, since APB is a semicircle, the angle APB is a right angle, and hence the straight lines (1) and (2) are at right angles.

Hence, by Art. 69, we have

$$\frac{k-y_1}{h-x_1} \cdot \frac{k-y_2}{h-x_2} = -1,$$

i.e.

$$(h-x_1)(h-x_2)+(k-y_1)(k-y_2)=0.$$

But this is the condition that the point (h, k) may lie on the curve whose equation is

$$(x-x_1)(x-x_2)+(y-y_1)(y-y_2)=0.$$

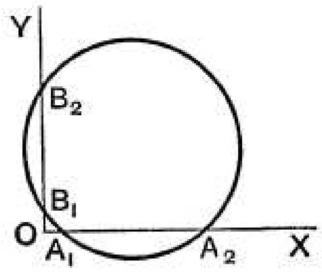
This therefore is the required equation.

146. *Intercepts made on the axes by the circle whose equation is*

$$ax^2+ay^2+2gx+2fy+c=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (1).$$

The abscissæ of the points where the circle (1) meets the axis of x, *i.e.* $y=0$, are given by the equation

$$ax^2+2gx+c=0 \quad \ldots\ldots\ldots\ldots (2).$$



The roots of this equation being x_1 and x_2, we have

$$x_1+x_2=-\frac{2g}{a},$$

and

$$x_1x_2=\frac{c}{a}. \qquad \text{(Art. 2.)}$$

Hence

$$A_1A_2=x_2-x_1=\sqrt{(x_1+x_2)^2-4x_1x_2}$$

$$=\sqrt{\frac{4g^2}{a^2}-\frac{4c}{a}}=2\frac{\sqrt{g^2-ac}}{a}.$$

Again, the roots of the equation (2) are both imaginary if $g^2<ac$. In this case the circle does not meet the axis of x in real points, *i.e.* geometrically it does not meet the axis of x at all.

The circle will touch the axis of x if the intercept A_1A_2 be just zero, *i.e.* if $g^2=ac$.

It will meet the axis of x in two points lying on opposite sides of the origin O if the two roots of the equation (2) are of opposite signs, *i.e.* if c be negative.

147. Ex. 1. *Find the equation to the circle which passes through the points* (1, 0), (0, −6), *and* (3, 4).

Let the equation to the circle be

$$x^2+y^2+2gx+2fy+c=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (1).$$

Since the three points, whose coordinates are given, satisfy this equation, we have

$$1+2g+c=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (2),$$

$$36-12f+c=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (3),$$

and

$$25+6g+8f+c=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (4).$$

Subtracting (2) from (3) and (3) from (4), we have

$$2g+12f=35,$$

and

$$6g+20f=11.$$

Hence $f=\frac{47}{8}$ and $g=-\frac{71}{4}$.

Equation (2) then gives $c=\frac{63}{2}$.

Substituting these values in (1) the required equation is

$$4x^2+4y^2-142x+47x+138=0.$$

Ex. 2. *Find the equation to the circle which touches the axis of y at a distance $+4$ from the origin and cuts off an intercept 6 from the axis of x.*

Any circle is $x^2+y^2+2gx+2fy+c=0.$

This meets the axis of y in points given by

$$y^2+2fy+c=0.$$

The roots of this equation must be equal and each equal to 4, so that it must be equivalent to $(y-4)^2=0$.

Hence $2f=-8$, and $c=16$.

The equation to the circle is then

$$x^2+y^2+2gx-8y+16=0.$$

This meets the axis of x in points given by

$$x^2+2gx+16=0,$$

i.e. at points distant

$$-g+\sqrt{g^2-16} \text{ and } -g-\sqrt{g^2-16}.$$

Hence

$$6=2\sqrt{g^2-16}.$$

Therefore $g=\pm 5$, and the required equation is

$$x^2+y^2\pm 10x-8y+16=0.$$

There are therefore two circles satisfying the given conditions. This is geometrically obvious.

EXAMPLES XVII

Find the equation to the circle

1. Whose radius is 3 and whose centre is $(-1, 2)$.
2. Whose radius is 10 and whose centre is $(-5, -6)$.
3. Whose radius is $a+b$ and whose centre is $(a, -b)$.
4. Whose radius is $\sqrt{a^2-b^2}$ and whose centre is $(-a, -b)$.

Find the coordinates of the centres and the radii of the circles whose equations are

5. $x^2+y^2-4x-8y=41$.
6. $3x^2+3y^2-5x-6y+4=0$.

7. $x^2+y^2=k(x+k)$. 8. $x^2+y^2=2gx-2fy$.

9. $\sqrt{1+m^2}(x^2+y^2)-2cx-2mcy=0$.

Draw the circles whose equations are

10. $x^2+y^2=2ay$. 11. $3x^2+3y^2=4x$.

12. $5x^2+5y^2=2x+3y$.

13. Find the equation to the circle which passes through the points $(1, -2)$ and $(4, -3)$ and which has its centre on the straight line $3x+4y=7$.

14. Find the equation to the circle passing through the points $(0, a)$ and (b, h), and having its centre on the axis of x.

Find the equations to the circles which pass through the points

15. $(0, 0)$, $(a, 0)$, and $(0, b)$. 16. $(1, 2)$, $(3, -4)$, and $(5, -6)$.

17. $(1, 1)$, $(2, -1)$, and $(3, 2)$. 18. $(5, 7)$, $(8, 1)$, and $(1, 3)$.

19. (a, b), $(a, -b)$, and $(a+b, a-b)$.

20. $ABCD$ is a square whose side is a; taking AB and AD as axes, prove that the equation to the circle circumscribing the square is

$$x^2+y^2=a(x+y).$$

21. Find the equation to the circle which passes through the origin and cuts off intercepts equal to 3 and 4 from the axes.

22. Find the equation to the circle passing through the origin and the points (a, b) and (b, a). Find the lengths of the chords that it cuts off from the axes.

23. Find the equation to the circle which goes through the origin and cuts off intercepts equal to h and k from the positive parts of the axes.

24. Find the equation to the circle, of radius a, which passes through the two points on the axis of x which are at a distance b from the origin.

Find the equation to the circle which

25. touches each axis at a distance 5 from the origin.

26. touches each axis and is of radius a.

27. touches both axes and passes through the point $(-2, -3)$.

28. touches the axis of x and passes through the two points $(1, -2)$ and $(3, -4)$.

29. touches the axis of y at the origin and passes through the point (b, c).

30. touches the axis of x at a distance 3 from the origin and intercepts a distance 6 on the axis of y.

31. Points (1, 0) and (2, 0) are taken on the axis of x, the axes being rectangular. On the line joining these points an equilateral triangle is described, its vertex being in the positive quadrant. Find the equations to the circles described on its sides as diameters.

32. If $y=mx$ be the equation of a chord of a circle whose radius is a, the origin of coordinates being one extremity of the chord and the axis of x being a diameter of the circle, prove that the equation of a circle of which this chord is the diameter is

$$(1+m^2)(x^2+y^2)-2a(x+my)=0.$$

33. Find the equation to the circle passing through the points (12, 43), (18, 39), and (42, 3) and prove that it also passes through the points $(-54, -69)$ and $(-81, -38)$.

34. Find the equation to the circle circumscribing the quadrilateral formed by the straight lines

$$2x+3y=2, \quad 3x-2y=4, \quad x+2y=3, \quad \text{and} \quad 2x-y=3.$$

35. Prove that the equation to the circle of which the points (x_1, y_1) and (x_2, y_2) are the ends of a chord of a segment containing an angle θ is

$$(x-x_1)(x-x_2)+(y-y_1)(y-y_2) \pm \cot\theta\,[(x-x_1)(y-y_2)-(x-x_2)(y-y_1)]=0.$$

36. Find the equations to the circles in which the line joining the points (a, b) and $(b, -a)$ is a chord subtending an angle of 45° at any point on its circumference.

ANSWERS

1. $x^2+y^2+2x-4y=4$. **2.** $x^2+y^2+10x+12y=39$.
3. $x^2+y^2-2ax+2by=2ab$. **4.** $x^2+y^2+2ax+2by+2b^2=0$.
5. $(2, 4)$; $\sqrt{61}$. **6.** $(\frac{5}{6}, 1)$; $\frac{1}{6}\sqrt{13}$. **7.** $\left(\frac{k}{2}, 0\right)$; $\frac{\sqrt{5}}{2}k$.
8. $(g, -f)$; $\sqrt{f^2+g^2}$. **9.** $\left(\frac{c}{\sqrt{1+m^2}}, \frac{mc}{\sqrt{1+m^2}}\right)$; c.
13. $15x^2+15y^2-94x+18y+55=0$.
14. $b(x^2+y^2-a^2)=x(b^2+h^2-a^2)$. **15.** $x^2+y^2-ax-by=0$.
16. $x^2+y^2-22x-4y+25=0$. **17.** $x^2+y^2-5x-y+4=0$.
18. $3x^2+3y^2-29x-19y+56=0$.
19. $b(x^2+y^2)-(a^2+b^2)x+(a-b)(a^2+b^2)=0$.
21. $x^2+y^2-3x-4y=0$.

22. $x^2+y^2-\frac{a^2+b^2}{a+b}(x+y)=0$; $\frac{a^2+b^2}{a+b}$.

23. $x^2+y^2-hx-ky=0$.

24. $x^2+y^2\pm 2y\sqrt{a^2-b^2}=b^2$.

25. $x^2+y^2-10x-10y+25=0$.

26. $x^2+y^2-2ax-2ay+a^2=0$.

27. $x^2+y^2+2(5\pm\sqrt{12})(x+y)+37\pm 10\sqrt{12}=0$.

28. $x^2+y^2-6x+4y+9=0$.

29. $b(x^2+y^2)=x(b^2+c^2)$.

30. $x^2+y^2\pm 6\sqrt{2}y-6x+9=0$.

31. $x^2+y^2-3x+2=0$; $2x^2+2y^2-5x-\sqrt{3}y+3=0$;

$$2x^2+2y^2-7x-\sqrt{3}y+6=0.$$

33. $(x+21)^2+(y+13)^2=65^2$.

34. $8x^2+8y^2-25x-3y+18=0$.

36. $x^2+y^2=a^2+b^2$; $x^2+y^2-2(a+b)x+2(a-b)y+a^2+b^2=0$.

SOLUTIONS/HINTS

1—4. Substitute in the equation of Art. 140.

5. The equation may be written

$$x^2-4x+4+y^2-8y+16=4+16+41,$$

or

$$(x-2)^2+(y-4)^2=61.$$

6. The equation may be written

$$x^2-\tfrac{5}{3}x+(\tfrac{5}{6})^2+y^2-2y+1=1+(\tfrac{5}{6})^2-\tfrac{4}{3},$$

or

$$(x-\tfrac{5}{6})^2+(y-1)^2=\tfrac{13}{36}.$$

7. The equation may be written

$$x^2-kx+\left(\frac{k}{2}\right)^2+y^2=k^2+\frac{k^2}{4},\ \text{or}\ \left(x-\frac{k}{2}\right)^2+y^2=\frac{5k^2}{4}.$$

8. The equation may be written

$$x^2-2gx+g^2+y^2+2fy+f^2=g^2+f^2,$$

i.e.

$$(x-g)^2+(y+f)^2=g^2+f^2.$$

9. The equation may be written

$$x^2-\frac{2c}{\sqrt{1+m^2}}\cdot x+\frac{c^2}{1+m^2}+y^2-\frac{2cm}{\sqrt{1+m^2}}\cdot y+\frac{c^2m^2}{1+m^2}$$
$$=\frac{c^2}{1+m^2}+\frac{c^2m^2}{1+m^2}=c^2,$$

i.e. $$\left\{x-\frac{c}{\sqrt{1+m^2}}\right\}^2+\left\{y-\frac{cm}{\sqrt{1+m^2}}\right\}^2=c^2.$$

13. Since the centre (h, k) is equally distant from the two given points, $\therefore (h-1)^2+(k+2)^2=(h-4)^2+(k+3)^2$.

$\therefore 3h-k=10$; also $3h+4k=7$; whence $h=\frac{47}{15}$, $k=-\frac{3}{5}$.

Hence the equation is of the form $(x-\frac{47}{15})^2+(y+\frac{3}{5})^3=$ a constant, or $15x^2+15y^2-94x+18y+c=0$.

Put $x=1$, $y=-2$; $\therefore 15+60-94-36+c=0$,

whence $$c=55.$$

14. Let its equation be $x^2+y^2+2gx+c=0$.

Since it passes through $(0, a)$ and (b, h),

$$\therefore a^2+c=0, \text{ and } b^2+h^2+2bg+c=0.$$

$$\therefore c=-a^2 \text{ and } 2g=\frac{a^2-b^2-h^2}{b}.$$

$\therefore$ the required equation is $bx^2+by^2+(a^2-b^2-h^2)x-a^2b=0$.

15. Let $x^2+y^2+2gx+2fy+c=0$ be the equation of the required circle. Since it is satisfied by $(0, 0)$; $(a, 0)$; $(0, b)$, we have $c=0$, $a^2+2ag=0$, and $b^2+2bf=0$. $\therefore 2g=-a$ and $2f=-b$, and the equation becomes $x^2+y^2-ax-by=0$.

16. Taking the same equation, we have

$$1+4+2g+4f+c=0,\ 9+16+6g-8f+c=0,$$

and $$25+36+10g-12f+c=0.$$

Whence, solving, $2g=-22$, $2f=-4$, $c=25$.

17. Taking the same equation, we have

$$2+2g+2f+c=0,\ 5+4y-2f+c=0,$$

and $$13+6g+4f+c=0.$$

Whence, solving, $2g=-5$, $2f=-1$, $c=4$.

18. Taking the same equation, we have

$$74+10g+14f+c=0,\ 65+16g+2f+c=0,$$

and $$10+2g+6f+c=0.$$

Whence, solving, $2g=-\frac{29}{3}$, $2f=-\frac{19}{3}$, $c=\frac{56}{3}$.

19. Since the circle passes through (a, b) and $(a, -b)$, its centre must lie on the axis of x. Let $(h, 0)$ be the centre

$$\therefore\ (h-a)^2 + b^2 = (h-a-b)^2 + (a-b)^2;$$

$$\therefore\ b(2h-2a-b) = a^2 - 2ab,\ i.e.\ 2h = \frac{a^2+b^2}{b}.$$

Hence this equation is of the form

$$bx^2 + by^2 - (a^2+b^2)x + C = 0.$$

Put $\quad x = a,\ y = b;\ \therefore\ C = (a-b)(a^2+b^2).$

20. The circle passes through the points $(0, 0)$; $(a, 0)$; $(0, a)$. Put $b = a$ in No. 15. The circle clearly passes through (a, a).

Aliter. The circle has its centre at the point $\left(\frac{a}{2}, \frac{a}{2}\right)$ and its radius equal to $\frac{a}{2}\sqrt{2}$.

Hence its equation is $\left(x - \frac{a}{2}\right)^2 + \left(y - \frac{a}{2}\right)^2 = \left(\frac{a}{2}\sqrt{2}\right)^2$, etc.

21. Put $a = 3$, $b = 4$ in No. 15.

22. Let $x^2 + y^2 + 2gx + 2fy = 0$ be the equation.

Then $a^2 + b^2 + 2ag + 2fb = 0$, and $b^2 + a^2 + 2bg + 2af = 0$.

Whence $2g = 2f = -\frac{a^2+b^2}{a+b}$, and the equation becomes

$$x^2 + y^2 - \frac{a^2+b^2}{a+b}(x+y) = 0.$$

To find the lengths of the chords cut off from the axes, put $x = 0$ and $y = 0$.

23. Put $a = h$, and $b = k$ in Ex. 15.

24. Let $(0, h)$ be the centre. Then

$$h^2 + b^2 = a^2. \quad \therefore\ h = \pm\sqrt{a^2 - b^2},$$

and the equation is

$$x^2 + (y-h)^2 = a^2,\ \text{or}\ x^2 + y^2 \pm 2\sqrt{a^2-b^2}\,y = b^2.$$

25. The circle cuts the axis of x in points given by $x^2+2gx+c=0$, which must be equivalent to $(x-5)^2=0$. $\therefore\ 2g=-10$, and $c=25$. Similarly $2f=-10$.

$\therefore$ the required equation is $x^2+y^2-10x-10y+25=0$.

26. As in the last example, both $x^2+2gx+c$ and $y^2+2fy+c$ must be perfect squares.

$\therefore\ c=f^2=g^2$. Also $a^2=f^2+g^2-c$. (Art. 142.)

$$\therefore\ a^2=f^2=g^2=c. \quad \therefore\ f=g=\pm a,$$

and the equation becomes $x^2+y^2\pm 2ax\pm 2ay+a^2=0$.

27. As in the last Ex., $c=f^2=g^2$, and the equation becomes $\quad x^2+y^2+2\sqrt{c}\,(x+y)+c=0$.

Since it passes through $(-2,-3)$, $\therefore\ c-10\sqrt{c}+13=0$.

$$\therefore\ \sqrt{c}=5\pm\sqrt{12}. \quad \therefore\ c=37\pm 10\sqrt{12}.$$

28. As in No. 26, $g=\sqrt{c}$.

Since it passes through $(1,-2)$ and $(3,-4)$, we have

$$5+2\sqrt{c}-4f+c=0, \text{ and } 25+6\sqrt{c}-8f+c=0.$$

$\therefore\ c-2\sqrt{c}-15=0$. $\therefore\ \sqrt{c}=5$ or -3, whence $f=10$ or 2, and the two circles are

$$x^2+y^2+10x+20y+25=0, \text{ and } x^2+y^2-6x+4y+9=0.$$

29. Since it touches the axis of y at the origin, $\therefore\ y^2+2fy+c=0$ must be equivalent to $y^2=0$. $\therefore\ f=0$ and $c=0$. Also $x^2+y^2+2gx=0$ passes through (b, c) if

$$b^2+c^2+2gb=0; \quad \therefore\ 2g=-\frac{b^2+c^2}{b}.$$

$\therefore$ the required equation is $b(x^2+y^2)-(b^2+c^2)x=0$.

