Chapter 5

Plastic Theory

CHAPTER HIGHLIGHTS

- Introduction
- Plastic bending of beams
- 🖙 Mechanism

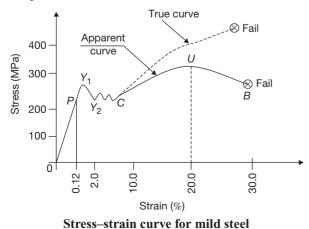
- Theorems of plastic analysis
- Methods of plastic analysis

INTRODUCTION

Plastic method of design, also known as 'limit design' or 'collapse method of design', or 'ultimate design', is based on ultimate load rather than working load. This is just because of the ductile nature of steel and, thus providing the large reserve strength beyond its yield point. This helps in reducing the size of sections than those designed by working stress method. Hence, the present chapter outlines the concept of plastic analysis and design of structural steel.

Stress-Strain Relation of Mild Steel

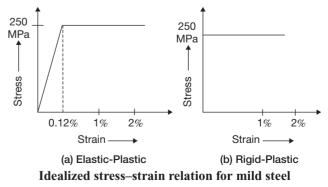
A simple stress-strain curve for mild steel is shown below:



NOTES

- 1. In the above stress-strain curve, P is the proportionality limit, Y_1 is the upper yield point, Y_2 is the lower yield point and U is the ultimate load point.
- 2. The different zones in the stress-strain curve are elastic zone (from *O* to point *P*: stress-strain relationship is linear), yield zone (from point Y_1 to Y_2), plastic zone (from point Y_2 to *C*), strain hardening zone (from point *C* to *U*) and Strain softening zone or necking zone (from point *U* to *B*).

But in plastic theory of structures, a simplified or idealized stress-strain curves are used.



• It is on the safer side to assume that the material to be perfectly elastic–plastic or rigid–plastic ignoring strain hardening.

PLASTIC BENDING OF BEAMS

Suppose a beam section subjected to an increase in bending moment as shown in the following figure. Due to this, the stress distribution will change as follows:

Elastic Stage $(M < M_{v})$

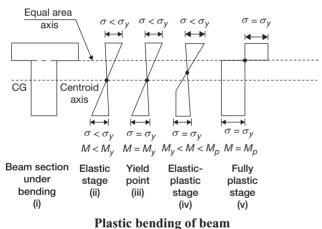
The maximum stress at extreme fibers will be within elastic range if applied moment is less than the yield moment or at low values of bending moment.

Elastic-Plastic Stage $(M_y < M < M_p)$

There will be an yielding at extreme fibers if there is a further increase in bending moment and there will be shifting of neutral axis, i.e., neutral axis no longer passes through the centroid of the section. Location of neutral axis in any case is such that the bending tensile and the compressive forces on the section are equal.

Plastic Stage $(M = M_p)$

In this stage, the entire cross-section will yield and act as a plastic hinge. The corresponding bending moment is called 'plastic moment of resistance' or simply 'plastic moment (M_p) '. The neutral axis of fully plastic section passes through the equal area axis.



Assumptions

The assumptions made in plastic analysis of beams are as follows:

- **1.** Plane sections normal to the axis of beam remain plane after bending.
- **2.** The material obeys the ideal stress–strain relationship. The increase in strength due to strain hardening is neglected.
- **3.** Beam is not subjected to axial load and shear strains are also neglected.

Plastic Moment

• The moment at which a plastic hinge is formed called 'plastic moment' is denoted by M_p .

• Plastic moment M_n is given by:

$$M_p = f_y Z_p$$

Where

 f_{v} = Yield stress in structural steel

 Z_p = Plastic section modulus of section

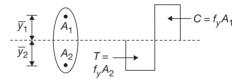
Plastic Hinge

It can be defined as a yielded zone due to flexure in a structural member in which infinite rotation can take place at a constant restraining moment M_n of the section.

