# 11

# Three Dimensional Geometry

## Short Answer Type Questions

**Q. 1** Find the position vector of a point $A$ in space such that $\vec{OA}$ is inclined at 60° to $OX$ and at 45° to $OY$ and $|\vec{OA}| = 10$ units.

**Sol.** Since, $\vec{OA}$ is inclined at 60° to $OX$ and at 45° to $OY$. Let $\vec{OA}$ makes angle $\alpha$ with $OZ$.

$$\therefore \quad \cos^2 60° + \cos^2 45° + \cos^2 \alpha = 1$$

$$\Rightarrow \quad \left(\frac{1}{2}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2 + \cos^2 \alpha = 1 \qquad [\because l^2 + m^2 + n^2 = 1]$$

$$\Rightarrow \quad \frac{1}{4} + \frac{1}{2} + \cos^2 \alpha = 1$$

$$\Rightarrow \quad \cos^2 \alpha = 1 - \left(\frac{1}{2} + \frac{1}{4}\right)$$

$$\Rightarrow \quad \cos^2 \alpha = 1 - \left(\frac{6}{8}\right)$$

$$\Rightarrow \quad \cos^2 \alpha = \frac{1}{4}$$

$$\Rightarrow \quad \cos \alpha = \frac{1}{2} = \cos 60°$$

$$\therefore \quad \alpha = 60°$$

$$\therefore \quad \vec{OA} = |\vec{OA}|\left(\frac{1}{2}\hat{\mathbf{i}} + \frac{1}{\sqrt{2}}\hat{\mathbf{j}} + \frac{1}{2}\hat{\mathbf{k}}\right)$$

$$= 10\left(\frac{1}{2}\hat{\mathbf{i}} + \frac{1}{\sqrt{2}}\hat{\mathbf{j}} + \frac{1}{2}\hat{\mathbf{k}}\right) \qquad [\because |\vec{OA}| = 10]$$

$$= 5\hat{\mathbf{i}} + 5\sqrt{2}\,\hat{\mathbf{j}} + 5\,\hat{\mathbf{k}}$$

**Q. 2** Find the vector equation of the line which is parallel to the vector $3\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + 6\hat{\mathbf{k}}$ and which passes through the point (1, −2, 3).

**Thinking Process**

*Here, we use the formula $\vec{\mathbf{r}} = \vec{\mathbf{b}} + \lambda \vec{\mathbf{a}}$, where $\vec{\mathbf{r}}$ is the equation of the line which passes through $\vec{\mathbf{b}}$ and parallel to $\vec{\mathbf{a}}$.*

**Sol.** Let $\vec{\mathbf{a}} = 3\hat{i} - 2\hat{j} + 6\hat{k}$ and $\vec{\mathbf{b}} = \hat{i} - 2\hat{j} + 3\hat{k}$

So, vector equation of the line, which is parallel to the vector $\vec{\mathbf{a}} = 3\hat{i} - 2\hat{j} + 6\hat{k}$ and passes through the vector $\vec{\mathbf{b}} = \hat{i} - 2\hat{j} + 3\hat{k}$ is $\vec{\mathbf{r}} = \vec{\mathbf{b}} + \lambda\vec{\mathbf{a}}$.

$$\therefore \quad \vec{\mathbf{r}} = (\hat{i} - 2\hat{j} + 3\hat{k}) + \lambda(3\hat{i} - 2\hat{j} + 6\hat{k})$$

$$\Rightarrow \quad (x\hat{i} + y\hat{j} + z\hat{k}) - (\hat{i} - 2\hat{j} + 3\hat{k}) = \lambda(3\hat{i} - 2\hat{j} + 6\hat{\mathbf{k}})$$

$$\Rightarrow \quad (x-1)\hat{\mathbf{i}} + (y+2)\hat{\mathbf{j}} + (z-3)\hat{\mathbf{k}} = \lambda(3\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + 6\hat{\mathbf{k}})$$

**Q. 3** Show that the lines $\dfrac{x-1}{2} = \dfrac{y-2}{3} = \dfrac{z-3}{4}$ and $\dfrac{x-4}{5} = \dfrac{y-1}{2} = z$ intersect. Also, find their point of intersection.

**Thinking Process**

*If shortest distance between the lines is zero, then they intersect.*

**Sol.** We have, $x_1 = 1, y_1 = 2, z_1 = 3$ and $a_1 = 2, b_1 = 3, c_1 = 4$

Also, $x_2 = 4, y_2 = 1, z_2 = 0$ and $a_2 = 5, b_2 = 2, c_2 = 1$

If two lines intersect, then shortest distance between them should be zero.

∴ Shortest distance between two given lines

$$= \frac{\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}}{\sqrt{(b_1c_2 - b_2c_1)^2 + (c_1a_2 - c_2a_1)^2 + (a_1b_2 - a_2b_1)^2}}$$

$$= \frac{\begin{vmatrix} 4-1 & 1-2 & 0-3 \\ 2 & 3 & 4 \\ 5 & 2 & 1 \end{vmatrix}}{\sqrt{(3\cdot 1 - 2\cdot 4)^2 + (4\cdot 5 - 1\cdot 2)^2 + (2\cdot 2 - 5\cdot 3)^2}}$$

$$= \frac{\begin{vmatrix} 3 & -1 & -3 \\ 2 & 3 & 4 \\ 5 & 2 & 1 \end{vmatrix}}{\sqrt{25 + 324 + 121}}$$

$$= \frac{3(3-8) + 1(2-20) - 3(4-15)}{\sqrt{470}}$$

$$= \frac{-15 - 18 + 33}{\sqrt{470}} = \frac{0}{\sqrt{470}} = 0$$

Therefore, the given two lines are intersecting.

For finding their point of intersection for first line,

$$\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-3}{4}=\lambda$$

$\Rightarrow$ $x=2\lambda+1,\ y=3\lambda+2$ and $z=4\lambda+3$

Since, the lines are intersecting. So, let us put these values in the equation of another line.

Thus, $$\frac{2\lambda+1-4}{5}=\frac{3\lambda+2-1}{2}=\frac{4\lambda+3}{1}$$

$\Rightarrow$ $$\frac{2\lambda-3}{5}=\frac{3\lambda+1}{2}=\frac{4\lambda+3}{1}$$

$\Rightarrow$ $$\frac{2\lambda-3}{5}=\frac{4\lambda+3}{1}$$

$\Rightarrow$ $2\lambda-3=20\lambda+15$

$\Rightarrow$ $18\lambda=-18=-1$

So, the required point of intersection is

$$x=2(-1)+1=-1$$
$$y=3(-1)+2=-1$$
$$z=4(-1)+3=-1$$

Thus, the lines intersect at $(-1,-1,-1)$.

**Q. 4 Find the angle between the lines**

**$\vec{r}=3\hat{i}-2\hat{j}+6\hat{k}+\lambda(2\hat{i}+\hat{j}+2\hat{k})$ and $\vec{r}=(2\hat{j}-5\hat{k})+\mu(6\hat{i}+3\hat{j}+2\hat{k})$.**

**Thinking Process**

*We know that,* $\cos\theta=\frac{|\vec{b}_1\cdot\vec{b}_2|}{|\vec{b}_1|\cdot|\vec{b}_2|}$, *where,* $\theta$ *is the angle between the lines* $\vec{a}_1+\lambda\vec{b}_1$ *and* $\vec{a}_2+\mu\vec{b}_2$.

**Sol.** We have, $\vec{r}=3\hat{i}-2\hat{j}+6\hat{k}+\lambda(2\hat{i}+\hat{j}+2\hat{k})$

and $\vec{r}=(2\hat{j}-5\hat{k})+\mu(6\hat{i}+3\hat{j}+2\hat{k})$

where, $\vec{a}_1=3\hat{i}-2\hat{j}+6\hat{k},\ \ \vec{b}_1=2\hat{i}+\hat{j}+2\hat{k}$

and $\vec{a}_2=2\hat{j}-5\hat{k},\ \vec{b}_2=6\hat{i}+3\hat{j}+2\hat{k}$

If $\theta$ is angle between the lines, then

$$\cos\theta=\frac{|\vec{b}_1\cdot\vec{b}_2|}{|\vec{b}_1|\cdot|\vec{b}_2|}$$
$$=\frac{|(2\hat{i}+\hat{j}+2\hat{k})\cdot(6\hat{i}+3\hat{j}+2\hat{k})|}{|2\hat{i}+\hat{j}+2\hat{k}||6\hat{i}+3\hat{j}+2\hat{k}|}$$
$$=\frac{|12+3+4|}{\sqrt{9}\sqrt{49}}=\frac{19}{21}$$

$\therefore$ $$\theta=\cos^{-1}\frac{19}{21}$$

**Q. 5** Prove that the line through $A(0, -1, -1)$ and $B(4, 5, 1)$ intersects the line through $C(3, 9, 4)$ and $D(-4, 4, 4)$.

**Sol.** We know that,the cartesian equation of a line that passes through two points $(x_1, y_1, z_1)$ and $(x_2, y_2, z_2)$ is

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}$$

Hence, the cartesian equation of line passes through $A(0, -1, -1)$ and $B(4, 5, 1)$ is

$$\frac{x - 0}{4 - 0} = \frac{y + 1}{5 + 1} = \frac{z + 1}{1 + 1}$$

$$\Rightarrow \quad \frac{x}{4} = \frac{y + 1}{6} = \frac{z + 1}{2} \quad \text{...(i)}$$

and cartesian equation of the line passes through $C(3, 9, 4)$ and $D(-4, 4, 4)$ is

$$\frac{x - 3}{-4 - 3} = \frac{y - 9}{4 - 9} = \frac{z - 4}{4 - 4}$$

$$\Rightarrow \quad \frac{x - 3}{-7} = \frac{y - 9}{-5} = \frac{z - 4}{0} \quad \text{...(ii)}$$

If the lines intersect, then shortest distance between both of them should be zero.

