5

PLASTIC ANALYSIS

5.1. Topics

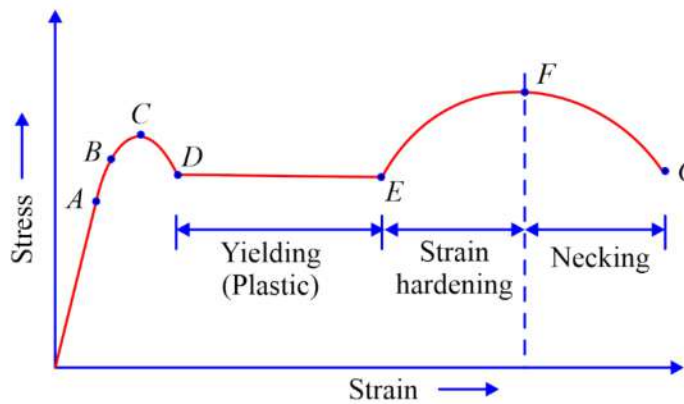
- Stress Strain Diagram
- Assumptions in plastic analysis
- Plastic Moment of a Section
- Shape factor
- Moment Curvature relation
- Determinacy and Indeterminacy
- Locations of plastic hinge
- Conditions in Plastic Analysis
- Mechanism in Structures
- Types of independent Mechanisms
- Theorems of Plastic Analysis
- Sequence of Plastic Hinge formation
- Collapse load determination
- Plastic Hinge length
- Load Factor

5.2. Introduction to Plastic Analysis

- Plastic design of a structure limits the structural usefulness of the material of the structure up to **ultimate load**.
- The load is found from the strength of steel in the **plastic range**.
- The method has its main application in steel structures as the strength of steel **beyond the yield stress** is fully utilized in this method.
- The method is **economical**
- The new steel code **IS 800 – 2007** utilizes this approach for Design

5.3. Stress Strain Curve - Plastic Analysis

- The concept of ductility forms the basis for the plastic theory.
- Mild steel is a ductile material.



5.3.1. Key Points of Mild Steel Curve

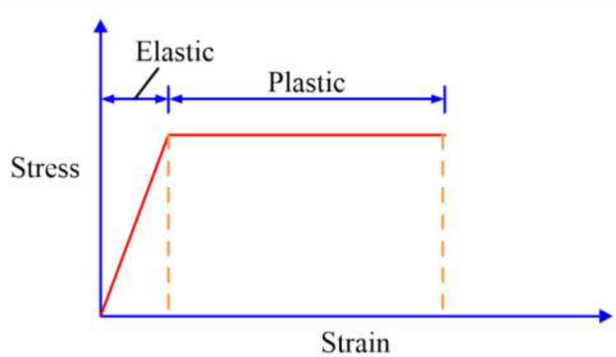
- Pt. A – Proportional limit
- Pt. B – Elastic limit
- Pt. C – upper yield point
- Pt. D – lower yield point
- DE zone of Plastic deformation
- Pt. F – ultimate stress point
- EF Zone - Strain Hardening
- Pt. G – Fracture point
- FG Zone – Necking

5.3.2. Elastic Vs. Plastic Theory

- **Elastic method** is based on **Hook's law.** Hence the structural usefulness of the material of the structure is limited to a stage when the stress in **extreme fiber reaches the yield stress** of material. The rest of the cross section remains unstressed, and the method do not take into account the strength beyond the yield stress point.
- **Plastic design** the structural usefulness if found from the **strength of steel in the plastic range**. Plastic Analysis is also known as **Ultimate load analysis**. The sections designed by this method are smaller in size.

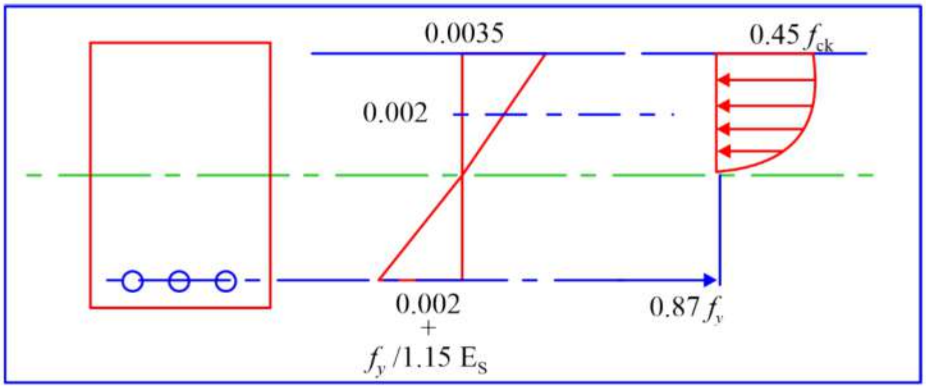
Important Note:

