CHAPTER

04

Differential Equations

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A differential equation can simply be said to be an equation involving derivatives of an unknown function. For example, consider the equation

$$\frac{dy}{dx}+xy=x^2$$

This is a differential equation since it involves the derivative of the funtion $y(x)$ which we may wish to determine. We must first understand why and how differnetial equations arise and why we need them at all. In general, we can say that a differential equation describes the behaviour of some continuously varying quantity.

Scenario 1 : A Freely Falling Body

A body is release at rest from a heigh h. How do we described the motion of this body?

The height x of the body is a function of time. Since the acceleration of the body is g, we have $\frac{d^2x}{dt^2}=g$

This is the differential equation describing the motion of the body. Along with the initial condition $x(0)=h$, it completely describes the motion of the body at all instants after the body starts falling.

Scenario 2 : Radioactive disintegration

Experimental evidence shows that the rate of decay of any ratioactive substance is proportional to the amount of the substance present,

i.e. $$\frac{dm}{dt}=-\lambda m$$

where m is the mass of the radioactive substance and a function of t. If we know $m(0)$, the initial mass, we can use this differential equation to determine the mass of the substance remaining at any later time instant.

Scenario 3 : Population Growth

The growth of population (of say, a biological culture) in a closed environment is dependent on the birth and death rates. The birth rate will contribute to increaseing the population while the death rate will contribute to its decrease. It has been found that for low populations, the birth rate is the dominant influence in population growth and the growth rate is linearly dependent on the current population. For high populations, there is a competition among the population for the limited resources available, and thus death rate becomes dominant. Also, the death rate shows a quadratic dependence on the current population.

Thus, if $N(t)$ represents the population at time t, the different equation describing the population variation is of the form

$$\frac{dN}{dt}=\lambda_1 N-\lambda_2 N^2$$

where λ_1 and λ_2 are constants.

Along with the initial population $N(0)$, this equation can tell us the population at any later time instant.

Session 1

Solution of a Differential Equation

These three examples should be sufficient for you to realise why and how differential equations arise and why they are important.

In all the three equations mentioned above, there is only independent variable (the time t in all the three cases). Such equations are termed **ordinary differential equations.** We might have equations involving more than one independent variable :

$$\frac{\partial f}{\partial x}+x\frac{\partial f}{\partial y}=x^2$$

where the notation $\frac{\partial}{\partial x}$ stands for the partial derivative, i.e. the term $\frac{\partial f}{\partial x}$ would imply that we differentiate the function f with respect to the independent variable x as the variable (while treating the other independent variable y as a constant). A similar interpretation can be attached to $\frac{\partial}{\partial y}$.

Such equations are termed **partial differential equations** but we shall not be concerned with them in this chapter.

Consider the ordinary differnetial equation

$$\frac{d^2y}{dx^2}+x\frac{dy}{dx}+x^2=c$$

The order of the highest derivative present in this equation is two; thus we shall call it a second order differential equation (*DE*, for convenience).

The **order** of a *DE* is the order of the highest derivative that occurs in the equation

Again, consider the *DE*

$$\frac{d^2y}{dx^3}+\frac{dy}{dx}=x^2y^2$$

The degree of the highest order derivative in this DE is two, so this is a DE of degree two (and order three).

The degree of a DE is the degree of the highest order derivative that occurs in the equation, **when all the derivatives in the equation are made of free of fractional powers.**

$$\sqrt{\left(\frac{dy}{dx}\right)^2-1+x\left(\frac{d^2y}{dx^2}\right)}=k$$

is not of degree two. When we make this equation free of fractional powers, by the following rearrangement,

$$\left(\frac{dy}{dx}\right)^2-1+\left\{k-x\left(\frac{d^2y}{dx^2}\right)^2\right\}^2$$

we see that the degree of the highest order derivative will become four. Thus, this is a DE of degree four (and order two).

Finally, an n^{th} **linear DE** (degree one) is an equation of the form

$$a_0\frac{d^ny}{dx^n}+a_1\frac{d^{n-1}}{dx^{n-1}}+\ldots+a_{n-1}\frac{dy}{dx}+a_ny=b$$

where the $a_i's$ and b are functions of x.