$\therefore$ Shortest distance between the lines

$$= \frac{\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}}{\sqrt{(b_1c_2 - b_2c_1)^2 + (c_1a_2 - c_2a_1)^2 + (a_1b_2 - a_2b_1)^2}}$$

$$= \frac{\begin{vmatrix} 3 - 0 & 9 + 1 & 4 + 1 \\ 4 & 6 & 2 \\ -7 & -5 & 0 \end{vmatrix}}{\sqrt{(6 \cdot 0 + 10)^2 + (-14 - 0)^2 + (-20 + 42)^2}}$$

$$= \frac{\begin{vmatrix} 3 & 10 & 5 \\ 4 & 6 & 2 \\ -7 & -5 & 0 \end{vmatrix}}{\sqrt{100 + 196 + 484}}$$

$$= \frac{3(0 + 10) - 10(14) + 5(-20 + 42)}{\sqrt{780}}$$

$$= \frac{30 - 140 + 110}{\sqrt{780}} = 0$$

So, the given lines intersect.

**Q. 6** Prove that the lines $x = py + q$, $z = ry + s$ and $x = p'y + q'$, $z = r'y + s'$ are perpendicular, if $pp' + rr' + 1 = 0$.

**Sol.** We have, $\quad x = py + q \Rightarrow y = \dfrac{x - q}{P} \quad$ ..(i)

and $\quad z = ry + s \Rightarrow y = \dfrac{z - s}{r} \quad$ ...(ii)

$$\Rightarrow \quad \frac{x - q}{p} = \frac{y}{1} = \frac{z - s}{r} \quad \text{[using Eqs. (i) and (ii)] ...(iii)}$$

Similarly, $$\frac{x - q'}{p'} = \frac{y}{1} = \frac{z - s'}{r'} \quad \text{...(iv)}$$

From Eqs. (iii) and (iv),

$$a_1 = p, b_1 = 1, c_1 = r$$

and $$a_2 = p', b_2 = 1, c_2 = r'$$

If these given lines are perpendicular to each other, then

$$a_1 a_2 + b_1 b_2 + c_1 c_2 = 0$$

$\Rightarrow$ $$pp' + 1 + rr' = 0$$

which is the required condition.

**Q. 7** Find the equation of a plane which bisects perpendicularly the line joining the points $A(2, 3, 4)$ and $B(4, 5, 8)$ at right angles.

**Sol.** Since, the equation of a plane is bisecting perpendicular the line joining the points $A(2, 3, 4)$ and $B(4, 5, 8)$ at right angles.

So, mid-point of $AB$ is $\left(\frac{2+4}{2}, \frac{3+5}{2}, \frac{4+8}{2}\right)$ *i.e.,* $(3, 4, 6)$.

Also, $$\vec{\mathbf{N}} = (4-2)\hat{\mathbf{i}} + (5-3)\hat{\mathbf{j}} + (8-4)\hat{\mathbf{k}} = 2\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 4\hat{\mathbf{k}}$$

So, the required equation of the plane is $(\vec{\mathbf{r}} - \vec{\mathbf{a}}) \cdot \vec{\mathbf{N}} = 0$.

$\Rightarrow$ $[(x-3)\hat{\mathbf{i}} + (y-4)\hat{\mathbf{j}} + (z-6)\hat{\mathbf{k}}] \cdot (2\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 4\hat{\mathbf{k}}) = 0$ $[\because \vec{\mathbf{a}} = 3\hat{\mathbf{i}} + 4\hat{\mathbf{j}} + 6\hat{\mathbf{k}}]$

$\Rightarrow$ $$2x - 6 + 2y - 8 + 4z - 24 = 0$$

$\Rightarrow$ $$2x + 2y + 4z = 38$$

$\therefore$ $$x + y + 2z = 19$$

**Q. 8** Find the equation of a plane which is at a distance $3\sqrt{3}$ units from origin and the normal to which is equally inclined to coordinate axis.

**Sol.** Since, normal to the plane is equally inclined to the coordinate axis.

Therefore, $$\cos\alpha = \cos\beta = \cos\gamma = \frac{1}{\sqrt{3}}$$

So, the normal is $\vec{\mathbf{N}} = \frac{1}{\sqrt{3}}\hat{\mathbf{i}} + \frac{1}{\sqrt{3}}\hat{\mathbf{j}} + \frac{1}{\sqrt{3}}\hat{\mathbf{k}}$ and plane is at a distance of $3\sqrt{3}$ units from origin.

The equation of plane is $\vec{\mathbf{r}} \cdot \hat{\mathbf{N}} = 3\sqrt{3}$ $\left[\because \hat{\mathbf{N}} = \frac{\vec{\mathbf{N}}}{|\mathbf{N}|}\right]$

[since, vector equation of the plane at a distance $p$ from the origin is $\vec{\mathbf{r}} \cdot \hat{\mathbf{N}} = p$]

$\Rightarrow$ $$(x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}) \cdot \frac{\left(\frac{1}{\sqrt{3}}\hat{\mathbf{i}} + \frac{1}{\sqrt{3}}\hat{\mathbf{j}} + \frac{1}{\sqrt{3}}\hat{\mathbf{k}}\right)}{1} = 3\sqrt{3}$$

$\Rightarrow$ $$\frac{x}{\sqrt{3}} + \frac{y}{\sqrt{3}} + \frac{z}{\sqrt{3}} = 3\sqrt{3}$$

$\therefore$ $$x + y + z = 3\sqrt{3} \cdot \sqrt{3} = 9$$

So, the required equation of plane is $x + y + z = 9$.

**Q. 9 If the line drawn from the point (–2, – 1, – 3) meets a plane at right angle at the point (1, – 3, 3), then find the equation of the plane.**

**Sol.** Since, the line drawn from the point (–2, – 1, – 3) meets a plane at right angle at the point (1, – 3, 3). So, the plane passes through the point (1, – 3, 3) and normal to plane is $(-3\hat{\mathbf{i}} + 2\hat{\mathbf{j}} - 6\hat{\mathbf{k}})$.

$$\Rightarrow \quad \vec{\mathbf{a}} = \hat{\mathbf{i}} - 3\hat{\mathbf{j}} + 3\hat{\mathbf{k}}$$

$$\text{and} \quad \vec{\mathbf{N}} = -3\hat{\mathbf{i}} + 2\hat{\mathbf{j}} - 6\hat{\mathbf{k}}$$

So, the equation of required plane is $(\vec{\mathbf{r}} - \vec{\mathbf{a}})\cdot\vec{\mathbf{N}} = 0$

$$\Rightarrow \quad [(x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}) - (\hat{\mathbf{i}} - 3\hat{\mathbf{j}} + 3\hat{\mathbf{k}})]\cdot(-3\hat{\mathbf{i}} + 2\hat{\mathbf{j}} - 6\hat{\mathbf{k}}) = 0$$

$$\Rightarrow \quad [(x-1)\hat{\mathbf{i}} + (y+3)\hat{\mathbf{j}} + (z-3)\hat{\mathbf{k}}]\cdot(-3\hat{\mathbf{i}} + 2\hat{\mathbf{j}} - 6\hat{\mathbf{k}}) = 0$$

$$\Rightarrow \quad -3x + 3 + 2y + 6 - 6z + 18 = 0$$

$$\Rightarrow \quad -3x + 2y - 6z = -27$$

$$\therefore \quad 3x - 2y + 6z - 27 = 0$$

**Q. 10 Find the equation of the plane through the points (2, 1, 0), (3, – 2, – 2) and (3, 1, 7).**

**Thinking Process**

*Here, apply the equation of the plane passing through the points $(x_1, y_1, z_1)$, $(x_2, y_2, z_2)$ and $(x_3, y_3, z_3)$ is given by* $\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ x_3-x_1 & y_3-y_1 & z_3-z_1 \end{vmatrix} = 0.$

**Sol.** We know that, the equation of a plane passing through three non-collinear points $(x_1, y_1, z_1)$, $(x_2, y_2, z_2)$ and $(x_3, y_3, z_3)$ is

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ x_3-x_1 & y_3-y_1 & z_3-z_1 \end{vmatrix} = 0$$

$$\Rightarrow \quad \begin{vmatrix} x-2 & y-1 & z-0 \\ 3-2 & -2-1 & -2-0 \\ 3-2 & 1-1 & 7-0 \end{vmatrix} = 0$$

$$\Rightarrow \quad \begin{vmatrix} x-2 & y-1 & z \\ 1 & -3 & -2 \\ 1 & 0 & 7 \end{vmatrix} = 0$$

$$\Rightarrow \quad (x-2)(-21+0) - (y-1)(7+2) + z(3) = 0$$

$$\Rightarrow \quad -21x + 42 - 9y + 9 + 3z = 0$$

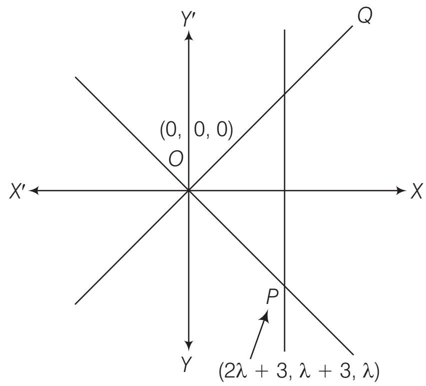
$$\Rightarrow \quad -21x - 9y + 3z = -51$$

$$\therefore \quad 7x + 3y - z = 17$$

So, the required equation of plane is $7x + 3y - z = 17$.

**Q. 11** Find the equations of the two lines through the origin which intersect the line $\frac{x-3}{2}=\frac{y-3}{1}=\frac{z}{1}$ at angles of $\frac{\pi}{3}$ each.

**Sol.** Given equation of the line is $\frac{x-3}{2}=\frac{y-3}{1}=\frac{z}{1}=\lambda$ ...(i)




So, DR's of the line are 2, 1, 1 and DC's of the given line are $\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{16}}$.

Also, the required lines make angle $\frac{\pi}{3}$ with the given line.