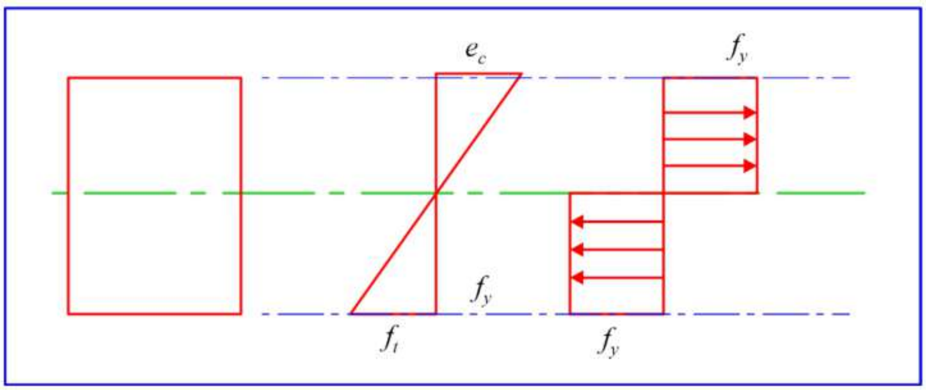
- Strain controlled
- Strain hardening and necking is neglected.
- Lower yield point is considered.
- Elasto Plastic curve forms the basis of design



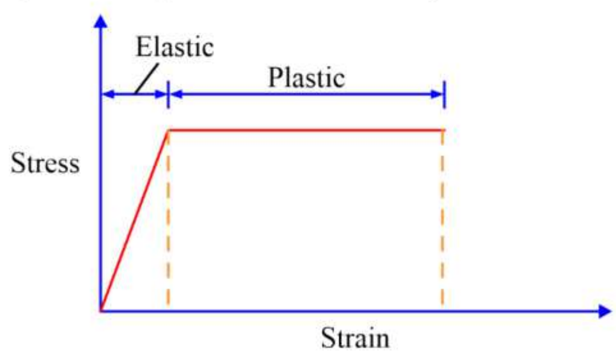
5.4. Assumptions - Plastic Analysis

5.4.1. Bernoulli's Assumption.





- Axial and shear deformations are neglected
- The cross section must be symmetrical w.r.t. the plane of loading
- The stress strain relationship is assumed to be **bi-linear** i.e. it consists of 2 straight lines.
- The material is **homogeneous**, and **isotropic** in both elastic and plastic state

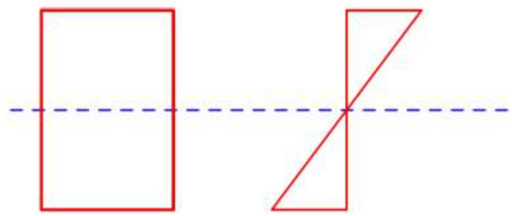


5.4.2. Applicability

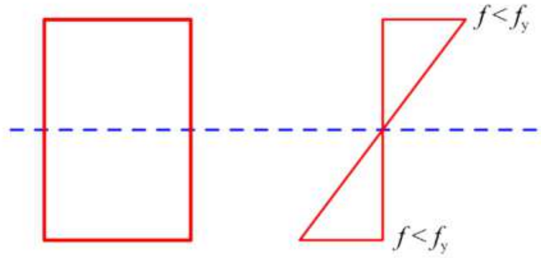
- Plastic theory is not suitable for structures subjected to impact and fatigue
- High tensile steel does not posses a defined yield point and the horizontal yielding part does not exist in the stress strain diagram hence its use is not applicable
- The Plastic theory is not applicable in brittle material but in RCC it can be applied with limitation to rotation exceptionally.

5.5. Plastic Moment - Plastic Analysis

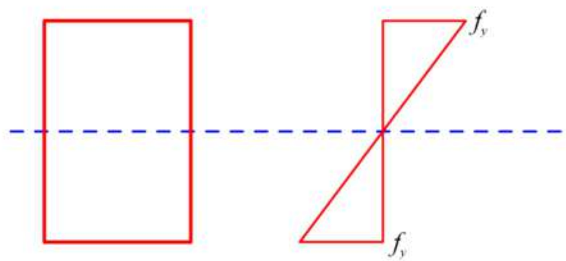
Bending strain distribution because **of Bernoulli's assumption**



