Exercise 8.R

Answer 1CC.

(a) The length of the curve C with equation $y = f(x), a \le x \le b$ is defined as the limit of the lengths of these inscribed polygons (if the limit exists):

i.e. $$L = \lim_{n\to\infty} \sum_{i=1}^{n} |P_{i-1}P_i|$$

(b) If f' is continuous on $[a,b]$ then the length of the curve $y = f(x), a \le x \le b$ is

$$L = \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$$

(c) If x is given as a function of y then the length of the curve $x = f(y), a \le y \le b$ is

$$L = \int_a^b \sqrt{1+\left(\frac{dx}{dy}\right)^2}\, dy$$

Answer 1E.

We have $y = \frac{1}{6}\left(x^2+4\right)^{3/2} \qquad 0 \le x \le 3$

Then $\frac{dy}{dx} = \frac{1}{6} \times \frac{3}{2}\left(x^2+4\right)^{1/2}.(2x),$ [by chain rule]

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2}x\left(x^2+4\right)^{1/2}$$

$$\Rightarrow \left(\frac{dy}{dx}\right)^2 = \frac{1}{4}x^2\left(x^2+4\right)$$

$$\Rightarrow 1+\left(\frac{dy}{dx}\right)^2 = 1+\frac{1}{4}x^4+x^2$$

$$\Rightarrow 1+\left(\frac{dy}{dx}\right)^2 = \left(1+\frac{1}{2}x^2\right)^2$$

$$\Rightarrow \sqrt{1+\left(\frac{dy}{dx}\right)^2} = \sqrt{\left(1+\frac{1}{2}x^2\right)^2}$$

$$= 1+\frac{1}{2}x^2$$

Then length of the curve is

$$L=\int_0^3\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

$$=\int_0^3\left(1+\frac{1}{2}x^2\right)dx$$

$$=\left(x+\frac{1}{2}\cdot\frac{x^3}{3}\right)\Bigg]_0^3$$

$$=\left(3+\frac{9}{2}-0-0\right)$$

$$=\frac{15}{2}$$

$$\Rightarrow \boxed{L=\frac{15}{2}}$$

Answer 1P.

(a) The surface area of the surface obtained by rotating the curve $y=f(x), a\le x\le b$, about the x-axis is

$$s=\int_a^b 2\pi f(x)\sqrt{1+\left[f'(x)\right]^2}\,dx$$

(b) If $x=g(y), c\le y\le d$ then

$$x=\int_c^d 2\pi y\sqrt{1+\left[g'(y)\right]^2}\,dy$$

(c) If the curve is rotated about the y-axis then

$$s=\int_a^b 2\pi x\sqrt{1+\left[f'(x)\right]^2}\,dx \text{ or}$$

$$s=\int_c^d 2\pi g(y)\sqrt{1+\left[g'(y)\right]^2}\,dy$$

Answer 2E.

We have $y=2\ln\sin\left(\frac{1}{2}x\right) \qquad \pi/3\le x\le\pi$

$$\Rightarrow \frac{dy}{dx}=2\frac{1}{\sin\left(\frac{1}{2}x\right)}\cos\left(\frac{1}{2}x\right)\cdot\frac{1}{2} \qquad \text{[Chain rule]}$$

$$=\frac{\cos\left(\frac{1}{2}x\right)}{\sin\left(\frac{1}{2}x\right)}=\cot\left(\frac{1}{2}x\right)$$

Therefore

$$\Rightarrow 1+\left(\frac{dy}{dx}\right)^2 = 1+\cot^2\left(\frac{1}{2}x\right)$$

$$\Rightarrow 1+\left(\frac{dy}{dx}\right)^2 = \csc^2\left(\frac{1}{2}x\right) \qquad \left[1+\cot^2\theta = \csc^2\theta\right]$$

$$\Rightarrow \sqrt{1+\left(\frac{dy}{dx}\right)^2} = \csc\left(\frac{1}{2}x\right)$$

Then length of the curve

$$L = \int_{\pi/3}^{\pi} \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

$$= \int_{\pi/3}^{\pi} \csc\left(\frac{1}{2}x\right)dx$$

Let $\frac{1}{2}x = t$

$$\Rightarrow \frac{1}{2}dx = dt$$

$$\Rightarrow dx = 2dt$$

When $x = \frac{\pi}{3}$, $t = \frac{\pi}{6}$ and when $x = \pi$, $t = \frac{\pi}{2}$

Then $L = 2\int_{\pi/6}^{\pi/2} \csc(t)\,dt$

Using $\int \csc u\, du = \ln\left|\csc u - \cot u\right| + c$

We have $L = 2\left[\ln\left|\csc t - \cot t\right|\right]_{\pi/6}^{\pi/2}$

$$= 2\left[\ln\left|\csc\frac{\pi}{2} - \cot\frac{\pi}{2}\right| - \ln\left|\csc\frac{\pi}{6} - \cot\frac{\pi}{6}\right|\right]$$

$$= 2\left[\ln 1 - \ln\left(2-\sqrt{3}\right)\right] = -2\ln\left(2-\sqrt{3}\right) \approx 2.63$$

$$\boxed{L \approx 2.63}$$

Answer 3CC.

Suppose a wall is submerged into a fluid vertically.
Suppose the top of the tall below the fluid level is at a and the bottom is at b

Let $c(x)$ be the cross-sectional length of the wall (measured perpendicular to the surface of the fluid).

Then the hydrostatic force against the wall is given by $F = \int_a^b \delta x c(x)\,dx$, δ is the weight density of the fluid.

Answer 3E.

(A) We have $y = \frac{x^4}{16} + \frac{1}{2x^2}$, $1 \le x \le 2$

Then $\frac{dy}{dx} = \left(\frac{x^3}{4} - \frac{1}{x^3}\right)$

$= \frac{x^6 - 4}{4x^3}$

$\Rightarrow \left(\frac{dy}{dx}\right)^2 = \frac{(x^6 - 4)^2}{16x^6}$

$\Rightarrow 1 + \left(\frac{dy}{dx}\right)^2 = 1 + \frac{(x^6 - 4)^2}{16x^6}$

$= \frac{16x^6 + x^{12} + 16 - 8x^6}{16x^6}$

$= \frac{x^{12} + 16 + 8x^6}{16x^6}$

$= \frac{(x^6 + 4)^2}{16x^6}$

$\Rightarrow \sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \frac{(x^6 + 4)}{4x^3}$

Then length of the curve

$$L = \int_1^2 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$= \int \frac{x^6 + 4}{4x^3} dx$$

$$= \frac{1}{4}\int_1^2 \left(x^3 + \frac{4}{x^3}\right) dx$$

$$= \frac{1}{4}\left[\frac{x^4}{4} + \left(\frac{-2}{x^2}\right)\right]_1^2$$

$$= \frac{1}{4}\left[4 + \left(-\frac{1}{2}\right) - \frac{1}{4} + 2\right]$$

$$= \frac{1}{4}\left(6 - \frac{3}{4}\right) = \frac{21}{16}$$

$$\Rightarrow \boxed{L = \frac{21}{16}}$$

(B) We have $y=\frac{x^4}{16}+\frac{1}{2x^2}$

$$\Rightarrow \frac{dy}{dx}=\frac{x^3}{4}-\frac{1}{x^3}$$

$$=\frac{x^6-4}{4x^3}$$

$$\Rightarrow 1+\left(\frac{dy}{dx}\right)^2=1+\frac{\left(x^6-4\right)^2}{16x^6}$$

$$=\frac{\left(x^6+4\right)^2}{16x^6}$$

$$\Rightarrow \sqrt{1+\left(\frac{dy}{dx}\right)^2}=\frac{x^6+4}{4x^3}$$

Then $ds=\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$

$$=\left(\frac{x^6+4}{4x^3}\right)dx$$

Then surface area obtained by rotating the curve from $x=1$ to $x=2$ about y – axis is

$$S=\int_1^2 2\pi x\,ds$$

$$=2\pi\int_1^2 x\frac{\left(x^6+4\right)}{4x^3}dx$$

$$=\frac{1}{2}\pi\int_1^2\left(\frac{x^6+4}{x^2}\right)dx$$

$$=\frac{1}{2}\pi\int_1^2\left(x^4+4x^{-2}\right)dx$$

$$=\frac{1}{2}\pi\left[\frac{x^5}{5}-4x^{-1}\right]_1^2$$

$$=\frac{1}{2}\pi\left[\frac{32}{5}-2-\frac{1}{5}+4\right]$$

$$=\frac{1}{2}\pi\times\frac{41}{5}$$

$$=\frac{41\pi}{10}$$

Answer 4CC.

(a) The center of mass of a thin plate is the point at which the plate balances horizontally.

(b) The center of mass of the plate is located at the point $(\bar{x},\bar{y})$, where

$$\bar{x}=\frac{1}{A}\int_a^b x\,f(x)\,dx,\ \bar{y}=\frac{1}{A}\int_a^b\frac{1}{2}\left[f(x)\right]^2dx$$

and $A=\int_a^b f(x)dx$ is the area of the

Answer 4E.

(A) We have the equation of the curve

$$y = x^2 \qquad 0 \le x \le 1$$

$$\Rightarrow \frac{dy}{dx} = 2x$$

$$\Rightarrow \left(\frac{dy}{dx}\right)^2 = 4x^2$$

$$\Rightarrow 1+\left(\frac{dy}{dx}\right)^2 = 1+4x^2$$

$$\Rightarrow \sqrt{1+\left(\frac{dy}{dx}\right)^2} = \sqrt{1+4x^2}$$

Then $ds = \sqrt{1+4x^2}\,dx$

Surface area of the region obtained by rotating the curve about y – axis is

$$S = \int_0^1 2\pi x\, ds = \int_0^1 2\pi\, x\sqrt{1+4x^2}\,dx$$

$$= \pi\int_0^1 2x\sqrt{1+4x^2}\,dx$$

Let $1+4x^2 = t$

$$\Rightarrow 8x\,dx = dt \Rightarrow 2x\,dx = \frac{dt}{4}$$

When x = 0 then t = 1 and when x = 1 then t = 5

Then $S = \pi\int_1^5 t^{1/2}\frac{dt}{4}$

$$= \frac{\pi}{4}\int_1^5 t^{1/2}dt$$

$$= \frac{\pi}{4}\left[\frac{2t^{3/2}}{3}\right]_1^5$$

$$= \frac{\pi}{4}\left[\frac{2}{3}.5\sqrt{5}-\frac{2}{3}\right] = \frac{\pi}{6}\left(5\sqrt{5}-1\right)$$

