Triangles

33. ABCD is a trapezium, in which AB is parallel to DC and its diagonals intersect each other at point O. show that $\frac{AO}{BO} = \frac{CO}{DO}$.

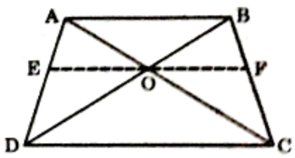
2014/2016 (2 Marks)

In the figure, ABCD is a trapezium with AB || DC .

Construction: Through O, draw EF || DC (see figure).

Now, in ΔADC, EO || DC, by BPT, we get:

$$\frac{AE}{DE} = \frac{AO}{OC} \text{-------------(1)}$$



Also, since AB || DC and EO || DC, we have:

EO || AB.

So, in ΔDAB, we have:

$\frac{DE}{AE} = \frac{DO}{BO}$ (By BPT)

$\Rightarrow \frac{AE}{DE} = \frac{BO}{DO}$ ------(2)

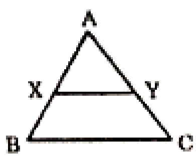
From (1) and (2) we get:

$\frac{AO}{OC} = \frac{BO}{DO} \Rightarrow \frac{AO}{BO} = \frac{OC}{DO}$ (Proved)

34. In ΔABC, X is the middle point of AB. If XY || BC, then prove that Y is the middle point of AC.

2015/2016 (3 Marks)

In figure, X is the mid-point of AB and XY || BC.



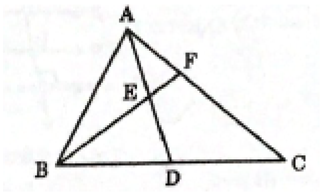
From BPT, $\frac{AX}{XB} = \frac{AY}{YC}$

$\Rightarrow \frac{AX}{AX} = \frac{AY}{YC}$ (X is the mid-point of AB)

$\Rightarrow \frac{AY}{YC} = 1 \Rightarrow AY = YC.$

So, Y is the mid-point of AC.

35. In the figure, AD is median of ΔABC and E is the mid-point of AD. If BE is produced to meet AC at F, then prove that $AF = \frac{1}{3}AC$.



2015/2016(3 Marks)

In the figure,

Draw DG || BF.

In ΔADG, we have:

AE = ED (Given)

EF || DG (DG || BF)

So, by BPT, $\frac{AE}{ED} = \frac{AF}{FG}$

$\Rightarrow \frac{AE}{AE} = \frac{AF}{FG}$ (AE = ED)

$\Rightarrow$ AF = FG -----(1)

Similarly, in ΔCBF, we have:

BD = DC

And DG || BF

So, by BPT,

$$\frac{CG}{FG} = \frac{CD}{BD}$$

$\Rightarrow \frac{CG}{FG} = \frac{CD}{CD}$ (CD = BD)

$\Rightarrow$ CG = FG ------(2)

From (1) and (2),

AF = FG = CG

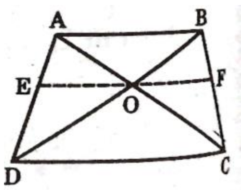
Also, AC = AF + FG + GC

So, AF = $\frac{1}{3}$ AC.

36. The diagonals of a quadrilateral ABCD intersect each other at the point O, such that $\frac{AO}{BO} = \frac{CO}{DO}$. Show that ABCD is a trapezium.

2014/2016 (3 Marks)

Through O, draw a parallel EF to DC. (See figure)



So, in ΔADC, we get

$\frac{AE}{ED} = \frac{AO}{OC}$ (By BPT) ------(1)

Again, $\frac{AO}{BO} = \frac{CO}{DO}$ (Given)

So, $\frac{AO}{OC} = \frac{BO}{DO}$ ------(2)

From (1) and (2), we get

$$\frac{AE}{ED} = \frac{BO}{DO}$$

$\Rightarrow \frac{ED}{AE} = \frac{DO}{BO}$

Hence, OE || AB (By converse of BPT) ----- (3)

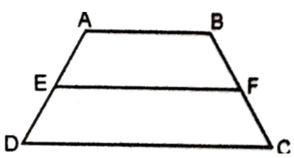
Also, OE || DC (By construction) ------ (4)

From (3) and (4), AB || DC

Hence ABCD is a trapezium.

---

37. In the given figure, ABCD is a trapezium with AB || DC, E and F are the points on non-parallel sides AD and BC respectively such that EF || AB. Prove that $\frac{AE}{ED} = \frac{BF}{FC}$.

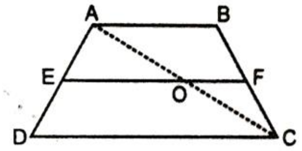


2011/2012/2013/2014/2016 (3 Marks)

Given AB || CD (Given)

And EF || AB (Given)

$\Rightarrow$ AB || DC || EF.



Join AC. It intersects EF at O.

In $\triangle$ADC, OE||CD as EF||CD.

Therefore, $\frac{AE}{ED} = \frac{AO}{OC}$ (By BPT) ....(1)

In $\triangle$ACB, OF||AB as EF||AB.

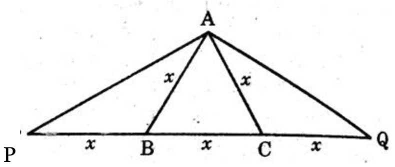
Therefore, $\frac{AO}{OC} = \frac{BF}{FC}$ (By BPT) ....(2)

From (1) and (2), we have:

$\frac{AE}{ED} = \frac{BF}{FC}$.