30. As in No. 25, $g=-3$ and $c=9$. Also the difference of the roots of the equation $y^2+2fy+9=0$ is 6.

Now $y_1+y_2=-2f$, and $y_1y_2=9$, so that

$$36=(y_1-y_2)^2=4f^2-36.$$

$\therefore\ f^2=18$, *i.e.* $f=\pm 3\sqrt{2}$, and the equation becomes

$$x^2+y^2-6x\pm 6\sqrt{2}\,y+9=0.$$

31. Let (a, b) be the vertex of the triangle so that b is positive.

Then $(a-1)^2+b^2=(a-2)^2+b^2=1^2$, whence $a=\frac{3}{2}$, and $b=\frac{\sqrt{3}}{2}$, and the three circles are (Art. 145)

$$(x-1)(x-2)+y^2=0, \quad (x-1)\left(x-\frac{3}{2}\right)+y\left(y-\frac{\sqrt{3}}{2}\right)=0,$$

and $$(x-2)\left(x-\frac{3}{2}\right)+y\left(y-\frac{\sqrt{3}}{2}\right)=0.$$

32. The given circle has its centre at $(a, 0)$ and is of radius a; therefore its equation is $x^2+y^2=2ax$.

It cuts the line $y=mx$ again where

$$x=\frac{2a}{1+m^2}, \quad y=\frac{2am}{1+m^2},$$

and the circle whose diameter is the line joining this point to the origin is (Art. 145)

$$x\left(x-\frac{2a}{1+m^2}\right)+y\left(y-\frac{2am}{1+m^2}\right)=0.$$

33. As in Art. 147, Ex. 1, the equations for g, f and c are

$$24g+86f+c=-1993, \quad \text{.................(i)}$$
$$36g+78f+c=-1845, \quad \text{................(ii)}$$
and $$84g+6f+c=-1773. \quad \text{..............(iii)}$$

Subtracting (i) from (ii), we have $3g-2f=37$, and subtracting (ii) from (iii), we have $2g-3f=3$. Whence $g=21, f=13$.

$\therefore$ the equation is of the form $(x+21)^2+(y+13)^2=r^2$.

Put $x=18$, $y=39$.

$$\therefore \quad r^2=39^2+52^2=13^2\{3^2+4^2\}=13^2 . 5^2=65^2.$$

The other two points satisfy the equation.

34. The lines (i) and (ii) are at right angles and also the lines (iii) and (iv).

From a figure it is easily seen that the intersection of (i) and (iv), and also that of (ii) and (iii), are at the ends of a diameter.

Solving, these points are $(\frac{11}{8}, -\frac{1}{4})$, and $(\frac{7}{4}, \frac{5}{8})$.

Hence the required equation is, (Art. 145),

$$(8x-11)(4x-7)+(8y-5)(4y+1)=0,$$

i.e. $$8x^2+8y^2-25x-3y+18=0.$$

35. Let (h, k) be any point on the circle.

If the chord joining (h, k) and (x_1, y_1) make an angle α with the axis of x, then $\tan\alpha = \frac{(k-y_1)}{h-x_1}$.

Similarly, if the chord joining (h, k) and (x_2, y_2) is inclined at β, $\tan\beta = \frac{k-y_2}{h-x_2}$.

But $$\theta = \alpha \sim \beta.$$

$$\therefore \pm\tan\theta = \frac{\frac{k-y_1}{h-x_1} - \frac{k-y_2}{h-x_2}}{1+\frac{k-y_1}{h-x_1}\cdot\frac{k-y_2}{h-x_2}},$$

which reduces to the required equation on substituting x for h and y for k.

36. In the last example put $x_1=a$, $y_1=b$, $x_2=b$, $y_2=-a$, and $\theta=45°$, and we obtain

$$(x-a)(x-b)+(y-b)(y+a)=\pm\{(x-a)(y+a)-(x-b)(y-b)\},$$

or $$x^2+y^2-2x(a+b)+2y(a-b)+a^2+b^2=0,$$

and $$x^2+y^2=a^2+b^2.$$

TANGENT

148. Tangent. The tangent at any point of a circle, and proves that it is always perpendicular to the radius drawn from the centre to the point of contact.

From this property may be deduced the equation to the tangent at any point (x', y') of the circle $x^2 + y^2 = a^2$.

For let the point P (Fig. Art. 139) be the point (x', y').

The equation to any straight line passing through P is, by Art. 62,

$$y - y' = m(x - x') \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (1).$$

Also the equation to OP is

$$y = \frac{y'}{x'}x \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (2).$$

The straight lines (1) and (2) are at right angles, *i.e.* the line (1) is a tangent, if

$$m \times \frac{y'}{x'} = -1, \qquad \text{(Art. 69)}$$

i.e. if

$$m = -\frac{x'}{y'}.$$

Substituting this value of m in (1), the equation of the tangent at (x', y') is

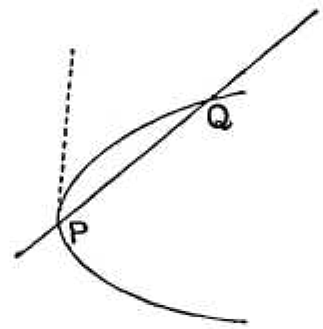
$$y - y' = -\frac{x'}{y'}(x - x'),$$

i.e. $$xx' + yy' = x'^2 + y'^2 \qquad (3).$$

But, since (x', y') lies on the circle, we have $x'^2 + y'^2 = a^2$, and the required equation is then

$$\mathbf{xx' + yy' = a^2}.$$

Tangent. Def. Let P and Q be any two points, near to one another, on any curve.



Join PQ; then PQ is called a secant.

The position of the line PQ when the point Q is taken indefinitely close to, and ultimately coincident with, the point P is called the tangent at P.

The student may better appreciate this definition, if he conceive the curve to be made up of a succession of very small points (much smaller than could be made by the finest conceivable drawing pen) packed close to one another along the curve. The tangent at P is then the straight line joining P and the next of these small points.

150. *To find the equation of the tangent at the point (x', y') of the circle $x^2 + y^2 = a^2$.*

Let P be the given point and Q a point (x'', y'') lying on the curve and close to P.

The equation to PQ is then

$$y - y' = \frac{y'' - y'}{x'' - x'}(x - x') \text{ .... (1).}$$

Since both (x', y') and (x'', y'') lie on the circle, we have

$$x'^2 + y'^2 = a^2,$$

and $$x''^2 + y''^2 = a^2.$$

By subtraction, we have

$$x''^2 - x'^2 + y''^2 - y'^2 = 0,$$

i.e. $$(x'' - x')(x'' + x') + (y'' - y')(y'' + y') = 0,$$

i.e. $$\frac{y'' - y'}{x'' - x'} = -\frac{x'' + x'}{y'' + y'}.$$

Substituting this value in (1), the equation to PQ is

$$y - y' = -\frac{x'' + x'}{y'' + y'}(x - x') \text{ .... (2).}$$

Now let Q be taken very close to P, so that it ultimately coincides with P, *i.e.* put $x'' = x'$ and $y'' = y'$.

Then (2) becomes

$$y - y' = -\frac{2x'}{2y'}(x - x'),$$

i.e. $$yy' + xx' = x'^2 + y'^2 = a^2.$$

The required equation is therefore

$$\mathbf{xx' + yy' = a^2} \text{ .... (3).}$$

It will be noted that the equation to the tangent found in this article coincides with the equation found from Euclid's definition in Art. 148.

Our definition of a tangent and Euclid's definition therefore give the same straight line in the case of a circle.

151. *To obtain the equation of the tangent at any point (x', y') lying on the circle*

$$x^2 + y^2 + 2gx + 2fy + c = 0.$$

Let P be the given point and Q a point (x'', y'') lying on the curve close to P.

The equation to PQ is therefore

$$y - y' = \frac{y'' - y'}{x'' - x'}(x - x') \quad \ldots\ldots\ldots\ldots\ldots (1).$$

Since both (x', y') and (x'', y'') lie on the circle, we have

$$x'^2 + y'^2 + 2gx' + 2fy' + c = 0 \quad \ldots\ldots\ldots\ldots (2),$$

and
$$x''^2 + y''^2 + 2gx'' + 2fy'' + c = 0 \quad \ldots\ldots\ldots (3).$$

By subtraction, we have

$$x''^2 - x'^2 + y''^2 - y'^2 + 2g(x'' - x') + 2f(y'' - y') = 0,$$

i.e. $$(x'' - x')(x'' + x' + 2g) + (y'' - y')(y'' + y' + 2f) = 0,$$

i.e. $$\frac{y'' - y'}{x'' - x'} = -\frac{x'' + x' + 2g}{y'' + y' + 2f}.$$

Substituting this value in (1), the equation to PQ becomes

$$y - y' = -\frac{x'' + x' + 2g}{y'' + y' + 2f}(x - x') \quad \ldots\ldots\ldots\ldots (4).$$

Now let Q be taken very close to P, so that it ultimately coincides with P, *i.e.* put $x'' = x'$ and $y'' = y'$.

The equation (4) then becomes

$$y - y' = -\frac{x' + g}{y' + f}(x - x'),$$

i.e. $$y(y' + f) + x(x' + g) = y'(y' + f) + x'(x' + g)$$
$$= x'^2 + y'^2 + gx' + fy'$$
$$= -gx' - fy' - c,$$

by (2).

This may be written

$$\mathbf{xx' + yy' + g(x + x') + f(y + y') + c = 0}$$

which is the required equation.

152. The equation to the tangent at (x', y') is therefore obtained from that of the circle itself by substituting xx' for x^2, yy' for y^2, $x + x'$ for $2x$, and $y + y'$ for $2y$.

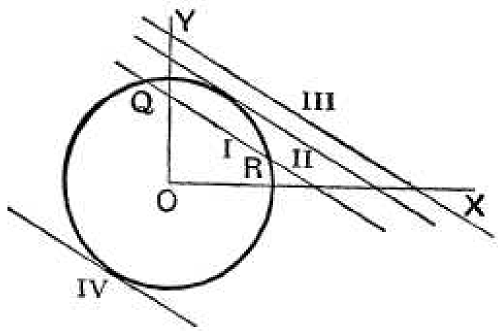
This is a particular case of a general rule which will be found to enable us to write down at sight the equation to the tangent at (x', y') to any of the curves with which we shall deal in this book.

153. *Points of intersection, in general, of the straight line*

$$y = mx + c \qquad \ldots\ldots(1),$$

with the circle

$$x^2 + y^2 = a^2 \qquad \ldots\ldots(2).$$



The coordinates of the points in which the straight line (1) meets (2) satisfy both equations (1) and (2).

If therefore we solve them as simultaneous equations we shall obtain the coordinates of the common point or points.

Substituting for y from (1) in (2), the abscissæ of the required points are given by the equation

$$x^2 + (mx + c)^2 = a^2,$$

i.e.

$$x^2(1 + m^2) + 2mcx + c^2 - a^2 = 0 \quad \ldots\ldots (3).$$

The roots of this equation are, by Art. 1, real, coincident, or imaginary, according as

$(2mc)^2 - 4(1 + m^2)(c^2 - a^2)$ is positive, zero, or negative,

i.e. according as

$a^2(1 + m^2) - c^2$ is positive, zero, or negative,

i.e. according as

c^2 is $<=$ or $> a^2(1 + m^2)$.

In the figure the lines marked I, II, and III are all parallel, *i.e.* their equations all have the same "m."

The straight line I corresponds to a value of c^2 which is $< a^2(1+m^2)$ and it meets the circle in two real points.

The straight line III which corresponds to a value of c^2, $> a^2(1+m^2)$, does not meet the circle at all, or rather, as in Art. 108, this is better expressed by saying that it meets the circle in imaginary points.

The straight line II corresponds to a value of c^2, which is equal to $a^2(1+m^2)$, and meets the curve in two coincident points, *i.e.* is a tangent.

154. We can now obtain the length of the chord intercepted by the circle on the straight line (1). For, if x_1 and x_2 be the roots of the equation (3), we have

$$x_1 + x_2 = -\frac{2mc}{1+m^2}, \text{ and } x_1 x_2 = \frac{c^2 - a^2}{1+m^2}.$$

Hence

$$x_1 - x_2 = \sqrt{(x_1+x_2)^2 - 4x_1x_2} = \frac{2}{1+m^2}\sqrt{m^2c^2 - (c^2-a^2)(1+m^2)}$$
$$= \frac{2}{1+m^2}\sqrt{a^2(1+m^2) - c^2}.$$

If y_1 and y_2 be the ordinates of Q and R we have, since these points are on (1),

$$y_1 - y_2 = (mx_1 + c) - (mx_2 + c) = m(x_1 - x_2).$$

Hence $$QR = \sqrt{(y_1-y_2)^2 + (x_1-x_2)^2} = \sqrt{1+m^2}\,(x_1 - x_2)$$
$$= 2\sqrt{\frac{a^2(1+m^2) - c^2}{1+m^2}}$$

In a similar manner we can consider the points of intersection of the straight line $y = mx + k$ with the circle

$$x^2 + y^2 + 2gx + 2fy + c = 0.$$

155. *The straight line*

$$y = mx + a\sqrt{1+m^2}$$

is always a tangent to the circle

$$x^2 + y^2 = a^2.$$

As in Art. 153 the straight line

$$y = mx + c$$

meets the circle in two points which are coincident if

$$c = a\sqrt{1 + m^2}.$$

But if a straight line meets the circle in two points which are indefinitely close to one another then, by Art. 149, it is a tangent to the circle.

The straight line $y = mx + c$ is therefore a tangent to the circle if

$$c = a\sqrt{1 + m^2},$$

i.e. the equation to any tangent to the circle is

$$\mathbf{y = mx + a\sqrt{1 + m^2}} \text{ .... (1).}$$

Since the radical on the right hand may have the + or − sign prefixed we see that corresponding to any value of m there are two tangents. They are marked II and IV in the figure of Art. 153.

156. The above result may also be deduced from the equation (3) of Art. 150, which may be written

$$y = -\frac{x'}{y'}x + \frac{a^2}{y'} \text{..............................(1).}$$

Put $-\frac{x'}{y'} = m$, so that $x' = -my'$, and the relation $x'^2 + y'^2 = a^2$ gives

$$y'^2(m^2 + 1) = a^2, \ i.e.\ \frac{a}{y'} = \sqrt{1 + m^2}.$$

The equation (1) then becomes

$$y = mx + a\sqrt{1 + m^2}.$$

This is therefore the tangent at the point whose coordinates are

$$\frac{-ma}{\sqrt{1 + m^2}} \text{ and } \frac{a}{\sqrt{1 + m^2}}.$$

157. If we assume that a tangent to a circle is always perpendicular to the radius vector to the point of contact, the result of Art. 155 may be obtained in another manner.

For a tangent is a line whose perpendicular distance from the centre is equal to the radius.

The straight line $y=mx+c$ will therefore touch the circle if the perpendicular on it from the origin be equal to a, *i.e.* if

$$\frac{c}{\sqrt{1+m^2}}=a,$$

i.e. if $$c=a\sqrt{1+m^2}.$$

This method is not however applicable to any other curve besides the circle.

158. Ex. *Find the equations to the tangents to the circle*

$$x^2+y^2-6x+4y=12$$

which are parallel to the straight line

$$4x+3y+5=0.$$

Any straight line parallel to the given one is

$$4x+3y+C=0 \quad \text{..............................(1)}.$$

The equation to the circle is

$$(x-3)^2+(y+2)^2=5^2.$$

The straight line (1), if it be a tangent, must be therefore such that its distance from the point $(3, -2)$ is equal to ± 5.

Hence $$\frac{12-6+C}{\sqrt{4^2+3^2}}=\pm 5, \qquad \text{(Art. 75)},$$

so that $$C=-6\pm 25=19 \text{ or } -31.$$

The required tangents are therefore

$$4x+3y+19=0 \text{ and } 4x+3y-31=0.$$

159. Normal. Def. The normal at any point P of a curve is the straight line which passes through P and is perpendicular to the tangent at P.

To find the equation to the normal at the point (x', y') *of* (1) *the circle*

$$x^2+y^2=a^2,$$

and (2) *the circle*

$$x^2+y^2+2gx+2fy+c=0.$$

(1) The tangent at (x', y') is

$$xx'+yy'=a^2,$$

i.e. $$y=-\frac{x'}{y'}x+\frac{a^2}{y'}.$$

The equation to the straight line passing through (x', y') perpendicular to this tangent is

$$y - y' = m(x - x'),$$

where $$m \times \left(-\frac{x'}{y'}\right) = -1, \qquad \text{(Art. 69)},$$

i.e. $$m = \frac{y'}{x'}.$$

The required equation is therefore

$$y - y' = \frac{y'}{x'}(x - x'),$$

i.e. $$x'y - xy' = 0.$$

This straight line passes through the centre of the circle which is the point (0, 0).

If we assume Euclid's propositions the equation is at once written down, since the normal is the straight line joining (0, 0) to (x', y').