Thus $\boxed{S = \frac{\pi}{6}\left(5\sqrt{5}-1\right)} \approx 5.33$

(B) We have $y = f(x) = x^2 \qquad 0 \le x \le 1$

$$\Rightarrow f'(x) = 2x$$

$$\Rightarrow \left[f'(x)\right]^2 = 4x^2$$

$$\Rightarrow 1+\left[f'(x)\right]^2 = 1+4x^2$$

Surface area of the region obtained by rotating the given curve about x – axis is

$$S = \int_0^1 2\pi f(x)\sqrt{1+\left[f'(x)\right]^2}\,dx$$

$$= \int_0^1 2\pi . x^2\sqrt{1+4x^2}\,dx$$

$$= 2\pi\int_0^1 x^2\sqrt{1+4x^2}\,dx$$

Let $2x = t \Rightarrow 2dx = dt$

When x = 0 then t = 0 and when x = 1 then t = 2

So $S = 2\pi\int_0^2 \frac{1}{4}t^2\sqrt{1+t^2}\,\frac{1}{2}dt$

$$= \frac{\pi}{4}\int_0^2 t^2\sqrt{1+t^2}\,dt$$

Using the integral

$$\int u^2\sqrt{a^2+u^2}\,du = \frac{u}{8}\left(a^2+2u^2\right)\sqrt{a^2+u^2}-\frac{a^4}{8}\ln\left(u+\sqrt{a^2+u^2}\right)+C$$

We have $S=\frac{\pi}{4}\left[\frac{t}{8}\left(1+2t^2\right)\sqrt{1+t^2}-\frac{1}{8}\ln\left|t+\sqrt{1+t^2}\right|\right]_0^2$

Or $S=\frac{\pi}{32}\left[2\times 9\sqrt{5}-\ln\left(2+\sqrt{5}\right)-0+\ln(1)\right]$

$$=\frac{\pi}{32}\left[18\sqrt{5}-\ln\left(2+\sqrt{5}\right)\right]$$

$$\Rightarrow \boxed{S=\frac{\pi}{32}\left[18\sqrt{5}-\ln\left(2+\sqrt{5}\right)\right]}$$

Answer 5CC.

If a plane region R that lies entirely on one side of a line l in its plane is rotated about l, then the volume of the resulting solid is the product of the area of R and the distance traveled by the centroid of R.

This is the relation between volume and the surface area multiplied by the distance travelled by the centroid.

So, Pappus theorem is useful to find the volume of the irregular shaped regions.

Answer 5E.

We have $y=f(x)=e^{-x^2}$, $0\le x\le 3$

Then $f'(x)=e^{-x^2}(-2x)$ (by chain rule)

$\Rightarrow\left[f'(x)\right]^2=4x^2e^{-2x^2}$

$\Rightarrow 1+\left[f'(x)\right]^2=1+4x^2e^{-2x^2}$

$\Rightarrow \sqrt{1+\left[f'(x)\right]^2}=\sqrt{1+4x^2e^{-2x^2}}$

Then length of the curve $L=\int_0^3\sqrt{1+4x^2e^{-2x^2}}\,dx$

Let $G(x)=\sqrt{1+4x^2e^{-2x^2}}$

Interval [0, 3] and $n=6$

Then $\Delta x=\frac{3}{6}$

$=0.5$

$=\frac{1}{2}$

Subintervals are $[0,0.5]$, $[0.5,1]$, $[1,1.5]$, $[1.5,2]$, $[2,2.5]$, $[2.5,3]$

Then we can estimate the length of the curve by Simpson's rule as

$$L\approx\frac{\Delta x}{3}\left[G(0)+4G(0.5)+2G(1)+4G(1.5)+2G(2)+4G(2.5)+G(3)\right]$$

$$=\frac{1}{6}\left[G(0)+4G(0.5)+2G(1)+4G(1.5)+2G(2)+4G(2.5)+G(3)\right]$$

$$\Rightarrow \boxed{L\approx 3.292287}$$

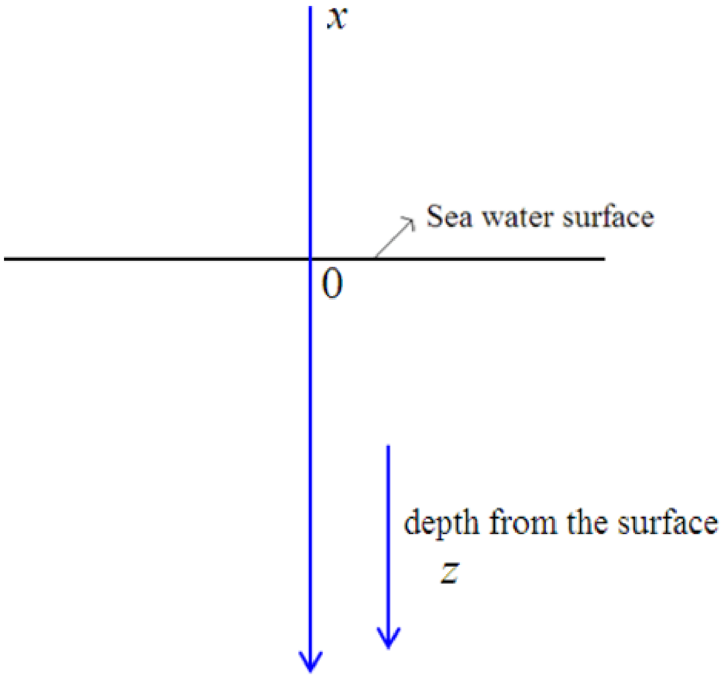
Answer 5P.

(a)

Consider the density of sea water, $\rho = \rho(z)$ where *z* is the depth below the surface.

Let the vertical line be $x-$ axis represents the depth below the surface and it is pointing towards down and center at the surface.

The statement can be express as a diagram.



To estimate the pressure at a depth of *z*, split $[0,z]$ as a *n* subintervals.

Choose $x_i^* \in [x_{i-1}, x_i]$ for each *i*.

Let the density of thin layer lies between the depths of x_{i-1}, and x_i be *k*.

That is $k = \rho(x_i^*)$.

Then, the weight of a piece of that layer *k* with unit cross- sectional area is $\rho(x_i^*)g\Delta x$.

The weight of the water at *z* is $\sum_{i=1}^{n} \rho(x_i^*)g\Delta x$.

Hence, the total weight of the water along with $x-$axis is

$$W = \lim_{n\to\infty}\left(\sum_{i=1}^{n} \rho(x_i^*)g\Delta x\right)$$

It can be write in integral form as $P(z) = \int_0^z \rho(x)g\,dx$.

$$P(z) = P_0 + \int_0^z \rho(x)g\,dx$$

Here P_0 is the pressure at the origin.

Differentiate $P(z) = P_0 + \int_0^z \rho(x)g\,dx$ with respect to *z*.

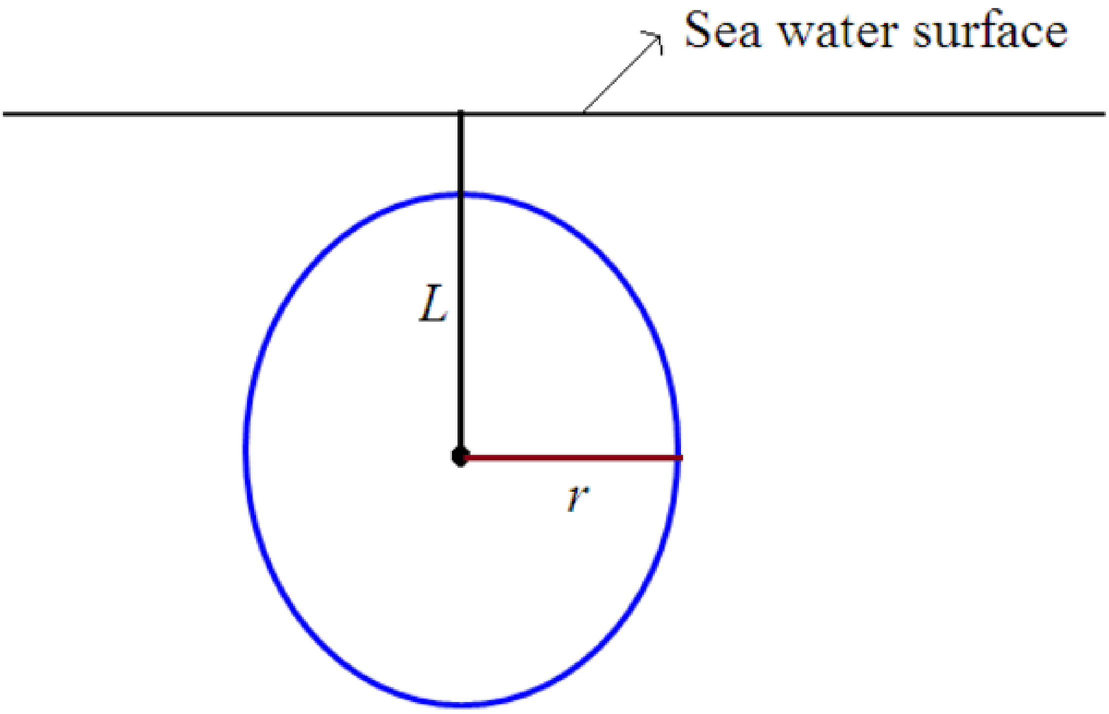
$$\frac{dP}{dz} = \rho(z)g$$

Hence, the pressure at a depth of *z* from the surface is $\boxed{\frac{dP}{dz} = \rho(z)g}$.

(b)

Consider a circle with radius *r* whose center is located at a distance $L > r$ below the surface.

Suppose the density of sea water at depth *z* is $\rho = \rho_0 e^{\frac{z}{H}}$.



