3. Trigonometric Functions

- **General solutions of some trigonometric equations:**
  - $\sin x = 0 \Rightarrow x = n\pi$, , where $n \in \mathbf{Z}$
  - $\cos x = 0 \Rightarrow x = (2n+1)\frac{\pi}{2}$,where $n \in \mathbf{Z}$
  - $\sin x = \sin y \Rightarrow x = n\pi + (-1)^n y$, where $n \in \mathbf{Z}$
  - $\cos x = \cos y \Rightarrow x = 2n\pi \pm y$, where $n \in \mathbf{Z}$
  - $\tan x = \tan y \Rightarrow x = n\pi + y$, where $n \in \mathbf{Z}$

Example 1: Solve $\cot x \cos^2 x = 2 \cot x$
Solution:

$$\cot x \cos^2 x = 2\cot x$$
$$\Rightarrow \cot x \cos^2 x - 2\cot x = 0$$
$$\Rightarrow \cot x(\cos^2 x - 2) = 0$$
$$\Rightarrow \cot x = 0 \text{ or } \cos^2 x = 2$$
$$\Rightarrow \frac{\cos x}{\sin x} = 0 \text{ or } \cos x = \pm\sqrt{2}$$
$$\Rightarrow \cos x = 0 \text{ or } \cos x = \pm\sqrt{2}$$
$$\text{Now, } \cos x = 0 \Rightarrow x = (2n+1)\frac{\pi}{2}, \text{ where } n \in \mathbf{Z}$$
$$\text{and } \cos x = \pm\sqrt{2}$$

But this is not possible as $-1 \le \cos x \le 1$

Thus, the solution of the given trigonometric equation is $x = (2n+1)\frac{\pi}{2}$ where $n \in$ Z.

Example 2: Solve $\sin 2x + \sin 4x + \sin 6x = 0$.
Solution:

$\sin 4x + (\sin 2x + \sin 6x) = 0$

$\Rightarrow \sin 4x + 2\sin\left(\frac{2x+6x}{2}\right)\cos\left(\frac{2x-6x}{2}\right) = 0$

$\Rightarrow \sin 4x + 2\sin 4x \cos 2x = 0$

$\Rightarrow \sin 4x(1 + 2\cos 2x) = 0$

$\Rightarrow \sin 4x = 0$ or $1 + 2\cos 2x = 0$

$\Rightarrow \sin 4x = 0$ or $\cos 2x = -\frac{1}{2}$

$\sin 4x = 0$

$\Rightarrow 4x = n\pi, n \in Z$

$\Rightarrow x = \frac{n\pi}{4}, n \in Z$

$\cos 2x = -\frac{1}{2}$

$\Rightarrow \cos 2x = \cos\frac{2\pi}{3}$

$\Rightarrow 2x = 2m\pi \pm \frac{2\pi}{3}, m \in Z$

$\Rightarrow x = m\pi \pm \frac{\pi}{3}, m \in Z$

Thus, $x = \frac{n\pi}{4}$ or $x = m\pi \pm \frac{\pi}{3}$, where $m, n \in \mathbf{Z}$

1. If A(x, y) is any point on the terminal arm OQ such that OA = $r = \sqrt{x^2 + y^2}$ and ∠POQ = q then:

$\sin q = \frac{y}{r}$

$\cos q = \frac{x}{r}$

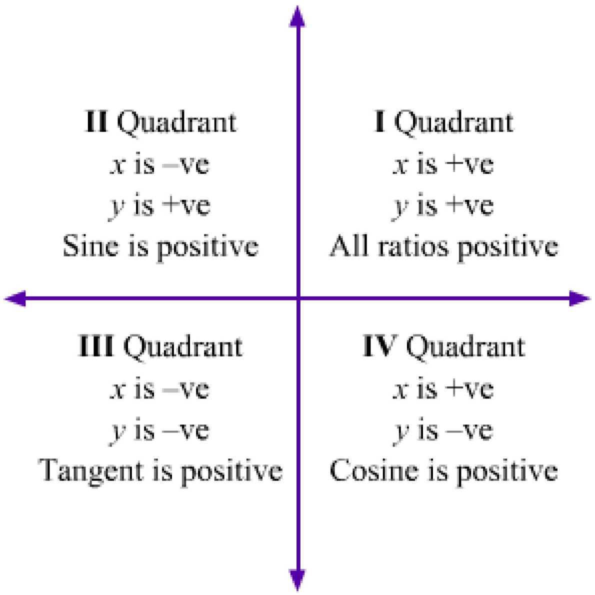
$\tan q = \frac{y}{x}$, where $x \neq 0$