- Plastic hinges are formed first at the sections subjected to the greatest deformation (curvature).
- The possible places for plastic hinges in a structure are at the points of concentrated loads, at fixed or rigid supports, at the change of cross-sections and at the point of zero shears.

Plastic Section Modulus

The plastic moment of beam section is shown below.



Plastic moment of a beam section

• At equilibrium condition:

Force in compression = Force in tension i.e., C = T

$$f_v A_1 = f_v A_2$$

$$A_1 = A_2 = \frac{A}{2}$$

• Therefore, areas above and below the neutral axis are equal and, hence, the neutral axis of plasticized section is called 'equal area axis'.

$$Z_p$$
 is given by, $Z_p = \frac{A}{2}(\overline{y_1} + \overline{y_2})$

Where

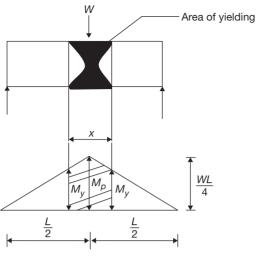
A = Area of cross-section of sections

 y_1, y_2 = Distance of centre of gravity of area above and below the neutral axis.

Hinge Length

- The value of moment at sections adjacent to the yield zone of a certain length is known as hinge length.
- It depends upon loading and geometry of section.

• Consider a simply supported rectangular beam with a central concentrated load 'W' as shown below.



Hinge length

From bending-moment diagram:

$$\frac{M_p}{\frac{L}{2}} = \frac{M_y}{\frac{L}{2} - \frac{x}{2}}$$

Where, $M_p = \frac{WL}{4}$

$$M_y = f_y Z_e = f_y \times \frac{bd^2}{6} = \frac{2}{3} f_y Z_p = \frac{2}{3} M_p.$$

 $\therefore (L-x) M_p = LM_y$ On solving,

$$x = \frac{L}{3}$$

Therefore, the hinge length of the plastic zone is equal to $\frac{1}{3}$ rd of the span.

NOTE

The plastic hinge length of a simply supported beam subjected to concentrated load is $\frac{L}{3}$, and due to distributed load it is $\frac{L}{\sqrt{3}}$.

Redistribution of Moments

- Plastic hinges are formed first at the highly stressed sections. The sections rotate without absorbing any more moment.
- The less stressed sections will be in equilibrium. Successive formation plastic hinges occurs at these sections by proportionate increase in moment.

- This process of formation of plastic hinges will continue till the ultimate load is reached.
- Therefore, the flexural members can sustain the ultimate loads only due to redistribution of moments.
- Redistribution of moments is the main contributing factor in reserving strength.

Shape Factor

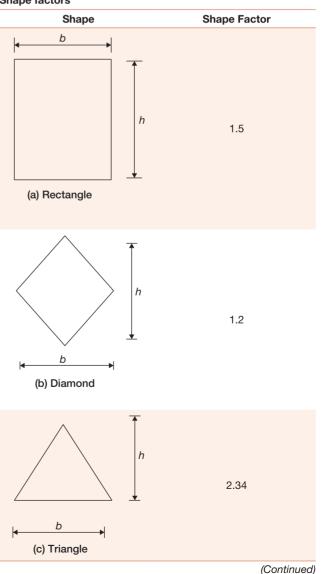
- It may be defined as the ratio of the plastic moment and the yield moment of the section.
- Denoted by S.

$$S = \frac{M_p}{M_y} = \frac{f_y Z_p}{f_y Z_e} = \frac{Z_p}{Z_e}$$

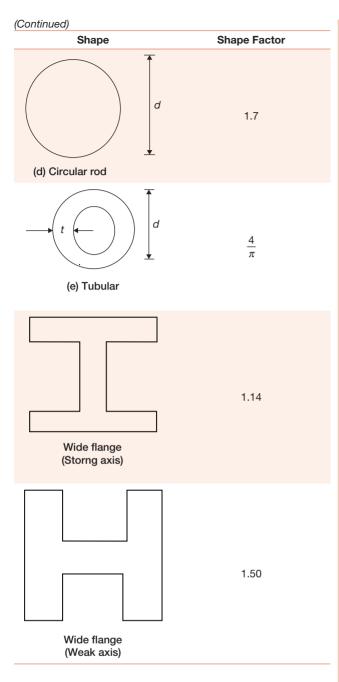
Where, $Z_{e} \mbox{ and } Z_{p}$ are the elastic and plastic section modulus.