Find the force.

$$F = \int_{-r}^{r} P(L+x)\cdot 2\sqrt{r^2 - x^2}\,dx$$

$$= \int_{-r}^{r}\left(P_0 + \int_0^{L+x} \rho_0 e^{\frac{z}{h}} g\,dz\right)\cdot 2\sqrt{r^2 - x^2}\,dx$$

Use $P(z) = P_0 + \int_0^z \rho(x)g\,dx$ and $\rho = \rho_0 e^{\frac{z}{H}}$

$$= P_0\int_{-r}^{r} 2\sqrt{r^2 - x^2}\,dx + \rho_0 gH\int_{-r}^{r}\left(e^{\frac{L+x}{H}} - e^0\right)\cdot 2\sqrt{r^2 - x^2}\,dx$$

$$= P_0\int_{-r}^{r} 2\sqrt{r^2 - x^2}\,dx + \rho_0 gH\int_{-r}^{r}\left(e^{\frac{L+x}{H}} - 1\right)\cdot 2\sqrt{r^2 - x^2}\,dx$$

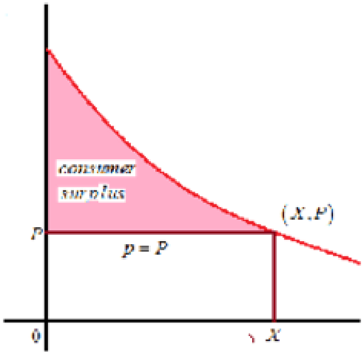
$$= 2\left(P_0 - \rho_0 gH\right)\int_{-r}^{r}\sqrt{r^2 - x^2}\,dx + \rho_0 gH\int_{-r}^{r} e^{\frac{L+x}{H}}\cdot 2\sqrt{r^2 - x^2}\,dx$$

$$= 2\left(P_0 - \rho_0 gH\right)\left(\pi r^2\right) + \rho_0 gHe^{\frac{L}{H}}\int_{-r}^{r} e^{\frac{x}{H}}\cdot 2\sqrt{r^2 - x^2}\,dx$$

Hence, the force is $\boxed{F = 2\left(P_0 - \rho_0 gH\right)\left(\pi r^2\right) + \rho_0 gHe^{\frac{L}{H}}\int_{-r}^{r} e^{\frac{x}{H}}\cdot 2\sqrt{r^2 - x^2}\,dx}$.

Answer 6CC.

The graph is



Amount of a commodity currently available $= X$
Current selling price is P
Demand function is $p(x)$

The consumer surplus is $\int_0^X [P(x) - P]dx$

This is nothing but the area enclosed by the curve $p(x)$ and the line $p = P$

The consumer surplus represents the amount of money saved by consumers in purchasing the commodity at price P, corresponding to an amount demanded of X. The above figure shows the interpretation of the consumer surplus as the area under the demand curve and above the line $P = P$.

Answer 6E.

We have $y = e^{-x^2}$, $0 \le x \le 3$

Then $\frac{dy}{dx} = -2xe^{-x^2}$

$$\Rightarrow \left(\frac{dy}{dx}\right)^2 = 1+4x^2 e^{-2x^2}$$

$$\Rightarrow 1+\left(\frac{dy}{dx}\right)^2 = 1+4x^2 e^{-2x^2}$$

$$\Rightarrow \sqrt{1+\left(\frac{dy}{dx}\right)^2} = \sqrt{1+4x^2\ e^{-2x^2}}$$

Then area of the surface obtained by rotating given curve about x-axis, is

$$S = \int_0^3 2\pi y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

$$\Rightarrow S = \int_0^3 2\pi . e^{-x^2}\sqrt{1+4x^2 e^{-2x^2}}\,dx$$

Let $f(x) = 2\pi e^{-x^2}\sqrt{1+4x^2 e^{-2x^2}}$ and interval is [0, 3]

When n = 6 then $\Delta x = 3/6 = \frac{1}{2} = 0.5$

Subintervals are $[0, 0.5]$, $[0.5, 1]$, $[1, 1.5]$, $[1.5, 2]$, $[2, 2.5]$, $[2.5, 3]$

Then we can estimate the area of the surface by Simpson's rule as

$$S \approx \frac{\Delta x}{3}\left[f(0)+4f(0.5)+2f(1)+4f(1.5)+2f(2)+4f(2.5)+f(3)\right]$$

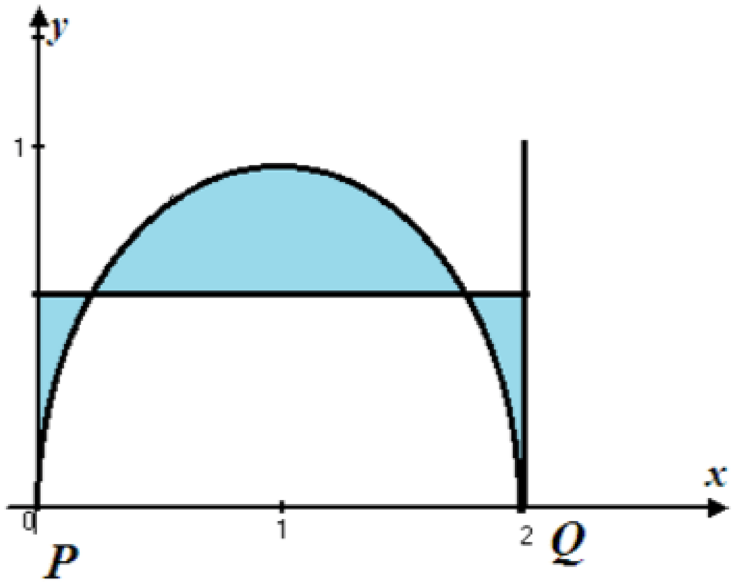
$$= \frac{1}{6}\left[f(0)+4f(0.5)+2f(1)+4f(1.5)+2f(2)+4f(2.5)+f(3)\right]$$

$$\approx 6.648327$$

So $\boxed{S = 6.648327}$

Answer 6P.

Consider the following figure:



As shown in the above figure, a semicircle with radius 1, horizontal diameter PQ, and tangent lines at P and Q.

Find at what height above the diameter the horizontal line should be placed so as to minimize the shaded area.

The equation of the circle with radius 1 is, $x^2 + y^2 = 1$ ...... (1)

Let the equation of the horizontal line is $y = c$, c is a constant ...... (2)

The equation (1) is written as,

$$x^2 + y^2 = 1$$
$$y^2 = 1 - x^2$$
$$y = \sqrt{1 - x^2}$$

Let $y_1 = \sqrt{1-x^2}, y_2 = c$.

Let,

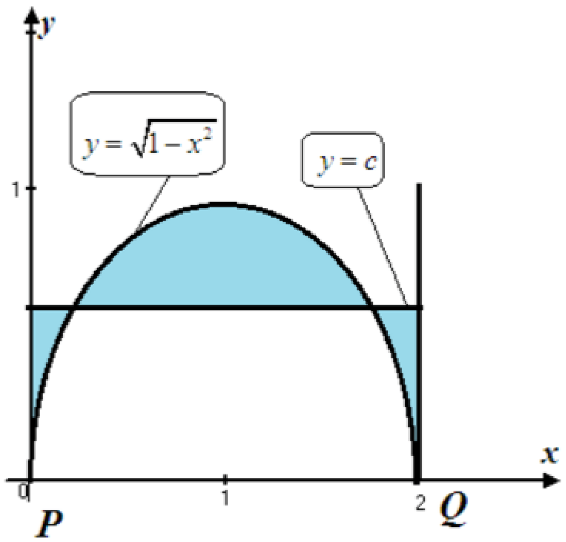
$$f(x) = y_1 - y_2$$
$$= \sqrt{1-x^2} - c$$

To minimize the shaded area, $f(x) = 0$.

$$\sqrt{1-x^2} - c = 0$$

That is, $c = \sqrt{1-x^2}$.

Here *c* is nothing but the height of the horizontal line above the diameter.



Answer 7CC.

(a) The cardiac output of the heart is the volume of blood pumped by the heart per unit time i.e; the rate of flow into the aorta.

(b) The dye dilution method is used to measure the cardiac output. Dye is injected into the right atrium and flows through the heart into the aorta. A probe inserted into the aorta measures the counteraction of the dye leaving the heart at equally spaced times over a time interval $[0, T]$ until the dye has cleared.

Let $c(t)$ be the concentration of the dye at time t.

If we divide $[0, T]$ into subintervals of equal length Δt, then the amount of dye that flows past the measuring point during the subinterval from $t = t_{i-1}$ to $t = t_i$ is approximately.

$$(\text{concentration})(\text{volume}) = c(t_i)(F\Delta t)$$

F is the rate of flow that we are trying to determine.

Thus the total amount of dye is approximately

$$\sum_{i=1}^{n} c(t_i) F \Delta t = F \sum_{i=1}^{n} c(t_i) \Delta t$$

And, letting $n \to \infty$, we find that the amount of dye is

$$A = F \int_0^T c(t)\, dt$$

Thus the cardiac output is given by

$F = A$ where the amount of dye A is known and the integral can be

$$\int_0^T c(t)\, dt$$

This can be approximated from the concentration readings.

Answer 7E.

We have $y = \int_1^x \sqrt{\sqrt{t}-1}\,dt$, $\qquad 1 \le x \le 16$

$$\Rightarrow \frac{dy}{dx} = \frac{d}{dx}\int_1^x \sqrt{\sqrt{t}-1}\,dt$$

Or $\quad \frac{dy}{dx} = \sqrt{\sqrt{x}-1} \qquad$ (By fundamental theorem part 1, $\frac{d}{dx}\int_1^x f(t)\,dt = f(x)$)

Then $\quad \left(\frac{dy}{dx}\right)^2 = \sqrt{x}-1$

$$\Rightarrow \sqrt{1+\left(\frac{dy}{dx}\right)^2} = \sqrt{\sqrt{x}}$$

$$= x^{1/4}$$

Length of the curve is

$$L = \int_1^{16} x^{1/4}\,dx \qquad \left[L = \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx\right]$$

$$= \left[\frac{4x^{5/4}}{5}\right]_1^{16}$$

$$= \frac{4}{5}\left[16^{5/4}-1\right]$$

$$= \frac{4}{5}\times\left[32-1\right]$$

$$= \frac{124}{5}$$

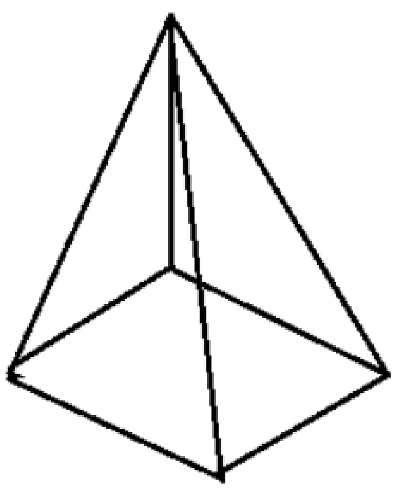
$$\Rightarrow \boxed{L = 124/5}$$

Answer 7P.

Consider the pyramid P with a square base of side $2b$, sphere S with its centre on the base of pyramid and it is tangent to all eight edges of the pyramid.

Find the height of the pyramid and volume of the intersection of the sphere and pyramid.

The shaped of pyramid with square base is as shown below.

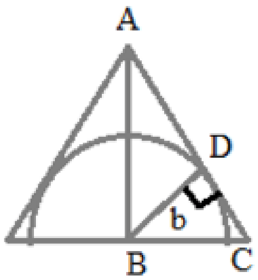


To find the height of the pyramid, we need to take up the triangle from the pyramid.