$\text{cosec } q = \frac{r}{y}$, where $y \neq 0$

$\sec q = \frac{r}{x}$, where $x \neq 0$

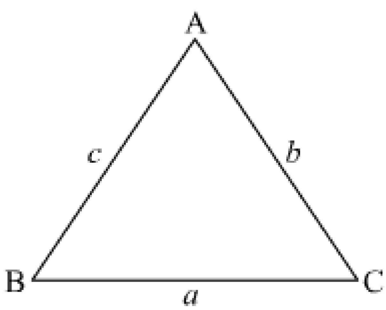
$\cot q = \frac{x}{y}$, where $y \neq 0$

2. The signs of various trigonometric ratios in different quadrants are as follows:



Properties and Solutions of Triangles

A triangle is a polygon having three sides and three angles. Consider a ΔABC whose lengths of the sides AB, BC and CA are c, a and b respectively.



Some geometrical properties related to ΔABC

1. $\angle A + \angle B + \angle C = 180° = \pi$ radians

2. Perimeter, $2s = a + b + c$

Semi-perimeter, $s = \dfrac{a+b+c}{2}$

3. Sum of any two sides of a triangle is always greater than the third side.

$a + b > c$, $b + c > a$ and $c + a > b$

4. Difference of any two sides of a triangle is always less than the third side.

$|a-b|<c, |b-c|<a$ and $|c-a|<b$

Sine Rule

In any ΔABC, $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$

Cosine Rule

In any ΔABC,

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$
$$\cos B = \frac{c^2 + a^2 - b^2}{2ca}$$
$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

Projection Formulae

In any ΔABC,

$$a = b\cos C + c\cos B$$
$$b = c\cos A + a\cos C$$
$$c = a\cos B + b\cos A$$

Napier's Analogy

In any ΔABC,

$$\tan\left(\frac{A-B}{2}\right) = \left(\frac{a-b}{a+b}\right)\cot\frac{C}{2}$$
$$\tan\left(\frac{B-C}{2}\right) = \left(\frac{b-c}{b+c}\right)\cot\frac{A}{2}$$
$$\tan\left(\frac{C-A}{2}\right) = \left(\frac{c-a}{c+a}\right)\cot\frac{B}{2}$$

Half-Angle Formulae

In any ΔABC,

$$\sin\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}},\ \sin\frac{B}{2} = \sqrt{\frac{(s-c)(s-a)}{ca}},\ \sin\frac{C}{2} = \sqrt{\frac{(s-a)(s-b)}{ab}}$$

$$\cos\frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}},\ \cos\frac{B}{2} = \sqrt{\frac{s(s-b)}{ca}},\ \cos\frac{C}{2} = \sqrt{\frac{s(s-c)}{ab}}$$

$$\tan\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}},\ \tan\frac{B}{2} = \sqrt{\frac{(s-c)(s-a)}{s(s-b)}},\ \tan\frac{C}{2} = \sqrt{\frac{(s-a)(s-b)}{s(s-c)}}$$

Area of triangle:

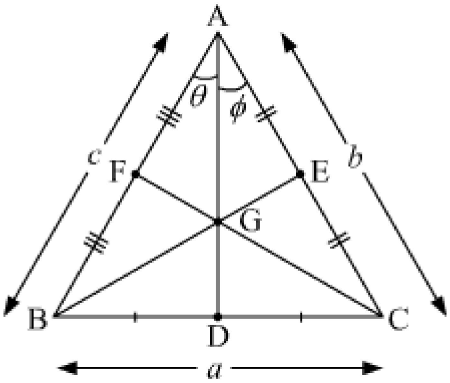
$$\Delta = \frac{1}{2}bc\sin A = \frac{1}{2}ca\sin B = \frac{1}{2}ab\sin C$$

$$\Delta = \frac{b^2\sin C\sin A}{2\sin B} = \frac{c^2\sin A\sin B}{2\sin C} = \frac{a^2\sin B\sin C}{2\sin A}$$

$\Delta = \frac{abc}{4R} = rs$, where r and R are the radius of the incircle and circumcircle respectively of ΔABC

Some terms related to a triangle:

Centroid, : The point of intersection of the medians of a triangle is known as the centroid of triangle.