---

38. In the given figure $\triangle$ABC is an equilateral Triangle, whose each side measures x units. P and Q are two points on BC produced such that PB = BC = CQ.



Prove that:

$$(a)\frac{PQ}{PA} = \frac{PA}{PB}(b)PA^2 = 3x^2$$

2015/2016 (3 Marks)

In$\triangle$PAB, PB = AB

So, $\angle$APB = $\angle$PAB

Also, $\angle$ABP = 180° - 60° = 120°

So, $\angle$APB = $\angle$PAB = $\frac{1}{2}(180° - 120°) = 30°$

Similarly, $\angle$QAC = $\angle$QCA = 30°

So, $\angle$PAQ = $\angle$PAB + $\angle$BAC + $\angle$QAC

= $30° + 60° + 30° = 120°$.

Now, in $\triangle$PQA and $\triangle$PAB, we have:

$\angle$APQ = $\angle$APB (Each 30°)

$\angle$PAQ = $\angle$PBA (Each 120°)

And $\angle$PQA = $\angle$PAB (Each 30°)

So, ΔPQA ~ΔPAB (By AAA similarity creation)

Hence, $\frac{PQ}{PA} = \frac{PA}{PB}$(Proved)

(b)

PQ = $3x$

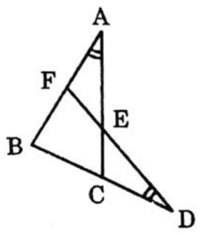
So, from $\frac{PQ}{PA} = \frac{PA}{PB}, we\ have$

$PA^2 = PQ \times PB$

$PA^2 = 3x \times x = 3x^2$. (Proved)

---

39. In the figure, if $\angle$A = $\angle$D, then prove that AE× $DC = DE \times AF$.



2014/2015/2016 (3 Marks)

In ΔAEC and ΔDEC, we have:

$\angle$A = $\angle$D (Given)

And $\angle$AEF = $\angle$DEC (Vertically opposite angles)

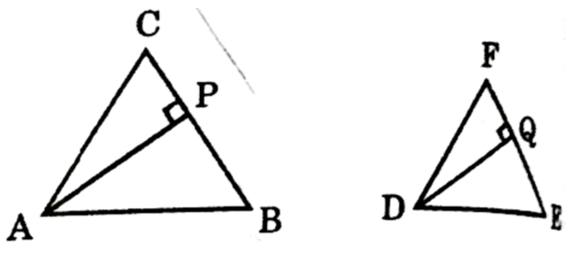
So, ΔAEF ~ΔDEC (By AA similarity criterion)

Therefore, $\frac{AE}{DE} = \frac{AF}{DC}$

⇒ AE × DC = DE × AF, Proved.

---

40. In the given figure, ΔABC ~ΔDEF, AP bisects $\angle$CAB and DQ bisects $\angle$FDE.



Prove that:

(a) $\frac{AP}{DQ} = \frac{AB}{DE}$

(b) $\Delta CAP \sim \Delta FDQ$

2015/2016 (3 Marks)

(a) $\Delta ABC \sim \Delta DEF$ (Given)

So, $\angle CAB = \angle FDE$ and $\angle B = \angle E$ ....(1)

Now, $\angle CAB = \angle FDE \Rightarrow \frac{1}{2}\angle CAB = \frac{1}{2}\angle FDE$.

$\Rightarrow$ $\angle PAB = \angle QDE$ ....(2)

So, $\Delta APB \sim \Delta DQE$ [From (1) and (2), AA similarity criterion]

$\Rightarrow$ $\frac{AP}{DQ} = \frac{AB}{DE}$.

(b)
Now, $\Delta ABC \sim \Delta DEF$
$\Rightarrow$ $\frac{AC}{DF} = \frac{AB}{DE}$
So, $\frac{AC}{DF} = \frac{AP}{DQ}$ (Because $\frac{AP}{DQ} = \frac{AB}{DE}$, Proved above) ....(1)

Also, since $\angle CAB = \angle FDE$, so $\frac{1}{2}\angle CAB = \frac{1}{2}\angle FDE$.

$\Rightarrow$ $\angle CAP = \angle FDQ$ ....(2)

From (1) and (2),
$\Delta CAP \sim \Delta FDQ$ (By SAS similarity criterion)

---

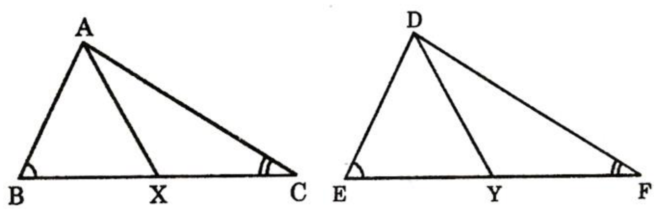
41. If $\Delta ABC \sim \Delta DEF$ and AX, DY are respectively the medians of ΔABC and ΔDEF. Then prove that:

(i) $\Delta ABX \sim \Delta DEY$

(ii) $\Delta ACX \sim \Delta DFY$

(iii) $\frac{AX}{DY} = \frac{BC}{EF}$

2014/2015 (4 Marks)



$\Delta ABC \sim \Delta DEF$ (Given)
So, $\frac{AB}{DE} = \frac{BC}{EF} = \frac{CA}{FD}$
And $\left.\begin{array}{l}\angle B = \angle E\\ \angle C = \angle F\end{array}\right\}$ (Corresponding sides are proportional and corresponding angles are equal)
From $\frac{AB}{DE} = \frac{BC}{EF}$, *we get*

$\frac{AB}{DE} = \frac{2BX}{2EY}$ (X and Y are mid points of BC and EF)

$\Rightarrow \frac{AB}{DE} = \frac{BX}{EY}$ ....(2)

(i)

Now, in ΔABX and ΔDEY, we have:

$\frac{AB}{DE} = \frac{BX}{EY}$ [From (2)]

And $\angle B = \angle E$ [From (1)]

So, ΔABX ~ΔDEY (By SAS similarity criterion), proved.