From Eq. (i), $x=(2\lambda+3), y=(\lambda+3)$ and $z=\lambda$

$$\because \quad \cos\theta=\frac{a_1a_2+b_1b_2+c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}}$$

$$\therefore \quad \cos\frac{\pi}{3}=\frac{(4\lambda+6)+(\lambda+3)+(\lambda)}{\sqrt{6}\sqrt{(2\lambda+3)^2+(\lambda+3)^2+\lambda^2}}$$

$$\Rightarrow \quad \frac{1}{2}=\frac{6\lambda+9}{\sqrt{6}\sqrt{(4\lambda^2+9+12\lambda+\lambda^2+9+6\lambda+\lambda^2)}}$$

$$\Rightarrow \quad \frac{\sqrt{6}}{2}=\frac{6\lambda+9}{\sqrt{6\lambda^2+18\lambda+18}}$$

$$\Rightarrow \quad 6\sqrt{(\lambda^2+3\lambda+3)}=2(6\lambda+9)$$

$$\Rightarrow \quad 36(\lambda^2+3\lambda+3)=36(4\lambda^2+9+12\lambda)$$

$$\Rightarrow \quad \lambda^2+3\lambda+3=4\lambda^2+9+12\lambda$$

$$\Rightarrow \quad 3\lambda^2+9\lambda+6=0$$

$$\Rightarrow \quad \lambda^2+3\lambda+2=0$$

$$\Rightarrow \quad \lambda(\lambda+2)+1(\lambda+2)=0$$

$$\Rightarrow \quad (\lambda+1)(\lambda+2)=0$$

$$\therefore \quad \lambda=-1,-2$$

So, the DC's are 1, 2, −1 and −1, 1, −2.

Also, both the required lines passes through origin.

So, the equations of required lines are $\frac{x}{1}=\frac{y}{2}=\frac{z}{-1}$ and $\frac{x}{-1}=\frac{y}{1}=\frac{z}{-2}$.

**Q. 12** **Find the angle between the lines whose direction cosines are given by the equation $l + m + n = 0$ and $l^2 + m^2 - n^2 = 0$.**

**Sol.** Eliminating $n$ from both the equations, we have

$$l^2 + m^2 - (l - m)^2 = 0$$

$\Rightarrow \quad l^2 + m^2 - l^2 - m^2 + 2ml = 0 \quad \Rightarrow \quad 2lm = 0$

$\Rightarrow \quad lm = 0 \quad \Rightarrow \quad (-m - n)m = 0 \qquad [\because l = -m - n]$

$\Rightarrow \quad (m + n)m = 0$

$\Rightarrow \quad m = -n \quad \Rightarrow \quad m = 0$

$\Rightarrow \quad l = 0, \; l = -n$

Thus, Dr's two lines are proportional to $0, -n, n$ and $-n, 0, n$ *i.e.*, $0, -1, 1$ and $-1, 0, 1$.

So, the vector parallel to these given lines are $\vec{\mathbf{a}} = -\hat{\mathbf{j}} + \hat{\mathbf{k}}$ and $\vec{\mathbf{b}} = -\hat{\mathbf{i}} + \hat{\mathbf{k}}$

Now, $\quad \cos\theta = \dfrac{\vec{\mathbf{a}}\,\vec{\mathbf{b}}}{|\vec{\mathbf{a}}||\vec{\mathbf{b}}|} = \dfrac{1}{\sqrt{2}} \cdot \dfrac{1}{\sqrt{2}} \quad \Rightarrow \quad \cos\theta = \dfrac{1}{2}$

$\therefore \quad \theta = \dfrac{\pi}{3} \qquad \left[\because \cos\dfrac{\pi}{3} = \dfrac{1}{2}\right]$

**Q. 13** **If a variable line in two adjacent positions has direction cosines $l, m, n$ and $l + \delta l, m + \delta m, n + \delta n$, then show that the small angle $\delta\theta$ between the two positions is given by $\delta\theta^2 = \delta l^2 + \delta m^2 + \delta n^2$.**

**Sol.** We have $l, m, n$ and $l + \delta l, m + \delta m, n + \delta n$ as direction cosines of a variable line in two different positions.

$\therefore \quad l^2 + m^2 + n^2 = 1 \qquad \text{... (i)}$

and $\quad (l + \delta l)^2 + (m + \delta m)^2 + (n + \delta n)^2 = 1 \qquad \text{...(ii)}$

$\Rightarrow \quad l^2 + m^2 + n^2 + \delta l^2 + \delta m^2 + \delta n^2 + 2(l\,\delta l + m\,\delta m + n\,\delta n) = 1$

$\Rightarrow \quad \delta l^2 + \delta m^2 + \delta n^2 = -2(l\,\delta l + m\,\delta m + n\,\delta n) \qquad [\because l^2 + m^2 + n^2 = 1]$

$\Rightarrow \quad l\,\delta l + m\,\delta m + n\,\delta n = \dfrac{-1}{2}(\delta l^2 + \delta m^2 + \delta n^2) \qquad \text{...(iii)}$

Now, $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ are unit vectors along a line with direction cosines $l, m, n$ and $(l + \delta l), (m + \delta m), (n + \delta n)$, respectively.

$\therefore \quad \vec{\mathbf{a}} = l\hat{\mathbf{i}} + m\hat{\mathbf{j}} + n\hat{\mathbf{k}}$ and $\vec{\mathbf{b}} = (l + \delta l)\hat{\mathbf{i}} + (m + \delta m)\hat{\mathbf{j}} + (n + \delta n)\hat{\mathbf{k}}$

$\Rightarrow \quad \cos\delta\theta = \dfrac{\vec{\mathbf{a}} \cdot \vec{\mathbf{b}}}{|\vec{\mathbf{a}}||\vec{\mathbf{b}}|} = \vec{\mathbf{a}} \cdot \vec{\mathbf{b}} \qquad [\because |\hat{\mathbf{a}}| = |\hat{\mathbf{b}}| = 1]$

$\Rightarrow \quad \cos\delta\theta = l(l + \delta l) + m(m + \delta m) + n(n + \delta n)$

$= (l^2 + m^2 + n^2) + (l\,\delta l + m\,\delta m + n\,\delta n)$

$= 1 - \dfrac{1}{2}(\delta l^2 + \delta m^2 + \delta n^2) \qquad$ [using Eq. (iii)]

$\Rightarrow \quad 2(1 - \cos\delta\theta) = (\delta l^2 + \delta m^2 + \delta n^2)$

$\Rightarrow \quad 2 \cdot 2\sin^2\dfrac{\delta\theta}{2} = \delta l^2 + \delta m^2 + \delta n^2 \qquad \left[\because 1 - \cos\theta = 2\sin^2\dfrac{\theta}{2}\right]$

$\Rightarrow \quad 4\left(\dfrac{\delta\theta}{2}\right)^2 = \delta l^2 + \delta m^2 + \delta n^2 \qquad \left[\text{since, } \dfrac{\delta\theta}{2} \text{ is small, then } \sin\dfrac{\delta\theta}{2} = \dfrac{\delta\theta}{2}\right]$

$\therefore \quad \delta\theta^2 = \delta l^2 + \delta m^2 + \delta n^2$

**Q. 14** If $O$ is the origin and $A$ is $(a, b, c)$, then find the direction cosines of the line $OA$ and the equation of plane through $A$ at right angle to $OA$.

**Sol.** Since, DC's of line $OA$ are $\frac{a}{\sqrt{a^2+b^2+c^2}}, \frac{b}{\sqrt{a^2+b^2+c^2}}$ and $\frac{c}{\sqrt{a^2+b^2+c^2}}$.

Also, $\vec{\mathbf{n}} = \overrightarrow{\mathbf{OA}} = \vec{\mathbf{a}} = a\hat{\mathbf{i}} + b\hat{\mathbf{j}} + c\hat{\mathbf{k}}$

The equation of plane passes through $(a, b, c)$ and perpendicular to $OA$ is given by

$$[\vec{\mathbf{r}} - \vec{\mathbf{a}}]\cdot\vec{\mathbf{n}} = 0$$

$$\Rightarrow \quad \vec{\mathbf{r}}\cdot\vec{\mathbf{n}} = \vec{\mathbf{a}}\cdot\vec{\mathbf{n}}$$

$$\Rightarrow \quad [(x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}})\cdot(a\hat{\mathbf{i}} + b\hat{\mathbf{j}} + c\hat{\mathbf{k}})] = (a\hat{\mathbf{i}} + b\hat{\mathbf{j}} + c\hat{\mathbf{k}}).(a\hat{\mathbf{i}} + b\hat{\mathbf{j}} + c\hat{\mathbf{k}})$$

$$\Rightarrow \quad ax + by + cz = a^2 + b^2 + c^2$$

**Q. 15** Two systems of rectangular axis have the same origin. If a plane cuts them at distances $a, b, c$ and $a', b', c'$, respectively from the origin, then prove that $\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} = \frac{1}{a'^2} + \frac{1}{b'^2} + \frac{1}{c'^2}$.

**Sol.** Consider $OX, OY, OZ$ and $ox, oy, oz$ are two system of rectangular axes.
Let their corresponding equation of plane be

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1 \qquad \text{...(i)}$$

and

$$\frac{x}{a'} + \frac{y}{b'} + \frac{z}{c'} = 1 \qquad \text{...(ii)}$$

Also, the length of perpendicular from origin to Eqs. (i) and (ii) must be same.

$$\therefore \quad \frac{\frac{0}{a} + \frac{0}{b} + \frac{0}{c} - 1}{\sqrt{\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}}} = \frac{\frac{0}{a'} + \frac{0}{b'} + \frac{0}{c'} - 1}{\sqrt{\frac{1}{a'^2} + \frac{1}{b'^2} + \frac{1}{c'^2}}}$$

$$\Rightarrow \quad \sqrt{\frac{1}{a'^2} + \frac{1}{b'^2} + \frac{1}{c'^2}} = \sqrt{\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}}$$

$$\Rightarrow \quad \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} = \frac{1}{a'^2} + \frac{1}{b'^2} + \frac{1}{c'^2}$$

# Long Answer Type Questions

**Q. 16** Find the foot of perpendicular from the point $(2, 3, -8)$ to the line $\frac{4-x}{2} = \frac{y}{6} = \frac{1-z}{3}$. Also, find the perpendicular distance from the given point to the line.