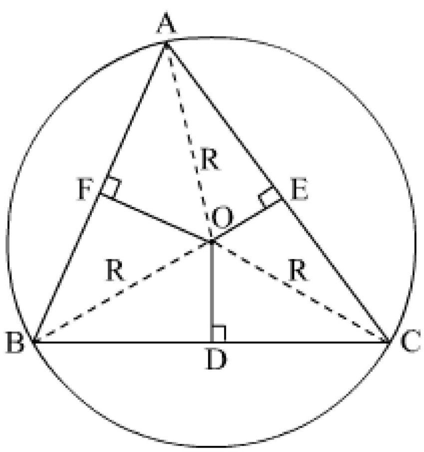
1. Lengths of the medians:

$$AD = \frac{1}{2}\sqrt{2b^2 + 2c^2 - a^2}$$

$$BE = \frac{1}{2}\sqrt{2c^2 + 2a^2 - b^2}$$

$$CF = \frac{1}{2}\sqrt{2a^2 + 2b^2 - c^2}$$

Circumcentre, (O): The point of intersection of perpendicular bisectors of all the three sides of triangle is called its circumcentre. This point is equidistant from the three vertices of the circle passing through them.



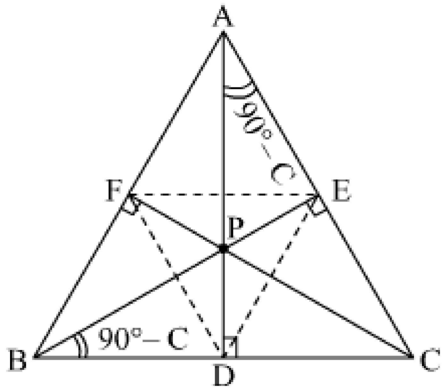
By sine Rule, we have

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$$

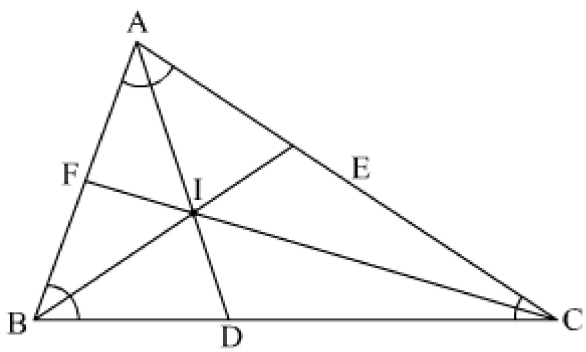
$$\Rightarrow a = 2R\sin A,\ b = 2R\sin B,\ c = 2R\sin C$$

$$R = \frac{abc}{4\Delta}$$

Orthocentre, (P): The point of intersection of the altitudes of a triangle is called the orthocentre of the triangle.



Incentre (I): The point of intersection of the interior angle bisectors of a triangle is called the incentre of the triangle. The circle drawn by taking it as the centre and touching all the sides of the triangle is called the incircle of the triangle. The radius of this circle is called inradius and is denoted by *r*.



In ΔABC, I is the incentre. Incentre always lies inside the triangle.