Solving an n^{th} order DE to evaluate the unknown function will essentially consists of doing n integrations on the DE. Each integration step will introduce an arbitrary constant. Thus, you can expect in general that the **solution of an n^{th} order DE will contain n independnet arbitrary constants.**

By n independent constants, we mean to say that the most general solution of the DE cannot be expressed in fewer that n constants. As an example, the second order DE

$$\frac{d^y}{dx^2}+y=0$$

has its most general solution of the form

$$y=A\cos x+B\sin x. \quad \ldots(1)$$

(verify that this is a solution by explicit substitution).

Thus, two arbitrary and independent constants must be included in the general solution. We cannot reduce (1) to a relation containing only one arbitrary constant. On the other hand, it can be verified that the function

$$y=ae^{x+b}$$

is a solution to the second-order DE

$$\frac{d^2y}{dx^2}=y$$

but even through it (seems to) contain two arbitrary constants, it is not the general solution to this DE. This is because it can be reduced to a relation involving only one arbitrary constant :

$$y=ae^{x+b}=ae^x.e^b=ce^x \quad (\text{where } c=a\cdot e^b)$$

Let us summarise what we have seen till now : the most general solution of an n^{th} order DE will consist of n orbitrary constants; conversely, from a functional relation involving n arbitrary constants, an n^{th} order DE can be generated (we shall soon see how to do this). We are generally interested in solutions of the DE satisfying some particular constraints (say, some initial values). Since the most general solution of the DE involves n arbitrary constant, we see that the maximum member of independent conditions which can be imposed on a solution of the DE is n. As a first example, consider the functional relation

$$y=x^2+c_1e^{2x}+c_2e^{3x} \quad \ldots(1)$$

This curve's equation contains two arbitrary constants; as we vary c_1 and c_2, we obtain different curves; those curves constitute a family of curves. All members of this family will satisfy the DE that we can generate from this general relation; this DE will be second order since the relation contains two arbitrary constants.

We now see how to generate the DE. Differentiate the given relation twice to obtain

$$y'=2x+2c_1e^{2x}+3c_2e^{3x} \quad (2)$$

$$y''=2+4c_1e^{2x}+9c_2e^{3x} \quad \ldots(3)$$

From (1), (2) and (3), c_1 and c_2 can be eliminated to obtain

$$\begin{vmatrix} e^{2x} & e^{3x} & x^2-y \\ 2e^{2x} & 3e^{3x} & 2x-y' \\ 4e^{2x} & 9e^{3x} & 2-y'' \end{vmatrix}$$

$$\Rightarrow \quad \begin{vmatrix} 1 & 1 & x^2-y \\ 2 & 3 & 2x-y' \\ 4 & 9 & 2-y'' \end{vmatrix}$$

$$\Rightarrow 6-3''-18x+9y'+8x-4y'-4+2y''+7x^2-6y=0$$

$$\Rightarrow \quad y''-5y'+6y=6x^2-10x+2 \quad \ldots(4)$$

This is the required DE; it corresponds to the family of curves given by (1). Differently put, the most general solution of this DE is given by (1).

As an exercise for the reader, show that the DE corrersponding to the general equation

$$y = Ae^{2x} + Be^{x} + C$$

where A, B, C are arbitrary constants, is

$$y''' - 3y'' + 2y' = 0$$

By expected, the three arbitrary constants cause the DE to the third order.

Example 1 Find the order and degree (if defined) of the following differential equations :

(i) $y = 1 + \left(\frac{dy}{dx}\right) + \frac{1}{2!}\left(\frac{dy}{dx}\right)^2 + \frac{1}{3!}\left(\frac{dy}{dx}\right)^3 + \dots$

(ii) $\left(\frac{d^3y}{dx^3}\right)^{2/3} = \frac{dy}{dx} + 2$ (iii) $\frac{d^2y}{dx^2} = x \ln\left(\frac{dy}{dx}\right)$

Sol. (i) The given differential equation can be rewritten as

$$y = e^{dy/dx}$$

$$\Rightarrow \quad \frac{dy}{dx} = \ln y.$$

Hence, its order is 1 and degree 1.