**Sol.** We have, equation of line as $\frac{4-x}{2} = \frac{y}{6} = \frac{1-z}{3}$

$$\Rightarrow \quad \frac{x-4}{-2} = \frac{y}{6} = \frac{z-1}{-3} = \lambda$$

$$\Rightarrow \quad x = -2\lambda + 4,\ y = 6\lambda \text{ and } z = -3\lambda + 1$$

Let the coordinates of $L$ be $(4-2\lambda, 6\lambda, 1-3\lambda)$ and direction ratios of $PL$ are proportional to $(4-2\lambda-2, 6\lambda-3, 1-3\lambda+8)$ *i.e.,* $(2-2\lambda, 6\lambda-3, 9-3\lambda)$.

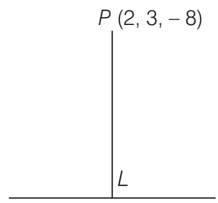
Also, direction ratios are proportional to $-2, 6, -3$. Since, $PL$ is perpendicular to give line.

$\therefore \quad -2(2-2\lambda)+6(6\lambda-3)-3(9-3\lambda)=0$

$\Rightarrow \quad -4+4\lambda+36\lambda-18-27+9\lambda=0$

$\Rightarrow \quad 49\lambda=49 \Rightarrow \lambda=1$

So, the coordinates of $L$ are $(4-2\lambda, 6\lambda, 1-3\lambda)$ *i.e.,* $(2, 6, -2)$.




$$\frac{4-x}{2}=\frac{y}{6}=\frac{1-z}{3}$$

Also, $\quad$ length of $PL=\sqrt{(2-2)^2+(6-3)^2+(-2+8)^2}$

$$=\sqrt{0+9+36}=3\sqrt{5} \text{ units}$$

**Q. 17** Find the distance of a point $(2, 4, -1)$ from the line

$$\frac{x+5}{1}=\frac{y+3}{4}=\frac{z-6}{-9}.$$

**Sol.** We have, equation of the line as $\dfrac{x+5}{1}=\dfrac{y+3}{4}=\dfrac{z-6}{-9}=\lambda$

$\Rightarrow \quad x=\lambda-5, y=4\lambda-3, z=6-9\lambda$

Let the coordinates of $L$ be $(\lambda-5, 4\lambda-3, 6-9\lambda)$, then Dr's of $PL$ are $(\lambda-7, 4\lambda-7, 7-9\lambda)$.

Also, the direction ratios of given line are proportional to $1, 4, -9$.

Since, $PL$ is perpendicular to the given line.

$\therefore \quad (\lambda-7)\cdot 1+(4\lambda-7)\cdot 4+(7-9\lambda)\cdot(-9)=0$

$\Rightarrow \quad \lambda-7+16\lambda-28+81\lambda-63=0$

$\Rightarrow \quad 98\lambda=98 \Rightarrow \lambda=1$

So, the coordinates of $L$ are $(-4, 1, -3)$.

$\therefore \quad$ Required distance, $PL=\sqrt{(-4-2)^2+(1-4)^2+(-3+1)^2}$

$$=\sqrt{36+9+4}=7 \text{ units}$$

**Q. 18** Find the length and the foot of perpendicular from the point $\left(1, \dfrac{3}{2}, 2\right)$ to the plane $2x-2y+4z+5=0$.

**Sol.** Equation of the given plane is $2x-2y+4z+5=0$ $\quad \ldots$ (i)

$\Rightarrow \quad \vec{\mathbf{n}}=2\hat{\mathbf{i}}-2\hat{\mathbf{j}}+4\hat{\mathbf{k}}$

So, the equation of line through $\left(1, \dfrac{3}{2}, 2\right)$ and parallel to $\vec{\mathbf{n}}$ is given by

$$\frac{x-1}{2}=\frac{y-3/2}{-2}=\frac{z-2}{4}=\lambda$$

$\Rightarrow \quad x=2\lambda+1, y=-2\lambda+\dfrac{3}{2}$ and $z=4\lambda+2$

If this point lies on the given plane, then

$$2(2\lambda + 1) - 2\left(-2\lambda + \frac{3}{2}\right) + 4(4\lambda + 2) + 5 = 0 \qquad \text{[using Eq. (i)]}$$

$$\Rightarrow \quad 4\lambda + 2 + 4\lambda - 3 + 16\lambda + 8 + 5 = 0$$

$$\Rightarrow \quad 24\lambda = -12 \Rightarrow \lambda = \frac{-1}{2}$$

$\therefore$ Required foot of perpendicular

$$= \left[2 \times \left(\frac{-1}{2}\right) + 1, -2 \times \left(\frac{-1}{2}\right) + \frac{3}{2}, 4 \times \left(\frac{-1}{2}\right) + 2\right] \text{ i.e., } \left(0, \frac{5}{2}, 0\right)$$

$\therefore$ Required length of perpendicular $= \sqrt{(1-0)^2 + \left(\frac{3}{2} - \frac{5}{2}\right)^2 + (2-0)^2}$

$$= \sqrt{1 + 1 + 4} = \sqrt{6} \text{ units}$$

**Q. 19 Find the equation of the line passing through the point (3, 0, 1) and parallel to the planes $x + 2y = 0$ and $3y - z = 0$.**

**Sol.** Equation of the two planes are $x + 2y = 0$ and $3y - z = 0$.

Let $\vec{\mathbf{n}}_1$ and $\vec{\mathbf{n}}_2$ are the normals to the two planes, respectively.

$$\therefore \quad \vec{\mathbf{n}}_1 = \hat{\mathbf{i}} + 2\hat{\mathbf{j}} \quad \text{and} \quad \vec{\mathbf{n}}_2 = 3\hat{\mathbf{j}} - \hat{\mathbf{k}}$$

Since, required line is parallel to the given two planes.

Therefore,
$$\vec{\mathbf{b}} = \vec{\mathbf{n}}_1 \times \vec{\mathbf{n}}_2 = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & 2 & 0 \\ 0 & 3 & -1 \end{vmatrix}$$

$$= \hat{\mathbf{i}}(-2) - \hat{\mathbf{j}}(-1) + \hat{\mathbf{k}}(3)$$

$$= -2\hat{\mathbf{i}} + \hat{\mathbf{j}} + 3\hat{\mathbf{k}}$$

So, the equation of the lines through the point (3, 0, 1) and parallel to the given two planes are

$$(x-3)\hat{\mathbf{i}} + (y-0)\hat{\mathbf{j}} + (z-1)\hat{\mathbf{k}} + \lambda(-2\hat{\mathbf{i}} + \hat{\mathbf{j}} + 3\hat{\mathbf{k}})$$

$$\Rightarrow \quad (x-3)\hat{\mathbf{i}} + y\hat{\mathbf{j}} + (z-1)\hat{\mathbf{k}} + \lambda(-2\hat{\mathbf{i}} + \hat{\mathbf{j}} + 3\hat{\mathbf{k}})$$

**Q. 20 Find the equation of the plane through the points (2, 1, – 1), (–1, 3, 4) and perpendicular to the plane $x - 2y + 4z = 10$.**

**Sol.** The equation of the plane passing through (2, 1, – 1) is

$$a(x-2) + b(y-1) + c(z+1) = 0 \qquad \text{...(i)}$$

Sicne, this passes through (–1, 3, 4).

$$\therefore \quad a(-1-2) + b(3-1) + c(4+1) = 0$$

$$\Rightarrow \quad -3a + 2b + 5c = 0 \qquad \text{... (ii)}$$

Since, the plane (i) is perpendicular to the plane $x - 2y + 4z = 10$.

$$\therefore \quad 1 \cdot a - 2 \cdot b + 4 \cdot c = 0$$

$$\Rightarrow \quad a - 2b + 4c = 0 \qquad \text{...(iii)}$$

On solving Eqs. (ii) and (iii), we get

$$\frac{a}{8+10} = \frac{-b}{-17} = \frac{c}{4} = \lambda$$

$$\Rightarrow \quad a = 18\lambda, b = 17\lambda, c = 4\lambda$$

From Eq. (i),

$$18\lambda(x-2)+17\lambda(y-1)+4\lambda(z+1)=0$$

$$\Rightarrow \quad 18x-36+17y-17+4z+4=0$$

$$\Rightarrow \quad 18x+17y+4z-49=0$$

$$\therefore \quad 18x+17y+4z=49$$

**Q. 21** Find the shortest distance between the lines gives by

$$\vec{r}=(8+3\lambda)\hat{i}-(9+16\lambda)\hat{j}+(10+7\lambda)\hat{k}$$

and $$\vec{r}=15\hat{i}+29\hat{j}+5\hat{k}+\mu(3\hat{i}+8\hat{j}-5\hat{k}).$$

**Sol.** We have, $\vec{r}=(8+3\lambda)\hat{i}-(9+16\lambda)\hat{j}+(10+7\lambda)\hat{k})$

$$=8\hat{i}-9\hat{j}+10\hat{k}+3\lambda\hat{i}-16\lambda\hat{j}+7\lambda\hat{k}$$

$$=8\hat{i}-9\hat{j}+10\hat{k}+\lambda(3\hat{i}-16\hat{j}+7\hat{k})$$

$$\Rightarrow \quad \vec{a}_1=8\hat{i}-9\hat{j}+10\hat{k} \text{ and } \vec{b}_1=3\hat{i}-16\hat{j}+7\hat{k} \quad \ldots\text{(i)}$$

Also $\vec{r}=15\hat{i}+29\hat{j}+5\hat{k}+\mu(3\hat{i}+8\hat{j}-5\hat{k})$

$$\Rightarrow \quad \vec{a}_2=15\hat{i}+29\hat{j}+5\hat{k} \text{ and } \vec{b}_2=3\hat{i}+8\hat{j}-5\hat{k} \quad \ldots\text{(ii)}$$

Now, shortest distance betwen two lines is given by $\left|\dfrac{(\vec{b}_1\times\vec{b}_2)\cdot(\vec{a}_2-\vec{a}_1)}{|\vec{b}_1\times\vec{b}_2|}\right|$

$$\therefore \quad \vec{b}_1\times\vec{b}_2=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 3 & -16 & 7\\ 3 & 8 & -5\end{vmatrix}$$

$$=\hat{i}(80-56)-\hat{j}(-15-21)+\hat{k}(24+48)$$

$$=24\hat{i}+36\hat{j}+72\hat{k}$$

Now, $$|\vec{b}_1\times\vec{b}_2|=\sqrt{(24)^2+(36)^2+(72)^2}$$

$$=12\sqrt{2^2+3^2+6^2}=84$$

and $$(\vec{a}_2-\vec{a}_1)=(15-8)\hat{i}+(29+9)\hat{j}+(5-10)\hat{k}$$

$$=7\hat{i}+38\hat{j}-5\hat{k}$$

$$\therefore \quad \text{Shortest distance}=\left|\frac{(24\hat{i}+36\hat{j}+72\hat{k})\cdot(7\hat{i}+38\hat{j}-5\hat{k})}{84}\right|$$

$$=\left|\frac{168+1368-360}{84}\right|=\left|\frac{1176}{84}\right|=14\text{ units}$$

**Q. 22** Find the equation of the plane which is perpendicular to the plane $5x+3y+6z+8=0$ and which contains the line of intersection of the planes $x+2y+3z-4=0$ and $2x+y-z+5=0$.