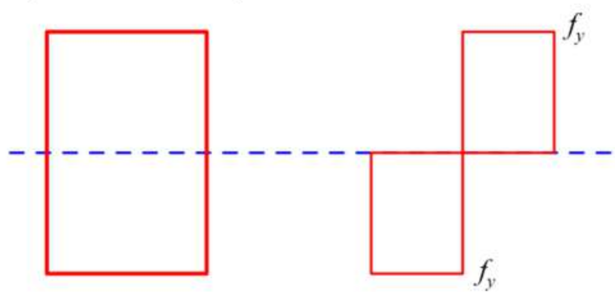
Bending stress (linear because of **Hook's law (Safe Moment)**



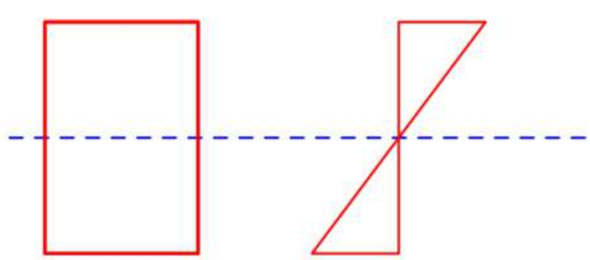
Bending stress diagram at yielding **(Yield Moment**)



Bending stress diagram at plastic state **(Plastic Moment)**



Bending strain distribution at **collapse condition (linear)**



5.6. Shape Factor- Plastic Analysis

- Shape factor represents **"Reserved Strength** " of beam section beyond yield moment to reach plastic state
- The more S.F. implies more reserve strength **beyond yield moment** to each plastic state

5.6.1. Shape factor for Different c/s

- Rectangular Section 1.5
- Circular Section 1.7
- I section 1.14
- H section 1.5
- Diamond section 2.0
- Triangular section 2.34

5.7. Moment Curvature Relation - Plastic Analysis

Curvature is proportional to M , i.e. as the moment increases curvature also increases

$$\frac{d^2y}{dx^2} = \frac{d\theta}{dx} = \frac{1}{R}$$

$$\frac{M}{I} = \frac{f}{y} = \frac{E}{R}$$

Where : M = Moment at Section

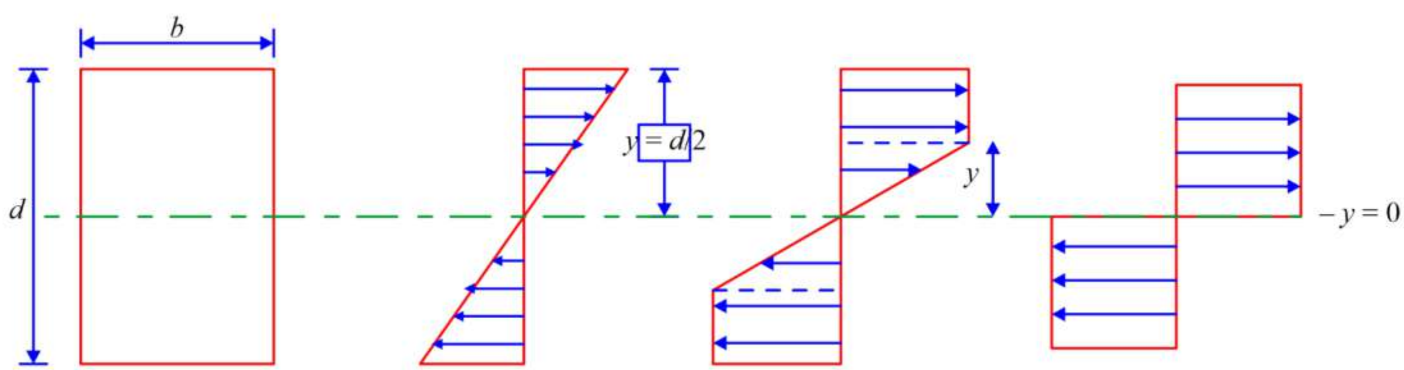
I = Moment of Inertia

Y = distance from NA to extreme fibre of elastic state

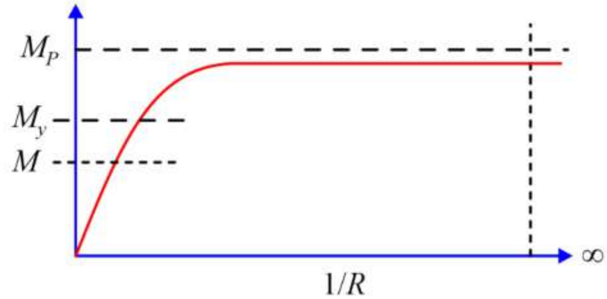
R = radius of curvature of bent up beam

1/R = curvature i.e rate of change of slope $\alpha \frac{1}{y}$

At fully plastic state curvature is infinity,



5.7.1. Moment Curvature Graph



Conclusion

- Mp depends on the area of c/s and distribution of the area.
- The most efficient c/s is I - section (because for a given c/s area, centroidal distance are maximum.