(2) The equation to the tangent at (x', y') to the circle

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

is $$y = -\frac{x' + g}{y' + f}x - \frac{gx' + fy' + c}{y' + f}. \qquad \text{(Art. 151.)}$$

The equation to the straight line, passing through the point (x', y') and perpendicular to this tangent, is

$$y - y' = m(x - x'),$$

where $$m \times \left(-\frac{x' + g}{y' + f}\right) = -1, \qquad \text{(Art. 69)},$$

i.e. $$m = \frac{y' + f}{x' + g}.$$

The equation to the normal is therefore

$$y - y' = \frac{y' + f}{x' + g}(x - x'),$$

i.e. $$y(x' + g) - x(y' + f) + fx' - gy' = 0.$$

EXAMPLES XVIII

Write down the equation of the tangent to the circle

1. $x^2+y^2-3x+10y=15$ at the point $(4, -11)$.

2. $4x^2+4y^2-16x+24y=117$ at the point $(-4, -\frac{11}{2})$.

Find the equations to the tangents to the circle

3. $x^2+y^2=4$ which are parallel to the line $x+2y+3=0$.

4. $x^2+y^2+2gx+2fy+c=0$ which are parallel to the line

$$x+2y-6=0.$$

5. Prove that the straight line $y=x+c\sqrt{2}$ touches the circle $x^2+y^2=c^2$, and find its point of contact.

6. Find the condition that the straight line $cx-by+b^2=0$ may touch the circle $x^2+y^2=ax+by$ and find the point of contact.

7. Find whether the straight line $x+y=2+\sqrt{2}$ touches the circle

$$x^2+y^2-2x-2y+1=0.$$

8. Find the condition that the straight line $3x+4y=k$ may touch the circle $x^2+y^2=10x$.

9. Find the value of p so that the straight line

$$x\cos\alpha+y\sin\alpha-p=0$$

may touch the circle

$$x^2+y^2-2ax\cos\alpha-2by\sin\alpha-a^2\sin^2\alpha=0.$$

10. Find the condition that the straight line $Ax+By+C=0$ may touch the circle

$$(x-a)^2+(y-b)^2=c^2.$$

11. Find the equation to the tangent to the circle $x^2+y^2=a^2$ which

(i) is parallel to the straight line $y=mx+c$,

(ii) is perpendicular to the straight line $y=mx+c$,

(iii) passes through the point $(b, 0)$,

and (iv) makes with the axes a triangle whose area is a^2.

12. Find the length of the chord joining the points in which the straight line

$$\frac{x}{a}+\frac{y}{b}=1$$

meets the circle $x^2+y^2=r^2$.

13. Find the equation to the circles which pass through the origin and cut off equal chords a from the straight lines $y=x$ and $y=-x$.

14. Find the equation to the straight lines joining the origin to the points in which the straight line $y=mx+c$ cuts the circle

$$x^2+y^2=2ax+2by.$$

Hence find the condition that these points may subtend a right angle at the origin.

Find also the condition that the straight line may touch the circle.

Find the equation to the circle which

15. has its centre at the point (3, 4) and touches the straight line

$$5x+12y=1.$$

16. touches the axes of coordinates and also the line

$$\frac{x}{a}+\frac{y}{b}=1,$$

the centre being in the positive quadrant.

17. has its centre at the point $(1, -3)$ and touches the straight line $2x-y-4=0$.

18. Find the general equation of a circle referred to two perpendicular tangents as axes.

19. Find the equation to a circle of radius r which touches the axis of y at a point distant h from the origin, the centre of the circle being in the positive quadrant.

Prove also that the equation to the other tangent which passes through the origin is

$$(r^2-h^2)x+2rhy=0.$$

20. Find the equation to the circle whose centre is at the point (α, β) and which passes through the origin, and prove that the equation of the tangent at the origin is

$$\alpha x+\beta y=0.$$

21. Two circles are drawn through the points $(a, 5a)$ and $(4a, a)$ to touch the axis of y. Prove that they intersect at an angle $\tan^{-1}\frac{40}{9}$.

22. A circle passes through the points $(-1, 1)$, $(0, 6)$, and $(5, 5)$. Find the points on this circle the tangents at which are parallel to the straight line joining the origin to its centre.

ANSWERS

1. $5x-12y=152$.

2. $24x+10y+151=0$.

3. $x+2y=\pm 2\sqrt{5}$.

4. $x+2y+g+2f=\pm\sqrt{5}\sqrt{g^2+f^2-c}$.

5. $\left(-\frac{c}{\sqrt{2}}, \frac{c}{\sqrt{2}}\right)$. 6. $c=a$; $(0, b)$. 7. Yes.

8. $k=40$ or -10. 9. $a\cos^2\alpha + b\sin^2\alpha \pm \sqrt{a^2+b^2\sin^2\alpha}$.

10. $Aa+Bb+C=\pm c\sqrt{A^2+B^2}$.

11. (1) $y=mx\pm a\sqrt{1+m^2}$; (2) $my+x=\pm a\sqrt{1+m^2}$;
(3) $ax\pm y\sqrt{b^2-a^2}=ab$; (4) $x+y=a\sqrt{2}$.

12. $2\sqrt{r^2-\frac{a^2b^2}{a^2+b^2}}$. 13. $x^2+y^2\pm\sqrt{2}ax=0$; $x^2+y^2\pm\sqrt{2}ay=0$.

14. $c=b-am$; $c=b-am\pm\sqrt{(1+m^2)(a^2+b^2)}$.

15. $x^2+y^2-6x-8y+\frac{381}{169}=0$.

16. $x^2+y^2-2cx-2cy+c^2=0$, where $2c=a+b\pm\sqrt{a^2+b^2}$.

17. $5x^2+5y^2-10x+30y+49=0$. 18. $x^2+y^2-2cx-2cy+c^2=0$.

19. $(x-r)^2+(y-h)^2=r^2$. 20. $x^2+y^2-2\alpha x-2\beta y=0$.

SOLUTIONS/HINTS

1 and **2.** Use the formula of Art. 151.

3. Put $a=2$ and $m=-\frac{1}{2}$ in the formula of Art. 155.

4. Any line parallel to the given one is $x+2y+C=0$, and the equation to the circle is

$$(x+g)^2+(y+f)^2=g^2+f^2-c.$$

As in Art. 158, $\frac{-g-2f+C}{\sqrt{5}}=\pm\sqrt{g^2+f^2-c}$;

$$\therefore\ C=g+2f\pm\sqrt{5}\,.\sqrt{g^2+f^2-c}.$$

5. Eliminating y, we have $x^2+(x+c\sqrt{2})^2=c^2$,

i.e. $$\{\sqrt{2}x+c\}^2=0,$$

which is a perfect square; therefore the given line is a tangent.

Also $x=-\frac{c}{\sqrt{2}}$, whence $y=\frac{c}{\sqrt{2}}$.

6. The equation of the lines joining the origin to the common points of the circle and the given line are (Art. 122)

$$b^2(x^2+y^2)+(ax+by)(cx-by)=0,$$

or $$(b^2+ac)x^2+(bc-ab)xy=0,$$

which must be a perfect square. $\therefore bc - ab = 0$. $\therefore c = a$, and the point of contact is where $x = 0$ cuts the circle, viz. $(0, b)$.

7. The circle may be written $(x-1)^2 + (y-1)^2 = 1^2$.

As in Art. 158, since $\frac{1+1-2-\sqrt{2}}{\sqrt{2}} = -1$, therefore the line is a tangent.

8. As in No. 6, the equation $k(x^2+y^2) = 10x(3x+4y)$, or $x^2(k-30) - 40xy + ky^2 = 0$ must be a perfect square;

$$\therefore\ 400 = (k-30)k, \text{ i.e. } k^2 - 30k - 400 = 0.$$

$$\therefore\ k = 40 \text{ or } -10.$$

9. The circle may be written

$$(x - a\cos\alpha)^2 + (y - b\sin\alpha)^2 = a^2 + b^2\sin^2\alpha.$$

As in Art. 158, we then have

$$a\cos^2\alpha + b\sin^2\alpha - p = \pm\sqrt{a^2 + b^2\sin^2\alpha}, \text{ etc.}$$

10. As in Art. 158, $\frac{Aa + Bb + C}{\sqrt{A^2+B^2}} = \pm c$.

11. (1) See Art. 155.

(2) Change m into $-\frac{1}{m}$ in (1).

(3) Any tangent is $y = mx \pm a\sqrt{1+m^2}$; this passes through the point $(b, 0)$ if $mb = \pm a\sqrt{1+m^2}$;

$\therefore\ m^2b^2 = a^2(1+m^2)$; $\therefore\ m = \pm\frac{a}{\sqrt{b^2-a^2}}$. Hence etc.

(4) Any tangent is $x\cos\alpha + y\sin\alpha = a$.

Its intercepts on the axes are $a\sec\alpha$, $a\operatorname{cosec}\alpha$.

$\therefore\ \frac{1}{2}a^2\sec\alpha \cdot \operatorname{cosec}\alpha = a^2$. $\therefore\ \sin 2\alpha = 1$; $\therefore\ \alpha = 45°$, and the equation becomes $x + y = a\sqrt{2}$.

12. Put $a = r$, $m = -\frac{b}{a}$, $c = b$ in the formula of Art. 154.

13. One of the circles passes through the points

$$(0, 0);\ \left(\frac{a}{\sqrt{2}}, \frac{a}{\sqrt{2}}\right), \left(-\frac{a}{\sqrt{2}}, \frac{a}{\sqrt{2}}\right).$$

Let its equation be $x^2 + y^2 + 2gx + 2fy = 0$.

$$\therefore\ a^2 + a\sqrt{2}\,(g+f) = 0, \text{ and } a^2 + (a\sqrt{2})(f-g) = 0.$$

$$\therefore\ g = 0 \text{ and } f = -\frac{a}{\sqrt{2}}. \quad \therefore\ x^2 + y^2 - \sqrt{2}ay = 0$$

is one of the circles. Similarly for the three other circles.

14. The required equation is (Art. 122)

$$c(x^2 + y^2) = 2(ax + by)(y - mx).$$

These lines are at right angles if, (Art. 111),

$$2c + 2am - 2b = 0, \text{ i.e. if } c = b - am.$$

The circle may be written $(x-a)^2 + (y-b)^2 = a^2 + b^2$.

As in Art. 158, the line will touch the circle if

$$\frac{b - am - c}{\sqrt{1+m^2}} = \pm\sqrt{a^2 + b^2}.$$

15. As in Art. 158, the line $5x + 12y - 1 = 0$ touches the circle $(x-3)^2 + (y-4)^2 = r^2$ if

$$r = \pm\frac{15 + 48 - 1}{\sqrt{5^2 + 12^2}}, \quad \text{i.e. if } r^2 = \frac{62^2}{13^2}.$$

Hence substituting, etc.

16. Let its radius be c, and its equation

$$(x-c)^2 + (y-c)^2 = c^2. \quad \text{[Ex. XVII, 26.]}$$

As in Art. 158 $\dfrac{bc + ac - ab}{\sqrt{a^2 + b^2}} = \pm c$;

$$\therefore\ c = \frac{ab}{a + b \pm\sqrt{a^2 + b^2}} = \frac{ab\{a + b \mp \sqrt{a^2 + b^2}\}}{2ab};$$

$$\therefore\ 2c = a + b \pm \sqrt{a^2 + b^2}.$$

17. As in Art. 158, the circles $(x-1)^2 + (y+3)^2 = r^2$ will touch the line $2x - y - 4 = 0$ if $\dfrac{2 + 3 - 4}{\sqrt{5}} = \pm r$, i.e. if $r^2 = \frac{1}{5}$.

Hence substituting, etc.

18. See Ex. XVII, 26.

19. From a figure it is easily seen that the centre is the point (r, h). $\therefore$ the required equation is

$$(x-r)^2+(y-h)^2=r^2.$$

The line $y+mx=0$ will be a tangent if, (Art. 158),

$$\frac{h+mr}{\sqrt{1+m^2}}=\pm r. \quad \therefore (h+mr)^2=r^2(1+m^2),$$

whence
$$m=\frac{r^2-h^2}{2rh}.$$

20. The required equation is (Art. 142)

$$x^2+y^2=2\alpha x+2\beta y.$$

See Art. 152 and put $x'=y'=0$.

21. Let (h, k) be the coordinates of the centre of either circle.

Then $(h-a)^2+(k-5a)^2=(h-4a)^2+(k-a)^2=h^2$.

Solving, we have

$$h=\frac{5a}{2} \text{ or } \frac{205a}{18} \text{ and } k=3a \text{ or } \frac{29a}{3}.$$

If c be the distance between the centres, then

$$c^2=(h_1-h_2)^2+(k_1-k_2)^2$$
$$=\left(\frac{205a}{18}-\frac{45a}{18}\right)^2+\left(\frac{20a}{3}\right)^2=\frac{10000a^2}{81}.$$

Also h_1 and h_2 are the radii, and $\cos\alpha=\dfrac{h_1{}^2+h_2{}^2-c^2}{2h_1h_2}$

$$=\frac{\frac{25}{4}+\left(\frac{205}{18}\right)^2-\frac{10000}{81}}{2\times\frac{5}{2}\times\frac{205}{18}}=\frac{9}{41}, \text{ on reduction.}$$

Hence
$$\tan\alpha=\frac{\sqrt{41^2-9^2}}{9}=\frac{40}{9}.$$

22. Let the equation of the circle be

$$x^2+y^2+2gx+2fy+c=0.$$

Then we have,

$$2-2g+2f+c=0, \; 36+12f+c=0,$$

and $50+10g+10f+c=0$. Whence $g=-2$, $f=-3$, $c=0$.

∴ the centre is the point (2, 3) and the radius $= \sqrt{13}$.

∴ the equation of the radius from the origin to the centre is $3x = 2y$. If α be the angle which the diameter perpendicular to this makes with the axis of x, then

$$\sin\alpha = \frac{2}{\sqrt{13}}, \quad \cos\alpha = -\frac{3}{\sqrt{13}},$$

and the required points are (Art. 86)

$$\{2 \pm \sqrt{13}\cos\alpha,\ 3 \pm \sqrt{13}\sin\alpha\}, \text{ or } (-1, 5) \text{ and } (5, 1).$$

160. *To shew that from any point there can be drawn two tangents, real or imaginary, to a circle.*

Let the equation to the circle be $x^2 + y^2 = a^2$, and let the given point be (x_1, y_1). [Fig. Art. 161.]

The equation to any tangent is, by Art. 155,

$$y = mx + a\sqrt{1+m^2}.$$

If this pass through the given point (x_1, y_1) we have

$$y_1 = mx_1 + a\sqrt{1+m^2} \quad\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

This is the equation which gives the values of m corresponding to the tangents which pass through (x_1, y_1).

Now (1) gives

$$y_1 - mx_1 = a\sqrt{1+m^2},$$

i.e. $$y_1^2 - 2mx_1y_1 + m^2x_1^2 = a^2 + a^2m^2,$$

i.e. $$m^2(x_1^2 - a^2) - 2mx_1y_1 + y_1^2 - a^2 = 0 \quad\ldots\ldots(2).$$

The equation (2) is a quadratic equation and gives therefore two values of m (real, coincident, or imaginary) corresponding to any given values of x_1 and y_1. For each of these values of m we have a corresponding tangent.

The roots of (2) are, by Art. 1, real, coincident or imaginary according as

$(2x_1y_1)^2 - 4(x_1^2 - a^2)(y_1^2 - a^2)$ is positive, zero, or negative,

i.e. according as

$$a^2(-a^2 + x_1^2 + y_1^2) \text{ is positive, zero, or negative,}$$

i.e. according as $\quad x_1^2 + y_1^2 \gtreqless a^2.$

If $x_1^2 + y_1^2 > a^2$, the distance of the point (x_1, y_1) from the centre is greater than the radius and hence it lies outside the circle.

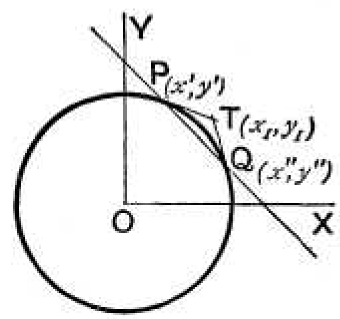
If $x_1^2 + y_1^2 = a^2$, the point (x_1, y_1) lies on the circle and the two coincident tangents become the tangent at (x_1, y_1).

If $x_1^2 + y_1^2 < a^2$, the point (x_1, y_1) lies within the circle, and no tangents can then be geometrically drawn to the circle. It is however better to say that the tangents are imaginary.

161. Chord of Contact. Def. If from any point T without a circle two tangents TP and TQ be drawn to the circle, the straight line PQ joining the points of contact is called the chord of contact of tangents from T.

To find the equation of the chord of contact of tangents drawn to the circle $x^2 + y^2 = a^2$ from the external point (x_1, y_1).

Let T be the point (x_1, y_1), and P and Q the points (x', y') and (x'', y'') respectively.



The tangent at P is

$$xx' + yy' = a^2 \text{ ....(1),}$$

and that at Q is

$$xx'' + yy'' = a^2 \text{ ....(2).}$$

Since these tangents pass through T, its coordinates (x_1, y_1) must satisfy both (1) and (2).

Hence $$x_1x' + y_1y' = a^2 \text{ ....(3)},$$
and $$x_1x'' + y_1y'' = a^2 \text{ ....(4)}.$$

The equation to PQ is then

$$\mathbf{xx_1 + yy_1 = a^2} \text{ ....(5)}.$$

For, since (3) is true, it follows that the point (x', y'), *i.e.* P, lies on (5).

Also, since (4) is true, it follows that the point (x'', y''), *i.e.* Q, lies on (5).

Hence both P and Q lie on the straight line (5), *i.e.* (5) is the equation to the required chord of contact.