(ii)

Again, $\frac{AC}{DF} = \frac{BC}{EF}$

$\Rightarrow \frac{AC}{DF} = \frac{2XC}{2YF} \Rightarrow \frac{AC}{DF} = \frac{XC}{YF}$ ....(3)

And $\angle C = \angle F$ [From (1)]

So, ΔACX ~ΔDFY (By SAS), Proved.

(iii)

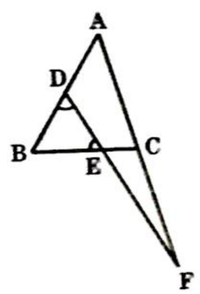
From (i) above,

$\frac{AX}{DY} = \frac{BX}{EY} \Rightarrow \frac{AX}{DY} = \frac{2BX}{2EY}$

$\Rightarrow \frac{AX}{DY} = \frac{BC}{EF}$, Proved.

---

42. In the figure, $\angle BED = \angle BDE$ and E is the middle point of BC. Prove that $\frac{AF}{CF} = \frac{AD}{BE}$.

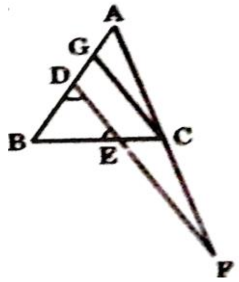


2014 /2015/ 2016 (4 Marks)

Construction: On AB, take a point G such that CG || DF.

In ΔBDE, $\angle E = \angle D$ (Given) ....(1)

So, BD = BE ....(2)



From ΔBCG, we have:

$DE || GC$

So, $\frac{BE}{EC} = \frac{BD}{DG}$

But BD = BE [From (2)]

So, EC = DG

$\Rightarrow$ BE = DG (E is mid-point of BC) ......(3)

Now, $CG || FD$ (By construction)

So, $\Delta ACG \sim \Delta AFD$

$\Rightarrow \frac{AC}{AF} = \frac{AG}{AD}$

So, $1 - \frac{AC}{AF} = 1 - \frac{AG}{AD}$

$\Rightarrow \frac{AF-AC}{AF} = \frac{AD-AG}{AD}$

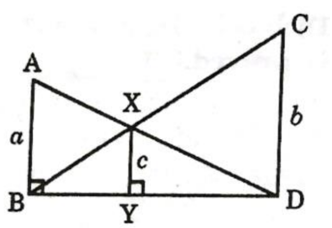
$\Rightarrow \frac{CF}{AF} = \frac{DG}{AD}$

$\Rightarrow$ $\frac{AF}{CF} = \frac{AD}{DG}$

$\Rightarrow \frac{AF}{CF} = \frac{AD}{BE}$. [From (3)], Proved.

---

43 In the figure, $\angle ABD = \angle XYD = \angle CDB = 90°, AB = a, XY = c\ and\ CD = b, then\ prove\ that\ c(a+b) = ab.$



2014/2015/2016 (4 Marks)

AB ⊥ BD and XY ⊥ BD (∠ABD = 90°, ∠XYD = 90°)

$\Rightarrow$ $AB || XY$

So, ∠BAX = ∠YXD

Hence, $\Delta DXY \sim \Delta DAB$ (By AA similarity criterion)

So, $\frac{DY}{DB} = \frac{c}{a} = \frac{DX}{DA}$ ......(1)

Also, by AA similarity criterion,

$\Delta BXY \sim \Delta BCD$

So, $\frac{BY}{DB} = \frac{c}{b} = \frac{BX}{BC}$ ....(2)

From (1), $\frac{DY}{BD} = \frac{c}{a} \Rightarrow 1 - \frac{DY}{DB} = 1 - \frac{c}{a}$.

$\Rightarrow \quad \frac{DB-DY}{DB} = \frac{a-c}{a}$

$\Rightarrow \quad \frac{BY}{DB} = \frac{a-c}{a}$.

So, from (2), we have:

$\frac{a-c}{a} = \frac{c}{b}$

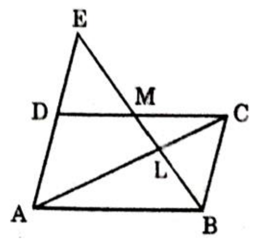
$\Rightarrow$ cb – bc = ac

$\Rightarrow$ ab = ac + bc

$\Rightarrow$ ab = c(a + b), proved

---

44. In the parallelogram ABCD, middle point of CD is M. A line segment BM is drawn which cuts AC at L and meets AD extended at E. Prove that EL = 2BL.