Observe the sphere is the centre of the base of the pyramid, the base of the pyramid is $2b$

This is also the diameter of the sphere.

Take the cross – section of the pyramid, passing through the top and through two opposite corners of the square base.



From the above diagram, the diameter of the sphere is $2b$ and the radius of the sphere is $BD = b$. The triangle ΔABD is isosceles triangle so the length of the side AD is b.

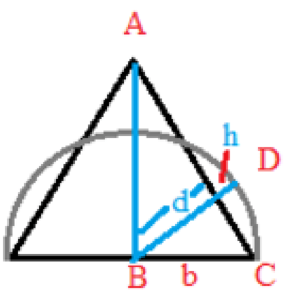
The height of AB is,

$$\begin{aligned} AB &= \sqrt{(AD)^2 + (BD)^2} \\ &= \sqrt{b^2 + b^2} \\ &= \sqrt{2b^2} \\ &= \sqrt{2}b \end{aligned}$$

Hence, the height of the pyramid is $\boxed{\sqrt{2}b}$.

To find the volume of the intersection of sphere and pyramid, we need to find the half the volume of sphere, find the sum of four equal volumes (cut of the triangular faces of the pyramid) and minus them.

Take the cross section of the pyramid, passing through the top and the midpoints of opposite sides of the square base.



First, find the value of d from the centre of the sphere to one of the triangular faces.

From the similar triangular,

$$\frac{d}{b}=\frac{AB}{AC} \quad ......(1)$$

The value of AC is,

$$\begin{aligned}AC&=\sqrt{(AB)^2+(BC)^2}\\&=\sqrt{\left(\sqrt{2}b\right)^2+(b)^2}\\&=\sqrt{2b^2+b^2}\\&=\sqrt{3}b\end{aligned}$$

Substitute the values $AB=\sqrt{2}b$ and $AC=\sqrt{3}b$ in equation – (1),

$$\begin{aligned}\frac{d}{b}&=\frac{AB}{AC}\\&=\frac{\sqrt{2}b}{\sqrt{3}b}\\&=\frac{\sqrt{6}}{3}\\d&=\frac{\sqrt{6}}{3}b\end{aligned}$$

From the diagram, the lengths of BD and BC are radius of the semi sphere i.e. $h+d=b$.

The value of h is,

$$\begin{aligned}h+d&=b\\h&=b-d\\&=b-\frac{\sqrt{6}}{3}b\\&=\frac{3-\sqrt{6}}{3}b\end{aligned}$$

We know that the volume (cut of the triangular faces of the pyramid) is $\pi h^2\left(r-\frac{h}{3}\right)$.

Substitute the values in the above volume formula,

$$\begin{aligned}\pi h^2\left(r-\frac{h}{3}\right)&=\pi\left(\frac{3-\sqrt{6}}{3}b\right)^2\left(b-\frac{3-\sqrt{6}}{9}b\right)\quad\left(\text{Since } h=\frac{3-\sqrt{6}}{3}b \text{ and } r=b\right)\\&=\pi\frac{5-2\sqrt{6}}{3}b^2\left(\frac{9-3+\sqrt{6}}{9}\right)b\\&=\pi\left(\frac{2}{3}-\frac{7\sqrt{6}}{27}\right)b^3\end{aligned}$$

The volume of the intersection of sphere and pyramid is,

$$\frac{1}{2}\begin{pmatrix}\text{volume}\\\text{of}\\\text{sphere}\end{pmatrix}-4\left(\text{Volume}\begin{pmatrix}\text{cut of the}\\\text{triangular faces of}\\\text{the pyramid}\end{pmatrix}\right)=\frac{1}{2}\left(\frac{4}{3}\pi b^3\right)-4\left(\pi\left(\frac{2}{3}-\frac{7\sqrt{6}}{27}\right)b^3\right)$$

$$=\left(\frac{4}{6}-\frac{8}{3}+\frac{28\sqrt{6}}{27}\right)\pi b^3$$

$$=\left(\frac{28\sqrt{6}}{27}-2\right)\pi b^3$$

Hence, the volume of the intersection of sphere and pyramid is $\boxed{\left(\frac{28\sqrt{6}}{27}-2\right)\pi b^3}$ cubic units.

Answer 8CC.

A probability density function f is a function on the domain of a continuous random variable X such that $\int_a^b f(x)\,dx$ measures the probability that X lies between a and b.

Such a function f has nonnegative values and satisfies the relation $\int_D f(x)\,dx = 1$, where D is the domain of the corresponding random variable X.

If $D = \mathbb{R}$ or if we define $f(x)=0$ for real numbers $x \notin D$ then $\int_{-\infty}^{\infty} f(x)\,dx = 1$

This can otherwise be followed as the total probability is 1

A probability density function f always satisfy $f(x) \geq 0 \, \forall x \in D$

Answer 8E.

We have $y=\int_1^x \sqrt{\sqrt{t}-1}\,dt$, $\quad 1 \leq x \leq 16$

Then $\frac{dy}{dx}=\frac{d}{dx}\int_1^x \sqrt{\sqrt{t}-1}\,dt$

$$\Rightarrow \frac{dy}{dx}=\sqrt{\sqrt{x}-1}$$

[By fundamental Theorem of calculus $\frac{d}{dx}\int_1^x f(t)\,dt = f(x)$]

Therefore

$$\Rightarrow \left(\frac{dy}{dx}\right)^2=\sqrt{x}-1$$

$$\Rightarrow 1+\left(\frac{dy}{dx}\right)^2=\sqrt{x}$$

$$\Rightarrow \sqrt{1+\left(\frac{dy}{dx}\right)^2}=x^{1/4}$$

Then $ds=\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$

$$= x^{1/4}dx$$

The area of the surface obtained by rotating the given curve about y-axis is

$$S=\int_1^{16} 2\pi x\, ds=\int_1^{16} 2\pi x . x^{1/4} dx$$

$$\Rightarrow S=2\pi\int_1^{16} x^{5/4} dx$$

$$\Rightarrow S=2\pi\left[\frac{4}{9}x^{9/4}\right]_1^{16}$$

$$=2\pi\left[\frac{4}{9}(16)^{9/4}-1.\frac{4}{9}\right]$$

$$\Rightarrow S=\frac{8\pi}{9}[512-1]$$

$$\Rightarrow \boxed{S=\frac{4088\pi}{9}}$$

Answer 9CC.

a) Suppose $f(x)$ is the probability function for the weight of a female college student, where x is measured in pounds.

$\int_0^{130} f(x)dx$ represents the probability that the weight of a randomly chosen female college students is less than 130 pounds.

(b) Mean of this density function

$$\mu=\int_{-\infty}^{\infty} x f(x)\, dx$$

$$=\int_0^{\infty} x f(x)\, dx$$

(c) The median of f is the number m such that $\int_m^{\infty} f(x)dx=\frac{1}{2}$

Answer 9E.

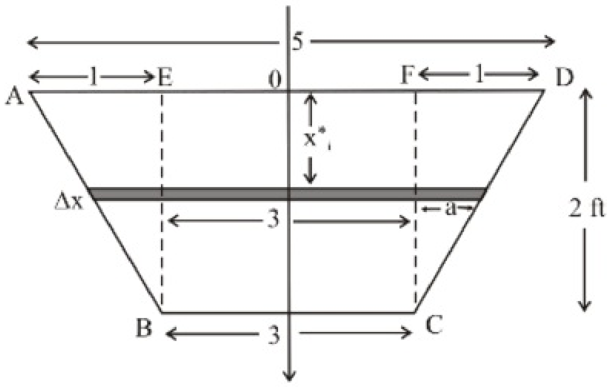


Fig. 1

We consider a vertical x – axis at the center of the gate such that origin is at the surface of water.

Let ABCD be the gate

We have $|EF|=|BC|=3\,ft$

And $|AD|=5\,ft=|EF|+|AE|+|FD|$

$=|EF|+2|AE|$ Since $[|AE|=|FD|]$

$=3+2|AE|$

$\Rightarrow |AE|=1\,ft$

Now we consider a strip with height Δx at the distance x_i^* from the origin.
By triangle property, we have

$$\frac{a}{(2-x_i^*)}=\frac{1}{2}$$

$$\Rightarrow a=\frac{1}{2}(2-x_i^*)$$

Then width of the strip $=3+2.a=3+(2-x_i^*)=(5-x_i^*)$

So area of the strip $A_i=(5-x_i^*)\Delta x$

And pressure on the strip $P_i=\delta d_i=62.5x_i^*$ since $\delta=62.5 \text{lb/ft}^3$

Force on the strip $F_i=P_i\ A_i$

Or $F_i=62.5x_i^*(5-x_i^*)\Delta x$

Then total force on the one side of the gate is

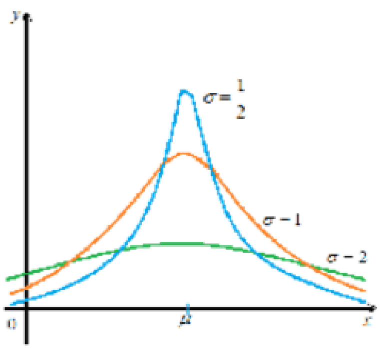
$$F=\lim_{n\to\infty}\sum_{i=1}^{n}62.5x_i^*(5-x_i^*)\Delta x$$
$$=\int_0^2 62.5x(5-x)dx$$
$$=62.5\int_0^2(5x-x^2)dx$$
$$=62.5\left[\frac{5x^2}{2}-\frac{x^3}{3}\right]_0^2$$
$$=62.5\left[10-\frac{8}{3}\right]$$
$$F=62.5\times\frac{22}{3}\text{ lb}$$
$$\Rightarrow \boxed{F\approx 458\ lb}$$

Answer 10CC.

The normal distribution is the probability distribution of a continuous random variable X, known as normal random variable or normal variable. It is given by

$$f(x)=\frac{1}{\sigma\sqrt{2\pi}}e^{\frac{-(x-\mu)^2}{2\sigma^2}}$$

The positive constant σ is called the standard deviation, it measures how spread out the values of X are.
From the bell-shaped graphs of members of the family in below figure, we see that for small values of σ the values of X are clustered about the mean, where as for larger values of σ the values of X are more spread out.



We see that for small values of σ the values of X are clustered about the mean, whereas for larger values of σ the values of X are more spread out.
So, μ occurs where σ is least.

Answer 10E.