**Sol.** The equation of a plane through the line of intersection of the planes $x+2y+3z-4=0$ and $2x+y-z+5=0$ is

$$(x+2y+3z-4)+\lambda(2x+y-z+5)=0$$

$$\Rightarrow \quad x(1+2\lambda)+y(2+\lambda)+z(-\lambda+3)-4+5\lambda=0 \quad \ldots\text{(i)}$$

Also, this is perpendicular to the plane $5x + 3y + 6z + 8 = 0$.

$\therefore \quad 5(1 + 2\lambda) + 3(2 + \lambda) + 6(3 - \lambda) = 0 \quad [\because a_1 a_2 + b_1 b_2 + c_1 c_2 = 0]$

$\Rightarrow \quad 5 + 10\lambda + 6 + 3\lambda + 18 - 6\lambda = 0$

$\therefore \quad \lambda = -29/7$

From Eq. (i),

$$x\left[1 + 2\left(\frac{-29}{7}\right)\right] + y\left(2 - \frac{29}{7}\right) + z\left(\frac{29}{7} + 3\right) - 4 + 5\left(\frac{-29}{7}\right) = 0$$

$\Rightarrow \quad x(7 - 58) + y(14 - 29) + z(29 + 21) - 28 - 145 = 0$

$\Rightarrow \quad -51x - 15y + 50z - 173 = 0$

So, the required equation of plane is $51x + 15y - 50z + 173 = 0$.

**Q. 23 If the plane $ax + by = 0$ is rotated about its line of intersection with the plane $z = 0$ through an angle $\alpha$, then prove that the equation of the plane in its new position is $ax + by \pm (\sqrt{a^2 + b^2} \tan \alpha)\, z = 0$.**

**Sol.** Equation of the plane is $ax + by = 0$ ... (i)

$\therefore$ Equation of the plane after new position is

$$\frac{ax \cos \alpha}{\sqrt{a^2 + b^2}} + \frac{by \cos \alpha}{\sqrt{b^2 + a^2}} \pm z \sin \alpha = 0$$

$$\Rightarrow \quad \frac{ax}{\sqrt{a^2 + b^2}} + \frac{by}{\sqrt{b^2 + a^2}} \pm z \tan \alpha = 0 \quad \text{[on dividing by } \cos \alpha]$$

$$\Rightarrow \quad ax + by \pm z \tan \alpha \sqrt{\alpha^2 + b^2} = 0 \quad \text{[on multiplying with } \sqrt{a^2 + b^2}]$$

**Alternate Method**

Given,planes are $ax + by = 0$ ...(i)

and $z = 0$ ...(ii)

Therefore, the equation of any plane passing through the line of intersection of planes (i) and (ii) may be taken as $ax + by + k = 0$. ...(iii)

Then, direction cosines of a normal to the plane (iii) are $\frac{a}{\sqrt{a^2 + b^2 + k^2}}, \frac{b}{\sqrt{a^2 + b^2 + k^2}}, \frac{c}{\sqrt{a^2 + b^2 + k^2}}$ and direction cosines of the normal to the plane (i) are $\frac{a}{\sqrt{a^2 + b^2}}, \frac{b}{\sqrt{a^2 + b^2}}$, 0.

Since, the angle between the planes (i) and (ii) is $\alpha$,

$$\therefore \quad \cos \alpha = \frac{a \cdot a + b \cdot b + k \cdot 0}{\sqrt{a^2 + b^2 + k^2}\sqrt{a^2 + b^2}}$$

$$= \sqrt{\frac{a^2 + b^2}{a^2 + b^2 + k^2}}$$

$$\Rightarrow \quad k^2 \cos^2 \alpha = a^2 (1 - \cos^2 \alpha) + b^2 (1 - \cos^2 \alpha)$$

$$\Rightarrow \quad k^2 = \frac{(a^2 + b^2) \sin^2 \alpha}{\cos^2 \alpha}$$

$$k = \pm \sqrt{a^2 + b^2} \tan \alpha$$

On putting this value in plane (iii), we get the equation of the plane as

$$ax + by + z\sqrt{a^2 + b^2} \tan \alpha = 0$$

**Q. 24** **Find the equation of the plane through the intersection of the planes $\vec{r} \cdot (\hat{i} + 3\hat{j}) - 6 = 0$ and $\vec{r} \cdot (3\hat{i} - \hat{j} - 4\hat{k}) = 0$, whose perpendicular distance from origin is unity.**

**Sol.** We have, $\vec{n}_1 = (\hat{i} + 3\hat{j}), d_1 = 6$ and $\vec{n}_2 = (3\hat{i} - \hat{j} - 4\hat{k}), d_2 = 0$

Using the relation, $\vec{r} \cdot (\vec{n}_1 + \lambda \vec{n}_2) = d_1 + d_2\lambda$

$\Rightarrow \quad \vec{r} \cdot [(\hat{i} + 3\hat{j}) + \lambda (3\hat{i} - \hat{j} - 4\hat{k})] = 6 + 0 \cdot \lambda$

$\Rightarrow \quad \vec{r} \cdot [(1 + 3\lambda)\hat{i} + (3 - \lambda)\hat{j} + \hat{k} (- 4\lambda)] = 6 \qquad \text{...(i)}$

On dviding both sides by $\sqrt{(1 + 3\lambda)^2 + (3 - \lambda)^2 + (-4\lambda)^2}$, we get

$$\frac{\vec{r} \cdot [(1 + 3\lambda)\hat{i} + (3 - \lambda)\hat{j} + \hat{k}(- 4\lambda)]}{\sqrt{(1 + 3\lambda)^2 + (3 - \lambda)^2 + (- 4\lambda)^2}} = \frac{6}{\sqrt{(1 + 3\lambda)^2 + (3 - \lambda)^2 + (- 4\lambda)^2}}$$

Since, the perpendicular distance from origin is unity.

$$\therefore \quad \frac{6}{\sqrt{(1 + 3\lambda)^2 + (3 - \lambda)^2 + (- 4\lambda)^2}} = 1$$

$$\Rightarrow \quad (1 + 3\lambda)^2 + (3 - \lambda)^2 + (- 4\lambda)^2 = 36$$

$$\Rightarrow \quad 1 + 9\lambda^2 + 6\lambda + 9 + \lambda^2 - 6\lambda + 16\lambda^2 = 36$$

$$\Rightarrow \quad 26\lambda^2 + 10 = 36$$

$$\Rightarrow \quad \lambda^2 = 1$$

$$\therefore \quad \lambda = \pm 1$$

Using Eq. (i), the required equation of plane is

$$\vec{r} \cdot [(1 \pm 3)\hat{i} + (3 \mp 1)\hat{j} + (\mp 4)\hat{k}] = 6$$

$$\Rightarrow \quad \vec{r} \cdot [(1 + 3)\hat{i} + (3 - 1)\hat{j} + (- 4)\hat{k}] = 6$$

$$\text{and} \quad \vec{r} \cdot [(1 - 3)\hat{i} + (3 + 1)\hat{j} + 4\hat{k}] = 6$$

$$\Rightarrow \quad \vec{r} \cdot (4\hat{i} + 2\hat{j} - 4\hat{k}) = 6$$

$$\text{and} \quad \vec{r} . (- 2\hat{i} + 4\hat{j} + 4\hat{k}) = 6$$

$$\Rightarrow \quad 4x + 2y - 4z - 6 = 0$$

$$\text{and} \quad - 2x + 4y + 4z - 6 = 0$$

**Q. 25** **Show that the points $(\hat{i} - \hat{j} + 3\hat{k})$ and $3(\hat{i} + \hat{j} + \hat{k})$ are equidistant from the plane $\vec{r} \cdot (5\hat{i} + 2\hat{j} - 7\hat{k}) + 9 = 0$ and lies on opposite side of it.**

**Sol.** To show that these given points $(\hat{i} - \hat{j} + 3\hat{k})$ and $3(\hat{i} + \hat{j} + \hat{k})$ are equidistant from the plane $\vec{r} \cdot (5\hat{i} + 2\hat{j} - 7\hat{k}) + 9 = 0$, we first find out the mid- point of the points which is $2\hat{i} + \hat{j} + 3\hat{k}$.

On substituting $\vec{r}$ by the mid-point in plane, we get

$$\text{LHS} = (2\hat{i} + \hat{j} + 3\hat{k}) \cdot (5\hat{i} + 2\hat{j} - 7\hat{k}) + 9$$

$$= 10 + 2 - 21 + 9 = 0$$

$$= \text{RHS}$$

Hence, the two points lie on opposite sides of the plane are equidistant from the plane.