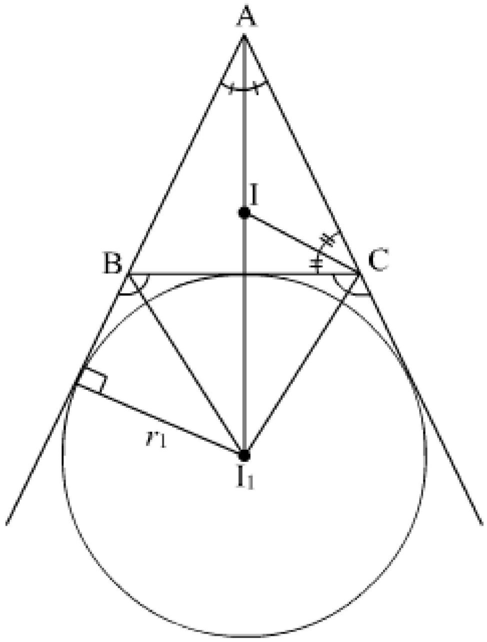
1. $r = \frac{\Delta}{s}$

2. $r = (s-a)\tan\frac{A}{2} = (s-b)\tan\frac{B}{2} = (s-c)\tan\frac{C}{2}$

3. $r = \frac{a\sin\frac{B}{2}\sin\frac{C}{2}}{\cos\frac{A}{2}} = \frac{b\sin\frac{C}{2}\sin\frac{A}{2}}{\cos\frac{B}{2}} = \frac{c\sin\frac{A}{2}\sin\frac{B}{2}}{\cos\frac{C}{2}}$

4. $r = 4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$

   Excentres (I_1, I_2, I_3): InΔABC, the bisectors of the exterior angles ∠B and ∠C, obtained on producing the sides AB and AC respectively, intersect each other at the point I_1. The circle with centre I_1 and touching the side BC and the extended sides AB and AC is called the ex-circle of the ΔABC.

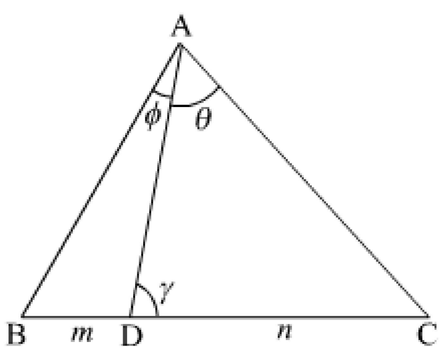


Distance between the circumcentre, O and the orthocentre, P $OP = R\sqrt{1-8\cos A\cos B\cos C}$

Distance between the circumcentre, O and the incentre, I $OI = R\sqrt{1-8\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}} = \sqrt{R^2-2rR}$

Ptolemy's theorem: In a cyclic quadrilateral PQRS,
PR × QS = PQ × RS + QR × PS

***m* − *n* theorem:** If D is a point on the side BC of ΔABC such that BD: DC = m: n and ∠BAD = Φ, ∠CAD = θ and ∠ADC = γ, then



$(m+n)\cot\gamma = m\cot\phi - n\cot\theta$
$(m+n)\cot\gamma = n\cot B - m\cot C$

- If $\sin y = x$, then $y = \sin^{-1}x$ (We read it as sine inverse x)

  Here, $\sin^{-1}x$ is an inverse trigonometric function. Similarly, the other inverse trigonometric functions are as follows:

  - If $\cos y = x$, then $y = \cos^{-1}x$
  - If $\tan y = x$, then $y = \tan^{-1}x$
  - If $\cot y = x$, then $y = \cot^{-1}x$

- If sec $y = x$, then $y = \sec^{-1}x$
- If cosec $y = x$, then $y = \text{cosec}^{-1}x$

- The domains and ranges (principle value branches) of inverse trigonometric functions can be shown in a table as follows:

Function	Domain	Range (Principle value branches)
$y = \sin^{-1}x$	[–1, 1]	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
$y = \cos^{-1}x$	[–1, 1]	$[0, \pi]$
$y = \tan^{-1}x$	**R**	$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$
$y = \cot^{-1}x$	**R**	$(0, \pi)$
$y = \sec^{-1}x$	**R** – (–1, 1)	$[0, \pi] - \left\{\frac{\pi}{2}\right\}$
$y = \text{cosec}^{-1}x$	**R** – (–1, 1)	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$

- Note that $y = \tan^{-1}x$ does not mean that $y = (\tan x)^{-1}$. This argument also holds true for the other inverse trigonometric functions.
- The principal value of an inverse trigonometric function can be defined as the value of inverse trigonometric functions, which lies in the range of principal branch.

Example 1: What is the principal value of $\tan^{-1}\left(-\sqrt{3}\right) + \sin^{-1}(1)$?