(ii) The given differential equation can be rewritten as

$$\left(\frac{d^3y}{dx^3}\right)^2 = \left(\frac{dy}{dx} + 2\right)^3.$$

Hence, its order is 3 and degree 2.

(iii) Its order is obviously 2.

Since, the given differential equation cannot be written as a polynomial in all the differential coefficients, the degree of the equation is not defined.

Example 2 Find the order and degree (if defined) of the following differential equations :

(i) $\sqrt{\frac{d^2y}{dx^2}} = \sqrt[3]{\frac{dy}{dx} + 3}$ (ii) $\frac{d^2y}{dx^2} = \sin\left(\frac{dy}{dx}\right)$

(iii) $\frac{dy}{dx} = \sqrt{3x + 5}$

Sol. (i) The given differential equation can be rewritten as

$$\left(\frac{d^2y}{dx^2}\right)^3 = \left(\frac{dy}{dx} + 3\right)^2$$

Hence, order is 2 and degree is 3.

(ii) The given differential equation has the order 2. Since, the given differential equation cannot be written as a polynomial in the differential coefficients, the degree of the equation is not defined.

(iii) Its order is obviously 1 and degree 1.

Linear and Non-linear Differential Equation

A differential equation is a linear differential equation if it is expressible in the form

$$a_0 \frac{d^n y}{dx^n} + a_1 \frac{d^{n-1}y}{dx^{n-1}} + a_2 \frac{d^{n-2}y}{dx^{n-2}} + \dots + a_{n-1}\frac{dy}{dx} + any = Q$$

where $a_0, a_1, a_2, \dots, a_n$ and Q are either constants or functions of independent variable x.

Thus, if a differential equation when expressed in the form of a polynomial involves the derivatives and dependent variable in the first power and there ae no product of these, and also the coefficient of the various terms are either constants or functions of the independent variable, then it is said to be linear differential equation. otherwise, it i a non-linear differential equation.

The differentiable equation $\left(\frac{d^3y}{dx^3}\right)^2 - 6\left(\frac{d^2y}{dx^2}\right) - 4y = 0$, is a non-linear differential equation, because its degree is 2, more than one.

e.g. The differential equation, $\left(\frac{d^3y}{dx^3}\right) + 2\left(\frac{dy}{dx}\right)^2 + 9y = x$, is non-linear differential equation, because differential coefficient $\frac{dy}{dx}$ has exponent 2.

e.g. The differential equation $(x^2 + y^2)\,dx - 2xydy = 0$ is a non-linear differential equation, because the exponent of dependent variable y is 2 and it involves the product of y and $\frac{dy}{dx}$. e.g. Consider the differential equation

$$\left(\frac{d^2y}{dx^2}\right) - 5\left(\frac{dy}{dx}\right) + 6y = \sin x$$

This is a linear differential equation of order 2 and degree.

Formation of Differential Equations

If an equation in independent and dependent variables involving some arbitrary constants is given, then a differential equation is obtained as follows :

(i) Differentiate the given equation w.r.t. the independent variable (say x) as many times as the number of arbitrary constants in it.

(ii) Eliminate the arbitrary constants.

(iii) The eliminant is the required differential equation. i.e. If we have an equation $f(x, y, c_1, c_2, \dots, c_n) = 0$

Containing n arbitrary constants $c_1, c_2, c_3, \ldots, c_n$, then by differentiating this n times, we shall get n-equations.

Now, among these n-equations and the given equation, in all $(n+1)$ equations, if the n arbitrary constants $c_1, c_2, c_3, \ldots, c_n$ are eliminated, we shall evidently get a differential equation of the nth order. For there being n differentiation, the resulting equation must contain a derivative of the nth order.

Algorithm for Formation of Differential Equations

Step I Write the given equation involving independent variable x (say), dependent variable y (say) and the arbitrary constant.

Step II Obtain the number of arbitrary constants in Step I. Let there be n arbitrary constants.