5.8. Indeterminacy - Plastic Analysis

How is Indeterminacy helpful in studying Plastic Analysis

Considering the planar structure we know that there are 3 Equations of Equilibrium under **general loading case.**

$$\sum F_x = 0, \sum Fy = 0, \sum Mz = 0$$

Also for the planar structure as **Beam** subjected to **vertical loading** there are only 2 Equations of Equilibrium.

$$\sum F_y = 0, \sum M_z = 0,$$ **horizontals are neglected**

In Regards to supports

- Roller has one Normal Reaction
- Hinged or Pinned has total of two reactions in Vertical and Horizontal
- Fixed support has total of three reactions in vertical , horizontal and moment
- **Internal Hinge** has zero moment at the location and if present in sufficient number it make the structure unstable. It gives an additional equation due to which the indeterminacy of the structure decreases
- **Plastic hinge** has Plastic Moment at the location and if present in sufficient number it make the structure unstable

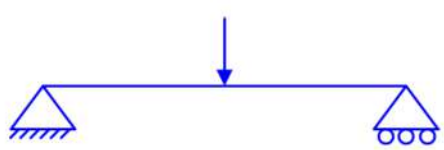
Static indeterminacy of any structure is given as

$$SI = \text{Reactions} - \text{Equilibrium equation} - \text{nos.of Hinge}$$

If SI < 0 Unstable, SI = 0 Determinate, SI>0 Indeterminate

- If a structure is statically stable then the number of additional hinges (Plastic) required for the complete collapse of the structure is

$$N = SI + 1$$



5.9 Plastic Hinge Location

- Under the Point or Concentrated load
- At the location of maximum moment
- At the change of cross section
- At location where material properties change
- At fixed or continuous support pts.

5.9.1. Condition - Plastic Analysis

There are 3 conditions in plastic equilibrium

- Equilibrium condition
- Mechanism condition
- Yield condition

Equilibrium Condition

$$\sum F_x = 0, \sum F_y = 0, \quad \sum M_z = 0$$

Mechanism condition

At collapse, sufficient no. of plastic hinges must be developed so that a part or entire structure must transform in to a mechanism leading to its collapse.

Number of mechanism = Probable Location – Static Indeterminacy

Yield Condition

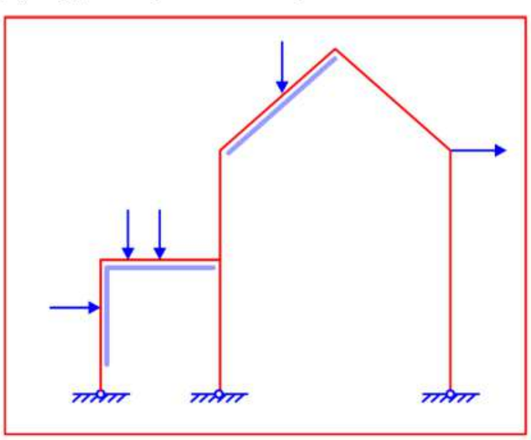
At collapse the bending moment at any section should not exceed the plastic moment capacity i.e. $M \leq M_p$

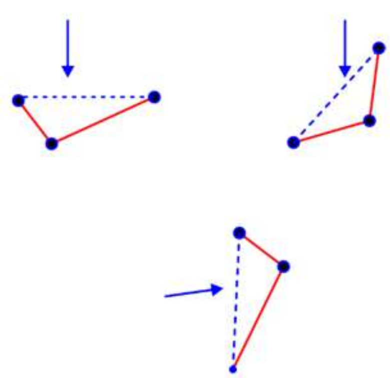
5.10. Mechanism - Plastic Analysis

- Beam mechanisms
- Sway mechanisms
- Gable mechanism
- Joint mechanism

5.10.1. Beam Mechanisms

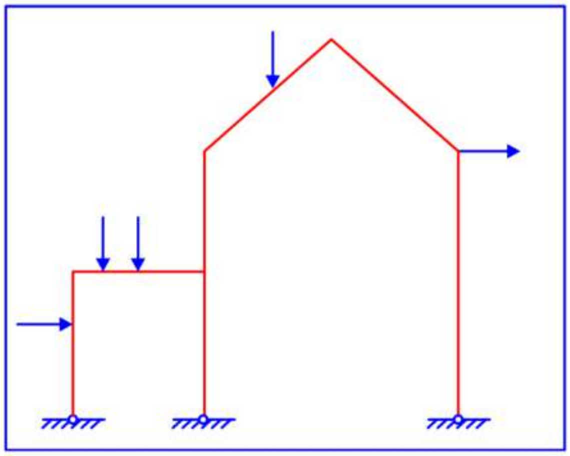
Takes place in simply supported, continuous, fixed beams etc.