**Q. 26** $\vec{AB} = 3\hat{i} - \hat{j} + \hat{k}$ and $\vec{CD} = -3\hat{i} + 2\hat{j} + 4\hat{k}$ are two vectors. The position vectors of the points *A* and *C* are $6\hat{i} + 7\hat{j} + 4\hat{k}$ and $-9\hat{i} + 2\hat{k}$, respectively. Find the position vector of a point *P* on the line *AB* and a point *Q* on the line *CD* such that $\vec{PQ}$ is perpendicular to $\vec{AB}$ and $\vec{CD}$ both.

**Sol.** We have, $\vec{AB} = 3\hat{i} - \hat{j} + \hat{k}$ and $\vec{CD} = -3\hat{i} + 2\hat{j} + 4\hat{k}$

Also, the position vectors of *A* and *C* are $6\hat{i} + 7\hat{j} + 4\hat{k}$ and $-9\hat{j} + 2\hat{k}$, respectively. Since, $\vec{PQ}$ is perpendicular to both $\vec{AB}$ and $\vec{CD}$.

So, *P* and *Q* will be foot of perpendicular to both the lines through *A* and *C*.

Now, equation of the line through A and parallel to the vector $\vec{AB}$ is,

$$\vec{r} = (6\hat{i} + 7\hat{j} + 4\hat{k}) + \lambda(3\hat{i} - \hat{j} + \hat{k})$$

and the line through *C* and parallel to the vector $\vec{CD}$ is given by

$$\vec{r} = -9\hat{j} + 2\hat{k} + \mu(-3\hat{i} + 2\hat{j} + 4\hat{k}) \quad \text{...(i)}$$

Let $\vec{r} = (6\hat{i} + 7\hat{j} + 4\hat{k}) + \lambda(3\hat{i} - \hat{j} + \hat{k})$

and $\vec{r} = -9\hat{j} + 2\hat{k} + \mu(-3\hat{i} + 2\hat{j} + 4\hat{k}) \quad \text{...(ii)}$

Let *P* $(6 + 3\lambda, 7 - \lambda, 4 + \lambda)$ is any point on the first line and *Q* be any point on second line is given by $(-3\mu, -9 + 2\mu, 2 + 4\mu)$.

$$\therefore \quad \vec{PQ} = (-3\mu - 6 - 3\lambda)\hat{i} + (-9 + 2\mu - 7 + \lambda)\hat{j} + (2 + 4\mu - 4 - \lambda)\hat{k}$$

$$= (-3\mu - 6 - 3\lambda)\hat{i} + (2\mu + \lambda - 16)\hat{j} + (4\mu - \lambda - 2)\hat{k}$$

If $\vec{PQ}$ is perpendicular to the first line, then

$$3(-3\mu - 6 - 3\lambda) - (2\mu + \lambda - 16) + (4\mu - \lambda - 2) = 0$$

$$\Rightarrow \quad -9\mu - 18 - 9\lambda - 2\mu - \lambda + 16 + 4\mu - \lambda - 2 = 0$$

$$\Rightarrow \quad -7\mu - 11\lambda - 4 = 0 \quad \text{...(iii)}$$

If $\vec{PQ}$ is perpendicular to the second line, then

$$-3(-3\mu - 6 - 3\lambda) + 2(2\mu + \lambda - 16) + 4(4\mu - \lambda - 2) = 0$$

$$\Rightarrow \quad 9\mu + 18 + 9\lambda + 4\mu + 2\lambda - 32 + 16\mu - 4\lambda - 8 = 0$$

$$\Rightarrow \quad 29\mu + 7\lambda - 22 = 0 \quad \text{...(iv)}$$

On solving Eqs. (iii) and (iv), we get

$$-49\mu - 77\lambda - 28 = 0$$

$$\Rightarrow \quad 319\mu + 77\lambda - 242 = 0$$

$$\Rightarrow \quad 270\mu - 270 = 0$$

$$\Rightarrow \quad \mu = 1$$

Using $\mu$ in Eq. (iii), we get

$$-7(1) - 11\lambda - 4 = 0$$

$$\Rightarrow \quad -7 - 11\lambda - 4 = 0$$

$$\Rightarrow \quad -11 - 11\lambda = 0$$

$$\Rightarrow \quad \lambda = -1$$

$$\therefore \quad \vec{PQ} = [-3(1) - 6 - 3(-1)]\hat{i} + [2(1) + (-1) - 16]\hat{j} + [4(1) - (-1) - 2]\hat{k}$$

$$= -6\hat{i} - 15\hat{j} + 3\hat{k}$$

**Q. 27** Show that the straight lines whose direction cosines are given by $2l + 2m - n = 0$ and $mn + nl + lm = 0$ are at right angles.

**Sol.** We have, $$2l + 2m - n = 0 \quad \ldots(i)$$
and $$mn + nl + lm = 0 \quad \ldots(ii)$$
Eliminating $m$ from the both equations, we get
$$m = \frac{n-2l}{2} \quad \text{[from Eq. (i)]}$$
$$\Rightarrow \quad \left(\frac{n-2l}{2}\right)n + nl + l\left(\frac{n-2l}{2}\right) = 0$$
$$\Rightarrow \quad \frac{n^2 - 2nl + 2nl + nl - 2l^2}{2} = 0$$
$$\Rightarrow \quad n^2 + nl - 2l^2 = 0$$
$$\Rightarrow \quad n^2 + 2nl - nl - 2l^2 = 0$$
$$\Rightarrow \quad (n + 2l)(n - l) = 0$$
$$\Rightarrow \quad n = -2l \text{ and } n = l$$
$$\therefore \quad m = \frac{-2l - 2l}{2}, m = \frac{l - 2l}{2}$$
$$\Rightarrow \quad m = -2l, m = \frac{-l}{2}$$
Thus, the direction ratios of two lines are proportional to $l, -2l, -2$ and $l, \frac{-l}{2}, l$.
$$\Rightarrow \quad 1, -2, -2 \text{ and } 1, \frac{-1}{2}, 1$$
$$\Rightarrow \quad 1, -2, -2 \text{ and } 2, -1, 2$$
Also, the vectors parallel to these lines are $\vec{a} = \hat{i} - 2\hat{j} - 2\hat{k}$ and $\vec{b} = 2\hat{i} - \hat{j} + 2\hat{k}$, respectively.
$$\therefore \quad \cos\theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|} = \frac{(\hat{i} - 2\hat{j} - 2\hat{k}) \cdot (2\hat{i} - \hat{j} + 2\hat{k})}{3 \cdot 3}$$
$$= \frac{2 + 2 - 4}{9} = 0$$
$$\therefore \quad \theta = \frac{\pi}{2} \quad \left[\because \cos\frac{\pi}{2} = 0\right]$$

**Q. 28** If $l_1, m_1, n_1, l_2, m_2, n_2$ and $l_3, m_3, n_3$ are the direction cosines of three mutually perpendicular lines, then prove that the line whose direction cosines are proportional to $l_1 + l_2 + l_3, m_1 + m_2 + m_3$ and $n_1 + n_2 + n_3$ makes equal angles with them.

**Sol.** Let $$\vec{a} = l_1\hat{i} + m_1\hat{j} + n_1\hat{k}$$
$$\vec{b} = l_2\hat{i} + m_2\hat{j} + n_2\hat{k}$$
$$\vec{c} = l_3\hat{i} + m_3\hat{j} + n_3\hat{k}$$
$$\vec{d} = (l_1 + l_2 + l_3)\hat{i} + (m_1 + m_2 + m_2)\hat{j} + (n_1 + n_2 + n_3)\hat{k}$$
Also, let $\alpha, \beta$ and $\gamma$ are the angles between $\vec{a}$ and $\vec{d}$, $\vec{b}$ and $\vec{d}$, $\vec{c}$ and $\vec{d}$.
$$\therefore \quad \cos\alpha = l_1(l_1 + l_2 + l_3) + m_1(m_1 + m_2 + m_3) + n_1(n_1 + n_2 + n_3)$$
$$= l_1^2 + l_1 l_2 + l_1 l_3 + m_1^2 + m_1 m_2 + m_1 m_3 + n_1^2 + n_1 n_2 + n_1 n_3$$

$$= (l_1^2 + m_1^2 + n_1^2) + (l_1 l_2 + l_1 l_3 + m_1 m_2 + m_1 m_3 + n_1 n_2 + n_1 n_3)$$
$$= 1 + 0 = 1$$
$$[\because l_1^2 + m_1^2 + n_1^2 = 1 \text{ and } l_1 \perp l_2, l_1 \perp l_3, m_1 \perp m_2, m_1 \perp m_3, n_1 \perp n_2, n_1 \perp n_3]$$

Similarly, $\cos\beta = l_2 (l_1 + l_2 + l_3) + m_2 (m_1 + m_2 + m_3) + n_2 (n_1 + n_2 + n_3)$
$= 1 + 0$ and $\cos\gamma = 1 + 0$

$\Rightarrow$ $\cos\alpha = \cos\beta = \cos\gamma$

$\Rightarrow$ $\alpha = \beta = \gamma$

So, the line whose direction cosines are proportional to $l_1 + l_2 + l_3, m_1 + m_2 + m_3, n_1 + n_2 + n_3$ makes equal angles with the three mutually perpendicular lines whose direction cosines are $l_1, m_1, n_1, l_2, m_2, n_2$ and $l_3, m_3, n_3$ respectively.

# Objective Type Questions

**Q. 29** Distance of the point $(\alpha, \beta, \gamma)$ from $Y$-axis is

(a) $\beta$ (b) $|\beta|$
(c) $|\beta| + |\gamma|$ (d) $\sqrt{\alpha^2 + \gamma^2}$

**Sol.** *(d)* Required distance $= \sqrt{(\alpha - 0)^2 + (\beta - \beta)^2 + (\gamma - 0)^2} = \sqrt{\alpha^2 + \gamma^2}$

**Q. 30** If the direction cosines of a line are $k$, $k$ and $k$, then

(a) $k > 0$ (b) $0 < k < 1$
(c) $k = 1$ (d) $k = \frac{1}{\sqrt{3}}$ or $-\frac{1}{\sqrt{3}}$

**Sol.** *(d)* Since, direction cosines of a line are $k$, $k$ and $k$.

$\therefore$ $l = k, m = k$ and $n = k$

We know that, $l^2 + m^2 + n^2 = 1$

$\Rightarrow$ $k^2 + k^2 + k^2 = 1$

$\Rightarrow$ $k^2 = \frac{1}{3}$

$\therefore$ $k = \pm \frac{1}{\sqrt{3}}$

**Q. 31** The distance of the plane $\vec{\mathbf{r}} \left( \frac{2}{7}\hat{\mathbf{i}} + \frac{3}{7}\hat{\mathbf{j}} - \frac{6}{7}\hat{\mathbf{k}} \right) = 1$ from the origin is

(a) 1 (b) 7
(c) $\frac{1}{7}$ (d) None of these

**Sol.** *(a)* The distance of the plane $\vec{\mathbf{r}} \left( \frac{2}{7}\hat{\mathbf{i}} + \frac{3}{7}\hat{\mathbf{j}} - \frac{6}{7}\hat{\mathbf{k}} \right) = 1$ from the origin is 1.