Solution:

Let $\tan^{-1}\left(-\sqrt{3}\right) = y$ and $\sin^{-1}(1) = z$

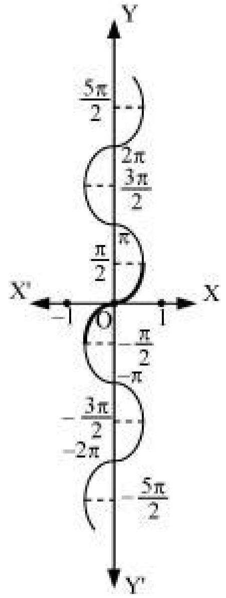
$\Rightarrow \tan y = -\sqrt{3} = -\tan\left(\frac{\pi}{3}\right) = \tan\left(-\frac{\pi}{3}\right)$ and $\sin z = 1 = \sin\frac{\pi}{2}$

We know that the ranges of principal value branch of $\tan^{-1}$ and $\sin^{-1}$ are $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ respectively. Also, $\tan\left(-\frac{\pi}{3}\right) = -\sqrt{3} \sin\left(\frac{\pi}{2}\right) = 1$

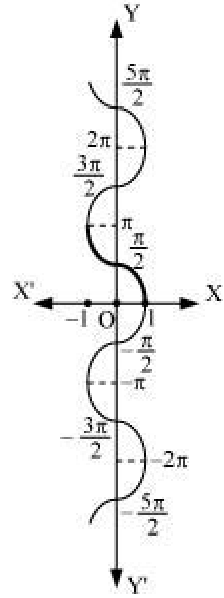
Therefore, principal values of $\tan^{-1}\left(-\sqrt{3}\right) = \frac{-\pi}{3}$ and $\sin^{-1}\left(1\right) = \frac{\pi}{2}$

$\therefore \tan^{-1}\left(-\sqrt{3}\right) + \sin^{-1}1 = \frac{-\pi}{3} + \frac{\pi}{2} = \frac{\pi}{6}$

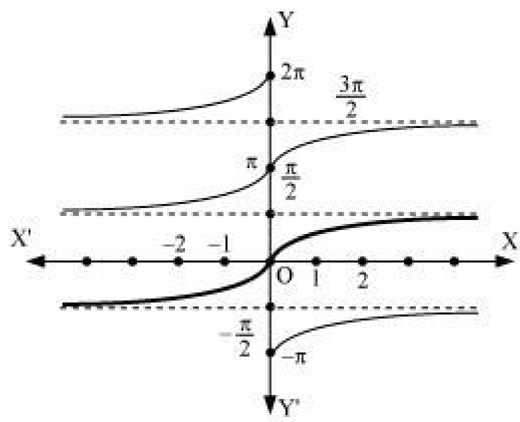
- Graphs of the six inverse trigonometric functions can be drawn as follows:



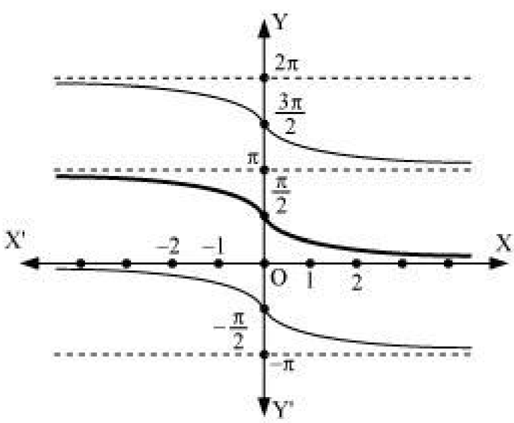
$y = \sin^{-1} x$



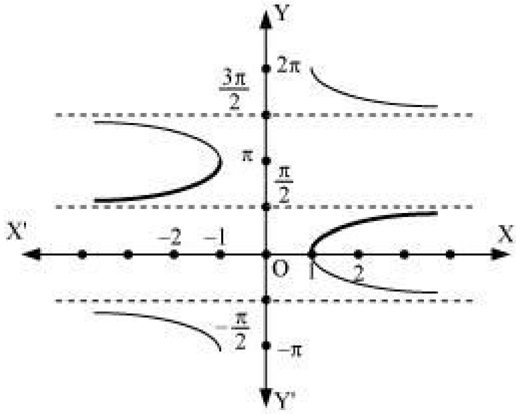
$y = \cos^{-1} x$



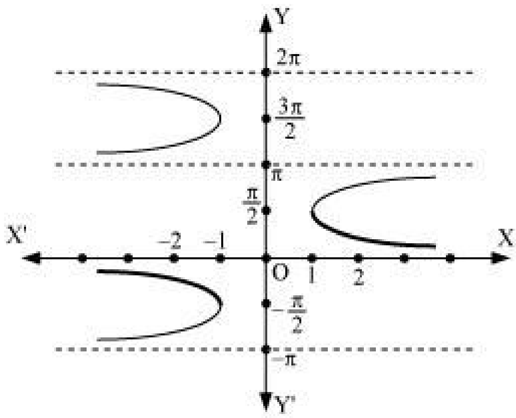
$y = \tan^{-1} x$



$y = \cot^{-1} x$



$y = \sec^{-1} x$



$y = \text{cosec}^{-1} x$

- The relation $\sin y = x \Rightarrow y = \sin^{-1} x$ gives $\sin(\sin^{-1} x) = x$, where $x \in [-1, 1]$; and $\sin^{-1}(\sin x) = x$, where $x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

  This property can be similarly stated for the other inverse trigonometric functions as follows:

  - $\cos(\cos^{-1} x) = x, x \in [-1, 1]$ and $\cos^{-1}(\cos x) = x, x \in [0, \pi]$
  - $\tan(\tan^{-1} x) = x, x \in \mathbf{R}$ and $\tan^{-1}(\tan x) = x, x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$
  - $\text{cosec}(\text{cosec}^{-1} x) = x, x \in \mathbf{R} - (-1, 1)$ and $\text{cosec}^{-1}(\text{cosec}\, x) = x, x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$
  - $\sec(\sec^{-1} x) = x, x \in \mathbf{R} - (-1, 1)$ and $\sec^{-1}(\sec x) = x, x \in [0, \pi] - \left\{\frac{\pi}{2}\right\}$
  - $\cot(\cot^{-1} x) = x, x \in \mathbf{R}$ and $\cot^{-1}(\cot x) = x, x \in (0, \pi)$

- For suitable values of domains;
  - $\sin^{-1}\left(\frac{1}{x}\right) = \text{cosec}^{-1} x, x \in \mathbf{R} - (-1, 1)$
  - $\cos^{-1}\left(\frac{1}{x}\right) = \sec^{-1} x, x \in \mathbf{R} - (-1, 1)$
  - $\tan^{-1}\left(\frac{1}{x}\right) = \begin{cases} \cot^{-1} x, & x > 0 \\ \cot^{-1} \pi, & x - 0x < \end{cases}$
  - $\text{cosec}^{-1}\left(\frac{1}{x}\right) = \sin^{-1} x, x \in [-1, 1]$
  - $\sec^{-1}\left(\frac{1}{x}\right) = \cos x, x \in [-1, 1]$

- $\cot^{-1}\left(\frac{1}{x}\right) = \begin{cases} \tan^{-1}x, & x > 0 \\ \pi + \tan^{-1}x, & x < 0 \end{cases}$

Note: While solving problems, we generally use the formulas $\tan^{-1}\left(\frac{1}{x}\right) = \cot^{-1}x$ and $\cot^{-1}\left(\frac{1}{x}\right) = \tan^{-1}x$ when the conditions for x (i.e., $x > 0$ or $x < 0$) are not given

- For suitable values of domains;
  - $\sin^{-1}(-x) = -\sin^{-1}x, x \in [-1, 1]$
  - $\cos^{-1}(-x) = \pi - \cos^{-1}x, x \in [-1, 1]$
  - $\tan^{-1}(-x) = -\tan^{-1}x,\ x \in \mathbf{R}$
  - $\mathbf{cosec^{-1}(-x) = -cosec^{-1}x, |x| \geq 1}$
  - $\sec^{-1}(-x) = \pi - \sec^{-1}x, |x| \geq 1$
  - $\cot^{-1}(-x) = \pi - \cot^{-1}x, x \in \mathbf{R}$