• It is a function of cross-section form or shape. It indicates a reserve capacity of section.





3.434 | Part III = Unit 6 = Steel Structures



Load Factor

- It is a factor of safety based upon collapse load.
- It may be defined as the ratio of the collapse load to the working load.
- The prime function of load factor is to ensure the safety of structure under service conditions.

• Denoted by F:
$$F = \frac{\text{Collapse load}(W_u)}{\text{Working load}(W_w)}$$

$$F = \frac{W_u}{W_w} = \frac{M_p}{M_y} = \frac{f_y \cdot Z_p}{fZ_e} = \frac{f_y}{f} S = \text{FOS} \times \text{S}$$

Where

 $f_v =$ Yield stress

f = Permissible stress

S = Shape factor

 $\frac{f_y}{c}$ = Factor of safety as in elastic design

- Therefore, the load factor may also be defined as the product of factor of safety and shape factor.
- It depends upon the nature of loading, the support conditions and the geometrical shape of structural members.

Load Factor (F) (IS: 800-1984)

Dead load	1.7
Dead load + imposed load	1.7
Dead load + wind/seismic load	1.7
Dead load + imposed load + wind/seismic load	1.3

• When structures are subjected to wind, the corresponding load factor for plastic design is reduced by 25%.

Mechanism

- The conditions of mechanism occur when sufficient number of plastic hinges is formed and segments of the beam between the plastic hinges are able to move without an increase of load.
- Number of plastic hinges required for mechanism are, $N = D_s + 1$.
 - $D_s =$ Degree of static Indeterminacy.

Types of Mechanisms

- Various possible Independent mechanisms are as follows:
 - (a) Beam mechanism: All the loaded spans behave as beam mechanism.
 - (b) Sway mechanism: It is a result of the lateral loads.
 - (c) Joint mechanism: It is due to action of moment and the number of members meeting at a joint should be three or more.
 - (d) Gable mechanism: It occurs in gable frames.
- Any of these two independent mechanisms may be combined to form composite (combined) mechanism.

Number of Independent Mechanisms

Number of Independent mechanisms, n = N - r

Where

- N = Number of possible plastic hinges.
- r = Number of redundancies.

Conditions in Plastic Analysis

The conditions to be satisfied in plastic methods of analysis are as follows:

- **1.** Equilibrium condition: $\Sigma F = 0$ and $\Sigma M = 0$
- **2.** Mechanism condition: Also called 'continuity condition', it arises due to formation of plastic hinges and structure at collapse is capable of deforming as a mechanism.
- **3.** Yield condition: The bending moment at any section should be less than or equal to the plastic moment of the section. It is called 'plastic moment' condition'.

THEOREMS OF PLASTIC ANALYSIS

Static or Lower Bound Theorem

- Satisfies the equilibrium and yield condition.
- In this, the value of the load 'W' must be less than or equal to the collapse load (W_u) . The moments should not be greater than M_p .

That is, $W \leq W_{u}$ and $M \geq M_{n}$

• Therefore, this method represents the lower limit to the true ultimate load and has a maximum factor of safety.

Kinematic or Upper Bound Theorem

- It satisfies the equilibrium and continuity conditions.
- In this, the value of load corresponding to any mechanism is greater than or equal to collapse load (W_u) , i.e. $W \ge W_u$
- It represents an upper limit to the true ultimate load and has a smaller factor of safety.

Uniqueness Theorem

- It satisfies all the three conditions of plastic analysis.
- A collapse load will be chosen in such a way that a bending moment will be equal to the fully plastic moment and is sufficient to cause the failure as a mechanism.