Step III Differentiate the relation in step I, n times with respect to x.

Step IV Eliminate arbitrary consstants with the help of n equations involving differential coefficients obtained in step III and an equation in step I.

The equation so obtained is the desired differential equation. The following examples will illustrate the above procedure.

Example 3 Form the differential equation, if $y^2 = 4a(x+b)$, where a, b are arbitrary constants.

Sol. Differentiating $y^2 = 4a(x+b)$ w.r.t. x,

$$2y \cdot \frac{dy}{dx} = 4a \quad \text{i.e.} \quad y\frac{dy}{dx} = 2a$$

Again, differentiating w.r.t. x, we get

$$y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = 0$$

which is the required differential equation. Thus, the elimination of arbitrary leads to the formation of a differential equation.

Example 4 Find the differential equation whose solution represents the family $xy = ae^x + be^{-x}$.

Sol.
$$xy = ae^x + be^{-x} \quad \ldots(\text{i})$$

Differentiating Eq. (i) w.r.t. x, we get

$$x\frac{dy}{dx} + y = ae^x - be^{-x} \quad \ldots(\text{ii})$$

Differentiating Eq. (ii) w.r.t. x, we get

$$x\frac{d^2y}{dx^2} + \frac{dy}{dx} \cdot 1 + \frac{dy}{dx} = ae^x + be^{-x} \quad \ldots(\text{iii})$$

Using Eqs. (i) and (iii), we get

$$x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = xy$$

Which is the required differential equation.

Example 5 Find the differential equation whose solution represents the family $c(y+c)^2 = x^3$.

Sol. Differentiating $c(y+c)^2 = x^3$ ...(i)

We get $c[2(y+c)]\frac{dy}{dx} = 3x^2$ but from Eq. (i), we have

$$\frac{2x^3}{(y+c)^2}(y+c)\frac{dy}{dx} = 3x^2$$

$$\Rightarrow \quad \frac{2x^3}{y+c} \cdot \frac{dy}{dx} = 3x^2 \quad \text{i.e.} \quad \frac{2x}{y+c} \cdot \frac{dy}{dx} = 3$$

$$\Rightarrow \quad \frac{2x}{3}\left[\frac{dy}{dx}\right] = y + c \quad \therefore \quad c = \frac{2x}{3}\left[\frac{dy}{dx}\right] - y$$

Substituting c in Eq. (i), we get

$$\left[\frac{2x}{3}\left(\frac{dy}{dx}\right) - y\right]\left[\frac{2x}{3}\frac{dy}{dx}\right]^2 = x^3$$

Which is the required differential equation.

Example 6 Find the differential equation whose solution represents the family $y = ae^{3x} + be^x$.

Sol.
$$y = ae^{3x} + be^x \quad \ldots(\text{i})$$

Differentiating the given equation twice, we get

$$\frac{dy}{dx} = 3ae^{3x} + be^x \quad \text{and} \quad \frac{d^2y}{dx^2} = 9ae^{3x} + be^x$$

From the three equations by eliminating a and b, we obtain

$$\frac{d^2y}{dx^2} - \frac{4dy}{dx} + 3y = 0$$

Remark

The order of the differential equation will be equal to number of independent parameters and is not equal to the number of all the parameters in the family of curves.

Example 7 Find the order of the family of curves $y = (c_1 + c_2)e^x + c_3 e^{x + c_4}$.

Sol. Here, the number of arbitrary parameters is 4 but the order of the corresponding differential equation will not be 4 as it can be rewritten as, $y = (c_1 + c_2 + c_3 e^{c_4})e^x$, which is of the form $y = Ae^x$. Hence, the corresponding differential equation will be of order 1.