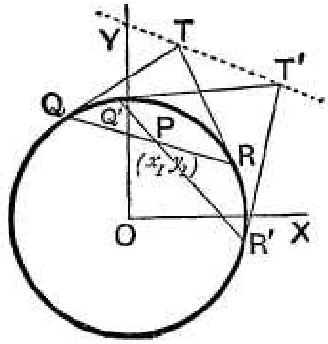
If the point (x_1, y_1) lie within the circle the argument of the preceding article will shew that the line joining the (imaginary) points of contact of the two (imaginary) tangents drawn from (x_1, y_1) is $xx_1 + yy_1 = a^2$.

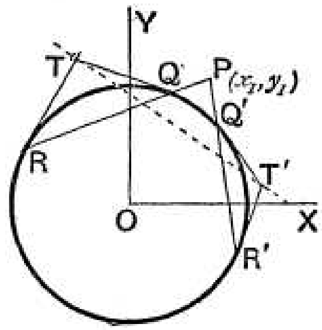
We thus see, since this line is always real, that we may have a real straight line joining the imaginary points of contact of two imaginary tangents.

162. Pole and Polar. Def. If through a point P (within or without a circle) there be drawn any straight line to meet the circle in Q and R, the locus of the point of intersection of the tangents at Q and R is called the polar of P; also P is called the pole of the polar.

In the next article the locus will be proved to be a straight line.

163. *To find the equation to the polar of the point* (x_1, y_1) *with respect to the circle* $x^2 + y^2 = a^2$.





Let QR be any chord drawn through P and let the tangents at Q and R meet in the point T whose coordinates are (h, k).

Hence QR is the chord of contact of tangents drawn from the point (h, k) and therefore, by Art. 161, its equation is $xh + yk = a^2$.

Since this line passes through the point (x_1, y_1) we have

$$x_1h + y_1k = a^2 \text{ ....(1).}$$

Since the relation (1) is true it follows that the variable point (h, k) always lies on the straight line whose equation is

$$\mathbf{xx_1 + yy_1 = a^2} \text{ ....(2).}$$

Hence (2) is the polar of the point (x_1, y_1).

In a similar manner it may be proved that the polar of (x_1, y_1) with respect to the circle

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

is
$$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0.$$

164. The equation (2) of the preceding article is the same as equation (5) of Art. 161. If, therefore, the point (x_1, y_1) be without the circle, as in the right-hand figure, the polar is the same as the chord of contact of the real tangents drawn through (x_1, y_1).

If the point (x_1, y_1) be on the circle, the polar coincides with the tangent at it. (Art. 150.)

If the point (x_1, y_1) be within the circle, then, as in Art. 161, the equation (2) is the line joining the (imaginary) points of contact of the two (imaginary) tangents that can be drawn from (x_1, y_1).

We see therefore that the polar might have been defined as follows:

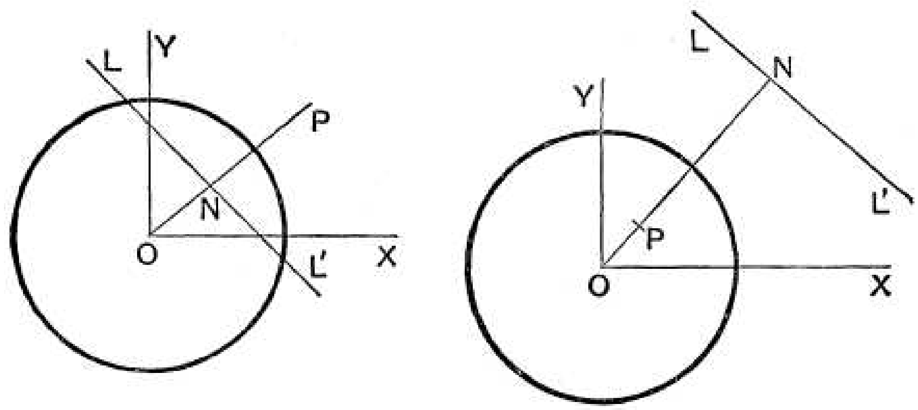
The polar of a given point is the straight line which passes through the (real or imaginary) points of contact of tangents drawn from the given point; also the pole of any straight line is the point of intersection of tangents at the points (real or imaginary) in which this straight line meets the circle.

165. *Geometrical construction for the polar of a point.*

The equation to OP, which is the line joining (0, 0) to (x_1, y_1), is

$$y = \frac{y_1}{x_1} x,$$

i.e. $$xy_1 - x_1 y = 0 \quad \text{....................(1)}.$$



Also the polar of P is

$$xx_1 + yy_1 = a^2 \quad \text{....................(2)}.$$

By Art. 69, the lines (1) and (2) are perpendicular to one another. Hence OP is perpendicular to the polar of P.

Also the length $\quad OP = \sqrt{x_1^2 + y_1^2}$, and the perpendicular, ON, from O upon (2)

$$=\frac{a^2}{\sqrt{x_1^2+y_1^2}}.$$

Hence the product $ON . OP = a^2$.

The polar of any point P is therefore constructed thus: Join OP and on it (produced if necessary) take a point N such that the rectangle $ON . OP$ is equal to the square of the radius of the circle.

Through N draw the straight line LL' perpendicular to OP; this is the polar required.

[It will be noted that the middle point N of any chord LL' lies on the line joining the centre to the pole of the chord.]

166. *To find the pole of a given line with respect to any circle.*

Let the equation to the given line be

$$Ax + By + C = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots(1).$$

(1) Let the equation to the circle be

$$x^2 + y^2 = a^2,$$

and let the required pole be (x_1, y_1).

Then (1) must be the equation to the polar of (x_1, y_1), *i.e.* it is the same as the equation

$$xx_1 + yy_1 - a^2 = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots(2).$$

Comparing equations (1) and (2), we have

$$\frac{x_1}{A} = \frac{y_1}{B} = \frac{-a^2}{C},$$

so that $$x_1 = -\frac{A}{C}a^2 \text{ and } y_1 = -\frac{B}{C}a^2.$$

The required pole is therefore the point

$$\left(-\frac{A}{C}a^2, -\frac{B}{C}a^2\right).$$

(2) Let the equation to the circle be

$$x^2 + y^2 + 2gx + 2fy + c = 0.$$

If (x_1, y_1) be the required pole, then (1) must be equivalent to the equation

$$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0, \qquad \text{(Art. 163)},$$

i.e. $$x(x_1 + g) + y(y_1 + f) + gx_1 + fy_1 + c = 0 \;\ldots\ldots (3).$$

Comparing (1) with (3), we therefore have

$$\frac{x_1 + g}{A} = \frac{y_1 + f}{B} = \frac{gx_1 + fy_1 + c}{C}.$$

By solving these equations we have the values of x_1 and y_1.

Ex. *Find the pole of the straight line*

$$9x + y - 28 = 0 \ldots\ldots\ldots\ldots (1)$$

with respect to the circle

$$2x^2 + 2y^2 - 3x + 5y - 7 = 0 \ldots\ldots\ldots\ldots (2).$$

If (x_1, y_1) be the required point the line (1) must coincide with the polar of (x_1, y_1), whose equation is

$$2xx_1 + 2yy_1 - \tfrac{3}{2}(x + x_1) + \tfrac{5}{2}(y + y_1) - 7 = 0,$$

i.e. $$x(4x_1 - 3) + y(4y_1 + 5) - 3x_1 + 5y_1 - 14 = 0 \ldots\ldots\ldots (3).$$

Since (1) and (3) are the same, we have

$$\frac{4x_1 - 3}{9} = \frac{4y_1 + 5}{1} = \frac{-3x_1 + 5y_1 - 14}{-28}.$$

Hence $$x_1 = 9y_1 + 12,$$

and $$3x_1 - 117y_1 = 126.$$

Solving these equations we have $x_1 = 3$ and $y_1 = -1$, so that the required point is $(3, -1)$.

167. *If the polar of a point P pass through a point T, then the polar of T passes through P.*

Let P and T be the points (x_1, y_1) and (x_2, y_2) respectively. (Fig. Art. 163.)

The polar of (x_1, y_1) with respect to the circle $x^2 + y^2 = a^2$ is

$$xx_1 + yy_1 = a^2.$$

This straight line passes through the point T if

$$x_2x_1 + y_2y_1 = a^2 \ldots\ldots\ldots\ldots (1).$$

Since the relation (1) is true it follows that the point (x_1, y_1), *i.e.* P, lies on the straight line $xx_2 + yy_2 = a^2$, which is the polar of (x_2, y_2), *i.e.* T, with respect to the circle.

Hence the proposition.

Cor. The intersection, T, of the polars of two points, P and Q, is the pole of the line PQ.

168. *To find the length of the tangent that can be drawn from the point* (x_1, y_1) *to the circles*

$$(1) \quad x^2 + y^2 = a^2,$$

and

$$(2) \quad x^2 + y^2 + 2gx + 2fy + c = 0.$$

If T be an external point (Fig. Art. 163), TQ a tangent and O the centre of the circle, then TQO is a right angle and hence

$$TQ^2 = OT^2 - OQ^2.$$

(1) If the equation to the circle be $x^2 + y^2 = a^2$, O is the origin, $OT^2 = x_1^2 + y_1^2$, and $OQ^2 = a^2$.

Hence $$TQ^2 = x_1^2 + y_1^2 - a^2.$$

(2) Let the equation to the circle be

$$x^2 + y^2 + 2gx + 2fy + c = 0,$$

i.e. $$(x + g)^2 + (y + f)^2 = g^2 + f^2 - c.$$

In this case O is the point $(-g, -f)$ and

$$OQ^2 = (\text{radius})^2 = g^2 + f^2 - c.$$

Hence $$OT^2 = [x_1 - (-g)]^2 + [y_1 - (-f)]^2 \quad \text{(Art. 20)}.$$
$$= (x_1 + g)^2 + (y_1 + f)^2.$$

Therefore $$TQ^2 = (x_1 + g)^2 + (y_1 + f)^2 - (g^2 + f^2 - c)$$
$$= x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c.$$

In each case we see that (the equation to the circle being written so that the coefficients of x^2 and y^2 are each unity) the square of the length of the tangent drawn to the circle from the point (x_1, y_1) is obtained by substituting x_1 and y_1 for the current coordinates in the left-hand member of the equation to the circle.

***169.** *To find the equation to the pair of tangents that can be drawn from the point* (x_1, y_1) *to the circle* $x^2 + y^2 = a^2$.

Let (h, k) be any point on either of the tangents from (x_1, y_1).

Since any straight line touches a circle if the perpendicular on it from the centre is equal to the radius, the perpendicular from the origin upon the line joining (x_1, y_1) to (h, k) must be equal to a.

The equation to the straight line joining these two points is

$$y - y_1 = \frac{k - y_1}{h - x_1}(x - x_1),$$

i.e. $$y(h - x_1) - x(k - y_1) + kx_1 - hy_1 = 0.$$

Hence $$\frac{kx_1 - hy_1}{\sqrt{(h - x_1)^2 + (k - y_1)^2}} = a,$$

so that $$(kx_1 - hy_1)^2 = a^2[(h - x_1)^2 + (k - y_1)^2].$$

Therefore the point (h, k) always lies on the locus

$$(x_1 y - xy_1)^2 = a^2[(x - x_1)^2 + (y - y_1)^2] \text{ .... } (1).$$

This therefore is the required equation.

The equation (1) may be written in the form

$$x^2(y_1^2 - a^2) + y^2(x_1^2 - a^2) - a^2(x_1^2 + y_1^2) = 2xyx_1y_1 - 2a^2xx_1 - 2a^2yy_1,$$

i.e. $$(x^2 + y^2 - a^2)(x_1^2 + y_1^2 - a^2) = x^2x_1^2 + y^2y_1^2 + a^4 + 2xyx_1y_1 - 2a^2xx_1 - 2a^2yy_1 = (xx_1 + yy_1 - a^2)^2 \text{ .... } (2).$$

***170.** In a later chapter we shall obtain the equation to the pair of tangents to any curve of the second degree in a form analogous to that of equation (2) of the previous article.

Similarly the equation to the pair of tangents that can be drawn from (x_1, y_1) to the circle

$$(x - f)^2 + (y - g)^2 = a^2$$

is $$\{(x - f)^2 + (y - g)^2 - a^2\}\{(x_1 - f)^2 + (y_1 - g)^2 - a^2\} = \{(x - f)(x_1 - f) + (y - g)(y_1 - g) - a^2\}^2 \text{ ...... } (1).$$

If the equation to the circle be given in the form

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

the equation to the tangents is, similarly,

$$(x^2 + y^2 + 2gx + 2fy + c)(x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c) = [xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c]^2 \text{ ...... } (2).$$

EXAMPLES XIX

Find the polar of the point

1. (1, 2) with respect to the circle $x^2+y^2=7$.

2. (4, −1) with respect to the circle $2x^2+2y^2=11$.

3. (−2, 3) with respect to the circle
$$x^2+y^2-4x-6y+5=0.$$

4. $(5, -\frac{1}{2})$ with respect to the circle
$$3x^2+3y^2-7x+8y-9=0.$$

5. $(a, -b)$ with respect to the circle
$$x^2+y^2+2ax-2by+a^2-b^2=0.$$

Find the pole of the straight line

6. $x+2y=1$ with respect to the circle $x^2+y^2=5$.

7. $2x-y=6$ with respect to the circle $5x^2+5y^2=9$.

8. $2x+y+12=0$ with respect to the circle
$$x^2+y^2-4x+3y-1=0.$$

9. $48x-54y+53=0$ with respect to the circle
$$3x^2+3y^2+5x-7y+2=0.$$

10. $ax+by+3a^2+3b^2=0$ with respect to the circle
$$x^2+y^2+2ax+2by=a^2+b^2.$$

11. Tangents are drawn to the circle $x^2+y^2=12$ at the points where it is met by the circle $x^2+y^2-5x+3y-2=0$; find the point of intersection of these tangents.

12. Find the equation to that chord of the circle $x^2+y^2=81$ which is bisected at the point (−2, 3), and its pole with respect to the circle.

13. Prove that the polars of the point (1, −2) with respect to the circles whose equations are
$$x^2+y^2+6y+5=0 \text{ and } x^2+y^2+2x+8y+5=0$$
coincide; prove also that there is another point the polars of which with respect to these circles are the same and find its coordinates.

14. Find the condition that the chord of contact of tangents from the point (x', y') to the circle $x^2+y^2=a^2$ should subtend a right angle at the centre.

15. Prove that the distances of two points, P and Q, each from the polar of the other with respect to a circle, are to one another inversely as the distances of the points from the centre of the circle.

16. Prove that the polar of a given point with respect to any one of the circles $x^2+y^2-2kx+c^2=0$, where k is variable, always passes through a fixed point, whatever be the value of k.

17. Tangents are drawn from the point (h, k) to the circle $x^2+y^2=a^2$; prove that the area of the triangle formed by them and the straight line joining their points of contact is

$$\frac{a(h^2+k^2-a^2)^{\frac{3}{2}}}{h^2+k^2}.$$

Find the lengths of the tangents drawn

18. to the circle $2x^2+2y^2=3$ from the point $(-2, 3)$.

19. to the circle $3x^2+3y^2-7x-6y=12$ from the point $(6, -7)$.

20. to the circle $x^2+y^2+2bx-3b^2=0$ from the point

$$(a+b,\ a-b).$$

21. Given the three circles

$$x^2+y^2-16x+60=0,$$
$$3x^2+3y^2-36x+81=0,$$

and

$$x^2+y^2-16x-12y+84=0,$$

find (1) the point from which the tangents to them are equal in length, and (2) this length.

22. The distances from the origin of the centres of three circles $x^2+y^2-2\lambda x=c^2$ (where c is a constant and λ a variable) are in geometrical progression; prove that the lengths of the tangents drawn to them from any point on the circle $x^2+y^2=c^2$ are also in geometrical progression.

23. Find the equation to the pair of tangents drawn

(1) from the point $(11, 3)$ to the circle $x^2+y^2=65$,

(2) from the point $(4, 5)$ to the circle

$$2x^2+2y^2-8x+12y+21=0.$$

ANSWERS

1. $x+2y=7$. **2.** $8x-2y=11$. **3.** $x=0$.

4. $23x+5y=57$. **5.** $by-ax=a^2$. **6.** $(5, 10)$.

7. $(\frac{3}{5}, -\frac{3}{10})$. **8.** $(1, -2)$. **9.** $(\frac{1}{2}, -\frac{1}{3})$.

10. $(-2a, -2b)$. **11.** $(6, -\frac{18}{5})$.

12. $3y-2x=13$; $\left(-\frac{162}{13}, \frac{243}{13}\right)$. **13.** $(2, -1)$. **14.** $x'^2+y'^2=2a^2$.

18. $\frac{1}{2}\sqrt{46}$. **19.** 9. **20.** $\sqrt{2a^2+2ab+b^2}$. **21.** $\left(\frac{33}{4}, 2\right)$; $\frac{1}{4}$.

23. (1) $28x^2+33xy-28y^2-715x-195y+4225=0$;

(2) $123x^2-64xy+3y^2-664x+226y+763=0$.

SOLUTIONS/HINTS

1—5. Use the equations of Art. 163.

6—10. See Ex. of Art. 166.