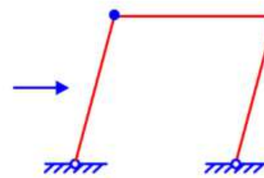




5.10.2. Sway Mechanisms

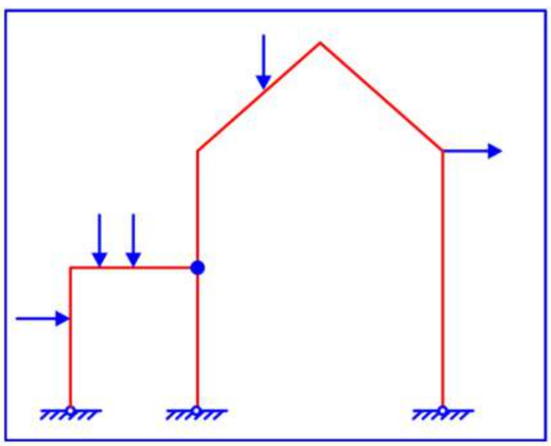
Takes place in Frames due to drifting of the column top joints.

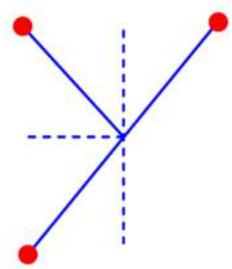




5.10.3. Joint Mechanism

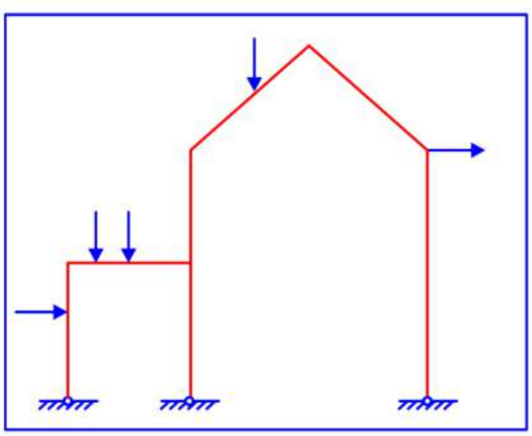
Occurs where **more than two structural members** meet. Plastic hinge is formed in all the members at that joint

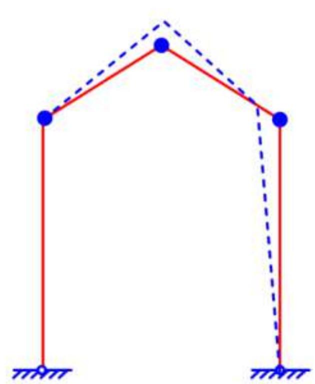




5.10.4. Gable Mechanism

Columns spread more at the top than at the base. This occurs in Gable frames of a warehouse or gable frames.





5.11. Theorems - Plastic Analysis

Static Theorem

- This theorem is based on the principles of statics
- It utilizes two conditions that are Equilibrium and Yield
- It is a lower bound theorem which states that for any distribution of moments the frame should be safe and admissible under the value of load W, where W is less than or equal to collapse load Wu
- Hence $W \leq W_u$, $M \leq M_p$

Kinematic Theorem

- It is based on the concept of work done and energy absorbed
- It utilizes two conditions that are Equilibrium and Mechanism
- It is a upper bound theorem which states that for a given frame subjected to load W, the W is found to correspond to any assumed mechanism, must be either greater or equal to the collapse load Wu
- Hence $W \geq W_u$

5.11.1. Static Theorem Procedure

- Draw the Normal BMD for the load on member
- Identify the locations where Plastic Moment capacity can be reached in a sequence.
- Relate the load W_u with the last plastic hinge location in terms of the bending moment M_p

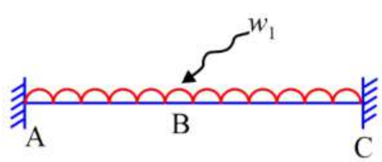
5.11.2. Kinematic Theorem Procedure

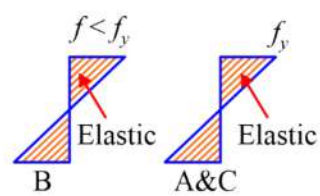
- Locate the possible location of plastic hinges
- Determine the number of possible independent and combined mechanisms
- Write the equation of equilibrium by the principal of virtual work i.e.