Example 8 The differential equation of all non-horizontal lines in a plane is given by

(a) $\frac{d^2y}{dx^2} = 0$ (b) $\frac{d^2x}{dy^2} = 0$

(c) $\frac{d^2y}{dx^2} = 0$ and $\frac{d^2x}{dy^2} = 0$ (d) All of these

Sol. The equation of the family of all non-horizontal lines in a plane is given by,

$$ax + by = 1 \quad \text{(where } a \neq 0) \quad ...(i)$$

Differentiating w.r.t. y, we get

$$a\frac{dx}{dy} + b = 0 \quad (\text{as } a \neq 0 \text{ and } b \in R)$$

Again, differentiating w.r.t. y, we get

$$a\frac{d^2x}{dy^2} = 0 \quad (\text{as } a \neq 0 \text{ and } b \in R)$$

$$\Rightarrow \quad \frac{d^2x}{dy^2} = 0 \quad (\text{as } a \neq 0)$$

$\therefore$ Differential equation of all non-horizontal lines in a plane is $\frac{d^2x}{dy^2} = 0$. Hence, (b) is the correct answer.

| Example 9 The differential equation of all non-vertical lines in a plane is given by

(a) $\frac{d^2y}{dx^2} = 0$ (b) $\frac{d^2x}{dy^2} = 0$

(c) $\frac{d^2x}{dy^2} = 0$ and $\frac{d^2y}{dx^2} = 0$ (d) All of these

Sol. The equation of the family of all non-vertical lines in a plane is given by $ax + by = 1$, where $b \neq 0$ and $a \in R$.

Differentiating both the sides w.r.t. x, we get

$$a + b\frac{dy}{dx} = 0 \quad (\text{as } b \neq 0 \text{ and } a \in R)$$

Again, differentiating both the sides w.r.t. x, we get

$$b\frac{d^2y}{dx^2} = 0 \quad (\text{as } b \neq 0 \text{ and } a \in R)$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = 0 \quad (\text{as } b \neq 0)$$

$\therefore$ Differential equation of all non-vertical lines in a plane.

$$\Rightarrow \quad \frac{d^2y}{dx^2} = 0$$

Hence, (a) is the correct answer.

| Example 10 The differential equation of all straight lines which are at a constant distance p from the origin, is

(a) $(y + xy_1)^2 = p^2(1 + y_1^2)$ (b) $(y - xy_1^2) = p^2(1 + y_1)^2$

(c) $(y - xy_1)^2 = p^2(1 + y_1^2)$ (d) None of these

Sol. As, we know $\quad x\cos\alpha + y\sin\alpha = p \quad ...(i)$

Represents the family of straight lines which are at a constant distance p from origin. Differentiating Eq. (i) w.r.t. x, we get

$$\cos\alpha + \sin\alpha \cdot \frac{dy}{dx} = 0$$

$$\Rightarrow \quad \tan\alpha = -\frac{1}{y_1} \text{ or } \sin\alpha = \frac{1}{\sqrt{1 + y_1^2}}$$

and
$$\cos\alpha = -\frac{y_1}{\sqrt{1 + y_1^2}} \quad ...(ii)$$

From Eqs. (i) and (ii), we get

$$\frac{-xy_1}{\sqrt{1 + y_1^2}} + \frac{y}{\sqrt{1 + y_1^2}} = p$$

$\Rightarrow (y - xy_1)^2 = p^2(1 + y_1^2)$ is required differential equations.

Hence, (c) is the correct answer.

| Example 11 The differential equation of all circles of radius r, is given by

(a) $\{1 + (y_1)^2\}^2 = r^2y_2^3$ (b) $\{1 + (y_1)^2\}^3 = r^2y_2^3$

(c) $\{1 + (y_1)^2\}^3 = r^2y_2^2$ (d) None of these

Sol. Equation of circle of radius r,

$$(x - a)^2 + (y - b)^2 = r^2 \quad ...(i)$$

(Here, a, b are two arbitrary constants)

Differentiating Eq. (i), we get

$$2(x - a) + 2(y - b)y_1 = 0 \quad ...(ii)$$

Again, differentiating Eq. (ii), we get

$$1 + (y - b)y_2 + y_1^2 = 0$$

$$\Rightarrow \quad (y - b) = -\left(\frac{1 + y_1^2}{y_2}\right) \quad ...(iii)$$