6. $x+2y=1$ and $xx_1+yy_1=5$ are identical if

$$\frac{x_1}{1}=\frac{y_1}{2}=5. \quad \therefore\ x_1=5,\ y_1=10.$$

7. $2x-y=6$ and $5xx_1+5yy_1=9$ are identical if

$$\frac{5x_1}{2}=\frac{5y_1}{-1}=\frac{9}{6}. \quad \therefore\ x_1=\frac{3}{5},\ y_1=-\frac{3}{10}.$$

8. $2x+y+12=0$ and $xx_1+yy_1-2(x+x_1)+\frac{3}{2}(y+y_1)-1=0$ are identical if $\dfrac{x_1-2}{2}=\dfrac{y_1+\frac{3}{2}}{1}=\dfrac{-2x_1+\frac{3}{2}y_1-1}{12}$, whence, solving, $x_1=1,\ y_1=-2$.

9. $48x-54y+53=0$ and

$$6xx_1+6yy_1+5(x+x_1)-7(y+y_1)+4=0$$

are identical if $\dfrac{6x_1+5}{48}=\dfrac{6y_1-7}{-54}=\dfrac{5x_1-7y_1+4}{53}$, whence, solving, $x_1=\frac{1}{2},\ y_1=-\frac{1}{3}$.

10. $ax+by+3a^2+3b^2=0$ and

$$xx_1+yy_1+a(x+x_1)+b(y+y_1)=a^2+b^2$$

are identical if $\dfrac{x_1+a}{a}=\dfrac{y_1+b}{b}=\dfrac{ax_1+by_1-a^2-b^2}{3a^2+3b^2}$. Whence, solving, $x=-2a,\ y=-2b$.

11. Subtracting the equations, we obtain $5x-3y-10=0$, which must be the equation of the straight line through the common points of the circles.

Now $5x-3y-10$ and $xx_1+yy_1=12$ are identical if $\frac{x_1}{5}=\frac{y_1}{-3}=\frac{12}{10}$, whence $x_1=6$, $y_1=-\frac{18}{5}$.

$\therefore$ $(6, -\frac{18}{5})$ is the pole of the common chord with respect to the first circle.

12. We want the line through $(-2, 3)$ perpendicular to $2y+3x=0$, viz. $y-3=\frac{2}{3}(x+2)$, *i.e.* $3y-2x=13$. If (x_1, y_1) be the pole of this line, the latter must be the same as $xx_1+yy_1=81$. $\therefore$ $\frac{x_1}{-2}=\frac{y_1}{3}=\frac{81}{13}$. $\therefore$ etc.

13. The polars of (x_1, y_1) with respect to the circles, viz. $xx_1+y(y_1+3)+3y_1+5=0$, and

$$x(x_1+1)+y(y_1+4)+x_1+4y_1+5=0,$$

are identical if

$$\frac{x_1+1}{x_1}=\frac{y_1+4}{y_1+3}=\frac{x_1+4y_1+5}{3y_1+5}, \text{ i.e. if } \frac{1}{x_1}=\frac{1}{y_1+3}=\frac{x_1+y_1}{3y_1+5}.$$

Whence, solving, $x_1=2$ or 1, and $y_1=-1$ or -2.

$\therefore$ the other point is $(2, -1)$.

14. The lines joining the origin to the common points of $x^2+y^2=a^2$ and $xx'+yy'=a^2$ are, (Art. 122),

$$a^2(x^2+y^2)=(xx'+yy')^2.$$

These are at right angles if (Art. 111) $x'^2+y'^2=2a^2$.

15. Take $x^2+y^2=a^2$ as the equation of the circle and (x_1, y_1), (x_2, y_2) the coordinates of P and Q.

$$\frac{\text{The perpendicular from } P \text{ on polar of } Q\,(xx_2+yy_2=a^2)}{\text{The perpendicular from } Q \text{ on polar of } P\,(xx_1+yy_1=a^2)}$$

$$=\frac{\dfrac{x_1x_2+y_1y_2-a^2}{\sqrt{x_2^2+y_2^2}}}{\dfrac{x_1x_2+y_1y_2-a^2}{\sqrt{x_1^2+y_1^2}}}=\frac{\sqrt{x_1^2+y_1^2}}{\sqrt{x_2^2+y_2^2}}=\frac{\text{distance of } P \text{ from the centre}}{\text{distance of } Q \text{ from the centre}}.$$

16. The equation to the polar of (x_1, y_1) is

$$xx_1 + yy_1 - k(x + x_1) + c^2 = 0,$$

which always passes through the intersection of the fixed lines $xx_1 + yy_1 - c^2 = 0$ and $x + x_1 = 0$.

17. The length of the chord intercepted by the circle $x^2 + y^2 = a^2$ on the line $xh + yk = a^2$ is (Art. 154)

$$2\sqrt{a^2 - \frac{a^4}{h^2 + k^2}},$$

and the length of the perpendicular from (h, k) upon its polar is $\dfrac{h^2 + k^2 - a^2}{\sqrt{h^2 + k^2}}$.

$$\therefore \ \Delta = \frac{1}{2} \cdot \frac{h^2 + k^2 - a^2}{\sqrt{h^2 + k^2}} \cdot 2a\sqrt{\frac{h^2 + k^2 - a^2}{h^2 + k^2}} = \frac{a(h^2 + k^2 - a^2)^{\frac{3}{2}}}{h^2 + k^2}.$$

18—20. Use Art. 168.

18. $t^2 = (-2)^2 + 3^2 - \frac{3}{2} = \text{etc.}$

19. $t^2 = 6^2 + (-7)^2 - \frac{7}{3} \times 6 - 2 \times (-7) - 4 = \text{etc.}$

20. $t^2 = (a + b)^2 + (a - b)^2 + 2b(a + b) - 3b^2$

$$= 2a^2 + 2b^2 + 2ab + 2b^2 - 3b^2 = 2a^2 + 2ab + b^2.$$

21. If (h, k) be the point, we have

$$h^2 + k^2 - 16h + 60 = h^2 + k^2 - 12h + 27 = h^2 + k^2 - 16h - 12k + 84.$$

$$\therefore \ 12k = 24 \text{ and } 4h = 33. \quad \therefore \ h = \tfrac{33}{4}, \quad k = 2,$$

and the square of the length

$$= \frac{33^2}{16} + 4 - 33.4 + 60 = \frac{1}{16}. \quad \therefore \text{ the length} = \frac{1}{4}.$$

22. Let $(\lambda_1, 0)$, $(\lambda_2, 0)$, $(\lambda_3, 0)$ be the centres, so that $\lambda_1\lambda_3 = \lambda_2^2$. Any point on $x^2 + y^2 = c^2$ is $(c\cos\theta, c\sin\theta)$. The squares of the lengths of the three tangents are $2\lambda_1 c\cos\theta$, $2\lambda_2 c\cos\theta$, $2\lambda_3 c\cos\theta$. Hence etc.

23. (1) Substitute in equation (1) of Art. 169.

$$\therefore \ (3x - 11y)^2 = 65\{(x - 11)^2 + (y - 3)^2\}, \text{ etc.}$$

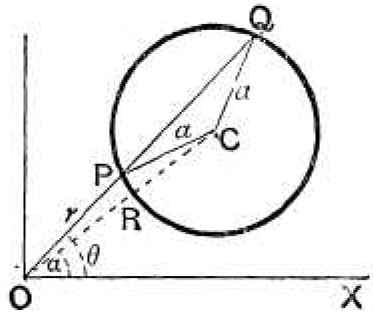
(2) Substitute in equation (2) of Art. 170.

$$(x^2 + y^2 - 4x + 6y + \tfrac{21}{2})(16 + 25 - 16 + 30 + \tfrac{21}{2}) = (2x + 8y + \tfrac{35}{2})^2,$$

i.e. $131(2x^2 + 2y^2 - 8x + 12y + 21) = (4x + 16y + 35)^2$, etc.

171. *To find the general equation of a circle referred to polar coordinates.*

Let O be the origin, or pole, OX the initial line, C the centre and a the radius of the circle.



Let the polar coordinates of C be R and α, so that $OC = R$ and $\angle XOC = \alpha$.

Let a radius vector through O at an angle θ with the initial line cut the circle in P and Q. Let OP, or OQ, be r.

Then (*Trig.* Art. 164) we have

$$CP^2 = OC^2 + OP^2 - 2OC \, . \, OP \cos COP,$$

i.e. $$a^2 = R^2 + r^2 - 2Rr \cos(\theta - \alpha),$$

i.e. $$r^2 - 2Rr \cos(\theta - \alpha) + R^2 - a^2 = 0 \text{ ....(1)}.$$

This is the required polar equation.

172. *Particular cases of the general equation in polar coordinates.*

(1) Let the initial line be taken to go through the centre C. Then $\alpha = 0$, and the equation becomes

$$r^2 - 2Rr\cos\theta + R^2 - a^2 = 0.$$

(2) Let the pole O be taken on the circle, so that

$$R = OC = a.$$

The general equation then becomes

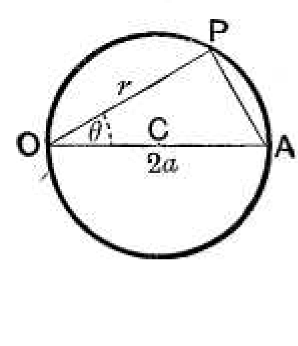
$$r^2 - 2ar\cos(\theta - \alpha) = 0,$$

i.e. $$r = 2a\cos(\theta - \alpha).$$

(3) Let the pole be on the circle and also let the initial line pass through the centre of the circle. In this case

$$\alpha = 0, \quad \text{and} \quad R = a.$$

The general equation reduces then to the simple form $r = 2a\cos\theta$.



This is at once evident from the figure.

For, if OCA be a diameter, we have

$$OP = OA\cos\theta,$$

i.e. $$r = 2a\cos\theta.$$

173. The equation (1) of Art. 171 is a quadratic equation which, for any given value of θ, gives two values of r. These two values in the figure are OP and OQ.

If these two values be called r_1 and r_2, we have, from equation (1),

$$r_1 r_2 = \text{product of the roots} = R^2 - a^2,$$

i.e. $$OP \,.\, OQ = R^2 - a^2.$$

The value of the rectangle $OP \,.\, OQ$ is therefore the same for all values of θ. It follows that if we drew any other line through O to cut the circle in P_1 and Q_1 we should have $OP \,.\, OQ = OP_1 \,.\, OQ_1$.

This is Euc. III. 36, Cor.

174. *Find the equation to the chord joining the points on the circle $r = 2a \cos\theta$ whose vectorial angles are θ_1 and θ_2, and deduce the equation to the tangent at the point θ_1.*

The equation to any straight line in polar coordinates is (Art. 88)

$$p = r\cos(\theta - \alpha) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

If this pass through the points $(2a\cos\theta_1, \theta_1)$ and $(2a\cos\theta_2, \theta_2)$, we have

$$2a\cos\theta_1 \cos(\theta_1 - \alpha) = p = 2a\cos\theta_2\cos(\theta_2 - \alpha) \ldots\ldots\ldots (2).$$

Hence $$\cos(2\theta_1 - \alpha) + \cos\alpha = \cos(2\theta_2 - \alpha) + \cos\alpha,$$

i.e. $$2\theta_1 - \alpha = -(2\theta_2 - \alpha),$$

since θ_1 and θ_2 are not, in general, equal.

Hence $$\alpha = \theta_1 + \theta_2,$$

and then, from (2), $p = 2a\cos\theta_1\cos\theta_2$.

On substitution in (1), the equation to the required chord is

$$r\cos(\theta - \theta_1 - \theta_2) = 2a\cos\theta_1\cos\theta_2 \quad \ldots\ldots\ldots\ldots\ldots (3).$$

The equation to the tangent at the point θ_1 is found, as in Art. 150, by putting $\theta_2 = \theta_1$ in equation (3).

We thus obtain as the equation to the tangent

$$r\cos(\theta - 2\theta_1) = 2a\cos^2\theta_1.$$

As in the foregoing article it could be shewn that the equation to the chord joining the points θ_1 and θ_2 on the circle $r = 2a\cos(\theta - \gamma)$ is

$$r\cos[\theta - \theta_1 - \theta_2 + \gamma] = 2a\cos(\theta_1 - \gamma)\cos(\theta_2 - \gamma)$$

and hence that the equation to the tangent at the point θ_1 is

$$r\cos(\theta - 2\theta_1 + \gamma) = 2a\cos^2(\theta_1 - \gamma).$$

EXAMPLES XX

1. Find the coordinates of the centre of the circle

$$r = A\cos\theta + B\sin\theta.$$

2. Find the polar equation of a circle, the initial line being a tangent. What does it become if the origin be on the circumference?

3. Draw the loci

(1) $r = a$; (2) $r = a\sin\theta$; (3) $r = a\cos\theta$; (4) $r = a\sec\theta$;
(5) $r = a\cos(\theta - \alpha)$; (6) $r = a\sec(\theta - \alpha)$.

4. Prove that the equations $r = a\cos(\theta - \alpha)$ and $r = b\sin(\theta - \alpha)$ represent two circles which cut at right angles.

5. Prove that the equation $r^2\cos\theta - ar\cos 2\theta - 2a^2\cos\theta = 0$ represents a straight line and a circle.

6. Find the polar equation to the circle described on the straight line joining the points (a, α) and (b, β) as diameter.

7. Prove that the equation to the circle described on the straight line joining the points $(1, 60°)$ and $(2, 30°)$ as diameter is

$$r^2 - r[\cos(\theta - 60°) + 2\cos(\theta - 30°)] + \sqrt{3} = 0.$$

8. Find the condition that the straight line

$$\frac{1}{r} = a\cos\theta + b\sin\theta$$

may touch the circle $r = 2c\cos\theta$.

ANSWERS

1. $\left(\frac{1}{2}\sqrt{A^2 + B^2},\ \tan^{-1}\frac{B}{A}\right)$.

2. $r^2 - 2ra\operatorname{cosec}\alpha \,.\, \cos(\theta - \alpha) + a^2\cot^2\alpha = 0$, $r = 2a\sin\theta$.

6. $r^2 - r[a\cos(\theta - \alpha) + b\cos(\theta - \beta)] + ab\cos(\alpha - \beta) = 0$.

8. $b^2c^2 + 2ac = 1$.

SOLUTIONS/HINTS

1. Put $B/A = \tan \alpha$ and the equation becomes

$$r = \sqrt{A^2 + B^2} \cos (\theta - \alpha),$$

which is a circle whose centre is $(\frac{1}{2} \sqrt{A^2 + B^2}, \alpha)$. (Art. 172.)

2. (1) In the figure of Art. 171, if the circle touches OX, $R = a \operatorname{cosec} \alpha$. Substitute in equation (1).

(2) Put $\alpha = 90°$.

3. (1) is a circle whose centre is the origin. For (2), (3) and (5) see Art. 172. For (4) and (6) see Art. 88.

4. For the circles intersect at the pole, and the vectorial angles of their centres are α and $\alpha + 90°$.

5. Changing to Cartesian Coordinates, we have

$$x (x^2 + y^2) - a (x^2 - y^2) - 2a^2 x = 0,$$

or

$$x (x - 2a) (x + a) + y^2 (x + a) = 0,$$

i.e.

$$(x + a) (x^2 + y^2 - 2ax) = 0.$$

The equation therefore represents the line $r \cos \theta + a = 0$, and the circle $r = 2a \cos \theta$.

6. Let (a, α) and (b, β) be the points Q and R, and (r, θ) the point P, any point on the circle.

Since $Q\hat{P}R =$ a right angle, $PQ^2 + PR^2 = QR^2$.

$$\therefore\ a^2 + r^2 - 2ar \cos (\theta - \alpha) + b^2 + r^2 - 2br \cos (\theta - \beta)$$
$$= a^2 + b^2 - 2ab \cos (\alpha - \beta), \text{ (Art. 33)},$$

i.e. $r^2 - r \{a \cos (\theta - \alpha) + b \cos (\theta - \beta)\} + ab \cos (\alpha - \beta) = 0.$

7. Substitute in the result of No. 6.

8. Eliminating r,

$$2c \cos \theta (a \cos \theta + b \sin \theta) = 1 = \sin^2 \theta + \cos^2 \theta,$$

or

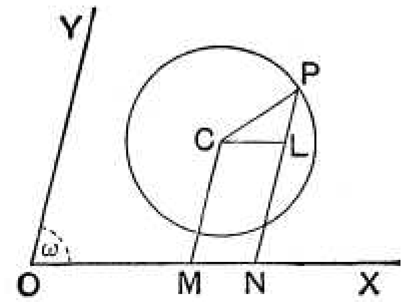
$$\tan^2 \theta - 2bc \tan \theta + 1 - 2ac = 0,$$

which has equal roots if $b^2 c^2 + 2ac = 1$.

175. *To find the general equation to a circle referred to oblique axes which meet at an angle* ω.

Let C be the centre and a the radius of the circle. Let the coordinates of C be (h, k) so that if CM, drawn parallel to the axis of y, meets OX in M, then

$$OM = h \text{ and } MC = k.$$



Let P be any point on the circle whose coordinates are x and y. Draw PN, the ordinate of P, and CL parallel to OX to meet PN in L.

Then $\qquad CL = MN = ON - OM = x - h,$

and $\qquad LP = NP - NL = NP - MC = y - k.$

Also $\angle CLP = \angle ONP = 180° - \angle PNX = 180° - \omega.$

Hence, since $\quad CL^2 + LP^2 - 2CL \cdot LP \cos CLP = a^2,$

we have $\mathbf{(x-h)^2 + (y-k)^2 + 2(x-h)(y-k)\cos\omega = a^2},$

i.e. $\quad x^2 + y^2 + 2xy\cos\omega - 2x(h + k\cos\omega) - 2y(k + h\cos\omega)$
$$+ h^2 + k^2 + 2hk\cos\omega = a^2.$$

The required equation is therefore found.

176. As in Art. 142 it may be shewn that the equation

$$x^2 + 2xy\cos\omega + y^2 + 2gx + 2fy + c = 0$$

represents a circle and its radius and centre found.