METHODS OF PLASTIC ANALYSIS

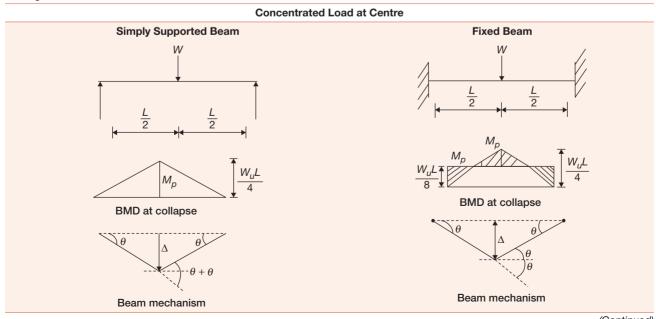
Static Method

The collapse load is determined by the following steps:

- **1.** First, calculate the degree of indeterminacy. If it is worked out as a redundant structure, then convert into determinate structure by removing the redundant forces.
- **2.** A bending-moment diagram (i.e., fixed BMD) for determinate structure is drawn.
- **3.** Bending-moment diagram (i.e., fixed BMD) for determinate structure with redundant forces is drawn.
- 4. The free BMD and fixed BMD are combined in such a way that the mechanism is formed.
- **5.** By applying equilibrium equations, the value of collapse load is worked out.

Kinematic or Mechanism or Upper Bound Methods

- 1. In this method, locating the possible places of hinges such as load points, frame joints, maximum bending moment points, etc is the first requirement.
- **2.** Select the independent and combined mechanism and the collapse load is worked out by using the virtual work principle.
- **3.** A bending-moment diagram corresponding to collapse mechanism is drawn, and it is to be checked that the bending moment at any point in a structure should be less than the plastic moment at that point.

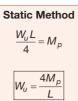


Collapse load of standard cases

3.436 | Part III • Unit 6 • Steel Structures

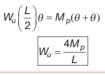
(Continued)

Concentrated Load at Centre



Kinematic Method

External work done = Internal work done



Static Method	
$\frac{W_{U}L}{4} = 2M_{p}$	
$W_{U} = \frac{8M_{P}}{L}$	

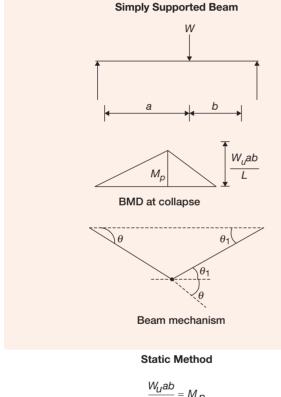
Kinematic Method



$$W_{u}\left(\frac{L}{2}\right)\theta = M_{p}\theta + M_{p}(\theta + \theta) + M_{p} \cdot \theta$$