Putting $(y - b)$ in Eq. (ii), we get

$$(x - a) = \frac{(1 + y_1^2)y_1}{y_2} \quad ...(iv)$$

From Eqs. (i), (iii) and (iv), we get

$$\frac{(1 + y_1^2)^2 \cdot y_1^2}{y_2^2} + \frac{(1 + y_1^2)^2}{y_2^2} = r^2$$

$$\Rightarrow \quad (1 + (y_1)^2)^3 = r^2y_2^2$$

Hence, (c) is the correct answer.

| Example 12 The differential equations of all circles touching the x-axis at origin is

(a) $(y^2 - x^2) = 2xy\left(\frac{dy}{dx}\right)$

(b) $(x^2 - y^2)\frac{dy}{dx} = 2xy$

(c) $(x^2 - y^2) = 2xy\left(\frac{dy}{dx}\right)$

(d) None of the above

Sol. The equation of circle touches x-axis at origin.

$$\Rightarrow \quad (x - 0)^2 + (y - a)^2 = a^2$$

or
$$x^2 + y^2 - 2ay = 0 \quad ...(i)$$

Differentiating w.r.t. x, we get

$$2x + 2y\frac{dy}{dx} - 2a\frac{dy}{dx} = 0$$

$$\Rightarrow \quad \frac{dy}{dx}(a - y) = x$$

$$\Rightarrow \quad a = \frac{x + y\left(\frac{dy}{dx}\right)}{\left(\frac{dy}{dx}\right)} \quad \text{...(ii)}$$

From Eqs. (i) and (ii), we get

$$x^2 + y^2 - 2y\left[\frac{x + y\left(\frac{dy}{dx}\right)}{\left(\frac{dy}{dx}\right)}\right] = 0$$

or $$(x^2 - y^2)\frac{dy}{dx} = 2xy$$

Hence, (b) is the correct answer.

Example 13 The differential equation of all circles in the first quadrant which touch the coordinate axes is

(a) $(x - y)^2 (1 + (y')^2) = (x + yy')^2$

(b) $(x + y)^2 (1 + (y')^2) = (x + y')^2$

(c) $(x - y)^2 (1 + y') = (x + yy')^2$

(d) None of these

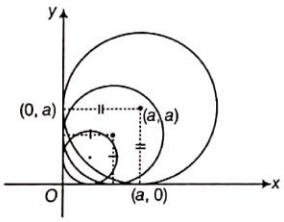
Sol. Equation of circles touching coordinate axes is

$$(x - a)^2 + (y - a)^2 = a^2 \quad \text{...(i)}$$

Differentiating, we get

$$2(x - a) + 2(y - a)y' = 0$$

$$\Rightarrow \quad a = \frac{x + yy'}{1 + y'}; \text{ where } y' = \frac{dy}{dx} \quad \text{...(ii)}$$



From Eqs. (i) and (ii),

$$\left(x - \frac{x + yy'}{1 + y'}\right)^2 + \left(y - \frac{x + yy'}{1 + y'}\right)^2 = \left(\frac{x + yy'}{1 + y'}\right)^2$$

$$\Rightarrow \quad \left(\frac{xy' - yy'}{1 + y'}\right)^2 + \left(\frac{y - x}{1 + y'}\right)^2 = \left(\frac{x + yy'}{1 + y'}\right)^2$$

$$\Rightarrow \quad (x - y)^2 (y')^2 + (y - x)^2 = (x + yy')^2$$

$$\Rightarrow \quad (x - y)^2 (1 + (y')^2) = (x + yy')^2$$

Hence, (a) is the correct answer.

Example 14 The differential equation satisfying the curve $\frac{x^2}{a^2 + \lambda} + \frac{y^2}{b^2 + \lambda} = 1$, where λ being arbitrary unknown, is

(a) $(x + yy_1)(xy_1 - y) = (a^2 - b^2) y_1$

(b) $(x + yy_1)(x - yy_1) = y_1$

(c) $(x - yy_1)(xy_1 + y) = (a^2 - b^2) y_1$

(d) None of these

Sol. Here, $$\frac{x^2}{a^2 + \lambda} + \frac{y^2}{b^2 + \lambda} = 1 \quad \text{...(i)}$$