2014/2015/2016 (4 Marks)

In ΔEDM and ΔBCM, we have

DM = CM (Given)

$\angle DME = \angle BME$ (Vertically opposite angles)

$\angle DEM = \angle CBM$ (Alternate interior angles, DE$||$BC)

So, $\Delta EDM = \Delta BCM$ (By AAS congruence criterion)

$\Rightarrow$ DE = BC (CPCT)

So, DE = AD (Because BC = AD)

Now, in ΔAEL and ΔCBL, we have:

$\angle ELA = \angle BLC$ (Vertically opposite angles)

$\angle DEL = \angle CBL$ (Vertically interior angles)

So, $\Delta AEL \sim \Delta CBL$ (By AA similarity criterion)

$\frac{AE}{EL} = \frac{CB}{BL}$ (Corresponding sides are proportional)

$\Rightarrow \frac{2AD}{EL} = \frac{BC}{BL}$ (Since AD = DE)

$\Rightarrow \frac{2AD}{EL} = \frac{AD}{BL}$ (BC = AD)

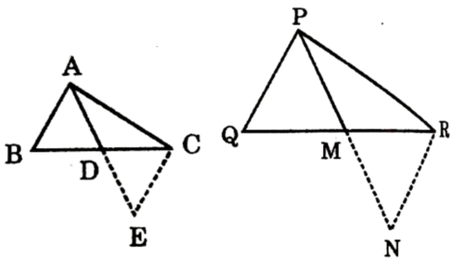
2BL = EL $\Rightarrow$EL = 2BL, proved.

---

45. Prove that if two sides and a median bisecting the third side of a triangle are respectively proportional to the corresponding sides and the median of another triangle, then the two triangles are similar.

2012/2013/2014/2016 (4 Marks)

Produce AD to E such that AD = DE and PM to N such that PM = MN.



Join CE and RN.

Now, $\Delta ABD \cong \Delta ECD$ (SAS)

And $\Delta PQM \cong \Delta NRM$ (SAS)

So, AB = CE and PQ = RN (By CPCT)

Now, in ΔACE and ΔPRN, $\frac{AB}{PQ} = \frac{AC}{PR} = \frac{2AD}{2PM}$

$\Rightarrow \frac{CE}{RN} = \frac{AC}{PR} = \frac{AE}{PN}$ ($\because$AB = CE and PQ = RN)

So, $\Delta ACE \sim \Delta PRN$ (By SSS similarity criterion)

So, $\angle CAD = \angle RPM$ ....(1)

Again, in ΔBAD and ΔQPM,

$\angle BAD = \angle CED$ ($\because \Delta ABD \cong \Delta ECD$)

$\angle QPM = \angle RNM$ ($\because \Delta PQM \cong \Delta NRM$)

$\angle AEC = \angle PNR$ ($\because \Delta AEC \sim \Delta PNR$)

Therefore, $\angle BAD = \angle QPM$ ....(2)

Adding (1) and (2), $\angle A = \angle P$.

Now, in ΔABC and ΔPQR,

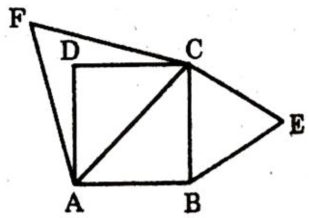
$\frac{AB}{PQ} = \frac{AC}{PR}$ $and\ \angle A = \angle P$

So, $\Delta ABC \sim \Delta PQR$ (By SAS similarity criterian)

46. Prove that the area of the equilateral triangle described on the side of a square is half the area of the equilateral triangle described on its diagonal.

2010/2011/2015/2016 (3 Marks)

Given: A square ABCD. Equilateral Δs BCE and ACF have been drawn on side BC and diagonal AC respectively.



To prove: $ar\ (\Delta BCE) = \frac{1}{2} \times ar\ (\Delta ACF)$

Proof: $\Delta BCE \sim \Delta ACF$ [Being equilateral, so similar by AAA criterion of similarity]

$\Rightarrow \quad \frac{ar\ (\Delta BCE)}{ar\ (\Delta ACF)} = \frac{BC^2}{AC^2}$

$\Rightarrow \quad \frac{ar\ (\Delta BCE)}{ar\ (\Delta ACF)} = \frac{BC^2}{\left(\sqrt{2}BC\right)^2}$ [Diagonal = $\sqrt{2}\ side \Rightarrow AC = \sqrt{2}\ BC$]

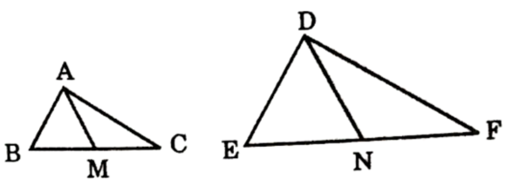
$\Rightarrow \quad \frac{ar\ (\Delta BCE)}{ar\ (\Delta ACF)} = \frac{1}{2} \Rightarrow ar\ (\Delta BCE) = \frac{1}{2} ar\ (\Delta ACF).$

---

47. Prove that the ratio of the areas of two similar triangles is equal to the square of the ratio of their corresponding medians

2012/2013/2015/2016 (3 Marks)

Given: ΔABC ~ΔDEF and AM and DN are medians of two triangles.



To prove: $\frac{ar\ (\Delta ABC)}{ar\ (\Delta DEF)} = \left(\frac{AM}{DN}\right)^2$

Proof: ΔABC ~ΔDEF (Given)

$\Rightarrow \quad \frac{ar\ (\Delta ABC)}{ar\ (\Delta DEF)} = \left(\frac{AB}{DE}\right)^2$ .....(1)

And $\quad \frac{AB}{DE} = \frac{BC}{EF}.$

Also, ∠B = ∠E.

Now, $\frac{AB}{DE} = \frac{BC}{EF} = \frac{2BM}{2EN} = \frac{BM}{EN}$

So, we have:

$\frac{AB}{DE} = \frac{BM}{EN}$and $\angle$B = $\angle$E.