$$W_{u} = \frac{8M_{p}}{L}$$

Eccentric Load



$$\frac{U^{m}}{L} = M_{p}$$
$$W_{u} = \frac{M_{p}L}{ab}$$

Kinetic Method

 $W_u = M_p \frac{L}{ab}$

 $W_{\mu}a\theta = M_{\mu}(\theta + \theta_{1})$

$$= M_{p}(\theta + \frac{a}{b}\theta)$$

Beam mechanism

 $\langle \theta \rangle$

Static Method

$$\frac{W_u ab}{I} = 2M_p$$

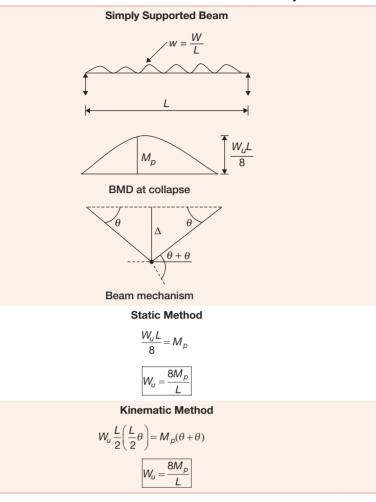
$$W_u = \frac{2M_pL}{ab}$$

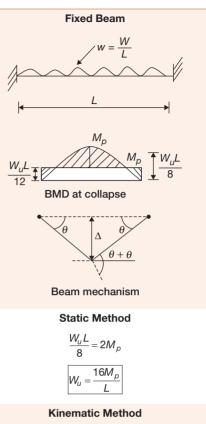
Kinematic Method

$$W_{u}a\theta = M_{p}\theta + M_{p}(\theta + \theta_{1}) + M_{p}\theta_{1}$$
$$W_{u}a\theta = M_{p}\theta + M_{p}(\theta + \frac{a}{b}\theta) + M_{p}(\frac{a}{b}\theta)$$
$$W_{u} = 2M_{p}\frac{L}{ab}$$

(Continued)

Uniformly Load Distributed





$$W_{u} \frac{L}{2} \left(\frac{L}{2} \theta \right) = M_{\rho} \theta + M_{\rho} (\theta + \theta) + M_{\rho} \theta$$
$$W_{u} = \frac{16M_{\rho}}{L}$$

Classification of Cross-sections

Classification of cross-sections is done based on moment– rotation characteristics assuming that the flange or web plate does not buckle locally.

The four different classes of cross-sections are:

Plastic Section (Class I)

- Used in plastic analysis and design.
- Can fully develop plastic hinges and failure of structure by formation of a plastic mechanism.
- Stress distribution for these sections is rectangular.

Compact Section (Class 2)

- Can develop plastic hinge, but do not have sufficient plastic hinge rotation capacity for formation of a plastic mechanism before buckling are called 'compact sections'.
- These may develop fully plastic stress distribution (i.e., rectangular) across the entire cross-section, but do not have adequate ductility.

Semi-compact Sections (Class 3)

- Sections which cannot develop fully plastic moment and stress at extreme fiber in compression can reach yield stress due to local buckling are called 'semi-compact or 'non-compact sections.
- These sections are used in elastic design and stress distribution for such sections is triangular.

Slender Sections (Class 4)

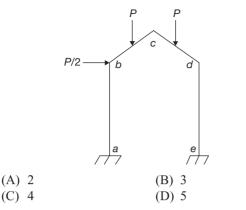
- Cross-sections in which the elements buckle locally even before attainment of yield stress are called 'slender sections'.
- Used in cold-formed members.

NOTE

Only plastic and compact sections should be used in limit state design and only plastic sections can be used in mechanism forming indeterminate frames.

Exercises

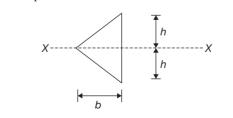
1. The number of independent mechanisms the gable frame will have when loaded as shown is



- 2. Equilibrium condition, yield conditions $(M \le M_p)$ and mechanism condition (formation of a plastic collapse mechanism) are the conditions to be satisfied by any correct plastic analysis results. Which of the above conditions does the statical method of plastic analysis consider?
 - (A) Equilibrium condition alone
 - (B) Equilibrium and mechanism conditions
 - (C) Yield and mechanism conditions
 - (D) Equilibrium and yield conditions
- 3. For a fixed beam with span L, having plastic moment capacity of M_p , the ultimate central concentrated load will be

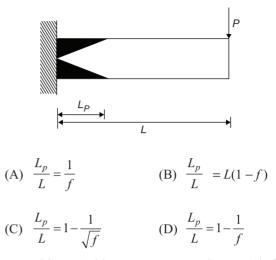
(A)
$$\frac{4M_p}{L}$$
 (B) $\frac{M_p}{8L}$
(C) $\frac{6M_p}{L}$ (D) $\frac{8M_p}{L}$