Differentiating both the sides, we get

$$\frac{2x}{a^2 + \lambda} + \frac{2y}{b^2 + \lambda} \cdot \frac{dy}{dx} = 0$$

$$\Rightarrow \quad x(b^2 + \lambda) + y(a^2 + \lambda) y_1 = 0$$

$$\Rightarrow \quad \lambda = -\left(\frac{xb^2 + a^2 yy_1}{x + yy_1}\right)$$

$$\therefore \quad a^2 + \lambda = a^2 - \frac{xb^2 + a^2 yy_1}{x + yy_1}$$

$$\Rightarrow \quad a^2 + \lambda = \frac{(a^2 - b^2) x}{x + yy_1} \quad \text{...(ii)}$$

Also, $$b^2 + \lambda = \frac{(a^2 - b^2) yy_1}{x + yy_1} \quad \text{...(iii)}$$

From Eqs. (i), (ii) and (iii), we get

$$\frac{x^2 (x + yy_1)}{(a^2 - b^2) x} + \frac{y^2 (x + yy_1)}{(a^2 - b^2) yy_1} = 1$$

$$\Rightarrow \quad (x + yy_1)(xy_1 - y) = (a^2 - b^2) y_1$$

Hence, (a) is the correct answer.

Example 15 The differential equation of all conics whose centre lies at origin, is given by

(a) $(3xy_2 + x^2 y_3)(y - xy_1) = 3xy_2 (y - xy_1 - x^2 y_2)$

(b) $(3xy_1 + x^2 y_2)(y_1 - xy_3) = 3xy_1 (y - xy_2 - x^2 y_3)$

(c) $(3xy_2 + x^2 y_3)(y_1 - xy) = 3xy_1 (y - xy_1 - x^2 y_2)$

(d) None of the above

Sol. Equation of all conics whose centre lies at origin, is

$$ax^2 + 2hxy + by^2 = 1 \quad \text{...(i)}$$

Differentiating Eq. (i) w.r.t. x, we get

$$2ax + 2hxy_1 + 2hy + 2byy_1 = 0$$

$$\Rightarrow \quad ax + h(y + xy_1) + byy_1 = 0$$

Multiplying by x equation becomes,

$$ax^2 + h(xy + x^2 y_1) + bxyy_1 = 0 \quad \text{...(ii)}$$

Subtracting Eqs. (i) and (ii), we get

$$h(xy - x^2 y_1) + b(y^2 - xyy_1) = 1$$

$\Rightarrow \quad (hx + by)\, y - xy_1 (hx + by) = 1$

$\Rightarrow \quad (hx + by)(y - xy_1) = 1$

$$\Rightarrow \quad hx + by = \frac{1}{y - xy_1} \quad \text{...(iii)}$$

Again, differentiating w.r.t. x, we get

$$h + by_1 = -\frac{(y_1 - xy_2 - y_1)}{(y - xy_1)^2}$$

or

$$h + by_1 = \frac{xy_2}{(y - xy_1)^2} \quad \text{...(iv)}$$

From Eqs. (iii) and (iv), we get

$$b(y - xy_1) = \frac{y - xy_1 - x^2y_2}{(y - xy_1)^2}$$

$$\Rightarrow \quad b = \frac{y - xy_1 - x^2y_2}{(y - xy_1)^3} \quad \text{...(v)}$$

Again, differentiating both the sides w.r.t. x, we get

$$0 = \frac{y_1 - y_1 - 3xy_2 - x^2y_3}{(y - xy_1)^3} + \frac{3(y - xy_1 - x^2y_2)\, xy_2}{(y - xy_1)^4}$$

$\Rightarrow \quad (3xy_2 + x^2y_3)(y - xy_1) = 3xy_2 (y - xy_1 - x^2y_2)$

Hence, (a) is the correct answer.

