PERMUTATION AND COMBINATION

1. FUNDAMENTAL PRINCIPLES OF COUNTING

1.1 Fundamental Principle of Multiplication

If an event can occur in m different ways following which another event can occur in n different ways following which another event can occur in p different ways. Then the total number of ways of simultaneous happening of all these events in a definite order is $m \times n \times p$.

1.2 Fundamental Principle of Addition

If there are two jobs such that they can be performed independently in m and n ways respectively, then either of the two jobs can be performed in (m + n) ways.

2. SOME BASIC ARRANGEMENTS AND SELECTIONS

2.1 Combinations

Each of the different selections made by taking some or all of a number of distinct objects or items, irrespective of their arrangements or order in which they are placed, is called a combination.

2.2 Permutations

Each of the different arrangements which can be made by taking some or all of a number of distinct objects is called a permutation.

Note...

1. Let r and n be positive integers such that $l \le r \le n$. Then, the number of all permutations of n distinct items or objects taken r at a time, is

$$^{n}P_{r} = {}^{n}C_{r} \times r\,!$$

Proof : Total ways = $n(n-1)(n-2)\ldots(n-\overline{r-1})$

$$= \frac{n(n-1)(n-2)\ldots(n-\overline{r-1})(n-r)!}{(n-r)!}$$

$$= \frac{n!}{(n-r)!}$$

$$= {}^{n}P_{r}.$$

So, the total no. of arrangements (permutations) of n-distinct items, taking r at a time is $^{n}P_{r}$ or P(n, r).

2. The number of all permutations (arrangements) of n distinct objects taken all at a time is n!.

3. The number of ways of selecting r items or objects from a group of n distinct items or objects, is

$$\frac{n!}{(n-r)!r!} = {}^{n}C_{r}.$$

3. GEOMETRIC APPLICATIONS OF ^nC_r

(i) Out of n non-concurrent and non-parallel straight lines, points of intersection are $^nC_2$.

(ii) Out of 'n' points the number of straight lines are (when no three are collinear) $^nC_2$.

(iii) If out of n points m are collinear, then No. of straight lines = $^nC_2 - {^mC_2} + 1$

(iv) In a polygon total number of diagonals out of n points (no three are collinear) = $^nC_2 - n = \frac{n(n-3)}{2}$.

(v) Number of triangles formed from n points is $^nC_3$. (when no three points are collinear)

(vi) Number of triangles out of n points in which m are collinear, is $^nC_3 - {^mC_3}$.

(vii) Number of triangles that can be formed out of n points (when none of the side is common to the sides of polygon), is $^nC_3 - {^nC_1} - {^nC_1} \cdot {^{n-4}C_1}$

(viii) Number of parallelograms in two systems of parallel lines (when 1st set contains m parallel lines and 2nd set contains n parallel lines), is = $^nC_2 \times {^mC_2}$

(ix) Number of squares in two system of perpendicular parallel lines (when 1st set contains m equally spaced parallel lines and 2nd set contains n same spaced parallel lines)

$$= \sum_{r=1}^{m-1} (m-r)(n-r); (m < n)$$

4. PERMUTATIONS UNDER CERTAIN CONDITIONS

The number of all permutations (arrangements) of n different objects taken r at a time :

(i) When a particular object is to be always included in each arrangement, is $^{n-1}C_{r-1} \times r\,!$.

(ii) When a particular object is never taken in each arrangement, is $^{n-1}C_r \times r!$.

5. DIVISION OF OBJECTS INTO GROUPS

5.1 Division of items into groups of unequal sizes

1. The number of ways in which (m + n) distinct items can be divided into two unequal groups containing m and n items, is $\frac{(m+n)!}{m!n!}$.

2. The number of ways in which (m+ n+ p) items can be divided into unequal groups containing m, n, p items, is

$$^{m+n+p}C_m \cdot {^{n+p}C_m} = \frac{(m+n+p)!}{m!n!p!}.$$

3. The number of ways to distribute (m + n+ p) items among 3 persons in the groups containing m, n and p items

= (No. of ways to divide) × (No. of groups)!

$$= \frac{(m+n+p)!}{m!n!p!} \times 3!.$$

5.2 Division of Objects into groups of equal size

The number of ways in which mn different objects can be divided equally into m groups, each containing n objects and the order of the groups is not important, is

$$\left(\frac{(mn)!}{(n!)^m}\right)\frac{1}{m\,!}$$

The number of ways in which mn different items can be divided equally into m groups, each containing n objects and the order of groups is important, is

$$\left(\frac{(mn)!}{(n!)^m} \times \frac{1}{m!}\right) m! = \frac{(mn)!}{(n!)^m}$$

6. PERMUTATIONS OF ALIKE OBJECTS

1. The number of mutually distinguishable permutations of n things, taken all at a time, of which p are alike of one kind, q alike of second kind such that p + q = n, is

$$\frac{n!}{p!q!}$$

2. The number of permutations of n things, of which p are alike of one kind, q are alike of second kind and remaining all are distinct, is $\frac{n!}{p!q!}$. Here $p + q \neq n$

3. The number of permutations of n things, of which p_1 are alike of one kind; p_2 are alike of second kind; p_3 are alike of third kind; ..... ; p_r are alike of r^{th} kind such that $p_1 + p_2 + \ldots + p_r = n$, is $\frac{n!}{p_1!p_2!p_3!\ldots p_r!}$.