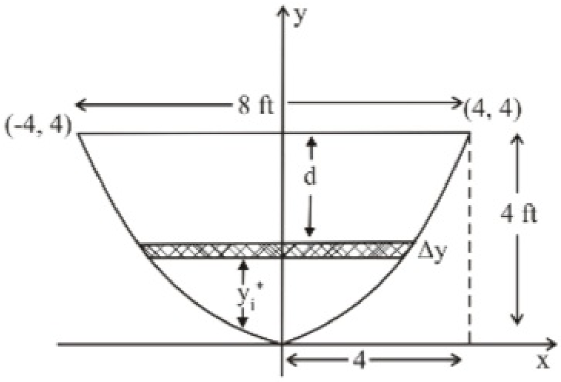


Fig. 1

We consider the vertex of the parabola at the origin.
Let equation of parabola be $y = ax^2$
Since (4, 4) is the point on the parabola so

$$4 = a(4)^2 \Rightarrow a = \frac{1}{4}$$

Thus the equation of the curve is $y = \frac{1}{4}x^2$

$$\Rightarrow x = \sqrt{4y}$$
$$= 2\sqrt{y} \qquad \text{..............(1)}$$

We consider a strip with height Δy at the distance y_i^* from the origin

Then width of the strip is $2\left(2\sqrt{y_i^*}\right) = 4\sqrt{y_i^*}$

Area of the strip is $A_i = 4\sqrt{y_i^*}\Delta y$

Pressure on the strip is $P_i = \delta . d_i$

$$= \delta\left(4 - y_i^*\right)$$

Force on the strip is $F_i = A_i \; P_i$

$$= \delta\left(4 - y_i^*\right).4\sqrt{y_i^*}\Delta y$$

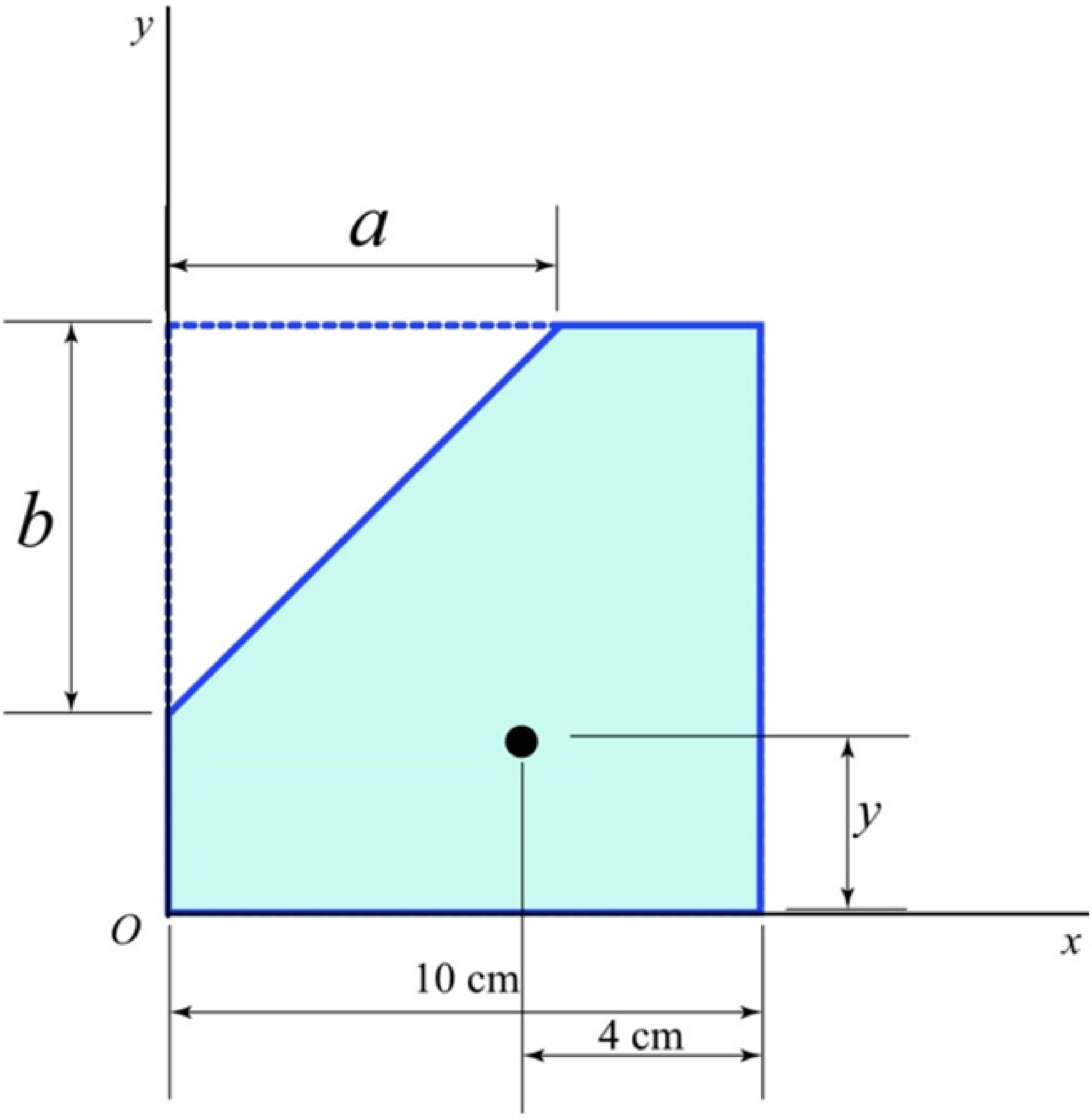
Total force on the end of the trough $F = \lim_{n\to\infty}\sum_{i=1}^{n} 4\delta\left(4 - y_i^*\right)\sqrt{y_i^*}\Delta y$

$$= \int_0^4 4\delta(4 - y)\sqrt{y}dy$$
$$= 4\delta\int_0^4\left(4y^{1/2} - y^{3/2}\right)dy$$
$$= 4\delta\left(\frac{8}{3}y^{3/2} - \frac{2}{5}y^{5/2}\right)_0^4$$
$$= 4\delta\left[\frac{8}{3}4^{3/2} - \frac{2}{5}4^{5/2}\right]$$
$$= 4\delta\left[\frac{64}{3} - \frac{64}{5}\right]$$
$$= 4\delta\left[\frac{320 - 192}{15}\right] = \frac{512\delta}{15}$$
$$= \frac{512\times 62.5}{15} \quad \left[\text{ since } \delta = 62.5 \text{ lb/ft}^3\right]$$

So $\boxed{F \approx 2133.3 \text{ lb}}$

Answer 10P.

Draw the figure represents the square, in which a part of triangle is cut,



Calculate the area of the square,

$$A_S = 10 \times 10$$
$$= 100 \text{ cm}^2$$

Calculate the area of the triangle,

$$A_T = \frac{1}{2} \times a \times b$$

Here, base of the triangle is *a*, and the height of the triangle is *b*.

Substitute 30cm^2 for A_T in the above equation,

$$A_T = \frac{1}{2} \times a \times b$$

$$30 = \frac{1}{2} \times a \times b$$

$$a \times b = 60 \quad \ldots\ldots (1)$$

Calculate the centroid of the shaded region from the origin of the above figure,

$$x = \frac{A_S \times x_1 - A_T \times x_2}{A_S - A_T}$$

Here, distance of the centroid of the square from the origin in *x*-direction is x_1, and distance of the centroid of the triangle from the origin in *x*-direction is x_2.

Substitute $6 \text{ cm for } x$, $(100 \text{ cm}^2) \text{ for } A_S$, $(30 \text{ cm}^2) \text{ for } A_T$, $5 \text{ cm for } x_1$, and $(a/3 \text{ cm}) \text{ for } x_2$ in the above equation,

$$6 = \frac{100 \times 5 - 30 \times \frac{a}{3}}{100 - 30}$$
$$a = 8 \text{ cm}$$

Substitute the respective values in the above equation (1),

$$8 \times b = 60$$
$$b = \frac{60}{8}$$
$$= 7.5 \text{ cm}$$

Calculate the centroid of the shaded region from the origin of the above figure,

$$y = \frac{A_S \times y_1 - A_T \times y_2}{A_S - A_T}$$

Here, distance of the centroid of the square from the origin in *y*-direction is y_1, and distance of the centroid of the triangle from the origin in *y*-direction is y_2.

Substitute $5 \text{ cm for } y_1$, $\left(10 - \frac{b}{3}\right) \text{ for } y_2$, and the respective values in the above equation,

$$y = \frac{100 \times 5 - 30 \times \left(10 - \frac{b}{3}\right)}{100 - 30}$$
$$= \frac{100 \times 5 - 30 \times \left(10 - \frac{7.5}{3}\right)}{100 - 30}$$
$$= 3.92 \text{ cm}$$

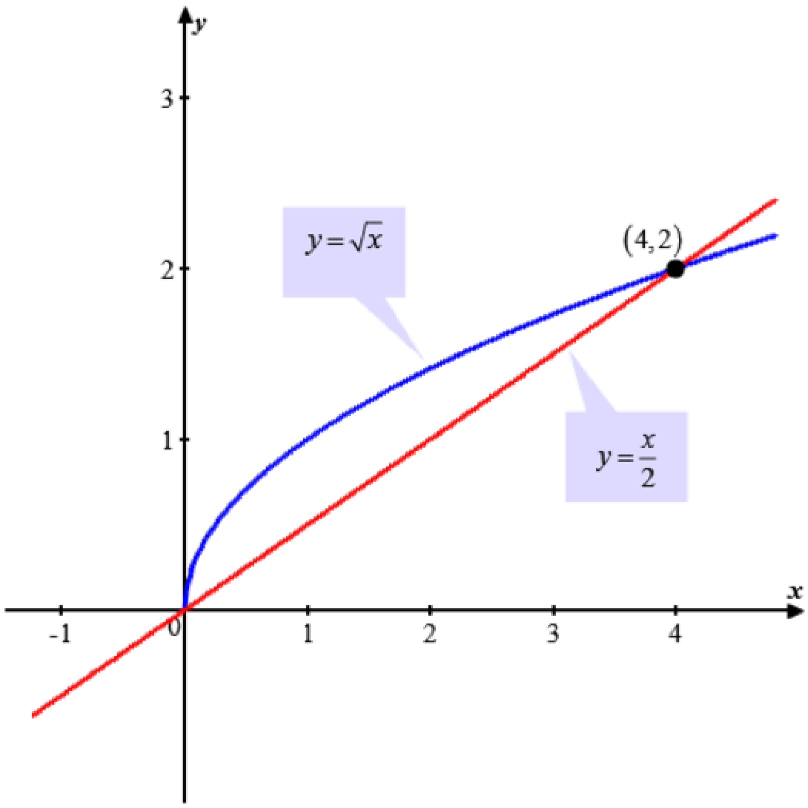
Therefore the centroid of the shaded region from the base is $\boxed{3.92 \text{ cm}}$.

Answer 11E.