So, ΔABM ~ΔDEN (By SAS similarity criterion)

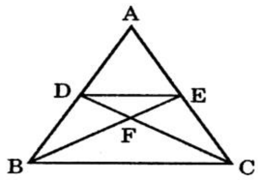
$\Rightarrow$ $\frac{AB}{DE} = \frac{AM}{DN}$ .....(2)

So, from (1) and (2),

$$\frac{ar\ (\Delta ABC)}{ar\ (\Delta DEF)} = \left(\frac{AM}{DN}\right)^2$$

---

48. In a ΔABC, DE||BC. If AD: DB = 3: 5, then find $\frac{ar(\Delta DFE)}{ar(\Delta CFB)}$.



2014/2015/2016 (4 Marks)

In the figure, DE||BC

So, $\left.\begin{array}{l}\angle FDE\ =\ \angle FCB\\ \angle FED = \ \angle FBC\end{array}\right\}$ (Alternate angles)

So, ΔDFE ~ΔCFB (AA similarity creation)

So, $\frac{ar\ (\Delta DFE)}{ar\ (\Delta CFB)} = \left(\frac{DE}{BC}\right)^2$ ....(1)

Now, we are given

$\frac{AD}{DB} = \frac{3}{5}$ ....(2)

$\Rightarrow$ $1 + \frac{AD}{DB} = 1 + \frac{3}{5}$

$\Rightarrow$ $\frac{DB+AD}{DB} = \frac{8}{5} \Rightarrow \frac{AB}{DB} = \frac{8}{5}$ ....(3)

So, from (2) and (3),

$\frac{AD}{DB} \times \frac{DB}{AB} = \frac{3}{5} \times \frac{5}{3} = \frac{3}{8} \Rightarrow \frac{AD}{AB} = \frac{3}{8}$ ....(4)

Now, from DE||BC, we also have:

$\angle$D = $\angle$B and $\angle$E = $\angle$C (Corresponding angles)

So, ΔADE ~ΔABC

$\Rightarrow$ $\frac{AD}{AB} = \frac{DE}{BC}$

So, from (4), we get

$\frac{DE}{BC} = \frac{3}{8}$

Putting $\frac{DE}{BC} = \frac{3}{8}$ in (1), we get

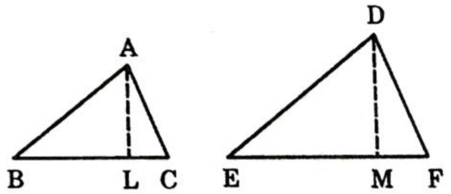
$$\frac{ar\ (\Delta DFE)}{ar\ (\Delta CFB)} = \left(\frac{3}{8}\right)^2 = \frac{9}{64}.$$

---

49. Prove that the ratio of the areas of two similar triangles is equal to the square of the ratio of their corresponding sides. 2012/2013/2015/2016 (4 Marks)

Given: Two ∆s ABC and DEF such that ∆ABC ~∆DEF.

To prove: $\frac{ar\ (\Delta ABC)}{ar\ (\Delta DEF)} = \frac{AB^2}{DE^2} = \frac{BC^2}{EF^2} = \frac{AC^2}{DF^2}$



Construction: Draw AL ⊥ BC and DM ⊥ EF.

Proof: Since similar triangles are equiangular and their corresponding sides are proportional, therefore

∆ABC ~∆DEF

⇒ ∠A = ∠D, ∠B = ∠E, ∠C =∠F

And $\frac{AB}{DE} = \frac{BC}{EF} = \frac{AC}{DF}$ ....(1)

Now, in ∆s ALB and DME, we have:

∠ALB = ∠DME [∵Each = 90°]

And ∠B = ∠E [From (1)]

∴ By AA criterion of similarly, we have:

$\Delta ALB \sim \Delta DME$

⇒ $\frac{AL}{DM} = \frac{AB}{DE}$ ....(2)

From (1) and (2), we get

$\frac{AB}{DE} = \frac{BC}{EF} = \frac{AC}{DF} = \frac{AL}{DM}$ ....(3)

Now, $\frac{ar(\Delta ABC)}{ar(\Delta DEF)} = \frac{\frac{1}{2}\times BC\times AL}{\frac{1}{2}\times EF\times DM} = \frac{BC}{EF}\times\frac{AL}{DM}$

$= \frac{BC}{EF}\times\frac{BC}{EF} = \frac{BC^2}{EF^2}$[From (3), $\frac{BC}{EF} = \frac{AL}{DM}$]

But, $\frac{BC}{EF} = \frac{AB}{DE} = \frac{AC}{DF} \Rightarrow \frac{BC^2}{EF^2} = \frac{AB^2}{DE^2} = \frac{AC^2}{DF^2}$

Hence, $\frac{ar(\Delta ABC)}{ar(\Delta DEF)} = \frac{AB^2}{DE^2} = \frac{BC^2}{EF^2} = \frac{AC^2}{DF^2}$

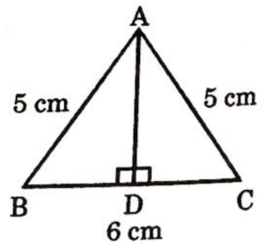
---

50. In an isosceles triangle, if the length of its sides are AB = 5cm, AC = 5cm and BC = 6cm, then find the length of its altitude drawn from A on BC.