Work done = Energy stored

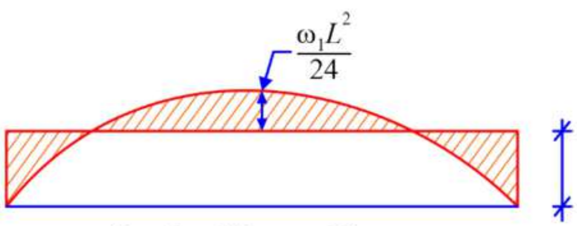
5.12. Sequence of Plastic Hinge Formation

(1) Load w_1





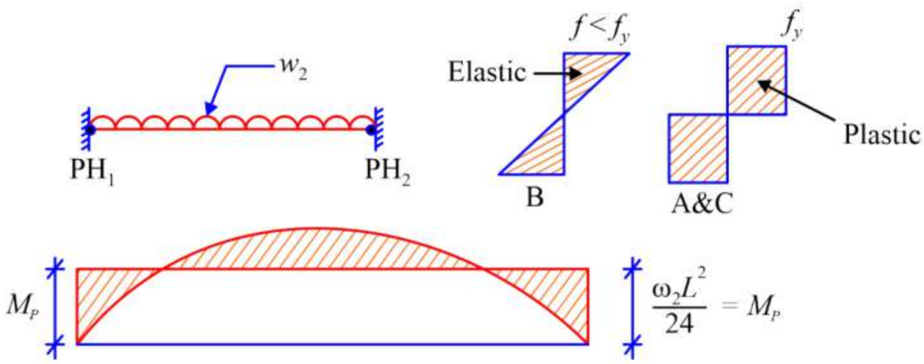
Bending Stress diagram



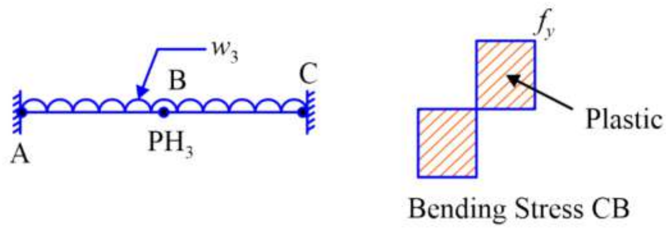
$\frac{\omega_1 L^2}{12} \cdot M_y$ or $\omega_1 = \frac{12 M_y}{L^2}$

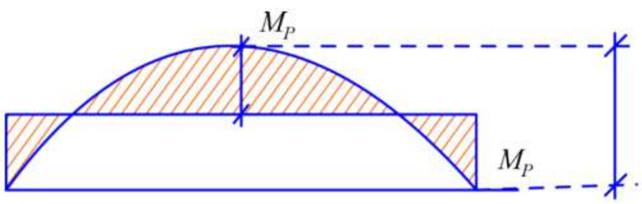
Bending Moment Diagram

(2) $w_2 > w_1$ Load Increases



(3) $w_3 > w_2$ Load Increased





$$\frac{\omega_3 L^2}{8} = 2 M_P \qquad \text{or} \qquad \omega_3 = \frac{16 M_P}{L^2}$$

Hence, If

$$\omega_1 = 10 \text{ kN/m}$$

Then find ω_2 and ω_3 for Rectangular Section.

(1) $$\frac{\omega_2}{\omega_1} = \frac{12 M_P / L^2}{12 M_y / L^2} = \frac{M_P}{M_y} = S_f = 1.5$$

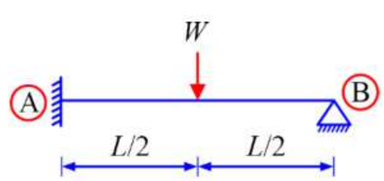
$\Rightarrow$ $$\omega_2 = 1.5, \ \omega_1 = 15 \text{ kN/m}$$

(2) $$\frac{\omega_3}{\omega_2} = \frac{16 M_P / L^2}{12 M_P / L^2} = \frac{4}{3} \qquad \Rightarrow \qquad \omega_3 = \frac{4}{3} \times 15 = 20 \text{ kN/m.}$$