Ex. *If the axes be inclined at* 60°, *prove that the equation*

$$x^2 + xy + y^2 - 4x - 5y - 2 = 0 \quad \dots\dots\dots\dots \quad (1)$$

represents a circle and find its centre and radius.

If ω be equal to 60°, so that $\cos\omega = \frac{1}{2}$, the equation of Art. 175 becomes

$$x^2 + xy + y^2 - x(2h + k) - y(2k + h) + h^2 + k^2 + hk = a^2.$$

This equation agrees with (1) if

$$2h+k=4\ldots\ldots\ldots\ldots(2),$$
$$2k+h=5\ldots\ldots\ldots\ldots(3),$$
and
$$h^2+k^2+hk-a^2=-2\ldots\ldots\ldots\ldots(4).$$

Solving (2) and (3), we have $h=1$ and $k=2$. Equation (4) then gives
$$a^2=h^2+k^2+hk+2=9,$$
so that
$$a=3.$$

The equation (1) therefore represents a circle whose centre is the point (1, 2) and whose radius is 3, the axes being inclined at 60°.

EXAMPLES XXI

Find the inclinations of the axes so that the following equations may represent circles, and in each case find the radius and centre ;

1. $x^2-xy+y^2-2gx-2fy=0.$

2. $x^2+\sqrt{3}xy+y^2-4x-6y+5=0.$

3. The axes being inclined at an angle ω, find the centre and radius of the circle
$$x^2+2xy\cos\omega+y^2-2gx-2fy=0.$$

4. The axes being inclined at 45°, find the equation to the circle whose centre is the point (2, 3) and whose radius is 4.

5. The axes being inclined at 60°, find the equation to the circle whose centre is the point $(-3,\ -5)$ and whose radius is 6.

6. Prove that the equation to a circle whose radius is a and which touches the axes of coordinates, which are inclined at an angle ω, is
$$x^2+2xy\cos\omega+y^2-2a(x+y)\cot\frac{\omega}{2}+a^2\cot^2\frac{\omega}{2}=0.$$

7. Prove that the straight line $y=mx$ will touch the circle
$$x^2+2xy\cos\omega+y^2+2gx+2fy+c=0$$
if
$$(g+fm)^2=c(1+2m\cos\omega+m^2).$$

8. The axes being inclined at an angle ω, find the equation to the circle whose diameter is the straight line joining the points
$$(x',\ y')\text{ and }(x'',\ y'').$$

ANSWERS

1. $120°$; $\left(\frac{4g+2f}{3},\ \frac{4f+2g}{3}\right)$; $\frac{2\sqrt{3}}{3}\sqrt{f^2+g^2+fg}.$

2. $30°$; $(8-6\sqrt{3},\ 12-4\sqrt{3})$; $\sqrt{47-24\sqrt{3}}$.

3. $\left(\dfrac{g-f\cos\omega}{\sin^2\omega},\ \dfrac{f-g\cos\omega}{\sin^2\omega}\right)$; $\dfrac{\sqrt{f^2+g^2-2fg\cos\omega}}{\sin\omega}$.

4. $x^2+\sqrt{2}xy+y^2-x(4+3\sqrt{2})-2y(3+\sqrt{2})+3(2\sqrt{2}-1)=0$.

5. $x^2+xy+y^2+11x+13y+13=0$.

8. $(x-x')(x-x'')+(y-y')(y-y'')+\cos\omega[(x-x')(y-y'')$
$+(x-x'')(y-y')]=0$.

SOLUTIONS/HINTS

1. Comparing with the equation of Art. 175,
$2\cos\omega=-1$, so that $\omega=120°$, $h+k\cos\omega=g$,

and $$k+h\cos\omega=f.$$

$$\therefore\ h-\frac{k}{2}=g \text{ and } k-\frac{h}{2}=f,$$

whence $$h=\frac{4g+2f}{3} \text{ and } k=\frac{4f+2g}{3}.$$

Also $a^2=h^2+k^2+2hk\cos\omega=gh+fk=\frac{4}{3}(g^2+f^2+gf)$.

2. Comparing with the equation of Art. 175,
$2\cos\omega=\sqrt{3}$, so that $\omega=30°$, $2=h+k\cos\omega=h+\dfrac{\sqrt{3}}{2}.k$,

and $$3=k+h\cos\omega=k+\frac{\sqrt{3}}{2}.h,$$

whence $h=8-6\sqrt{3}$ and $k=12-4\sqrt{3}$.

Also $a^2=h^2+k^2+2hk\cos\omega-5=2h+3k-5=47-24\sqrt{3}$.

3. Here $h+k\cos\omega=g$, and $k+h\cos\omega=f$,

whence $$h=\frac{g-f\cos\omega}{\sin^2\omega}, \text{ and } k=\frac{f-g\cos\omega}{\sin^2\omega}.$$

$$\therefore\ a^2=h^2+k^2+2hk\cos\omega=gh+fk=\frac{g^2+f^2-2fg\cos\omega}{\sin^2\omega}.$$

4, 5. Substitute in the equation of Art. 175.

6. In the equation of Art. 175, when we put $y=0$, the resulting equation must be a perfect square.

$$\therefore\ h^2+k^2\cos^2\omega+2hk\cos\omega=h^2+k^2+2hk\cos\omega-a^2.$$

$$\therefore\ k=a\operatorname{cosec}\omega, \text{ and similarly } h=a\operatorname{cosec}\omega.$$

$$\therefore\ h+k\cos\omega=a\operatorname{cosec}\omega\,(1+\cos\omega)=a\cot\frac{\omega}{2}.$$

Similarly $$k+h\cos\omega=a\cot\frac{\omega}{2}.$$

$$\therefore\ h^2+k^2+2hk\cos\omega-a^2=a^2\cot^2\frac{\omega}{2}.$$

Hence, substituting, etc.

7. Putting $y=mx$ in the equation of the circle, the common points are given by

$$x^2(m^2+2m\cos\omega+1)+2x(g+fm)+c=0,$$

which is a perfect square if $(g+fm)^2=c\,(1+2m\cos\omega+m^2)$.

8. See Art. 145. The condition that the lines (1) and (2) should be at right angles when the axes are oblique is (Art. 93) $\dfrac{k-y_1}{h-x_1}\cdot\dfrac{k-y_2}{h-x_2}+\left(\dfrac{k-y_1}{h-x_1}+\dfrac{k-y_2}{h-x_2}\right)\cos\omega+1=0.$

Hence etc.

Coordinates of a point on a circle expressed in terms of one single variable.

177. If, in the figure of Art. 139, we put the angle MOP equal to α, the coordinates of the point P are easily seen to be $a\cos\alpha$ and $a\sin\alpha$.

These equations clearly satisfy equation (1) of that article.

The position of the point P is therefore known when the value of α is given, and it may be, for brevity, called "the point α."

With the ordinary Cartesian coordinates we have to give the values of *two* separate quantities x' and y' (which are however connected by the relation $x' = \sqrt{a^2 - y'^2}$) to express the position of a point P on the circle. The above substitution therefore often simplifies solutions of problems.

178. *To find the equation to the straight line joining two points, α and β, on the circle $x^2 + y^2 = a^2$.*

Let the points be P and Q, and let ON be the perpendicular from the origin on the straight line PQ; then ON bisects the angle POQ, and hence

$$\angle XON = \tfrac{1}{2}(\angle XOP + \angle XOQ) = \tfrac{1}{2}(\alpha + \beta).$$

Also $$ON = OP \cos NOP = a\cos\frac{\alpha - \beta}{2}.$$

The equation to PQ is therefore (Art. 53),

$$x\cos\frac{\alpha+\beta}{2} + y\sin\frac{\alpha+\beta}{2} = a\cos\frac{\alpha-\beta}{2}.$$

If we put $\beta = \alpha$ we have, as the equation to the tangent at the point α,

$$x\cos\alpha + y\sin\alpha = a.$$

This may also be deduced from the equation of Art. 150 by putting $x' = a\cos\alpha$ and $y' = a\sin\alpha$.

179. If the equation to the circle be in the more general form

$$(x-h)^2 + (y-k)^2 = a^2, \qquad \text{(Art. 140)},$$

we may express the coordinates of P in the form

$$(h + a\cos\alpha,\ k + a\sin\alpha).$$

For these values satisfy the above equation.

Here α is the angle LCP [Fig. Art. 140].

The equation to the straight line joining the points α and β can be easily shewn to be

$$(x-h)\cos\frac{\alpha+\beta}{2} + (y-k)\sin\frac{\alpha+\beta}{2} = a\cos\frac{\alpha-\beta}{2},$$

and so the tangent at the point α is

$$(x-h)\cos\alpha + (y-k)\sin\alpha = a.$$

***180. Common tangents to two circles.** If O_1 and O_2 be the centres of two circles whose radii are r_1 and r_2, and if one pair of common tangents meet O_1O_2 in T_1 and the other pair meet it in T_2, then, by similar triangles, we have $\frac{O_1T_2}{T_2O_1} = \frac{r_1}{r_2} = \frac{O_1T_1}{O_2T_1}$. The points T_1 and T_2 therefore divide O_1O_2 in the ratio of the radii.

The coordinates of T_1 having been found, the corresponding tangents are straight lines passing through it, such that the perpendiculars on them from O_1 are each equal to r_1. So for the other pair which pass through T_2.

Ex. *Find the four common tangents to the circles*

$$x^2+y^2-22x+4y+100=0, \text{ and } x^2+y^2+22x-4y-100=0.$$

The equations may be written

$$(x-11)^2+(y+2)^2=5^2, \text{ and } (x+11)^2+(y-2)^2=15^2.$$

The centre of the first is the point $(11, -2)$ and its radius is 5.

The centre of the second is the point $(-11, 2)$ and its radius is 15.

Then T_2 is the point dividing internally the line joining the centres in the ratio 5 : 15 and hence (Art. 22) its coordinates are

$$\frac{15\times 11+5\times(-11)}{15+5} \text{ and } \frac{15\times(-2)+5\times 2}{15+5},$$

that is, T_2 is the point $(\frac{11}{2}, -1)$.

Similarly T_1 is the point dividing this line externally in the ratio 5 : 15, and hence its coordinates are

$$\frac{15\times 11-5\times(-11)}{15-5} \text{ and } \frac{15\times(-2)-5\times 2}{15-5},$$

that is, T_1 is the point (22, -4).

Let the equation to either of the tangents passing through T_2 be

$$y+1=m\left(x-\tfrac{11}{2}\right)\text{.............................(1).}$$

Then the perpendicular from the point (11, -2) on it is equal to ± 5, and hence

$$\frac{m\left(11-\frac{11}{2}\right)-(-2+1)}{\sqrt{1+m^2}}=\pm 5.$$

On solving, we have $m=-\frac{24}{7}$ or $\frac{4}{3}$.

The required tangents through T_2 are therefore

$$24x+7y=125, \text{ and } 4x-3y=25.$$

Similarly the equations to the tangents through T_1 are

$$y+4=m(x-22) \text{ ....(2),}$$

where
$$\frac{m(11-22)-(-2+4)}{\sqrt{1+m^2}}=\pm 5.$$

On solving, we have $m=\frac{7}{24}$ or $-\frac{3}{4}$.

On substitution in (2), the required equations are therefore

$$7x-24y=250 \text{ and } 3x+4y=50.$$

The four common tangents are therefore found.

181. We shall conclude this chapter with some miscellaneous examples on loci.

Ex. 1. *Find the locus of a point P which moves so that its distance from a given point O is always in a given ratio (n : 1) to its distance from another given point A.*

Take O as origin and the direction of OA as the axis of x. Let the distance OA be a, so that A is the point $(a, 0)$.

If (x, y) be the coordinates of any position of P we have

$$OP^2=n^2 \,.\, AP^2,$$

i.e.
$$x^2+y^2=n^2\left[(x-a)^2+y^2\right],$$

i.e.
$$(x^2+y^2)(n^2-1)-2an^2x+n^2a^2=0\text{..................(1).}$$

Hence, by Art. 143, the locus of P is a circle.

Let this circle meet the axis of x in the points C and D. Then OC and OD are the roots of the equation obtained by putting y equal to zero in (1).

Hence
$$OC=\frac{na}{n+1} \text{ and } OD=\frac{na}{n-1}.$$

We therefore have

$$CA = \frac{a}{n+1} \text{ and } AD = \frac{a}{n-1}.$$

Hence
$$\frac{OC}{CA} = \frac{OD}{AD} = n.$$

The points C and D therefore divide the line OA in the given ratio, and the required circle is on CD as diameter.

Ex. 2. *From any point on one given circle tangents are drawn to another given circle; prove that the locus of the middle point of the chord of contact is a third circle.*

Take the centre of the first circle as origin and let the axis of x pass through the centre of the second circle. Their equations are then

$$x^2 + y^2 = a^2 \quad \text{.............................} \quad (1),$$

and
$$(x-c)^2 + y^2 = b^2 \quad \text{..........................} \quad (2),$$

where a and b are the radii, and c the distance between the centres, of the circles.

Any point on (1) is $(a\cos\theta,\ a\sin\theta)$ where θ is variable. Its chord of contact with respect to (2) is

$$(x-c)(a\cos\theta - c) + ya\sin\theta = b^2 \quad \text{.................} \quad (3).$$

The middle point of this chord of contact is the point where it is met by the perpendicular from the centre, *viz.* the point $(c,\ 0)$.

The equation to this perpendicular is (Art. 70)

$$-(x-c)\,a\sin\theta + (a\cos\theta - c)\,y = 0 \quad \text{..............} \quad (4).$$

Any equation deduced from (3) and (4) is satisfied by the coordinates of the point under consideration. If we eliminate θ from them, we shall have an equation always satisfied by the coordinates of the point, whatever be the value of θ. The result will thus be the equation to the required locus.

Solving (3) and (4), we have

$$a\sin\theta = \frac{b^2 y}{y^2 + (x-c)^2},$$

and
$$a\cos\theta - c = \frac{b^2(x-c)}{y^2 + (x-c)^2},$$

so that
$$a\cos\theta = c + \frac{b^2(x-c)}{y^2 + (x-c)^2}.$$

Hence

$$a^2 = a^2\cos^2\theta + a^2\sin^2\theta = c^2 + 2cb^2\frac{x-c}{y^2+(x-c)^2} + \frac{b^4}{y^2+(x-c)^2}.$$

The required locus is therefore

$$(a^2 - c^2)\left[y^2 + (x-c)^2\right] = 2cb^2(x-c) + b^4.$$

This is a circle and its centre and radius are easily found.

Ex. 3. *Find the locus of a point P which is such that its polar with respect to one circle touches a second circle.*

Taking the notation of the last article, the equations to the two circles are

$$x^2+y^2=a^2 \quad \dots\dots\dots (1),$$

and

$$(x-c)^2+y^2=b^2 \quad \dots\dots\dots (2).$$

Let (h, k) be the coordinates of any position of P. Its polar with respect to (1) is

$$xh+yk=a^2 \quad \dots\dots\dots (3).$$

Also any tangent to (2) has its equation of the form (Art. 179)

$$(x-c)\cos\theta+y\sin\theta=b \quad \dots\dots\dots (4).$$

If then (3) be a tangent to (2) it must be of the form (4).

Therefore $$\frac{\cos\theta}{h}=\frac{\sin\theta}{k}=\frac{c\cos\theta+b}{a^2}.$$

These equations give

$$\cos\theta\,(a^2-ch)=bh, \text{ and } \sin\theta\,(a^2-ch)=bk.$$

Squaring and adding, we have

$$(a^2-ch)^2=b^2(h^2+k^2) \quad \dots\dots\dots (5).$$

The locus of the point (h, k) is therefore the curve

$$b^2(x^2+y^2)=(a^2-cx)^2.$$

Aliter. The condition that (3) may touch (2) may be otherwise found.

For, as in Art. 153, the straight line (3) meets the circle (2) in the points whose abscissæ are given by the equation

$$k^2(x-c)^2+(a^2-hx)^2=b^2k^2,$$

i.e. $$x^2(h^2+k^2)-2x(ck^2+a^2h)+(k^2c^2+a^4-b^2k^2)=0.$$

The line (3) will therefore touch (2) if

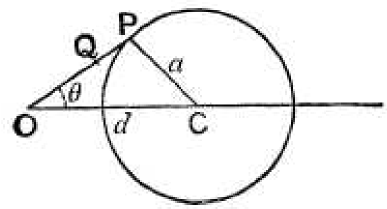
$$(ck^2+a^2h)^2=(h^2+k^2)(k^2c^2+a^4-b^2k^2),$$

i.e. if $$b^2(h^2+k^2)=(ch-a^2)^2,$$

which is equation (5).

Ex. 4. *O is a fixed point and P any point on a given circle; OP is joined and on it a point Q is taken so that OP . OQ = a constant quantity k^2; prove that the locus of Q is a circle which becomes a straight line when O lies on the original circle.*

Let O be taken as pole and the line through the centre C as the initial line. Let $OC=d$, and let the radius of the circle be a.



The equation to the circle is then $a^2=r^2+d^2-2rd\cos\theta$, (Art. 171), where $OP=r$ and $\angle POC=\theta$.

Let OQ be ρ, so that, by the given condition, we have $r\rho=k^2$ and hence $r=\frac{k^2}{\rho}$.