2014/2015/2016 (1 Mark)

AD ⊥ BC

So, ΔABD ≅ΔACD (RHS)



⇒ BD = CD = $\frac{6}{2}$ = 3 cm

From right triangle ABD, we have:

$AB^2 = BD^2 + AD^2 \Rightarrow 25 = 9 + AD^2$

⇒ $AD^2 = 16 \Rightarrow AD = 4$

Thus, AD = 4 cm.

---

51. Prove that in an equilateral triangle, three times of the square of one of the sides is equal to four times of the square of one of its altitudes.

2013/2015/2016 (2 Marks)

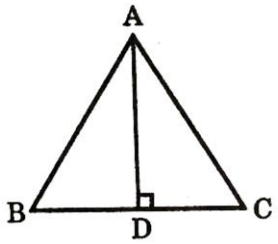
ΔABC is an equilateral triangle.

So, AB = BC = CA

Also, AD ⊥ BC

So, AD divides BC into two equal parts,

i.e. BD = DC



Now, in rt. ΔADC,

$AC^2 = AD^2 + DC^2$

$$= AD^2 + \left(\frac{BC}{2}\right)^2$$

Or $\quad AC^2 - \frac{BC^2}{4} = AD^2$

Or $\quad AB^2 - \frac{AB^2}{4} = AD^2 \qquad (\because AB = BC = AC)$

Or $\quad \frac{4AB^2 - AB^2}{4} = AD^2$

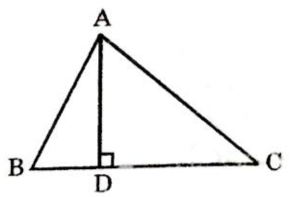
Or $\quad \frac{3AB^2}{4} = AD^2$

Or $\quad 3\,AB^2 = 4AD^2$

i.e. three times the square of a side of an equilateral triangle is equal to four times the square of its altitude.

---

52. In the figure, in ΔABC, AD ⊥ BC. Prove that $AC^2 = AB^2 + BC^2 - 2BC.BD$.



2013/2015/2016 (2 Marks)

In rt. ΔADB, $\quad AB^2 = AD^2 + BD^2$

$\Rightarrow \quad AD^2 = AB^2 - BD^2 \qquad$ ....(1)

In rt. ΔADC, $\quad AC^2 = AD^2 + DC^2$

$\Rightarrow AD^2 = AC^2 - DC^2$

$= AC^2 - (BC - BD)^2$

$= AC^2 - (BC^2 + BD^2 - 2BC.BD) \quad$ ....(2)

From (1) and (2),

$$AB^2 - BD^2 = AC^2 - (BC^2 + BD^2 - 2BC.BD)$$

$$\Rightarrow \qquad AC^2 = AB^2 - BD^2 + BC^2 + BD^2 - 2BC,BD$$

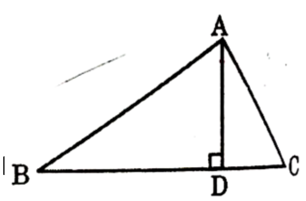
$$\Rightarrow \qquad AC^2 = AB^2 + BC^2 - 2BC.BD$$

---

53. The perpendicular from A on the side BC of aΔABC intersects BC at D such that DB = 3CD. Prove that$2AB^2 = 2AC^2 + BC^2$.

2013/2015/2016 (2 Marks)

BD = 3CD $\;\Rightarrow$ BD – CD = 2 CD



Now, $AB^2 = AD^2 + BD^2$ *and* $AC^2 = AD^2 + CD^2$

So, $AB^2 - AC^2 = BD^2 - CD^2$

$\Rightarrow$ $2(AB^2) - 2(AC^2) = 2(BD^2 - CD^2)$

$\Rightarrow$ $2(AB^2) - 2(AC^2)$ = 2(BD + CD)(BD - CD)

$\Rightarrow$ $2(AB^2) - 2(AC^2) = 2BC \times 2CD = 2BC \times 2\left(\frac{BC}{4}\right)$ [$\because$BC = 4CD from (1)]

$\Rightarrow$ $2(AB^2) = 2(AC^2) + BC^2$

---

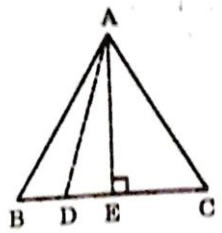
54. In an equilateral triangle ABC , D is a point on side BC such that 3BD = BC. Prove that $9AD^2 = 7AB^2$.

2010/2011/2013/2016 (2 Marks)

Let ABC be an equilateral triangle and let D be a point on BC such that

$$3BD = BC \Rightarrow BD = \frac{1}{3}BC$$

Draw AE $\perp$ BC. Join AD.



In Δs AEB and AEC, we have:

$\angle$AEB = $\angle$AEC ($\because$ Each = 90°)

And AE = AE (common)

$\therefore$ By RHS congruence criterion, we have:

ΔAEB $\cong\Delta$AEC

$\Rightarrow$ BE = EC (CPCT)

Now, we have:

$$BD = \frac{1}{3}BC,\ DC = BC - BD \Rightarrow BC - \frac{1}{3}BC = \frac{3BC-BC}{3} = \frac{2}{3}BC$$

So, DE = DC – EC = $\frac{2}{3}BC - \frac{BC}{2} = \frac{4BC-3BC}{6} = \frac{BC}{6}$ ....(1)

And $BE = EC = \frac{1}{2}BC$ ....(2)