Consider the curves $y=\frac{x}{2}$ and $y=\sqrt{x}$

Find the centroid of region bounded by these curves

Sketch the region bounded by the curves $y=\frac{x}{2}$ and $y=\sqrt{x}$



Recollect the formula for finding centroid of the region

$$(\bar{x},\bar{y})=\left(\frac{1}{A}\int_a^b x\left[f(x)-g(x)\right]dx,\frac{1}{A}\int_a^b \frac{1}{2}\left\{\left[f(x)\right]^2-\left[g(x)\right]^2\right\}dx\right)$$

Where A is the area of the region

Here $f(x)=\sqrt{x}$, $g(x)=\frac{x}{2}$, $a=0$, and $b=4$

First, find the area of region bounded by the curves $y=\frac{x}{2}$ and $y=\sqrt{x}$

$$A=\int_0^4\left(\sqrt{x}-\frac{x}{2}\right)dx$$

$$=\int_0^4\left(\sqrt{x}\right)dx-\int_0^4\left(\frac{x}{2}\right)dx$$

$$=\left[\frac{x^{3/2}}{3/2}-\frac{x^2}{4}\right]_0^4$$

$$=\left[\frac{2}{3}x^{3/2}-\frac{1}{4}x^2\right]_0^4$$

$$=\left[\frac{16}{3}-4\right]$$

$$=\frac{4}{3}$$

Thus, $A=\frac{4}{3}$

Now, find the $\bar{x}$ coordinates of centroid

$$\begin{aligned}\bar{x} &= \frac{1}{A}\int_a^b x\left[f(x)-g(x)\right]dx \\ &= \frac{3}{4}\int_0^4 x\left[\sqrt{x}-\frac{x}{2}\right]dx \\ &= \frac{3}{4}\int_0^4\left[x^{3/2}-\frac{x^2}{2}\right]dx \\ &= \frac{3}{4}\left[\frac{2}{5}x^{5/2}-\frac{1}{6}x^3\right]_0^4 \\ &= \frac{3}{4}\left[\frac{64}{5}-\frac{64}{6}\right] \\ &= \frac{8}{5}\end{aligned}$$

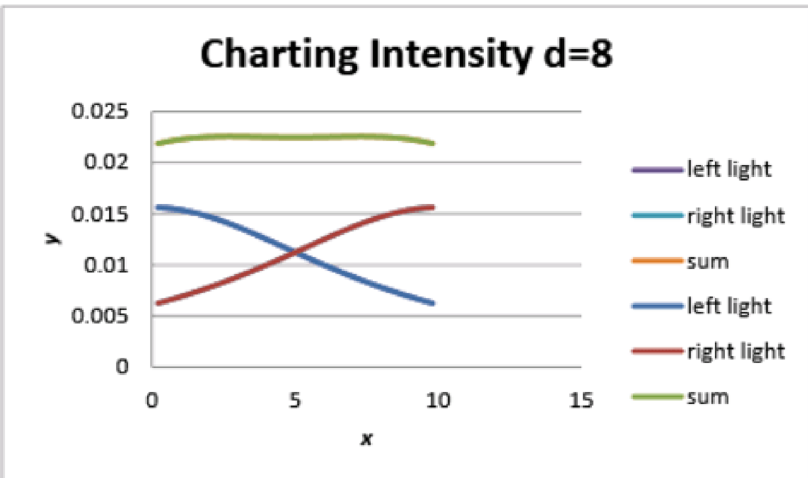
Thus, $\bar{x}=\frac{8}{5}$

Now, find the $\bar{y}$ coordinates of centroid

$$\begin{aligned}\bar{y} &= \frac{1}{A}\int_a^b \frac{1}{2}\left\{\left[f(x)\right]^2-\left[g(x)\right]^2\right\}dx \\ &= \frac{3}{4}\int_0^4 \frac{1}{2}\left\{\left[\sqrt{x}\right]^2-\left[\frac{x}{2}\right]^2\right\}dx \\ &= \frac{3}{8}\int_0^4\left\{x-\frac{x^2}{4}\right\}dx \\ &= \frac{3}{8}\left[\frac{x^2}{2}-\frac{x^3}{12}\right]_0^4 \\ &= \frac{3}{8}\left[\frac{16}{2}-\frac{64}{12}\right] \\ &= 1\end{aligned}$$

Therefore the centroid is $\boxed{\left(\bar{x},\bar{y}\right)=\left(\frac{8}{5},1\right)}$

For $d=8$, graph is as follows.



Now find the exact value of d which is a minimum. Take the derivative of the intensity function with respect to x. Then find the second derivative, to see where d has a sudden change, and a point of inflection.

$$I = \frac{F_1}{x_1^2 + d^2} + \frac{F_1}{100 + x_1^2 - 20x_1 + d^2}$$

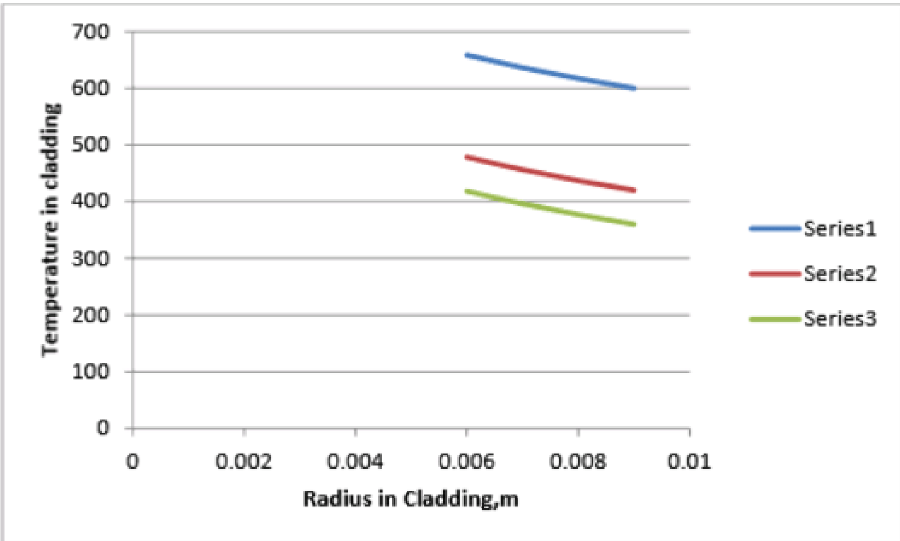
$$I'(x,d) = \frac{-2x_1F_1}{(x_1^2 + d^2)^2} - \frac{2F_1(x_1 - 10)}{((x_1 - 10)^2 + d^2)^2}$$

$$I''(x,d) = \frac{+4x_1^2F_1}{(x_1^2 + d^2)^3} + \frac{4F_1(x_1 - 10)^2}{((x_1 - 10)^2 + d^2)^3}$$

Continue to solve for d.

$$\boxed{0 = \frac{x_1^2}{(x_1^2 + d^2)^3} + \frac{(x_1 - 10)^2}{((x_1 - 10)^2 + d^2)^3}}$$

Plot the temperature distribution in the cladding for different values of heat transfer coefficient,



Answer 11P.

Consider y be the distance from the southern end of the needle to the nearest line to the north.

Let θ be the angle that the needle makes with ray extending eastward from the southern end.

Then $0 \le y \le L$ and $0 \le \theta \le \pi$.

The needle intersects one of the lines only when $y < h\sin\theta$.

The rectangular region is $0 \le y \le L$, $0 \le \theta \le \pi$.

Find the probability that the needle will intersect a line if $h = L$ and if $h = \frac{1}{2}L$.

If $h = L$, the probability is,

$$P = \frac{\text{area under } y = h\sin\theta}{\text{area of rectangle}}$$

$$= \frac{\text{area under } y = L\sin\theta}{\text{area of rectangle}}$$

$$= \frac{\int_0^{\pi} L\sin\theta}{\pi L}$$

$$= \frac{L\left[-\cos\theta\right]_0^{\pi}}{\pi L}$$

The above is simplified to,

$$= \frac{\left[-\cos\pi - \left(-\cos 0\right)\right]}{\pi}$$

$$= \frac{-(-1)-(-1)}{\pi}$$

$$= \frac{1+1}{\pi}$$

$$= \frac{2}{\pi}$$

Therefore, if $h = L$, the probability is, $\boxed{\frac{2}{\pi}}$.

If $h = \frac{1}{2}L$, the probability is,

$$P = \frac{\text{area under } y = h\sin\theta}{\text{area of rectangle}}$$

$$= \frac{\text{area under } y = \frac{1}{2}L\sin\theta}{\text{area of rectangle}}$$

$$= \frac{\int_0^{\pi} \frac{L}{2}\sin\theta}{\pi L}$$

$$= \frac{\frac{L}{2}\left[-\cos\theta\right]_0^{\pi}}{\pi L}$$

The above is simplified to,

$$= \frac{1}{2}\left(\frac{[-\cos\pi - (-\cos 0)]}{\pi}\right)$$

$$= \frac{1}{2}\left(\frac{-(-1)-(-1)}{\pi}\right)$$

$$= \frac{1}{2}\left(\frac{1+1}{\pi}\right)$$

$$= \frac{1}{2}\left(\frac{2}{\pi}\right)$$

$$= \frac{1}{\pi}$$

Therefore, if $h = \frac{1}{2}L$, the probability is, $\boxed{\frac{1}{\pi}}$.

Answer 12E.