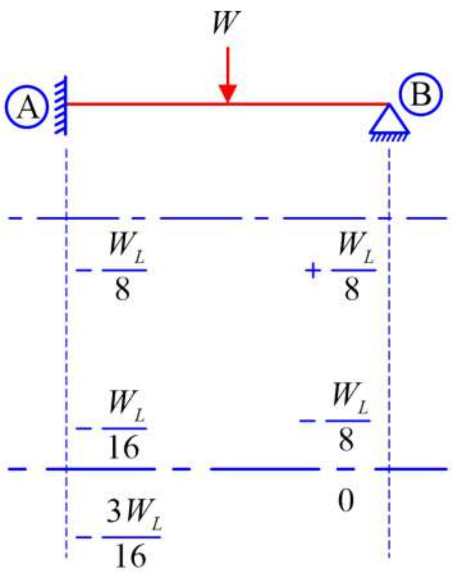
Ques. Find the Reaction at the Propped.

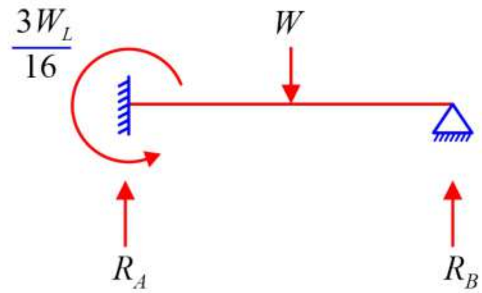
(a) Elastic Condition

(b) Collapse or Plastic Condition



Elastic Condition



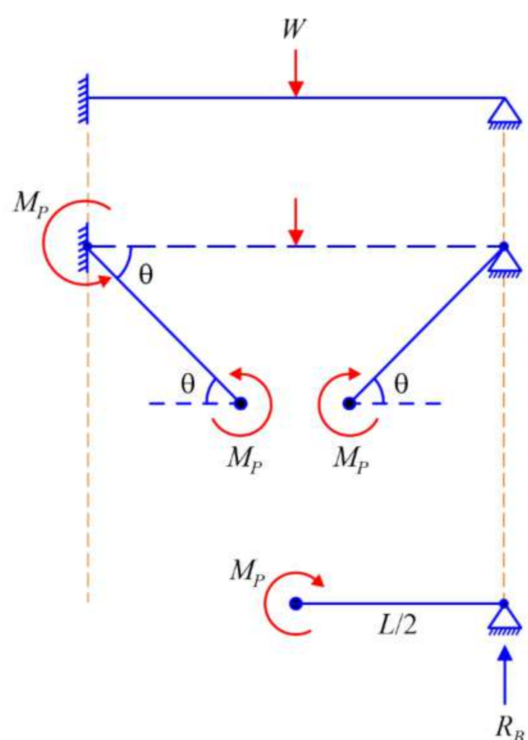


Taking $\Sigma M = 0$ about (A)

$$-\frac{3W_L}{16} + \frac{W \cdot L}{2} - R_3 . L = 0$$

Hence, $$R_B = \frac{5}{16}W$$

Plastic Condition



Considering the equilibrium of right side of beam.