In rt. ΔAED,

$$AD^2 = AE^2 + DE^2 \quad ....(3)$$

And in rt. ∆AEB,

$$AE^2 = AB^2 - BE^2 \qquad \text{.........(4)}$$

From (3) and (4),

$$AD^2 = AB^2 - BE^2 + DE^2$$

$$= BC^2 - \left(\frac{BC}{2}\right)^2 + \left(\frac{BC}{6}\right)^2 \qquad \text{[Using (1) and (2)]}$$

$$= BC^2 - \frac{BC^2}{} + \frac{BC^2}{}$$

$$= \frac{36BC^2 - 9BC^2 + BC^2}{36} = \frac{28BC^2}{36}$$

⇒ $$AD^2 = \frac{7BC^2}{9} \Rightarrow AD^2 = \frac{7AB^2}{9} \qquad (\because AB = BC)$$

⇒ $$9\,AD^2 = 7\,AB^2$$

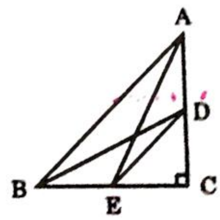
---

55. D and E are points on the sides CA and CB respectively of ∆ABC, right angled at C. Prove that $AE^2 + BD^2 = AB^2 + DE^2$.

2012/2013/2015/2016 (2 Marks)

We have: $AE^2 = AC^2 + CE^2$

And $BD^2 = BC^2 + CD^2$



$$\Rightarrow AE^2 + BD^2 = AC^2 + CE^2 + BC^2 + CD^2$$

$$= (AC^2 + BC^2) + (CE^2 + CD^2)$$

$$= AB^2 + DE^2$$

---

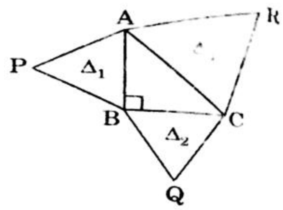
56. Prove that the equilateral triangles described on the two sides of a right angled triangle are together equal to the equilateral triangle described on the hypotenuse in terms of their areas.

2010/2011/2012/2015/2016 (2 Marks)

Given: A right angled ∆ABC with right angle at B. Equilateral ∆s PAB, QBC and RAC are described on the sides AB, BC and CA respectively.

To prove: ar (∆PAB) + ar (∆QBC) = ar (∆RAC).

Proof: Since ∆s PAB, QBC and RAC are equilateral, therefore they are equiangular and hence similar.



$\therefore \frac{ar(\Delta PAB)}{ar(\Delta RAC)}+\frac{ar(\Delta QBC)}{ar(\Delta RAC)} = \frac{AB^2}{AC^2}+\frac{BC^2}{AC^2}$

$=\frac{AB^2+BC^2}{AC^2} = \frac{AC^2}{AC^2} = 1$

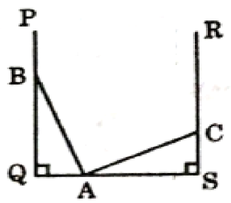
[$\because$∆ABC is right angled with ∠B=90^0 $\therefore AC^2 = AB^2 + BC^2$]

$\Rightarrow \frac{ar(\Delta PAB)+ar(\Delta QBC)}{ar(\Delta RAC)} = 1$

$\Rightarrow ar(\Delta PAB) + ar(\Delta QBC) = ar(\Delta RAC)$

---

57. As shown in the figure, a 26m long ladder is placed at A. If it is placed along wall PQ, it reaches a height of 24m, whereas it reaches a height of 10m, if it is placed against wall RS. Find the distance between the walls.



2014/2015/2016 (2 Marks)

From ∆ABQ, $AB^2 = AQ^2 + BQ^2$

$\Rightarrow \quad (26)^2 = AQ^2 + (24)^2 \Rightarrow 676 = AQ^2 + 576$

$\Rightarrow \quad AQ^2 = 100 \quad \Rightarrow AQ = \sqrt{100}\,m = 10m$

From ∆ASC, $AC^2 = AS^2 + CS^2$

$\Rightarrow \quad (26)^2 = AS^2 + (10)^2 \Rightarrow \quad 676 = AS^2 + 100$

$\Rightarrow \quad AS^2 = 676 - 100 = 576 \quad \Rightarrow AS = \sqrt{576} = 24m$

So, distance between the walls

= QS

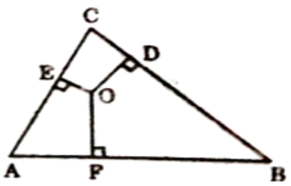
= AQ+ AS = 10 + 24 = 34 m.

---

58. In aΔABC, from any interior point O, OD $\perp$ BC, OE $\perp$ AC and OF $\perp$ AB are drawn. Prove that:

(i) $OA^2 + OB^2 + OC^2 - OD^2 - OE^2 - OF^2 = AE^2 + CD^2 + BF^2$

(ii) A$E^2 + CD^2 + BF^2 = AF^2 + BD^2 + CE^2$



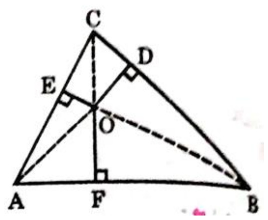
2014/2015/2016 (4 Marks)

Join OA, OB and OC.