We have equation of the curve $y = \sin x$

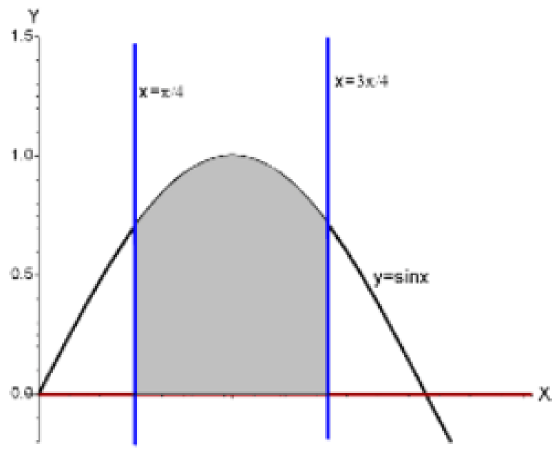


Fig.1

Area of the bounded region is

$$A = \int_{\pi/4}^{3\pi/4} \sin x \, dx = [-\cos x]_{\pi/4}^{3\pi/4}$$

$$= \left[-\cos\frac{3\pi}{4} + \cos\frac{\pi}{4}\right]$$

$$= \left[-\left(\frac{-1}{\sqrt{2}}\right) + \frac{1}{\sqrt{2}}\right]$$

$$= 2/\sqrt{2}$$

$$= \sqrt{2}$$

Let $\quad f(x) = \sin x$

Then $\quad \Rightarrow \bar{x} = \frac{1}{A}\int_a^b x f(x)\,dx$

$$= \frac{1}{\sqrt{2}}\int_{\pi/4}^{3\pi/4} x\sin x\,dx$$

$$= \frac{1}{\sqrt{2}}\left[\left[-x\cos x\right]_{\pi/4}^{3\pi/4} + \int_{\pi/4}^{3\pi/4}\cos x\,dx\right] \quad \text{(Integration by part)}$$

$$= \frac{1}{\sqrt{2}}\left[\left[-\frac{3\pi}{4}\cos\frac{3\pi}{4} + \frac{\pi}{4}\cos\frac{\pi}{4}\right] + \left[\sin x\right]_{\pi/4}^{3\pi/4}\right]$$

$$= \frac{1}{\sqrt{2}}\left[-\frac{3\pi}{4}\left(-\frac{1}{\sqrt{2}}\right) + \frac{\pi}{4}\left(\frac{1}{\sqrt{2}}\right) + \left[\sin\frac{3\pi}{4} - \sin\frac{\pi}{4}\right]\right]$$

$$= \frac{1}{\sqrt{2}}\left[\frac{3\pi}{4\sqrt{2}} + \frac{\pi}{4\sqrt{2}}\right]$$

$$= \frac{4\pi}{8}$$

$$= \frac{\pi}{2}$$

Now $\quad \bar{y} = \frac{1}{A}\int_a^b \frac{1}{2}\{f(x)\}^2\,dx$

$$= \frac{1}{\sqrt{2}}\int_{\pi/4}^{3\pi/4} \frac{1}{2}\sin^2 x\,dx$$

Since $\cos 2\theta = 1 - 2\sin^2\theta \Rightarrow \sin^2\theta = \frac{1-\cos 2\theta}{2}$

So $\quad \bar{y} = \frac{1}{4\sqrt{2}}\int_{\pi/4}^{3\pi/4}(1-\cos 2x)\,dx$

$$= \frac{1}{4\sqrt{2}}\left[x - \frac{\sin 2x}{2}\right]_{\pi/4}^{3\pi/4}$$

$$= \frac{1}{4\sqrt{2}}\left[\frac{3\pi}{4} - \frac{1}{2}\sin\frac{3\pi}{2} - \frac{\pi}{4} + \frac{1}{2}\sin\frac{\pi}{2}\right]$$

$$= \frac{1}{4\sqrt{2}}\left[\frac{2\pi}{4} + 1\right]$$

$$= \frac{1}{8\sqrt{2}}(\pi + 2)$$

Thus centroid is $\boxed{\left(\frac{\pi}{2}, \frac{1}{8\sqrt{2}}(\pi+2)\right)}$

Answer 13E.

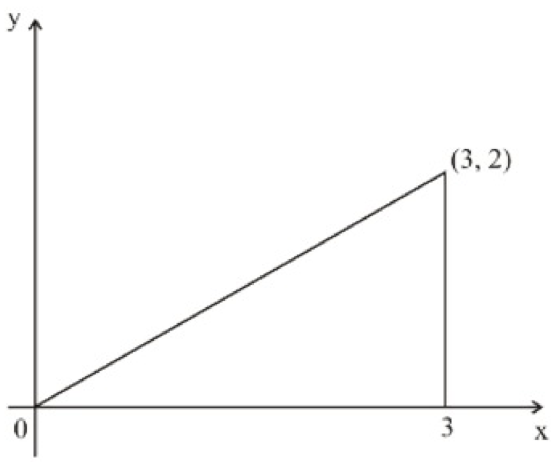


Fig. 1

Equation of the line passing through the points (0, 0) and (3, 2) is

$$(y-0)=\frac{(2-0)}{(3-0)}(x-0)$$

$$\Rightarrow y=\frac{2}{3}x$$

Let $f(x)=\frac{2}{3}x$

Area of the triangle $A=\frac{1}{2}\times \text{base}\times \text{height}$

$$=\frac{1}{2}\times 3\times 2=3$$

Then $\bar{x}=\frac{1}{A}\int_0^3 x f(x)dx=\frac{1}{3}\int_0^3 \frac{2}{3}x^2\,dx$

$$=\frac{2}{9}\int_0^3 x^2\,dx$$

$$=\frac{2}{9}\left[\frac{x^3}{3}\right]_0^3=\frac{2}{9}\times 9$$

$$\Rightarrow \bar{x}=2$$

And $\bar{y}=\frac{1}{A}\int_0^3 \frac{1}{2}\left[f(x)\right]^2 dx$

$$=\frac{1}{6}\int_0^3\left(\frac{2}{3}x\right)^2 dx$$

$$=\frac{4}{54}\int_0^3 x^2\,dx$$

$$=\frac{4}{54}\left[\frac{x^3}{3}\right]_0^3$$

$$=\frac{4}{54}\times 9=\frac{4}{6}=\frac{2}{3}$$

The centroid is $\boxed{(2,\ 2/3)}$

Answer 14E.

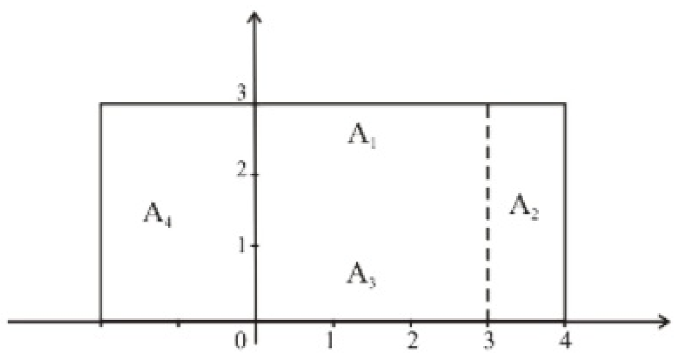


Fig. 1

For the rectangle labeled A_1

Area of the rectangle $A_1 = 3\times1 = 3$

This is the area bounded by lines $y = 3$ and $y = 2$ from $x = 0$ to $x = 3$

$$\begin{aligned}\bar{x}_1 &= \frac{1}{A}\int_0^3 x(3-2)\,dx \\ &= \frac{1}{3}\int_0^3 x\,dx \\ &= \frac{1}{3}\left[\frac{x^2}{2}\right]_0^3 \\ &= \frac{3}{2}\end{aligned}$$

And $$\begin{aligned}\bar{y}_1 &= \frac{1}{A}\int_0^3 \frac{1}{2}\left(3^2-2^2\right)dx \\ &= \frac{1}{3}\cdot\frac{5}{2}[3-0] \\ &= \frac{5}{2}\end{aligned}$$

For the rectangle labeled A_1, centroid is $(3/2,\ 5/2)$

For the rectangle leveled A_2

Area of $A_2 = 3\times1 = 3$

This is the area bounded by lines $y = 3$ and $y = 0$ from $x = 3$ and $x = 4$

$$\begin{aligned}\bar{x}_2 &= \frac{1}{A}\int_3^4 x(3)\,dx \\ &= \left[\frac{x^2}{2}\right]_3^4 \\ &= \frac{7}{2}\end{aligned}$$

$$\begin{aligned}\bar{y}_2 &= \frac{1}{A}\int_3^4 \frac{1}{2}(9)\,dx \\ &= \frac{9}{6}[4-3] \\ &= \frac{3}{2}\end{aligned}$$

For the rectangle leveled A_2, centroid is $(7/2,\ 3/2)$

Similarly

For the rectangle leveled A_3, area of $A_3 = 3\times1 = 3$

And for the rectangle leveled A_4, area of $A_4 = 3\times2 = 6$

The centroid of the rectangles A_3 and A_4 are $\left(\frac{3}{2},\frac{1}{2}\right)$ and $\left(-1,\frac{3}{2}\right)$

Then we find the common centroid of the total region

$$\begin{aligned}\bar{x} &= \frac{\bar{x}_1.A_1+\bar{x}_2.A_2+\bar{x}_3A_3+\bar{x}_4A_4}{A_1+A_2+A_3+A_4} \\ &= \frac{\frac{3}{2}\times3+\frac{7}{2}\times3+\frac{3}{2}\times3+(-1)\times6}{15} \\ &= \frac{9+21+9-12}{30} \\ &= \frac{27}{30} \\ &= \frac{9}{10}\end{aligned}$$

And $\bar{y} = \dfrac{\bar{y}_1 A_1 + \bar{y}_2 A_2 + \bar{y}_3 A_3 + \bar{y}_3 A_4}{A_1 + A_2 + A_3 + A_4}$

$$= \frac{\frac{5}{2}\times 3 + \frac{3}{2}\times 3 + \frac{1}{2}\times 3 + \frac{3}{2}\times 6}{15}$$

$$= \frac{15+9+3+18}{30}$$

$$= \frac{45}{30}$$

$$= \frac{3}{2}$$

Then centroid is $\boxed{\left(\frac{9}{10}, \frac{3}{2}\right)}$

Answer 15E.

Center of the circle is (1, 0)
Since centroid of any circle be at the center of the circle
So centroid of the circle is (1, 0)
Then $\bar{x} = 1$
Then distance traveled by the centroid during one rotation about y-axis is

$$d = 2\pi\bar{x}$$
$$= 2\pi$$

Area of the circle with radius 1 is $A = \pi(1)^2$
$= \pi$
Then by the Theorem of Pappus, the volume of the sphere obtained by rotating the circle about y-axis is

$$V = Ad$$
$$= \pi(2\pi)$$
$$= 2\pi^2$$
$$\Rightarrow \boxed{V = 2\pi^2}$$

Answer 16E.