Substituting this value in the equation to the circle, we have

$$a^2=\frac{k^4}{\rho^2}+d^2-2\frac{k^2d}{\rho}\cos\theta \text{.......................}(1),$$

so that the equation to the locus of Q is

$$r^2-2\frac{k^2d}{d^2-a^2}r\cos\theta=-\frac{k^4}{d^2-a^2} \text{..................} (2).$$

But the equation to a circle, whose radius is a' and whose centre is on the initial line at a distance d', is

$$r^2-2rd'\cos\theta=a'^2-d'^2 \text{......................}(3).$$

Comparing (1) and (2), we see that the required locus is a circle, such that

$$d'=\frac{k^2d}{d^2-a^2} \text{ and } a'^2-d'^2=-\frac{k^4}{d^2-a^2}.$$

Hence $$a'^2=\frac{k^4}{d^2-a^2}\left[\frac{d^2}{d^2-a^2}-1\right]=\frac{k^4a^2}{(d^2-a^2)^2}.$$

The required locus is therefore a circle, of radius $\frac{k^2a}{d^2-a^2}$, whose centre is on the same line as the original centre at a distance $\frac{k^2d}{d^2-a^2}$ from the fixed point.

When O lies on the original circle the distance d is equal to a, and the equation (1) becomes $k^2=2dr\cos\theta$, *i.e.*, in Cartesian coordinates,

$$x=\frac{k^2}{2d}.$$

In this case the required locus is a straight line perpendicular to OC.

When a second curve is obtained from a given curve by the above geometrical process, the second curve is said to be the **inverse** of the first curve and the fixed point O is called the centre of inversion.

The inverse of a circle is therefore a circle or a straight line according as the centre of inversion is not, or is, on the circumference of the original circle.

Ex. 5. *PQ is a straight line drawn through O, one of the common points of two circles, and meets them again in P and Q; find the locus of the point S which bisects the line PQ.*

Take O as the origin, let the radii of the two circles be R and R', and let the lines joining their centres to O make angles α and α' with the initial line.

The equations to the two circles are therefore, {Art. 172 (2)},

$$r = 2R\cos(\theta - \alpha), \text{ and } r = 2R'\cos(\theta - \alpha').$$

Hence, if S be the middle point of PQ, we have

$$2OS = OP + OQ = 2R\cos(\theta - \alpha) + 2R'\cos(\theta - \alpha').$$

The locus of the point S is therefore

$$r = R\cos(\theta - \alpha) + R'\cos(\theta - \alpha')$$
$$= (R\cos\alpha + R'\cos\alpha')\cos\theta + (R\sin\alpha + R'\sin\alpha')\sin\theta$$
$$= 2R''\cos(\theta - \alpha'') \quad \ldots\ldots\ldots\ldots (1),$$

where $$2R''\cos\alpha'' = R\cos\alpha + R'\cos\alpha',$$

and $$2R''\sin\alpha'' = R\sin\alpha + R'\sin\alpha'.$$

Hence $$R'' = \tfrac{1}{2}\sqrt{R^2 + R'^2 + 2RR'\cos(\alpha - \alpha')},$$

and $$\tan\alpha'' = \frac{R\sin\alpha + R'\sin\alpha'}{R\cos\alpha + R'\cos\alpha'}.$$

From (1) the locus of S is a circle, whose radius is R'', which passes through the origin O and is such that the line joining O to its centre is inclined at an angle α'' to the initial line.

EXAMPLES XXII

1. A point moves so that the sum of the squares of its distances from the four sides of a square is constant; prove that it always lies on a circle.

2. A point moves so that the sum of the squares of the perpendiculars let fall from it on the sides of an equilateral triangle is constant; prove that its locus is a circle.

3. A point moves so that the sum of the squares of its distances from the angular points of a triangle is constant; prove that its locus is a circle.

4. Find the locus of a point which moves so that the square of the tangent drawn from it to the circle $x^2 + y^2 = a^2$ is equal to c times its distance from the straight line $lx + my + n = 0$.

5. Find the locus of a point whose distance from a fixed point is in a constant ratio to the tangent drawn from it to a given circle.

6. Find the locus of the vertex of a triangle, given (1) its base and the sum of the squares of its sides, (2) its base and the sum of m times the square of one side and n times the square of the other.

7. A point moves so that the sum of the squares of its distances from n fixed points is given. Prove that its locus is a circle.

8. Whatever be the value of α, prove that the locus of the intersection of the straight lines

$$x \cos \alpha + y \sin \alpha = a \text{ and } x \sin \alpha - y \cos \alpha = b$$

is a circle.

9. From a point P on a circle perpendiculars PM and PN are drawn to two radii of the circle which are not at right angles; find the locus of the middle point of MN.

10. Tangents are drawn to a circle from a point which always lies on a given line; prove that the locus of the middle point of the chord of contact is another circle.

11. Find the locus of the middle points of chords of the circle $x^2+y^2=a^2$ which pass through the fixed point (h, k).

12. Find the locus of the middle points of chords of the circle $x^2+y^2=a^2$ which subtend a right angle at the point $(c, 0)$.

13. O is a fixed point and P any point on a fixed circle; on OP is taken a point Q such that OQ is in a constant ratio to OP; prove that the locus of Q is a circle.

14. O is a fixed point and P any point on a given straight line; OP is joined and on it is taken a point Q such that $OP \, . \, OQ=k^2$; prove that the locus of Q, *i.e.* the inverse of the given straight line with respect to O, is a circle which passes through O.

15. One vertex of a triangle of given species is fixed, and another moves along the circumference of a fixed circle; prove that the locus of the remaining vertex is a circle and find its radius.

16. O is any point in the plane of a circle, and OP_1P_2 any chord of the circle which passes through O and meets the circle in P_1 and P_2. On this chord is taken a point Q such that OQ is equal to (1) the arithmetic, (2) the geometric, and (3) the harmonic mean between OP_1 and OP_2; in each case find the equation to the locus of Q.

17. Find the locus of the point of intersection of the tangent to any circle and the perpendicular let fall on this tangent from a fixed point on the circle.

18. A circle touches the axis of x and cuts off a constant length $2l$ from the axis of y; prove that the equation of the locus of its centre is $y^2-x^2=l^2 \operatorname{cosec}^2 \omega$, the axes being inclined at an angle ω.

19. A straight line moves so that the product of the perpendiculars on it from two fixed points is constant. Prove that the locus of the feet of the perpendiculars from each of these points upon the straight line is a circle, the same for each.

20. O is a fixed point and AP and BQ are two fixed parallel straight lines; BOA is perpendicular to both and POQ is a right angle. Prove that the locus of the foot of the perpendicular drawn from O upon PQ is the circle on AB as diameter.

21. Two rods, of lengths a and b, slide along the axes, which are rectangular, in such a manner that their ends are always concyclic; prove that the locus of the centre of the circle passing through these ends is the curve $4(x^2-y^2)=a^2-b^2$.

22. Shew that the locus of a point, which is such that the tangents from it to two given concentric circles are inversely as the radii, is a concentric circle, the square of whose radius is equal to the sum of the squares of the radii of the given circles.

23. Shew that if the length of the tangent from a point P to the circle $x^2+y^2=a^2$ be four times the length of the tangent from it to the circle $(x-a)^2+y^2=a^2$, then P lies on the circle

$$15x^2+15y^2-32ax+a^2=0.$$

Prove also that these three circles pass through two points and that the distance between the centres of the first and third circles is sixteen times the distance between the centres of the second and third circles.

24. Find the locus of the foot of the perpendicular let fall from the origin upon any chord of the circle $x^2+y^2+2gx+2fy+c=0$ which subtends a right angle at the origin.

Find also the locus of the middle points of these chords.

25. Through a fixed point O are drawn two straight lines OPQ and ORS to meet the circle in P and Q, and R and S, respectively. Prove that the locus of the point of intersection of PS and QR, as also that of the point of intersection of PR and QS, is the polar of O with respect to the circle.

26. A, B, C, and D are four points in a straight line; prove that the locus of a point P, such that the angles APB and CPD are equal, is a circle.

27. The polar of P with respect to the circle $x^2+y^2=a^2$ touches the circle $(x-\alpha)^2+(y-\beta)^2=b^2$; prove that its locus is the curve given by the equation $(\alpha x+\beta y-a^2)^2=b^2(x^2+y^2)$.

28. A tangent is drawn to the circle $(x-a)^2+y^2=b^2$ and a perpendicular tangent to the circle $(x+a)^2+y^2=c^2$; find the locus of their point of intersection, and prove that the bisector of the angle between them always touches one or other of two fixed circles.

29. In any circle prove that the perpendicular from any point of it on the line joining the points of contact of two tangents is a mean proportional between the perpendiculars from the point upon the two tangents.

30. From any point on the circle

$$x^2+y^2+2gx+2fy+c=0$$

tangents are drawn to the circle

$$x^2+y^2+2gx+2fy+c\sin^2\alpha+(g^2+f^2)\cos^2\alpha=0;$$

prove that the angle between them is 2α.

31. The angular points of a triangle are the points

$$(a\cos\alpha,\ a\sin\alpha),\ (a\cos\beta,\ a\sin\beta),\ \text{and}\ (a\cos\gamma,\ a\sin\gamma);$$

prove that the coordinates of the orthocentre of the triangle are

$$a(\cos\alpha+\cos\beta+\cos\gamma)\ \text{and}\ a(\sin\alpha+\sin\beta+\sin\gamma).$$

Hence prove that if A, B, C, and D be four points on a circle the orthocentres of the four triangles ABC, BCD, CDA, and DAB lie on a circle.

32. A variable circle passes through the point of intersection O of any two straight lines and cuts off from them portions OP and OQ such that $m\,.\,OP+n\,.\,OQ$ is equal to unity; prove that this circle always passes through a fixed point.

33. Find the length of the common chord of the circles, whose equations are $(x-a)^2+y^2=a^2$ and $x^2+(y-b)^2=b^2$, and prove that the equation to the circle whose diameter is this common chord is

$$(a^2+b^2)(x^2+y^2)=2ab(bx+ay).$$

34. Prove that the length of the common chord of the two circles whose equations are

$$(x-a)^2+(y-b)^2=c^2\ \text{and}\ (x-b)^2+(y-a)^2=c^2$$

is

$$\sqrt{4c^2-2(a-b)^2}.$$

Hence find the condition that the two circles may touch.

35. Find the length of the common chord of the circles

$$x^2+y^2-2ax-4ay-4a^2=0\ \text{and}\ x^2+y^2-3ax+4ay=0.$$

Find also the equations of the common tangents and shew that the length of each is $4a$.

36. Find the equations to the common tangents of the circles

(1) $x^2+y^2-2x-6y+9=0$ and $x^2+y^2+6x-2y+1=0$,

(2) $x^2+y^2=c^2$ and $(x-a)^2+y^2=b^2$.

ANSWERS

4. A circle. 5. A circle. 6. A circle.

9. $x^2+y^2-2xy\cos\omega=\frac{a^2\sin^2\omega}{4}$, the given radii being the axes.

11. A circle. 12. A circle.

16. (1) A circle; (2) A circle; (3) The polar of O.

17. The curve $r=a+a\cos\theta$, the fixed point O being the origin and the centre of the circle on the initial line.

24. The same circle in each case.

33. $2ab\div\sqrt{a^2+b^2}$. 35. $a\sqrt{\frac{47}{45}}$; $x=4a$; $63x+16y+100a=0$.

36. (i) $x=0$, $3x+4y=10$, $y=4$, and $3y=4x$.

(ii) $y=mx+c\sqrt{1+m^2}$, where

$$m=\frac{\pm(b+c)}{\sqrt{a^2-(b+c)^2}}, \text{ or } \frac{\pm(b-c)}{\sqrt{a^2-(b-c)^2}}.$$

SOLUTIONS/HINTS

1. Take parallels to the sides of the square through its centre as axes, and let the side of the square $=2a$.

Then $(x-a)^2+(y-a)^2+(x+a)^2+(y+a)^2=c^2$.

$\therefore\ 2x^2+2y^2+4a^2=c^2$, which is a circle.

2. Let $x=0$, $y=0$, $x+y=a$ be the equations of the sides of the equilateral triangle.

Then $x^2\sin^2 60^\circ+y^2\sin^2 60^\circ+\frac{(x+y-a)^2}{2-2\cos 60^\circ}\sin^2 60^\circ=\text{cons.}$ (Art. 96).

$\therefore\ 2x^2+2y^2+2xy-2a(x+y)+a^2=\text{cons.}$, which is a circle by Art. 176.

3. Let $(a, 0)$, $(-a, 0)$, (h, k) be the coordinates of the angular points of the triangle.

Then $(x-a)^2+y^2+(x+a)^2+y^2+(x-h)^2+(y-k)^2=\text{cons.}$

$\therefore\ 3x^2+3y^2-2hx-2ky=\text{cons.}$, which is the equation of a circle.

4. $x^2+y^2-a^2=c.\dfrac{lx+my+n}{\sqrt{l^2+m^2}}$, which is the equation of a circle.

5. Take $(0, 0)$ for fixed point, and the general equation for the circle.

Then, by Art. 168, $x^2+y^2+2gx+2fy+c=\lambda(x^2+y^2)$; giving a circle.

6. (1) Let $(a, 0)$, $(-a, 0)$, (x, y) be the coordinates of the angular points.

Then $$(x-a)^2+y^2+(x+a)^2+y^2=\text{cons.}$$

$$\therefore\ 2x^2+2y^2=\text{cons.};\ \text{which is a circle.}$$

(2) $$m\{(x-a)^2+y^2\}+n\{(x+a)^2+y^2\}=\text{cons.},$$

i.e. $$(m+n)(x^2+y^2)-2ax(m-n)=\text{cons.},$$

which is the equation to a circle.

7. Let (a_1, b_1), (a_2, b_2), ... (a_n, b_n) be the coordinates of the fixed points.

Then $$\Sigma_{r=1}^{r=n}\{(x-a_r)^2+(y-b_r)^2\}=\text{cons.}$$

$\therefore\ n(x^2+y^2)-2\Sigma a_r x-2\Sigma b_r y=\text{cons.}$, which is a circle.

8. Square and add the given equations;

$$\therefore\ x^2+y^2=a^2+b^2;\ \text{the equation to a circle.}$$

9. Take the given radii for axes, and (h, k) as the coordinates of P, a as the radius of the circle, and (x, y), the coordinates of the middle point of MN. Then, since

$$OM=h+k\cos\omega \text{ and } ON=k+h\cos\omega,$$

we have $2x=h+k\cos\omega$ and $2y=k+h\cos\omega$; also

$$h^2+k^2+2hk\cos\omega=a^2.$$

Eliminating (h, k), we have

$$4(x-y\cos\omega)^2+4(y-x\cos\omega)^2$$
$$+8(x-y\cos\omega)(y-x\cos\omega)\cos\omega=a^2\sin^4\omega,$$

i.e. $$x^2+y^2-2xy\cos\omega=\tfrac{1}{4}a^2\sin^2\omega;\ \text{which is a circle.}$$

10. Let $x^2+y^2=a^2$ be the equation to the circle, and (h, k) the coordinates of the point which lies on the line $gx+fy=c$. The required locus is the intersection of $xh+yk=a^2$ with $\frac{x}{h}=\frac{y}{k}$, with the condition $gh+fk=c$. Whence, eliminating (h, k), we have $c(x^2+y^2)=a^2(gx+fy)$; the equation to a circle.

11. Let (x, y) be the coordinates of the middle point of the chord joining the points α and β. (Art. 178.)

Then $x=\frac{1}{2}a(\cos\alpha+\cos\beta)=a\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$, ...(i)

and $y=\frac{1}{2}a(\sin\alpha+\sin\beta)=a\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$. .....(ii)

Also $h\cos\frac{\alpha+\beta}{2}+k\sin\frac{\alpha+\beta}{2}=a\cos\frac{\alpha-\beta}{2}$. ...(iii)

Multiply (iii) by $a\cos\frac{\alpha-\beta}{2}$; then

$$xh+yk=a^2\cos^2\frac{\alpha-\beta}{2}=x^2+y^2, \text{ from (i) and (ii).}$$

12. Using the same equations, the condition gives

$$\frac{a\sin\alpha}{a\cos\alpha-c}\cdot\frac{a\sin\beta}{a\cos\beta-c}+1=0,$$

i.e. $$a^2\cos(\alpha-\beta)-ac(\cos\alpha+\cos\beta)+c^2=0,$$

i.e. $$2a^2\cos^2\frac{\alpha-\beta}{2}-a^2-2cx+c^2=0,$$

i.e. $$2(x^2+y^2)-a^2-2cx+c^2=0;$$

which is the equation to a circle.

13. Change r into μr in the general polar equation to a circle. (Art. 171.)

14. Change r into k^2/r in the general polar equation to a straight line.

15. Take A for origin and the line joining A to the centre of the locus of B as initial line; and let the equation of the locus of B be $c^2 + r^2 - 2cr\cos\theta = a^2$.

Let (ρ, ϕ) be the coordinates of B, and (r, θ) those of C. Then $r = \rho/\lambda$, where λ is a constant,

$$\theta = \phi + A\text{; also } \rho^2 + c^2 - 2c\rho\cos\phi = a^2.$$

$$\therefore\ \lambda^2 r^2 + c^2 - 2c\lambda r\cos(\theta - A) = a^2,$$

or, in Cartesians, $\quad (\lambda x - c\cos A) + (\lambda y - c\sin A)^2 = a^2.$

$$\therefore\ \text{the radius} = a/\lambda.$$

16. Let $r^2 - 2r\rho\cos(\theta - \alpha) + \rho^2 - a^2 = 0$ be the equation to the circle.

(i) $\quad 2r = r_1 + r_2 = 2\rho\cos(\theta - \alpha)$; which is a circle.

(ii) $\quad r^2 = r_1 r_2 = \rho^2 - a^2$; which is a circle.