- For suitable values of domains;
  - $\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}, x \in [-1, 1]$
  - $\tan^{-1}x + \cot^{-1}x = \frac{\pi}{2}, x \in \mathbf{R}$
  - $\sec^{-1}x + \text{cosec}^{-1}x = \frac{\pi}{2}, |x| \geq \mathbf{1}$

- For suitable values of domains;
  - $\tan^{-1}x + \tan^{-1}y = \begin{cases} \tan^{-1}\frac{x+y}{1-xy}, & xy < 1 \\ \pi + \tan^{-1}\frac{x+y}{1-xy}, & xy > 1 \end{cases}$
  - $\tan^{-1}x + \tan^{-1}y = \tan^{-1}\frac{x-y}{1+xy}$

Note: While solving problems, we generally use the formula $\tan^{-1}x + \tan^{-1}y = \tan^{-1}\frac{x+y}{1-xy}$ when the condition for xy is not given.

- For $x \in [-1, 1]$, $2\tan^{-1}x = \sin^{-1}\frac{2x}{1+x^2}$
- For $x \in (-1, 1)$, $2\tan^{-1}x = \tan^{-1}\frac{2x}{1-x^2}$
- For x ³ 0, $2\tan^{-1}x = \cos^{-1}\frac{1-x^2}{1+x^2}$

Example: 2

For $x, y \in [-1, 1]$, show that: $\sin^{-1}x + \sin^{-1}y = \sin^{-1}\left(x\sqrt{1-y^2} + y\sqrt{1-x^2}\right)$

Solution:

We know that $\sin^{-1}x$ and $\sin^{-1}y$ can be defined only for $x, y \in [-1, 1]$

Let $\sin^{-1}x = a$ and $\sin^{-1}y = b$

$\Rightarrow x = \sin a$ and $y = \sin b$

Also, cos a = $\sqrt{1-x^2}$ and cos b = $\sqrt{1-y^2}$

We know that, sin (a + b) = sin a cos b + cos a sin b

$\Rightarrow a + b = \sin^{-1}\left[x\sqrt{1-y^2} + y\sqrt{1-x^2}\right]$

$\Rightarrow \sin^{-1}x + \sin^{-1}y = \sin^{-1}\left[x\sqrt{1-y^2} + y\sqrt{1-x^2}\right]$

Example: 3

If $\tan^{-1}\left(\frac{5}{6}\right) + \tan^{-1}\left(\frac{1}{11}\right) = x$, then find sec x.

Solution:

We have $x = \tan^{-1}\left(\frac{5}{6}\right) + \tan^{-1}\left(\frac{1}{11}\right) = \tan^{-1}\left[\frac{\frac{5}{6}+\frac{1}{11}}{1-\frac{5}{6}\times\frac{1}{11}}\right]$

$\left[\text{Using the identity } \tan^{-1}x + \tan^{-1}y \tan^{-1}\left(\frac{x+y}{1-xy}\right), \text{ where } x = \frac{5}{6} \text{ and } y = \frac{1}{11}\right]$

$\therefore x = \tan^{-1}\left[\frac{\frac{55+6}{66}}{\frac{66-5}{66}}\right]$

$= \tan^{-1}1$

$= \frac{\pi}{4}$

$\sec x = \sec\frac{\pi}{4} = \sqrt{2}$

Example: 4

Show that: $3\tan^{-1}x = \tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right)$ where $x \in \left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$

Solution:

We know that,

$3\tan^{-1}x = \tan^{-1}x + 2\tan^{-1}x$

$$= \tan^{-1} x + \tan^{-1} \frac{2x}{1-x^2}$$

$$= \tan^{-1} \left[\frac{x + \frac{2x}{1-x^2}}{1 - x \times \frac{2x}{1-x^2}} \right]$$

$$= \tan^{-1} \left[\frac{\frac{3x - x^3}{1-x^2}}{\frac{1-3x^2}{1-x^2}} \right]$$

$$= \tan^{-1} \left(\frac{3x - x^3}{1-3x^2} \right)$$