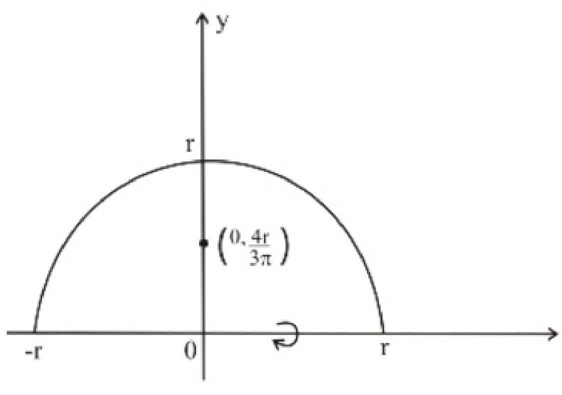


Fig. 1

The equation of semicircle is $y = \sqrt{r^2 - x^2}$
The center of the semicircle is at the origin. If we rotate this semicircle about x- axis, we get a sphere with radius r.
Let the centroid of the semicircle be $(\bar{x}, \bar{y})$
The distance traveled by centroid during one rotation about x-axis is $d = 2\pi\bar{y}$
And the area of semicircle $A = \frac{\pi}{2}(r^2) = \frac{\pi r^2}{2}$

Then by the theorem of Pappus, volume of the resulting sphere is

$$V = A.d = \frac{\pi r^2}{2} \times 2\pi \bar{y} = \pi^2 r^2 \bar{y}$$

But we have been given that the volume of the sphere is $\frac{4}{3}\pi r^3$

So $\quad \pi^2 r^2 \bar{y} = \frac{4}{3}\pi r^3$

$$\Rightarrow \bar{y} = \frac{4\pi r^3}{3\pi^2 r^2} = \frac{4r}{3\pi}$$

$$\Rightarrow \boxed{\bar{y} = 4r/3\pi}$$

And $\quad \bar{x} = \frac{1}{A}\int_{-r}^{r} x\sqrt{r^2 - x^2}\,dx = 0$

(Since $f(x) = x\sqrt{r^2 - x^2}$ is an odd function so $\int_{-a}^{a} f(x)\,dx = 0$)

Thus centroid is $\boxed{\left(0,\ \frac{4r}{3\pi}\right)}$

Answer 17E.

We have, demand function $p = 2000 - 0.1x - 0.01x^2$

Sales level is $x = 100$

Then price is $P = p(100) = 2000 - 0.01\times 100 - 0.01\times(100)^2$

$$= 2000 - 10 - 100$$
$$= 2000 - 110$$
$$= \$1890$$

Therefore consumer surplus is $= \int_0^{100}\left[p(x) - P\right]dx$

$$= \int_0^{100}\left(2000 - 0.1x - 0.01x^2 - 1890\right)dx$$
$$= \int_0^{100}\left(110 - 0.1x - 0.01x^2\right)dx$$
$$= \left[110x - \frac{0.1}{2}x^2 - \frac{0.01}{3}x^3\right]_0^{100}$$

Therefore the consumer surplus is

$$= \left[11000 - \frac{0.1\times100\times100}{2} - \frac{0.01\times100\times100\times100}{3}\right]$$
$$= \left[11000 - 500 - \frac{10000}{3}\right]$$
$$= 10500 - \frac{10000}{3}$$
$$= \frac{31500 - 10000}{3}$$
$$= \frac{21500}{3}$$
$$\boxed{\approx \$7166.67}$$

Answer 18E.

The readings of dye concentration at two-second intervals are given as

t	0	2	4	6	8	10	12	14	16	18	20	22	24
c(t)	0	1.9	3.3	5.1	7.6	7.1	5.8	4.7	3.3	2.1	1.1	0.5	0

We have $\Delta t = 2$

Interval is $[0,\ 24]$, $A = 6mg$

So we can approximate the integral $\int_0^{24} c(t)dt$ as follows

$$\int_0^{24} c(t)dt \approx \frac{\Delta t}{3}\left[c(0)+4c(2)+2c(4)+4c(6)+2c(8)+4c(10)+2c(12)\right.$$
$$\left.+4c(14)+2c(16)+4c(18)+2c(20)+4c(22)+c(24)\right]$$
$$=\frac{2}{3}[0+4\times1.9+2\times3.3+4\times5.1+7.6\times2+4\times7.1+2\times5.8+4\times4.7+2\times3.3$$
$$+4\times2.1+2\times1.1+4\times0.5+0]$$
$$=\frac{2}{3}[7.6+6.6+20.4+15.2+28.4+11.6+18.8+6.6+8.4+2.2+3]$$
$$=\frac{2}{3}\times(127.8)\approx 85.2 \text{ mg. s/L}$$

Then cardiac output $F=\dfrac{A}{\int_0^{24} c(t)dt}\approx\dfrac{6}{85.2}$

$$=0.0704 \text{ L/s}=4.225 \text{ L/min}$$

$$\boxed{F=4.225 \text{ L/min}}$$

Answer 19E.

(A) We have the function $f(x)=\begin{cases}\dfrac{\pi}{20}\sin\left(\dfrac{\pi x}{10}\right) & \text{if } 0\le x\le 10\\ 0 & \text{if } x<0 \text{ or } x>10\end{cases}$

Since $\sin x\ge 0$ for $0\le x\le\pi$

So $\sin\left(\dfrac{\pi x}{10}\right)\ge 0$ for $0\le x\le 10$

And so $f(x)\ge 0$ for all x

$$\text{Now} \int_0^{10} f(x)dx=\int_0^{10}\frac{\pi}{20}\sin\left(\frac{\pi x}{10}\right)dx$$
$$=\frac{\pi}{20}\int_0^{10}\sin\left(\frac{\pi x}{10}\right)dx$$
$$=\frac{\pi}{20}\left[-\frac{\cos\left(\frac{\pi x}{10}\right)}{\left(\frac{\pi}{10}\right)}\right]_0^{10}$$
$$=\frac{\pi}{20}\left[-\frac{10}{\pi}\cos(\pi)+\frac{10}{\pi}\cos 0\right]$$
$$=\frac{\pi}{20}\left[-\frac{10}{\pi}(-1)+\frac{10}{\pi}\right]$$
$$=\frac{\pi}{20}\times\frac{20}{\pi}$$
$$=1$$

Thus $f(x)$ is a probability density function

(C) Mean $\mu = \int_{-\infty}^{\infty} x\ f(x)dx$

$$= \int_0^{10} \frac{x\pi}{20} \sin\left(\frac{\pi x}{10}\right) dx$$

$$= \frac{\pi}{20} \int_0^{10} x \sin\left(\frac{\pi x}{10}\right) dx$$

$$= \frac{\pi}{20}\left[\left[-\frac{x\cos(\pi x/10)}{(\pi/10)}\right]_0^{10} + \int_0^{10} \frac{10}{\pi}\cos\left(\frac{\pi x}{10}\right)dx\right] \quad \text{(Integration by part)}$$

$$= \frac{\pi}{20}\left[\left\{-10\cos(\pi)\times\frac{10}{\pi}\right\} + \frac{10}{\pi}\left[\frac{\sin(\pi x/10)}{(\pi/10)}\right]_0^{10}\right]$$

$$= \frac{\pi}{20}\left[\left(\frac{100}{\pi}\right) + \frac{100}{\pi^2}\left[\sin(\pi) - \sin 0\right]\right]$$

$$= \frac{\pi}{20}\left[\frac{100}{\pi}\right]$$

$$= 5$$

$$\boxed{\mu = 5}$$

This is same as our expectation since the graph of *f(x)* is symmetric about the line $x = 5$.

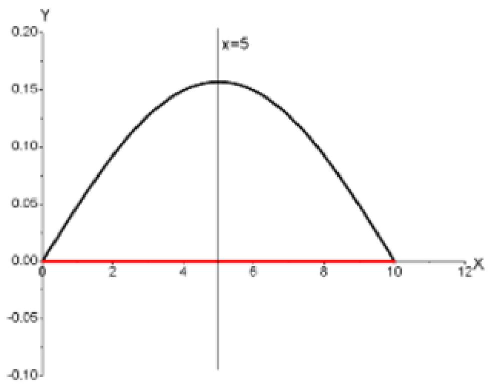


Fig.1

Answer 20E.

We have $\mu = 268$ and $\sigma = 15$

The probability of the pregnancies between 250 days and 280 days is

$$P(250 \le x \le 280) = \int_{250}^{280} \frac{1}{15\sqrt{2\pi}} e^{-(x-268)^2/(2\times 15^2)} dx$$

Let $f(x) = \dfrac{e^{(x-268)^2/(2\times 15^2)}}{15\sqrt{2\pi}}$

Intervals is $[250, 280]$,

Taking $n = 30$, we have $\Delta x = \dfrac{280-250}{30} = 1$

Then subintervals are $[250, 251], [251, 252], , ..., [279, 280]$

We can approximate the probability by Simpson's rule as

$$P(250 \le x \le 280) \approx \frac{1}{3}\left[\begin{array}{l} f(250) + 4f(251) + 2f(252) + 4f(253) + ... \\ \qquad .... + 4f(279) + f(280) \end{array}\right]$$

$$\approx 0.673075$$

Thus percentage of pregnancies between 250 days and 280 days is $\boxed{\approx 67.3\%}$

Answer 21E.

(A) We have been given $\mu = 8$

Then probability density function becomes

$$f(t) = \begin{cases} 0 & \text{if } t < 0 \\ \frac{1}{8}e^{-t/8} & \text{if } t \geq 0 \end{cases}$$

Waiting time interval is [0, 3]

So the probability that a customer is served in the first 3 minutes is

$$P(t \leq 3) = \int_0^3 \frac{1}{8}e^{-t/8}dt$$
$$= \frac{1}{8}\left[-8e^{-t/8}\right]_0^3$$
$$= \left[-e^{-3/8} + e^0\right]$$
$$= \boxed{1 - e^{-3/8}}$$
$$\boxed{P(t \leq 3) \approx 0.3127}$$

(B) The probability that a customer has to wait more than 10 minutes

$$P(t > 10) = \int_{10}^{\infty} \frac{1}{8}e^{-t/8}dt = \lim_{x\to\infty}\int_{10}^{x} \frac{1}{8}e^{-t/8}dt$$
$$= \lim_{x\to\infty}\left[-e^{-t/8}\right]_{10}^{x}$$
$$= \lim_{x\to\infty}\left[-e^{-x/8} + e^{-10/8}\right]$$
$$= e^{-5/4} - \lim_{x\to\infty} e^{-x/8}$$
$$= e^{-5/4} - 0$$
$$\Rightarrow \boxed{P(t > 10) = e^{-5/4} \approx 0.2865}$$

(C) If median is m

Then $\int_m^{\infty} f(t)\,dt = \frac{1}{2}$

$$\Rightarrow \int_m^{\infty} \frac{1}{8}e^{-t/8}dt = \frac{1}{2}$$
$$\Rightarrow \lim_{x\to\infty}\int_m^{x} \frac{1}{8}e^{-t/8}dt = \frac{1}{2}$$
$$\Rightarrow \lim_{x\to\infty}\left[-e^{-t/8}\right]_m^{x} = \frac{1}{2}$$
$$\Rightarrow \lim_{x\to\infty}\left[-e^{-x/8} + e^{-m/8}\right] = \frac{1}{2}$$
$$\Rightarrow e^{-m/8} - 0 = \frac{1}{2}$$
$$\Rightarrow -m/8 = \ln(1/2)$$
$$\Rightarrow -m = -8\ln 2$$
$$\Rightarrow \boxed{m = 8\ln 2 \approx 5.55 \quad \text{minutes}}$$