- 4. The plastic modulus of a section is 4.8 × 10⁻⁴ m³. The shape factor is 1.2. The plastic moment capacity of the section is 120 kN-m. The yield stress of the material is (A) 100 MPa
 (B) 240 MPa
 (C) 250 MPa
 (D) 300 MPa
- 5. The shape factor of the section shown in the figure is

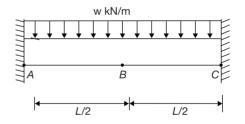


- (A) 1.5 (B) 1.12 (C) 2 (D) 1.7
- 6. A cantilever beam of length L and a cross-section with shape factor 'f' supports a concentrated load P as shown in the following figure:

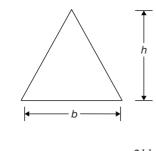
The length L_p of the plastic zone, when the maximum bending moment, equals the plastic moment M_p , given by



7. A steel beam (with a constant *EI*, and span *L*) is fixed at both ends and carries a uniformly distributed load (*w* kN-m), which is gradually increased till the beam reaches the stage of plastic collapse (refer to the following figure). Assuming '*B*' to be at mid-span, which of the following is true.



- (A) Hinges are formed at A, B and C together.
- (B) Hinges are formed at *B* and then at *A* and *C* together.
- (C) Hinges are formed at *A* and *C* together and then at *B*.
- (D) Hinges are formed at A and C only.
- 8. A cantilever beam of length *l*, width *b* and depth *d* is loaded with a concentrated vertical load at the tip. If yielding starts at a load *P*, the collapse load shall be
 - (A) 2.0*P*
 - (B) 1.5P
 - (C) 1.2P
 - (D) *P*
- 9. When the triangular section of a beam as shown in the following figure becomes a plastic hinge, the compressive force acting on the section (with σ_y denoting the yield stress) becomes



(A)
$$\frac{bh\sigma_y}{4}$$
 (B) $\frac{2bh\sigma_y}{9}$

(D) -

10. At the location of plastic hinge

. .

(C) -

- (A) radius of curvature is infinite.
- (B) curvature is infinite.
- (C) moment is infinite.
- (D) flexural stress is infinite.

11. Which one of the following is the load factor?

(A)
$$\frac{\text{Live load}}{\text{Dead load}}$$
(B) $\frac{\text{Failure load}}{\text{Working load}}$ (C) $\frac{\text{Total load}}{\text{Dead load}}$ (D) $\frac{\text{Dynamic load}}{\text{Static load}}$

12. The collapse load of a simply supported beam of span L and fully plastic moment M_p subjected to central concentrated load is given by

(A)
$$\frac{4M_p}{L}$$
 (B) $\frac{6M_p}{L}$
(C) $\frac{8M_p}{L}$ (D) $\frac{2M_p}{L}$

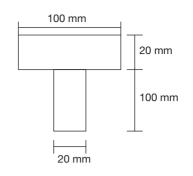
13. A structure has two degrees of indeterminacy. The number of plastic hinges that would be formed at complete collapse is

(A)	0	(B) 1
(C)	2	(D) 3

14. A propped cantilever beam AB of length 'L' fixed at 'A' and propped at B is subjected to a concentrated load 'w' at its centre. By kinematic approach calculate the ultimate collapse load (w) in terms of M_p .

(A)
$$w = \frac{2M_p}{L}$$
 (B) $w = \frac{6M_p}{L}$
(C) $w = \frac{4M_p}{L}$ (D) $w = \frac{8M_p}{L}$

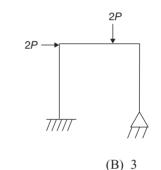
15. The distance of plastic neutral axis from top of T-section shown in the following figure:



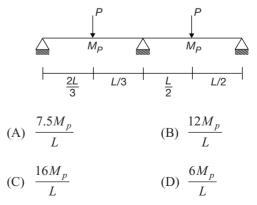
- (A) 15 mm
- (B) 60 mm
- (C) 20 mm
- (D) 40 mm