[since, $\vec{\mathbf{r}} \cdot \vec{\mathbf{n}} = d$ is the form of above equation, where $d$ represents the distance of plane from the origin *i.e.*, $d = 1$]

**Q. 32** The sine of the angle between the straight line $\frac{x-2}{3}=\frac{y-3}{4}=\frac{z-4}{5}$ and the plane $2x-2y+z=5$ is

(a) $\frac{10}{6\sqrt{5}}$ (b) $\frac{4}{5\sqrt{2}}$ (c) $\frac{2\sqrt{3}}{5}$ (d) $\frac{\sqrt{2}}{10}$

**Sol.** *(d)* We have, the equation of line as

$$\frac{x-2}{3}=\frac{y-3}{4}=\frac{z-4}{5}$$

Now, the line passes through point (2, 3, 4) and having direction ratios (3, 4, 5).

Since, the line passes through point (2, 3, 4) and parallel to the vector $(3\hat{\mathbf{i}}+4\hat{\mathbf{j}}+5\hat{\mathbf{k}})$.

$$\therefore \quad \vec{\mathbf{b}}=3\hat{\mathbf{i}}+4\hat{\mathbf{j}}+5\hat{\mathbf{k}}$$

Also, the cartesian form of the given plane is $2x-2y+z=5$.

$$\Rightarrow \quad (x\hat{\mathbf{i}}+y\hat{\mathbf{j}}+z\hat{\mathbf{k}})(2\hat{\mathbf{i}}-2\hat{\mathbf{j}}+\hat{\mathbf{k}})=5$$

$$\therefore \quad \vec{\mathbf{n}}=(2\hat{\mathbf{i}}-2\hat{\mathbf{j}}+\hat{\mathbf{k}})$$

We know that, $\sin\theta=\frac{|\vec{\mathbf{b}}\cdot\vec{\mathbf{n}}|}{|\vec{\mathbf{b}}|\cdot|\vec{\mathbf{n}}|}=\frac{\left|(3\hat{\mathbf{i}}+4\hat{\mathbf{j}}+5\hat{\mathbf{k}})\cdot(2\hat{\mathbf{i}}-2\hat{\mathbf{j}}+\hat{\mathbf{k}})\right|}{\sqrt{3^2+4^2+5^2}\cdot\sqrt{4+4+1}}$

$$=\frac{|6-8+5|}{\sqrt{50}\cdot 3}=\frac{3}{15\sqrt{2}}=\frac{1}{5\sqrt{2}}$$

$$\sin\theta=\frac{\sqrt{2}}{10}$$

**Q. 33** The reflection of the point $(\alpha,\beta,\gamma)$ in the $XY$-plane is

(a) $(\alpha,\beta,0)$ (b) $(0,0,\gamma)$ (c) $(-\alpha,-\beta,\gamma)$ (d) $(\alpha,\beta,-\gamma)$

**Sol.** *(d)* In $XY$-plane, the reflection of the point $(\alpha,\beta,\gamma)$ is $(\alpha,\beta,-\gamma)$.

**Q. 34** The area of the quadrilateral $ABCD$ where $A$ (0,4,1), $B$ (2, 3, −1), $C$ (4, 5, 0), and $D$ (2, 6, 2) is equal to

(a) 9 sq units (b) 18 sq units

(c) 27 sq units (d) 81 sq units

**Sol.** *(a)* We have,

$$\vec{\mathbf{AB}}=(2-0)\hat{\mathbf{i}}+(3-4)\hat{\mathbf{j}}+(-1-1)\hat{\mathbf{k}}=2\hat{\mathbf{i}}-\hat{\mathbf{j}}-2\hat{\mathbf{k}}$$

$$\vec{\mathbf{BC}}=(4-2)\hat{\mathbf{i}}+(5-3)\hat{\mathbf{j}}+(0+1)\hat{\mathbf{k}}=2\hat{\mathbf{i}}+2\hat{\mathbf{j}}+\hat{\mathbf{k}}$$

$$\vec{\mathbf{CD}}=(2-4)\hat{\mathbf{i}}+(6-5)\hat{\mathbf{j}}+(2-0)\hat{\mathbf{k}}=-2\hat{\mathbf{i}}+\hat{\mathbf{j}}+2\hat{\mathbf{k}}$$

$$\vec{\mathbf{DA}}=(0-2)\hat{\mathbf{i}}+(4-6)\hat{\mathbf{j}}+(1-2)\hat{\mathbf{k}}=-2\hat{\mathbf{i}}-2\hat{\mathbf{j}}-\hat{\mathbf{k}}$$

$$\therefore \quad \text{Area of quadrilateral } ABCD=|\vec{\mathbf{AB}}\times\vec{\mathbf{BC}}|=\begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ 2 & -1 & -2\\ 2 & 2 & 1\end{vmatrix}$$

$$=\left|\hat{\mathbf{i}}(-1+4)-\hat{\mathbf{j}}(2+4)+\hat{\mathbf{k}}(4+2)\right|$$

$$=\left|3\hat{\mathbf{i}}-6\hat{\mathbf{j}}+6\hat{\mathbf{k}}\right|$$

$$=\sqrt{9+36+36}=9 \text{ sq units}$$

**Q. 35** The locus represented by $xy + yz = 0$ is

(a) a pair of perpendicular lines
(b) a pair of parallel lines
(c) a pair of parallel planes
(d) a pair of perpendicular planes

**Sol.** *(d)* We have, $xy + yz = 0$

$\Rightarrow xy = -yz$

So, a pair of perpendicular planes.

**Q. 36** If the plane $2x - 3y + 6z - 11 = 0$ makes an angle $\sin^{-1}\alpha$ with $X$-axis, then the value of $\alpha$ is

(a) $\frac{\sqrt{3}}{2}$
(b) $\frac{\sqrt{2}}{3}$
(c) $\frac{2}{7}$
(d) $\frac{3}{7}$

**Sol.** *(c)* Since, $2x - 3y + 6z - 11 = 0$ makes an angle $\sin^{-1}\alpha$ with $X$-axis.

$$\vec{\mathbf{b}} = (1\hat{\mathbf{i}} + 0\hat{\mathbf{j}} + 0\hat{\mathbf{k}}) \text{ and } \vec{\mathbf{n}} = 2\hat{\mathbf{i}} - 3\hat{\mathbf{j}} + 6\hat{\mathbf{k}}$$

We know that, $$\sin\theta = \frac{|\vec{\mathbf{b}} \cdot \vec{\mathbf{n}}|}{|\vec{\mathbf{b}}| \cdot |\vec{\mathbf{n}}|}$$

$$= \frac{|(1\hat{\mathbf{i}}) \cdot (2\hat{\mathbf{i}} - 3\hat{\mathbf{j}} + 6\hat{\mathbf{k}})|}{\sqrt{1}\sqrt{4 + 9 + 36}} = \frac{2}{7}$$

## Fillers

**Q. 37** If a plane passes through the points (2, 0,0) (0, 3, 0) and (0, 0, 4) the equation of plane is ......... .

**Sol.** We know that, equation of a the plane that cut the coordinate axes at $(a, 0, 0)$ $(0, b, 0)$ and $(0, 0, c)$ is $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$.

Hence, the equation of plane passes through the points (2, 0, 0), (0, 3, 0) and (0, 0, 4) is $\frac{x}{2} + \frac{y}{3} + \frac{z}{4} = 1$.

**Q. 38** The direction cosines of the vector $(2\hat{\mathbf{i}} + 2\hat{\mathbf{j}} - \hat{\mathbf{k}})$ are ......... .

**Sol.** Direction cosines of $(2\hat{\mathbf{i}} + 2\hat{\mathbf{j}} - \hat{\mathbf{k}})$ are $\frac{2}{\sqrt{4+4+1}}, \frac{2}{\sqrt{4+4+1}}, \frac{-1}{\sqrt{4+4+1}}$ *i.e.,* $\frac{2}{3}, \frac{2}{3}, \frac{-1}{3}$.