(i) $OA^2 = AE^2 + OE^2$ ....(1)

$OB^2 = BF^2 + OF^2$ ....(2)

and $OC^2 = CD^2 + OD^2$ ....(3)



Adding (1), (2) and (3), we get:

$$OA^2 + OB^2 + OC^2 = AE^2 + BF^2 + CD^2 + OE^2 + OF^2 + OD^2$$

$\Rightarrow$ $OA^2 + OB^2 + OC^2 - OD^2 - OE^2 - OF^2 = AF^2 + BF^2 + CD^2$ ....(4)

(ii) Similarly, we can find that:

$OA^2 + OB^2 + OC^2 - OD^2 - OE^2 - OF^2 = AF^2 + BD^2 + CE^2$ ....(5)

So, from (4) and (5), we get:

$$AE^2 + BF^2 + CD^2 = AF^2 + BD^2 + CE^2$$

---

59. ΔABC is right angled at C. If BC = a, CA = b, AB = c and p is length of perpendicular drawn from C on AB, then prove that:

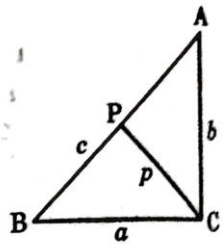
(i) cp = ab

(ii) $\frac{1}{p^2} = \frac{1}{a^2} + \frac{1}{b^2}$

2014/2015/2016 (2 Marks)

In the figure, we have:

CP ⊥ PB and CP = p



(i) Area of ΔABC = $\frac{1}{2} \times BC \times AC = \frac{1}{2}ab$

Also, Area of ΔABC = $\frac{1}{2} \times AB \times CP = \frac{1}{2}cp$

So, we have:

$$\frac{1}{2}cp = \frac{1}{2}ab \quad \Rightarrow \quad cp = ab \qquad \text{Proved.}$$

(ii) $AB^2 = BC^2 + AC^2$

$\Rightarrow$ $c^2 = a^2 + b^2$

$\Rightarrow$ $\left(\frac{ab}{p}\right)^2 = a^2 + b^2$ [From $cp = ab$]

$\Rightarrow$ $\frac{a^2b^2}{p^2} = a^2 + b^2 \Rightarrow \frac{1}{p^2} = \frac{a^2 + b^2}{a^2b^2} \Rightarrow \frac{1}{p^2} = \frac{a^2}{a^2b^2} + \frac{b^2}{a^2b^2}$

$\Rightarrow$ $\frac{1}{p^2} = \frac{1}{a^2} + \frac{1}{b^2}$ Proved.

---

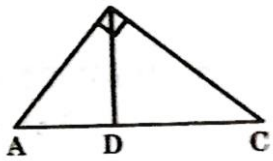
60. Prove that in a right triangle, the square of the hypotenuse is equal to the sum of the squares of the other two sides or state and prove Pythagoras theorem.

2010/2011/2012/2013/2014/2016 (2 Marks)

Given: A right angled ΔABC, in which ∠B = 90°

To prove: $AC^2 = AB^2 + BC^2$

Construction: From B, draw BD ⊥ AC



Proof: In Δs ADB and ABC, we have:

∠ADB = ∠ABC [∵Each =90°]

And $\angle A = \angle A$ [Common]

$\therefore$ By AA similarity criterion, we have:

$\Delta ADB \sim \Delta ABC$

$\Rightarrow \quad \frac{AD}{AB} = \frac{AB}{AC}$ [$\because$ Corresponding sides are proportional]

$\Rightarrow \quad AB^2 = AD \times AC$ ....(1)

In Δs BDC and ABC, we have:

$\angle CDB = \angle ABC$ [$\because$ Each = 90°]

And $\angle C = \angle C$ [Common]

So, by AA similarity criterion, we have:

$\Delta BDC \sim \Delta ABC$

$\Rightarrow \quad \frac{DC}{BC} = \frac{BC}{AC}$ [$\because$ Corresponding sides are proportional]

$\Rightarrow \quad BC^2 = AC \times DC$ ....(2)

Adding (1) and (2) we get

$AB^2 + BC^2 = AD \times AC + AC \times DC$

$\Rightarrow \quad AB^2 + BC^2 = AC(AD + DC)$

$\Rightarrow \quad AB^2 + BC^2 = AC \times AC$

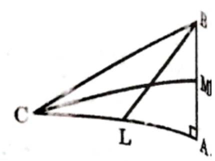
$\Rightarrow \quad AB^2 + BC^2 = AC^2$

$\Rightarrow \quad AC^2 = AB^2 + BC^2$

---

61. In the figure, BL and CM are the medians of a triangle right angled at A. Prove that:

$$4(BL^2 + CM^2) = 5BC^2.$$



2010/2011/2013/2015/2016 (2 Marks)

Given that M is the mid-point of AB and L is the mid-point of AC.

In rt. ΔABC,

$BC^2 = AB^2 + AC^2$ ......(1)

In rt. ΔABL,

$BL^2 = AB^2 + AL^2$ ......(2)

In rt. ΔAMC,

$$MC^2 = AM^2 + AC^2 \qquad ......(3)$$

Adding (2) and (3) and subtracting (1) from the result, we get

$$BL^2 + MC^2 - BC^2 = AL^2 + AM^2$$

$$= \left(\frac{AC}{2}\right)^2 + \left(\frac{AB}{2}\right)^2 \qquad (\because AM = MB \text{ and } AL = LC)$$

$$BL^2 + MC^2 - BC^2 = \frac{AC^2}{4} + \frac{AB^2}{4} = \frac{AC^2+AB^2}{4} = \frac{BC^2}{4} \qquad \text{[From (1)]}$$

$\Rightarrow$ $4(BL^2 + MC^2) - 4BC^2 = BC^2$

Or $4(BL^2 + MC^2) = 5BC^2$