(A) 1

16. The number of possible independent mechanisms for a portal frame shown in the figure is



- (C) 4 (D) 2
 17. The plastic modulus of a section is 4.5 × 10⁻⁴ m³. The shape factor is 1.5. The plastic moment capacity of the
 - section is 150 kN-m. The yield stress of the material in MPa is _____.
 - (A) 120(B) 300(C) 330(D) 250
- **18.** A continuous beam with constant *EI* is shown in the given figure. Collapse load for this beam will be equal to

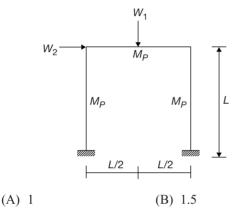


19. Match List I (Beam) with List II (collapse load) and select the correct answer using the codes given below the lists:

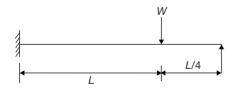
3.440 | Part III Unit 6 Steel Structures

	L	ist I								List II
a.	Simply supported beam with a central point load.								 1.	$\frac{8M_p}{L}$
b.	F	Fixed beam with a central point load 2.								
c.	Propped cantilever with a central point load							3.	$\frac{4M_p}{L}$	
Cod (A) (C)	a 3	b	2			(B) (D)	a 3 1	b 2 2	c 1 3	

20. Find the ratio of W_1 to W_2 for beam and sway mechanism for a portal frame shown below.



- (C) 2 (D) 3
- **21.** A propped cantilever beam of uniform moment capacity M_p is shown in the given figure:

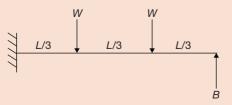


What is the collapse load *W*?

1. The plastic collapse load W_p for the propped cantilever supporting two point loads as shown in the figure in terms of plastic moment capacity, M_p is given by

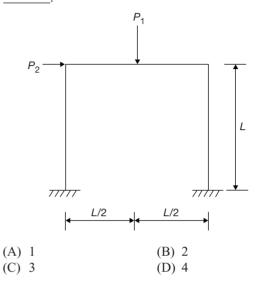
[GATE, 2007]

PREVIOUS YEARS' QUESTIONS

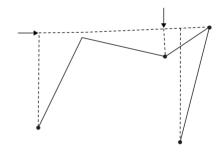


(A)
$$\frac{12}{L}Mp$$
 (B) $\frac{8}{L}Mp$
(C) $\frac{6}{L}Mp$ (D) $\frac{3}{L}Mp$

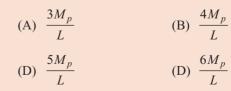
22. Given figure shows a portal frame with load. All members have the same plastic moment of resistance Mp. The ratio P_1 to P_2 for beam and sway mechanism is



23. A portal frame has a collapse mechanism as shown below. What is the type?



- (A) Pure portal mechanism
- (B) Panel mechanism
- (C) Dual beam mechanism
- (D) Combined mechanism



- 2. The shape of the cross-section, which has the largest shape factor, is [GATE, 2008]
 - (A) rectangular (B) I-section
 - (C) diamond (D) solid circular

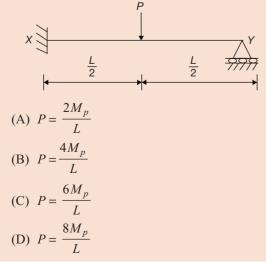
3. In the theory of plastic bending of beams, the ratio of plastic moment to yield moment is called

[GATE, 2009]

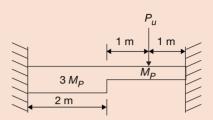
- (A) shape factor
- (B) plastic section modulus
- (C) modulus of resilience
- (D) rigidity modulus
- As per IS:800–2007 the cross-section in which extreme fiber can reach the yield stress but cannot develop the plastic moment of resistance due to local buckling is classified as [GATE, 2013]
 - (A) plastic section
 - (B) compact section
 - (C) semi-compact section
 - (D) shear section
- 5. Match the information given in List I with those in List II. [GATE, 2014]