**Q. 39** The vector equation of the line $\frac{x-5}{3} = \frac{y+4}{7} = \frac{z-6}{2}$ is ......... .

**Sol.** We have, $\vec{a} = 5\hat{i} - 4\hat{j} + 6\hat{k}$ and $\vec{b} = 3\hat{i} + 7\hat{j} + 2\hat{k}$

So, the vector equation will be

$$\vec{r} = (5\hat{i} - 4\hat{j} + 6\hat{k}) + \lambda(3\hat{i} + 7\hat{j} + 2\hat{k})$$

$$\Rightarrow \quad (x\hat{i} + y\hat{j} + z\hat{k}) - (5\hat{i} - 4\hat{j} + 6\hat{k}) = \lambda(3\hat{i} + 7\hat{j} + 2\hat{k})$$

$$\Rightarrow \quad (x-5)\hat{i} + (y+4)\hat{j} + (z-6)\hat{k} = \lambda(3\hat{i} + 7\hat{j} + 2\hat{k})$$

**Q. 40** The vector equation of the line through the points (3, 4, −7) and (1, −1, 6) is ......... .

**Sol.** We know that, vector equation of a line passes through two points is represented by $\vec{r} = \vec{a} + \lambda(\vec{b} - \vec{a})$

Here, $\vec{r} = x\hat{i} + y\hat{j} + 3\hat{k}$, $\vec{a} = 3\hat{i} + 4\hat{j} - 7\hat{k}$

and $\quad \vec{b} = \hat{i} - \hat{j} + 6\hat{k}$

$$\Rightarrow \quad (\vec{b} - \vec{a}) = -2\hat{i} - 5\hat{j} + 13\hat{k}$$

So, the required equation is

$$x\hat{i} + y\hat{j} + z\hat{k} = 3\hat{i} + 4\hat{j} - 7\hat{k} + \lambda(-2\hat{i} - 5\hat{j} + 13\hat{k})$$

$$\Rightarrow \quad (x-3)\hat{i} + (y-4)\hat{j} + (z+7)\hat{k} = \lambda(-2\hat{i} - 5\hat{j} + 13\hat{k})$$

**Q. 41** The cartesian equation of the plane $\vec{r} \cdot (\hat{i} + \hat{j} - \hat{k}) = 2$ is ......... .

**Sol.** We have, $\quad \vec{r} \cdot (\hat{i} + \hat{j} - \hat{k}) = 2$

$$\Rightarrow \quad (x\hat{i} + y\hat{j} + z\hat{k}) \cdot (\hat{i} + \hat{j} - \hat{k}) = 2$$

$$\Rightarrow \quad x + y - z = 2$$

which is the required form

## True/False

**Q. 42** The unit vector normal to the plane $x + 2y + 3z - 6 = 0$ is

$$\frac{1}{\sqrt{14}}\hat{i} + \frac{2}{\sqrt{14}}\hat{j} + \frac{3}{\sqrt{14}}\hat{k}.$$

**Sol.** ***True***

We have, $\quad \vec{n} = \hat{i} + 2\hat{j} + 3\hat{k}$

$$\therefore \quad \hat{n} = \frac{\hat{i} + 2\hat{j} + 3\hat{k}}{\sqrt{1^2 + 2^2 + 3^2}} = \frac{\hat{i}}{\sqrt{14}} + \frac{2\hat{j}}{\sqrt{14}} + \frac{3\hat{k}}{\sqrt{14}}$$

**Q. 43** The intercepts made by the plane $2x - 3y + 5z + 4 = 0$ on the coordinate axis are $-2, \frac{4}{3}$ and $-\frac{4}{5}$.

**Sol.** ***True***

We have, $2x - 3y + 5z + 4 = 0$

$\Rightarrow \quad 2x - 3y + 5z = -4$

$\Rightarrow \quad \frac{2x}{-4} - \frac{3y}{-4} + \frac{5z}{-4} = 1$

$\Rightarrow \quad \frac{x}{-2} + \frac{y}{\frac{4}{3}} - \frac{z}{\frac{4}{5}} = 1$

$\Rightarrow \quad \frac{x}{-2} + \frac{y}{\frac{4}{3}} + \frac{z}{\left(-\frac{4}{5}\right)} = 1$

So, the intercepts are $-2, \frac{4}{3}$ and $-\frac{4}{5}$.

**Q. 44** The angle between the line $\vec{r} = (5\hat{i} - \hat{j} - 4\hat{k}) + \lambda(2\hat{i} - \hat{j} + \hat{k})$ and the plane $\vec{r}\,(3\hat{i} - 4\hat{j} - \hat{k}) + 5 = 0$ is $\sin^{-1}\left(\frac{5}{2\sqrt{91}}\right)$.

**Sol.** ***False***

We have, $\vec{b} = 2\hat{i} - \hat{j} + \hat{k}$ and $\vec{n} = 3\hat{i} - 4\hat{j} - \hat{k}$

Let $\theta$ is the angle between line and plane.

Then, $\sin\theta = \frac{|\vec{b}\cdot\vec{n}|}{|\vec{b}|\cdot|\vec{n}|} = \frac{|(2\hat{i} - \hat{j} + \hat{k})\cdot(3\hat{i} - 4\hat{j} - \hat{k})|}{\sqrt{6}\cdot\sqrt{26}}$

$= \frac{|6 + 4 - 1|}{\sqrt{156}} = \frac{9}{2\sqrt{39}}$

$\therefore \quad \theta = \sin^{-1}\frac{9}{2\sqrt{39}}$

**Q. 45** The angle between the planes $\vec{r}\,(2\hat{i} - 3\hat{j} + \hat{k}) = 1$ and $\vec{r}\,(\hat{i} - \hat{j}) = 4$ is $\cos^{-1}\left(\frac{-5}{\sqrt{58}}\right)$.

**Sol.** ***False***

We know that, the angle between two planes is given by $\cos\theta = \frac{|\vec{n}_1\cdot\vec{n}_2|}{|\vec{n}_1||\vec{n}_2|}$

Here, $\vec{n}_1 = (2\hat{i} - 3\hat{j} + \hat{k})$ and $\vec{n}_2 = (\hat{i} - \hat{j})$

$\therefore \quad \cos\theta = \frac{|(2\hat{i} - 3\hat{j} + \hat{k})\;(\hat{i} - \hat{j})|}{\sqrt{4 + 9 + 1}\;\sqrt{1 + 1}}$

$\Rightarrow \quad \cos\theta = \frac{|2 + 3|}{\sqrt{14}\cdot\sqrt{2}} = \frac{5}{2\sqrt{7}}$

$\therefore \quad \theta = \cos^{-1}\left(\frac{5}{2\sqrt{7}}\right)$

**Q. 46** The line $\vec{r} = 2\hat{i} - 3\hat{j} - \hat{k} + \lambda(\hat{i} - \hat{j} + 2\hat{k})$ lies in the plane $\vec{r}(3\hat{i} + \hat{j} - \hat{k}) + 2 = 0$.

**Sol.** ***False***

We have, $\vec{r} = 2\hat{i} - 3\hat{j} - \hat{k} + \lambda(\hat{i} - \hat{j} + 2\hat{k})$

$\Rightarrow (x\hat{i} + y\hat{j} + z\hat{k}) = \hat{i}(2+\lambda) + \hat{j}(-3-\lambda) + \hat{k}(-1+2\lambda)$

Since, $x = (2+\lambda)$, $y = (-3-\lambda)$ and $z = (-1+2\lambda)$ are coordinates of general point which should satisfy the equation of the given plane.

$\therefore \quad [(2+\lambda)\hat{i} + (-3-\lambda)\hat{j} + (2\lambda-1)\hat{k}]\cdot[\hat{i} + \hat{j} - \hat{k}] = 2$

$\Rightarrow \quad (2+\lambda) - 3 - \lambda - 2\lambda + 1 = 2$

$\Rightarrow \quad -2\lambda = 2$

$\Rightarrow \quad \lambda = -1$

$\therefore \quad \vec{r} = (2-1)\hat{i} + (-3+1)\hat{j} + (-2-1)\hat{k}$

$= \hat{i} - 2\hat{j} - 3\hat{k}$

Again, from the equation of the plane

$\vec{r}\cdot(3\hat{i} + \hat{j} - \hat{k}) + 2 = 0$

$\Rightarrow \quad (\hat{i} - 2\hat{j} - 3\hat{k})\ (3\hat{i} + \hat{j} - \hat{k}) + 2 = 0$

$\Rightarrow \quad (3 - 2 + 3) + 2 = 0$

$\Rightarrow \quad 6 \neq 0$

which is not true.

So, the line $\vec{r} = 2\hat{i} - 3\hat{j} - \hat{k} + \lambda(\hat{i} - \hat{j} + 2\hat{k})$ does not lie in a plane.

**Q. 47** The vector equation of the line $\frac{x-5}{3} = \frac{y+4}{7} = \frac{z-6}{2}$ is

$$\vec{r} = 5\hat{i} - 4\hat{j} + 6\hat{k} + \lambda(3\hat{i} + 7\hat{j} + 2\hat{k})$$

**Sol.** ***True***

We have, $x = 5, y = -4, z = 6$

and $a = 3, b = 7, c = 2$

$\therefore \quad \vec{r} = (5\hat{i} - 4\hat{j} + 6\hat{k}) + \lambda(3\hat{i} + 7\hat{j} + 2\hat{k})$

**Q. 48** The equation of a line, which is parallel to $2\hat{i} + \hat{j} + 3\hat{k}$ and which passes through the point $(5, -2, 4)$ is $\frac{x-5}{2} = \frac{y+2}{-1} = \frac{z-4}{3}$.

**Sol.** ***False***

Here, $x_1 = 5, y_1 = -2, z_1 = 4$

and $a = 2, b = 1, c = 3$

$\Rightarrow \quad \frac{x-5}{2} = \frac{y+2}{1} = \frac{z-4}{3}$

**Q. 49** If the foot of perpendicular drawn from the origin to a plane is (5, − 3, − 2), then the equation of plane is $\vec{r}$ $(5\hat{i} - 3\hat{j} - 2\hat{k}) = 38$.

**Sol.** ***True***

Since, the required plane passes through the point $P$ (5, − 3, − 2) and is perpendicular to $\overrightarrow{OP}$.

$\therefore \quad \vec{a} = 5\hat{i} - 3\hat{j} - 2\hat{k}$

and $\quad \vec{n} = \overrightarrow{OP} = 5\hat{i} - 3\hat{j} - 2\hat{k}$

Now, the equation of the plane is

$$(\vec{r} - \vec{a}) \cdot \vec{n} = 0$$

$$\Rightarrow \quad \vec{r} \cdot \vec{n} = \vec{a} \cdot \vec{n}$$

$$\Rightarrow \quad \vec{r} \cdot (5\hat{i} - 3\hat{j} - 2\hat{k}) = (5\hat{i} - 3\hat{j} - 2\hat{k}) \cdot (5\hat{i} - 3\hat{j} - 2\hat{k})$$

$$\Rightarrow \quad \vec{r} . (5\hat{i} - 3\hat{j} - 2\hat{k}) = 25 + 9 + 4$$

$$\Rightarrow \quad \vec{r} \cdot (5\hat{i} - 3\hat{j} - 2\hat{k}) = 38$$