$$\Sigma M = 0 \quad \text{about P.H.}$$

$$+M_P - R_B \cdot \frac{L}{2} = 0$$

$\Rightarrow$ $$R_B = \frac{2M_P}{L}$$

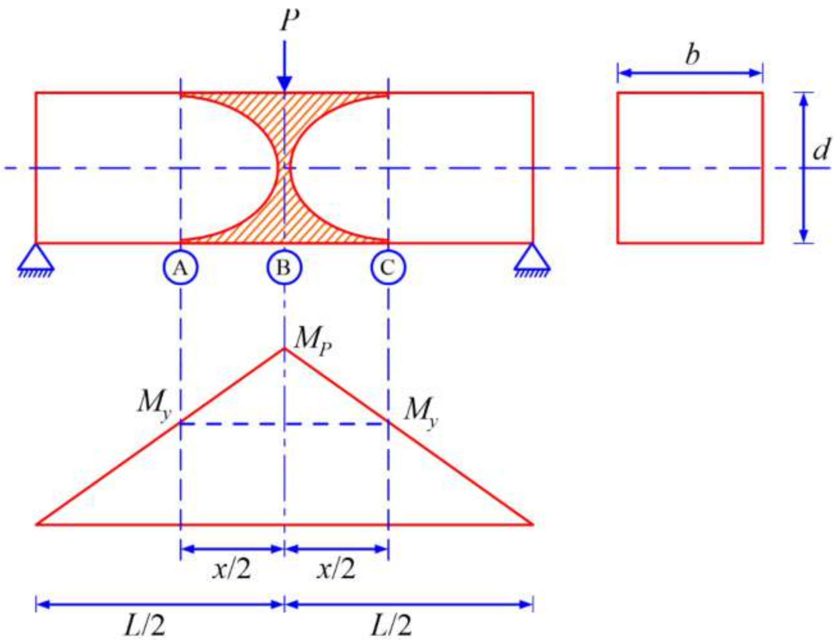
From standard case,

We know, $$W = \frac{6M_P}{L}$$

Hence, $$R_B = \frac{2}{L} \cdot \frac{W_L}{6} = \frac{W}{3}$$

$$R_B = \frac{W}{3}$$

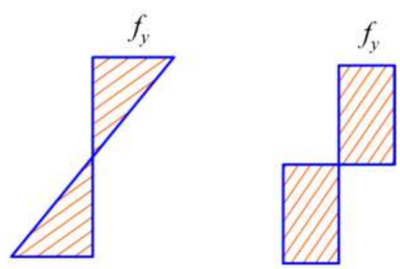
5.13. Length of Plastic Hinge for Rectangular Section



5.13.1. Bending Stress Diagram

$$M_P = \frac{P \cdot L}{4}$$

For Rectangular Section, $\frac{M_P}{M_y} = \text{SF} = 1.5$



From Bending Moment Dia.

$$\frac{M_P}{L/2} = \frac{M_y}{\frac{L}{2} \cdot \frac{x}{2}}$$

$\Rightarrow$ $$(L - x) = \frac{2}{3}L$$

$\Rightarrow$ $$x = \frac{L}{3}$$

Therefore, the hinge length of the plastic zone in equal to 1/3rd of the span.

$$L_f = \frac{P_u}{P_w} = \frac{M_P}{M}$$

$P_u =$ Collapse Load

$P_w =$ Working Load

$M_P =$ Plastic Moment

$M =$ Service Moment

Hence, $$L_f = \frac{f_y \cdot Z_p}{f \cdot Z_e} = \frac{f_y}{f} \cdot S_f$$

⇒ $$L_f = (\text{Fos}) \cdot S_f$$

$$\frac{f_u}{f} = \text{Fos in elastic design.}$$

$$f = \text{permissible bending stress}$$

$$= 0.66 f_y \ \text{[As per 15800 – 1984]}$$

5.14. Collapse Diagram - Plastic Analysis

5.14.1. Collapse diagram due to plastic hinge formation

Rigid body motion is considered i.e. structure becomes unstable and Mechanism has formed.

Hence to form a Mechanism in a statically indeterminate structure the **nos. of plastic hinge** required is equal to

$$N = SI + 1$$

There are 3 types of structural collapse based on number of plastic hinges formed

- **Partial collapse :** the number of plastic hinges formed is less than that required for complete collapse. Hence only a part of the structure becomes unstable
- **Complete collapse :** the number of plastic hinges formed is SI + 1
- **Over Complete collapse :** the number of plastic hinges formed is greater than that required for complete collapse. Hence there are chances of multiple mechanism occurring simultaneously.

Plastic Hinge Length

- Zone of yielding at which infinite rotations may take place.
- Large changes of slope occur over small length of member at a position
- The zone acts as if it was hinged with a constant moment Mp.
- The length of the yielded zone is called the hinge length.
- Plastic hinges are formed first at the sections subjected to greatest deformation(curvature).

Load Factor - Plastic Analysis

- It is defined as the ratio of collapse load to the working load
- It can also be defined as the product of the factor of safety and shape factor
- In practice a load factor varying from 1.7 to 2.0 is assumed depending upon the engineer's judgment
- IS800 – 2007 is silent on Load factor.
- Load factor has been defined in IS800 – 1984
- The prime function of load factor is to ensure that the structure is safe under working condition.

$$L_f = \frac{P_u}{P_w} = \frac{M_P}{M}$$

Hence,

$$L_f = \frac{f_y \cdot Z_p}{f \cdot Z_e} = \frac{f_y}{f} \cdot S_f$$

$\Rightarrow$

$$L_f = (\text{Fos}) \cdot S_f$$

P_u = Collapse Load

P_w = Working Load

M_P = Plastic Moment

M = Service Moment

$\frac{f_u}{f}$ = Fos in elastic design.

f = permissible bending stress

= $0.66 f_y$ [As per 15800 – 1984]

Example

Load factor for Rectangular Section,

$\Rightarrow$

$$L_f = \left(\frac{f_y}{0.66 f_y}\right) \times 1.5 = 2.26$$

Load factor for I section,

$\Rightarrow$

$$L_f = \left(\frac{f_y}{0.66 f_y}\right) \times 1.14 = 1.70$$

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