	List I		List II
P.	Factor to decrease ultimate strength to design strength	1.	Upper bound on ultimate load
Q.	Factor to increase working load to ultimate load for design	2.	Lower bound on ultimate load
R.	Statical method of ultimate load analysis	3.	Material partial safety factor
S.	Kinematical mechanism method of ultimate load analysis	4.	Load factor

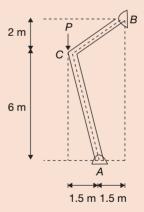
- (A) P-1; Q-2; R-3; S-4
- (B) P-2; Q-1; R-4; S-3
- (C) P-3; Q-4; R-2; S-1
- (D) P-4; Q-3; R-2; S-1
- 6. The ultimate collapse load (P) in terms of plastic moment M_p by kinematic approach for a propped cantilever of length L with P acting at its mid-span as shown in the figure, would be [GATE, 2014]



7. For formation of *collapse mechanism* in the following figure, the minimum value of P_u is cM_p/L . M_p and $3M_p$ denote the plastic moment capacities of beam sections as shown in this figure. The value of c is _____. [GATE, 2015]



- The semi-compact section of a laterally unsupported steel beam has an elastic section modulus, plastic section modulus and design bending compressive stress of 500 cm³, 650 cm³ and 200 MPa, respectively. The design flexural capacity (expressed in kN-m) of the section is ______. [GATE, 2016]
- **9.** A rigid member ACB is shown in the figure. The member is supported at A and B by pinned and guided roller supports, respectively. A force P acts at C as shown. Let R_{Ah} and R_{Bh} be the horizontal reactions at supports A and B, respectively, and R_{Av} be the vertical reaction at support A. Self-weight of the member may be ignored. [GATE, 2016]



Which one of the following sets gives the correct magnitudes of R_{Av} , R_{Bh} and R_{Ah} ?

- (A) $R_{Av} = 0; R_{Bh} = \frac{1}{3}P; \text{ and } R_{Ah} = \frac{2}{3}P$
- (B) $R_{Av} = 0; R_{Bh} = \frac{2}{3}P; \text{ and } R_{Ah} = \frac{1}{3}P$
- (C) $R_{Av} = P$; $R_{Bh} = \frac{3}{8}P$; and $R_{Ah} = \frac{1.5}{8}P$
- (D) $R_{Av} = P$; $R_{Bh} = \frac{1.5}{8}P$; and $R_{Ah} = \frac{1.5}{8}P$

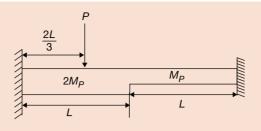
3.442 | Part III = Unit 6 = Steel Structures

10. A propped cantilever of span L carries a vertical concentrated load at the mid-span. If the plastic moment capacity of the section is M_p , the magnitude of the collapse load is **[GATE, 2016]**

(A)
$$\frac{8M_p}{L}$$
 (B) $\frac{6M_p}{L}$

(C)
$$\frac{4M_p}{L}$$
 (D) $\frac{2M_p}{L}$

11. A fixed-end beam is subjected to a concentrated load (P) as shown in the figure. The beam has two different segments having different plastic moment capacities $(M_p, 2M_p)$ as shown. [GATE, 2016]



The minimum value of load (*P*) at which the beam would collapse (ultimate load) is

(A)	$7.5 M_{p}/L$	(B)	$5.0 M_{p}/L$
(C)	$4.5 M_{p}^{f}/L$	(D)	$2.5M_{p}^{P}/L$

Answer Keys									
Exerci	Exercises								
1. C 11. B 21. C	 D 12. A 22. B 	3. D 13. D 23. D	4. C 14. B	5. C 15. C	6. D 16. D	7. C 17. C	8. B 18. D	9. A 19. A	10. B 20. C
Previo	us Years'	Questio	ns						
1. B 9. D	2. C 10. B	3. A 11. A	4. C	5. C	6. C	7. 0.09	to 0.10	8. 90.9	1

(

(