(iii) $\quad \frac{2}{r} = \frac{1}{r_1} + \frac{1}{r_2} = \frac{2\rho\cos(\theta - \alpha)}{\rho^2 - a^2}$; which is a straight line.

17. Take the fixed point for origin, and $r = 2a\cos\theta$, the equation to the circle. The required locus is the intersection of $r\cos(\theta - 2\alpha) = 2a\cos^2\alpha$ (Art. 174) with

$$\theta = 2\alpha. \quad \therefore\ r = 2a\cos^2\frac{\theta}{2} = a(1 + \cos\theta).$$

18. Taking the general equation of Art. 175, since the circle touches the axis of x,

$$\therefore\ (h + k\cos\omega)^2 - (h^2 + k^2 + 2hk\cos\omega - a^2) = 0.$$

$$\therefore\ k^2\sin^2\omega - a^2 = 0.$$

Also $\quad (k + h\cos\omega)^2 - (h^2 + k^2 + 2hk\cos\omega - a^2) = l^2.$

$\therefore\ a^2 - h^2\sin^2\omega = l^2. \quad \therefore\ k^2 - h^2 = l^2\operatorname{cosec}^2\omega.$ Hence etc.

19. Let the two fixed points be $(a, 0)$ and $(-a, 0)$, and let the straight line be

$$x\cos\omega + y\sin\omega = p. \ \ldots\ldots\ldots\ldots\ldots(1)$$

Since the product of the perpendiculars is constant $(= c^2$, say)

$$\therefore\ (a\cos\omega - p)(-a\cos\omega - p) = c^2,$$

i.e. $$p^2 = a^2\cos^2\omega + c^2. \ldots\ldots\ldots\ldots\ldots\ldots(2)$$

The equation to the perpendicular from $(a, 0)$ upon (1) is, by Art. 70,

$$(x-a)\sin\omega - y\cos\omega = 0,$$

i.e. $$x\sin\omega - y\cos\omega = a\sin\omega. \quad\ldots\ldots\ldots\ldots(3)$$

We have to eliminate p, ω from (1), (2), (3). Squaring and adding (1) and (3), we have

$$x^2 + y^2 = p^2 + a^2\sin^2\omega = a^2 + c^2, \text{ from (2)}.$$

Hence the locus is a circle, which remains unaltered when a is changed into $-a$. Hence, etc.

20. Take O as origin, and let the equation to AP be $x = a$, and that to BQ be $x = -b$.

Let $x\cos\omega + y\sin\omega = p$, be the equation to PQ, so that P is the point $\left(a, \dfrac{p - a\cos\omega}{\sin\omega}\right)$ and Q is the point

$$\left(-b, \frac{p + b\cos\omega}{\sin\omega}\right).$$

Since OP, OQ are at right angles,

$$\therefore \frac{p - a\cos\omega}{a\sin\omega} \times \frac{p + b\cos\omega}{-b\sin\omega} = -1. \quad \text{(Art. 69.)}$$

$$\therefore p^2 - (b-a)p\cos\omega - ab = 0.$$

Now p, ω are the polar coordinates of the foot of the perpendicular from O on PQ, so that the polar equation to the locus required is $r^2 + (b-a)r\cos\theta - ab = 0$, or, in Cartesians, $x^2 + y^2 + (b-a)x - ab = 0$, which is a circle on AB as diameter.

21. Let (h, k) be the coordinates of the centre of the circle for any position of the rods.

Then the intercepts of the rods on the axis of x, will be $h - \dfrac{a}{2}$, $h + \dfrac{a}{2}$, and on the axis of y they are $k - \dfrac{b}{2}$, $k + \dfrac{b}{2}$.

Hence $$\left(h - \frac{a}{2}\right)\left(h + \frac{a}{2}\right) = \left(k - \frac{b}{2}\right)\left(k + \frac{b}{2}\right).$$

Therefore the required locus is $4(x^2 - y^2) = a^2 - b^2$.

22. Let $x^2+y^2=a^2$ and $x^2+y^2=b^2$ be the equation to the two given circles.

Then $a^2(x^2+y^2-a^2)=b^2(x^2+y^2-b^2)$, *i.e.* $x^2+y^2=a^2+b^2$.

23. We have $x^2+y^2-a^2=16(x-a)^2+16y^2-16a^2$,

i.e. $$15x^2+15y^2-32ax+a^2=0.$$

The coordinates of the centres are $(0, 0)$; $(a, 0)$; $\left(\frac{16a}{15}, 0\right)$, which are all on the axis of x.

24. (i) The lines joining the common points of the circle and the chord $x\cos\alpha+y\sin\alpha=p$ to the origin are (Art. 122)

$$p^2(x^2+y^2)+2(gx+fy)(x\cos\alpha+y\sin\alpha)p + c(x\cos\alpha+y\sin\alpha)^2=0$$

which are at right angles if (Art. 111)

$$2p^2+2p(g\cos\alpha+f\sin\alpha)+c=0.$$

Hence, for the required locus, we have

$$2(x^2+y^2)+2gx+2fy+c=0,$$

since p, α are the polar coordinates of the foot of the perpendicular from $(0, 0)$.

(ii) We have to eliminate x_1, x_2, y_1, y_2 between

$$x_1^2+y_1^2+2gx_1+2fy_1+c=0,\quad x_2^2+y_2^2+2gx_2+2fy_2+c=0,$$
$$x_1x_2+y_1y_2=0,\quad 2x=x_1+x_2,\text{ and } 2y=y_1+y_2.$$

From the first three of these, we have

$$(x_1+x_2)^2+(y_1+y_2)^2+2g(x_1+x_2)+2f(y_1+y_2)+2c=0.$$
$$\therefore\ 2x^2+2y^2+2gx+2fy+c=0.$$

25. Take OPQ, ORS as the axes, and let

$$OP=x_1,\quad OQ=x_2,\quad OR=y_1 \text{ and } OS=y_2.$$

Let the equation to the circle be

$$x^2+2xy\cos\omega+y^2+2gx+2fy+c=0. \quad\ldots\ldots (1)$$

Then, as in Art. 146,

$$x_1+x_2=-2g,\quad y_1+y_2=-2f, \quad\ldots\ldots\ldots (2)$$

and $$x_1x_2=y_1y_2=c. \quad\ldots\ldots\ldots\ldots\ldots\ldots (3)$$

The equations to PR and QS are

$$\frac{x}{x_1}+\frac{y}{y_1}=1 \text{ and } \frac{x}{x_2}+\frac{y}{y_2}=1.$$

On addition, their point of intersection lies on

$x\dfrac{x_1+x_2}{x_1x_2}+y\dfrac{y_1+y_2}{y_1y_2}=2$, *i.e.* by equations (2) and (3) on

$$-x\frac{2g}{c}-y\frac{2f}{c}=2, \text{ i.e. on } gx+fy+c=0,$$

which, as in Art. 163, is the polar of (i) with respect to the origin.

The same would be true for any other pair of lines drawn through O. Hence in all cases the corresponding point of intersection lies on the polar of O. Hence, etc.

The same proposition may similarly be shewn for the intersection of PS and QR.

26. See Ex. XI. 18.

27. Let P be the point (x', y'). Its polar with regard to the first circle, viz. $xx'+yy'=a^2$, will be identical with $(x-\alpha)\cos\theta+(y-\beta)\sin\theta=b$, (any tangent to the second circle), if

$$\frac{\cos\theta}{x'}=\frac{\sin\theta}{y'}=\frac{b+\alpha\cos\theta+\beta\sin\theta}{a^2}=\frac{b}{a^2-\alpha x'-\beta y'}.$$

Therefore, eliminating θ, the locus of P is given by

$$(\alpha x+\beta y-a^2)^2=b^2(x^2+y^2).$$

28. Any tangent to the first circle is

$$(x-a)\cos\alpha+y\sin\alpha=b;$$

A perpendicular tangent to the second circle is

$$(x+a)\cos\left(\alpha\pm\frac{\pi}{2}\right)+y\sin\left(\alpha\pm\frac{\pi}{2}\right)=c, \text{ (Art. 179)},$$

or

$$y\cos\alpha-(x+a)\sin\alpha=\pm c;$$

$$\therefore \frac{\cos\alpha}{\mp cy-b(x+a)}=\frac{\sin\alpha}{-by\pm c(x-a)}=\frac{1}{a^2-x^2-y^2}.$$

Hence for the required locus, we have

$$\{cy \pm b(x+a)\}^2 + \{by \mp c(x-a)\}^2 = \{x^2+y^2-a^2\}^2.$$

The equation to the bisectors is (Art. 84)

$$(x-a)\cos\alpha + y\sin\alpha - b \pm \{y\cos\alpha - (x+a)\sin\alpha \mp c\} = 0,$$

or $$x\cos\left(\alpha - \frac{\pi}{4}\right) + (y+a)\sin\left(\alpha - \frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}(b \mp c),$$

and $$x\cos\left(\alpha + \frac{\pi}{4}\right) + (y-a)\sin\left(\alpha + \frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}(b \pm c),$$

which are tangents to the circles whose equations are

$x^2+(y+a)^2 = \frac{1}{2}(b \mp c)^2$, and $x^2+(y-a)^2 = \frac{1}{2}(b \pm c)^2$. (Art. 179.)

29. Let α, β be the points of contact of the tangents to the circle $x^2+y^2=a^2$, and ϕ the point from which the perpendiculars are drawn.

Then, if p_1, p_2, p are the perpendiculars,

$p_1 =$ the perpendicular from $(a\cos\phi,\ a\sin\phi)$ upon

$$x\cos\alpha + y\sin\alpha = a,$$

and $\therefore = a - a\cos(\phi-\alpha) = 2a\sin^2\dfrac{\phi-\alpha}{2}$;

$p_2 =$ the perpendicular from $(a\cos\phi,\ a\sin\phi)$ upon

$$x\cos\beta + y\sin\beta = a,$$

and $\therefore = a - a\cos(\phi-\beta) = 2a\sin^2\dfrac{\phi-\beta}{2}$;

and $p =$ the perpendicular from $(a\cos\phi,\ a\sin\phi)$ upon

$$x\cos\frac{\alpha+\beta}{2} + y\sin\frac{\alpha+\beta}{2} = a\cos\frac{\alpha-\beta}{2},$$

and $\therefore = a\cos\dfrac{\alpha-\beta}{2} - a\cos\left(\phi - \dfrac{\alpha+\beta}{2}\right) = 2a\sin\dfrac{\phi-\alpha}{2}\cdot\sin\dfrac{\phi-\beta}{2}$.

$$\therefore\ p_1p_2 = p^2.$$

30. The circles are concentric. If O be the centre and TP, TQ the tangents, then

$$OT = \sqrt{g^2+f^2-c} \text{ and } OP = \sqrt{g^2+f^2-c}\,.\sin\alpha.$$

$$\therefore\ O\hat{T}P = \alpha, \text{ and } P\hat{T}Q = 2\alpha.$$

31. As in Ex. 2, Page 56, the equations to the perpendiculars from the first two angular points upon the opposite sides are

$$(x - a\cos\alpha)(\cos\beta - \cos\gamma) + (y - a\sin\alpha)(\sin\beta - \sin\gamma) = 0,$$

and

$$(x - a\cos\beta)(\cos\gamma - \cos\alpha) + (y - a\sin\beta)(\sin\gamma - \sin\alpha) = 0.$$

These are clearly satisfied by

$$x = a(\cos\alpha + \cos\beta + \cos\gamma), \quad y = a(\sin\alpha + \sin\beta + \sin\gamma),$$

so that these coordinates give the orthocentre.

If A, B, C, D be the points α, β, γ, δ, then, on writing down the coordinates of the orthocentres of the four triangles, we easily see that they lie on the circle

$$[x - a(\cos\alpha + \cos\beta + \cos\gamma + \cos\delta)]^2 + [y - a(\sin\alpha + \sin\beta + \sin\gamma + \sin\delta)]^2 = a^2.$$

32. Taking the two straight lines for axes, let

$$x^2 + y^2 + 2xy\cos\omega = 2ax + 2by$$

be the equation to the circle. Then $ma + nb = \frac{1}{2}$.

If (h, k) be the coordinates of the centre, then

$$m(h + k\cos\omega) + n(k + h\cos\omega) = \tfrac{1}{2} \quad \text{(Art. 175)},$$

so that the centre lies on the straight line

$$x(m + n\cos\omega) + y(n + m\cos\omega) = \tfrac{1}{2}.$$

Hence, if ON be drawn perpendicular to this line and produced to O' so that $O'N = ON$, the circle will pass through the point O'.

33. The circles pass through the origin, and the other common point is $\left(\dfrac{2ab^2}{a^2 + b^2}, \dfrac{2a^2b}{a^2 + b^2}\right)$.

$$\therefore\ (\text{chord})^2 = \frac{4(a^2b^4 + a^4b^2)}{(a^2+b^2)^2};\quad \therefore\ \text{chord} = \frac{2ab}{\sqrt{a^2+b^2}}.$$

The equation to the circle is (Art. 145)

$$x\{(a^2+b^2)x - 2ab^2\} + y\{(a^2+b^2)y - 2a^2b\} = 0,$$

i.e.

$$(a^2+b^2)(x^2+y^2) = 2ab(bx+ay).$$

34. Let P be a common point, A and B the centres and C the middle point of AB. Then since $PA = PB = c$,

$\therefore$ common chord $= 2PC = \sqrt{4PA^2 - 4AC^2} = \sqrt{4c^2 - 2(a-b)^2}$.

If the circles touch, $PC = 0$; $\therefore\ a - b = \pm\sqrt{2}\,.\,c$.

35. The equations may be written,

$$(x-a)^2 + (y-2a)^2 = (3a)^2, \text{ and } \left(x - \frac{3a}{2}\right)^2 + (y+2a)^2 = \left(\frac{5a}{2}\right)^2.$$

Let A and B be the centres, C a common point, $A\hat{C}B = \alpha$, and d the length of common chord.

Then $\quad AB^2 = AC^2 + CB^2 - 2AC\,.\,CB\,.\cos\alpha$.

$\therefore\ \frac{65}{4} = 9 + \frac{25}{4} - 15\,.\cos\alpha$; $\therefore\ \cos\alpha = -\frac{1}{15}$ and $\sin\alpha = \frac{\sqrt{224}}{15}$.

Now $d\,.\,AB = 4\triangle ACB = 2AC\,.\,CB\sin\alpha$

$$= 2(3a)\left(\frac{5a}{2}\right)\frac{\sqrt{224}}{15} = \sqrt{224}\,a^2.$$

$$\therefore\ d = 2\sqrt{\tfrac{224}{65}}\,.\,a = 8\sqrt{\tfrac{14}{65}}\,.\,a.$$

Let $x\cos\alpha + y\sin\alpha - p = 0$ be the equation to a common tangent.

Then $\quad a\cos\alpha + 2a\sin\alpha - p = \pm 3a$,

and $\quad \frac{3}{2}a\cos\alpha - 2a\sin\alpha - p = \pm\frac{5a}{2}$.

Subtracting, the only equation giving real values of α is

$$-\tfrac{1}{2}\cos\alpha + 4\sin\alpha = \tfrac{1}{2}.$$

Whence, solving,

$$\left.\begin{aligned}\sin\alpha&=0,\\ \cos\alpha&=-1,\\ p&=-4a\end{aligned}\right\}\text{ or }\left\{\begin{aligned}\sin\alpha&=\tfrac{16}{65},\\ \cos\alpha&=\tfrac{63}{65},\\ p&=-\frac{100a}{65}\end{aligned}\right\},$$

so that the equations of the common tangents are

$$x=4a,\ \text{ and }\ 63x+16y+100a=0.$$

$$\therefore\ (\text{length})^2=AB^2-\{\text{difference of radii}\}^2=\tfrac{65}{4}a^2-\frac{a^2}{4}=16a^2.$$

$$\therefore\ \text{length}=4a.$$

36. (1) The equations are

$$(x-1)^2+(y-3)^2=1^2,\ \text{ and }\ (x+3)^2+(y-1)^2=3^2.$$

If $x\cos\alpha+y\sin\alpha=p$ is a common tangent

then $\cos\alpha+3\sin\alpha-p=\pm 1$, and $-3\cos\alpha+\sin\alpha-p=\pm 3$.

Subtracting, $2\cos\alpha+\sin\alpha=\pm 1$ or ± 2;

$$\therefore\ 3\cos^2\alpha+4\cos\alpha\sin\alpha=0,\ \text{ or }\ 3\sin^2\alpha-4\cos\alpha\sin\alpha=0.$$

Whence $\cos\alpha=0,\ -\tfrac{4}{5},\ 1,$ or $\tfrac{3}{5}$.

$\sin\alpha=1,\ \tfrac{3}{5},\ 0,$ or $\tfrac{4}{5}$,

and $p=4,\ 0,\ 0,$ or $\tfrac{6}{5}$.

Hence the four common tangents are

$$y=4,\quad 4x-3y=0,\quad x=0,\ \text{ and }\ 3x+4y=10.$$

(2) The line $y=mx+c\sqrt{1+m^2}$, which is a tangent to the first circle, will touch the second if

$$\{cm\sqrt{1+m^2}-a\}^2=(1+m^2)\left[a^2-b^2+c^2(1+m^2)\right],$$

which becomes

$$m^4\{a^4+(c^2-b^2)^2-2a^2(c^2+b^2)\}$$
$$+2m^2\{(c^2-b^2)^2-a^2(c^2+b^2)]+(c^2-b^2)^2=0,$$

whence
$$m^2=\frac{(b+c)^2}{a^2-(b+c)^2}\ \text{ or }\ \frac{(b-c)^2}{a^2-(b-c)^2}.$$

Substituting for m, we have the required equations.