4. Suppose there are r things to be arranged, allowing repetitions. Let further $p_1, p_2, \ldots, p_r$ be the integers such that the first object occurs exactly p_1 times, the second occurs exactly p_2 times subject, etc. Then the total number of permutations of these r objects to the above condition, is

$$\frac{(p_1 + p_2 + \ldots + p_r)!}{p!p_2!p_3!\ldots p_r!}.$$

7. DISTRIBUTION OF ALIKE OBJECTS

(i) The total number of ways of dividing n identical items among r persons, each one of whom, can receive 0, 1, 2, or more items ($\leq n$), is ${}^{n+r-1}C_{r-1}$.

OR

The total number of ways of dividing n identical objects into r groups, if blank groups are allowed, is ${}^{n+r-1}C_{r-1}$.

(ii) The total number of ways of dividing n identical items among r persons, each of whom, receives at least one item is ${}^{n-1}C_{r-1}$.

OR

The number of ways in which n identical items can be divided into r groups such that blank groups are not allowed, is ${}^{n-1}C_{r-1}$.

(iii) The number of ways in which n identical items can be divided into r groups so that no group contains less than k items and more than m (m < k) is

The coefficient of x^n in the expansion of

$(x^m + x^{m+1} + \ldots x^k)^r$

8. NO. OF INTEGRAL SOLUTIONS OF LINEAR EQUATIONS AND INEQUATIONS

Consider the eqn. $x_1 + x_2 + x_3 + x_4 + \ldots + x_r = n$ ...(i)

where $x_1, x_2, \ldots, x_r$ and n are non–negative integers. This equation may be interpreted as that n identical objects are to be divided into r groups.

1. The total no. of non-negative integral solutions of the equation $x_1 + x_2 + \ldots + x_r = n$ is ${}^{n+r-1}C_{r-1}$.

2. The total number of solutions of the same equation in the set N of natural numbers is ${}^{n-1}C_{r-1}$.

3. In order to solve inequations of the form

$$x_1 + x_2 + \ldots + x_m \leq n$$

we introduce a dummy (artificial) variable x_{m+1} such that $x_1 + x_2 + \ldots + x_m + x_{m+1} = n$, where $x_{m+1} \geq 0$.

The no. of solutions of this equation are same as the no. of solutions of in Eq. (i).

9. CIRCULAR PERMUTATIONS

1. The number of circular permutations of n distinct objects is (n – l)!.
2. If anti-clockwise and clockwise order of arrangements are not distinct then the number of circular permutations of n distinct items is 1/2 {(n – 1)!}

   e.g., arrangements of beads in a necklace, arrangements of flowers in a garland etc.

10. SELECTION OF ONE OR MORE OBJECTS

1. The number of ways of selecting one or more items from a group of n distinct items is $2^n - 1$.

Proof : Out of n items, 1 item can be selected in ${}^nC_1$ ways; 2 items can be selected in ${}^nC_2$ ways; 3 items can be selected in ${}^nC_3$ ways and so on......

Hence, the required number of ways

$= {}^nC_1 + {}^nC_2 + {}^nC_3 + \dots + {}^nC_n$

$= ({}^nC_0 + {}^nC_1 + {}^nC_2 + \dots + {}^nC_n) - {}^nC_0$

$= 2^n - 1.$

2. The number of ways of selecting r items out of n identical items is 1 .

3. The total number of ways of selecting zero or more items from a group of n identical items is (n + 1).

4. The total number of selections of some or all out of p + q + r items where p are alike of one kind, q are alike of second kind and rest are alike of third kind, is

[(p + 1)(q + 1)(r+ 1)]– 1.

5. The total number of ways of selecting one or more items from p identical items of one kind; q identical items of second kind; r identical items of third kind and n different items, is $(p + 1)(q + 1)(r + 1)2^n - 1$

11. THE NUMBER OF DIVISORS AND THE SUM OF THE DIVISORS OF A GIVEN NATURAL NUMBER

Let $N = p_1^{n_1}.p_2^{n_2}.p_3^{n_3}.....p_k^{n_k}$ ...(1)

where $p_1, p_2, \dots, p_k$ are distinct prime numbers and $n_1, n_2, \dots, n_k$ are positive integers.

1. Total number of divisors of $N = (n_1 + 1)(n_2 + 1)\dots(n_k + 1)$.
2. This includes 1 and n as divisors. Therefore, number of divisors other than 1 and n, is

   $(n_1 + 1)(n_2 + 1)(n_3 + 1)\dots(n_k + 1) - 2.$
3. The sum of all divisors of (1) is given by

$$= \left\{\frac{p_1^{n_1+1}-1}{p_1-1}\right\}\left\{\frac{p_2^{n_2+1}-1}{p_2-1}\right\}\left\{\frac{p_3^{n_3+1}-1}{p_3-1}\right\}\dots\left\{\frac{p_k^{n_k+1}-1}{p_k-1}\right\}.$$

12. DEARRANGEMENTS

If n distinct objects are arranged in a row, then the no. of ways in which they can be dearranged so that none of them occupies its original place, is

$$n!\left\{1-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}-\dots+(-1)^n\frac{1}{n!}\right\}$$

and it is denoted by D (n).

If r (0 ≤ r ≤ n) objects occupy the places assigned to them i.e., their original places and none of the remaining (n - r) objects occupies its original places, then the no. of such ways, is

$$D(n - r) = {}^nC_r \,.\, D(n - r)$$

$$= {}^nC_r \,.\, (n - r)\ !\left\{1-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}+\dots+(-1)^{n-r}\frac{1}{(n-r)!}\right\}.$$