Exercise for Session 1

1. The differential equation of all parabolas whose axis of symmetry is along X-axis is of order.

(a) 2 (b) 3 (c) 1 (d) None of these

2. The order and degree of the differential equation of all tangent lines to the parabola $x^2 = 4y$ is

(a) 1, 2 (b) 2, 2 (c) 3, 1 (d) 4, 1

3. The degree of the differential equation $\frac{d^2y}{dx^2} + 3\left(\frac{dy}{dx}\right)^2 = x^2 \log\left(\frac{d^2y}{dx^2}\right)$ is

(a) 1 (b) 2 (c) 3 (d) Not defined

4. The degree of the defferential equation satisfying the relation $\sqrt{1+x^2} + \sqrt{1+y^2} = \lambda(x\sqrt{1+y^2} - y\sqrt{1+x^2})$ is

(a) 1 (b) 2 (c) 3 (d) 4

5. The degree of the differential equation $\left(\frac{d^2y}{dx^2}\right)^2 + \left(\frac{dy}{dx}\right)^2 = x \sin\left(\frac{d^2y}{dx^2}\right)$ is

(a) 1 (b) 2 (c) 3 (d) Not defined

6. The differential equation of all circles touching the y-axis at origin, is

(a) $y^2 - x^2 = 2xy\frac{dy}{dx}$ (b) $y^2 - x^2 = 2xy\frac{dx}{dy}$ (c) $x^2 - y^2 = 2xy\frac{dy}{dx}$ (d) $x^2 - y^2 = 2xy\frac{dx}{dy}$

7. The differential equation of all parabolas having their axes of symmetry coincident with the axes of x, is

(a) $yy_2 + y_1^2 = y + y_1$ (b) $yy_2 + y_1^2 = 0$ (c) $yy_2 + y_1^2 = y_1$ (d) None of these

8. The differential equation of all conics whose axes coincide with the coordinate axes, is

(a) $xyy_2 + xy_1^2 - yy_1 = 0$ (b) $yy_2 + y_1^2 - yy_1 = 0$

(c) $xyy_2 + (x - y)\, y_1 = 0$ (d) None of these

9. The differential equation having $y = (\sin^{-1} x)^2 + A(\cos^{-1} x) + B$, where A and B are arbitrary constant, is

(a) $(1 - x^2)\, y_2 - xy_1 = 2$ (b) $(1 - x^2)\, y_2 + yy_1 = 0$

(c) $(1 - x)\, y_2 + xy_1 = 0$ (d) None of these

10. The differential equation of circles passing through the points of intersection of unit circle with centre at the origin and the line bisecting the first quadrant, is

(a) $y_1(x^2 + y^2 - 1) + (x + yy_1) = 0$ (b) $(y_1 - 1)(x^2 + y^2 - 1) + (x + yy_1)\, 2(x - y) = 0$

(c) $(x^2 + y^2 - 1) + yy_2 = 0$ (d) None of these

Session 2

Solving of Variable Seperable Form, Homogeneous Differential Equation

Solving of Variable Seperable Form

Solution of a Differential Equation

The solution of the differential equation is a relation between the variables of the equation not containing the derivatives, but satisfying the given differential equation (i.e from which the given differential equation can be derived).

Thus, the solution of $\frac{dy}{dx} = e^x$ could be obtained by simply integrating both the sides, i.e. $y = e^x + C$ and that of, $\frac{dy}{dx} = px + q$ is $y = p\frac{x^2}{2} + qx + C$, where C is arbitrary constant.

(i) **A general solution or an integral** of a differential equation is a relation between the variables (not involving the derivatives) which contains the same number of the arbitrary constants as the order of the differential equation. For example, a general solution of the differential equation $\frac{d^2x}{dt^2} = -4x$ is

$x = A\cos 2t + B\sin 2t$, where A and B are the arbitrary constants.

(ii) **Particular solution or particular integral** is that solution of the differential equation obtained from the general solution by assigning particular values to the arbitrary constant in the general solution.

For example, $x = 10\cot 2t + 5\sin 2t$ is a particular solution of differential equation $\frac{d^2x}{dt^2} = -4x$.

Differential Equations of the First Order and First Degree

In this section we shall discuss the differential equations which are of first order and first degree only.

A differential equation of first order and first degree is of the form $\frac{dy}{dx} = f(x, y)$.

Remark

All the differential equations, even of first order and first degree, cannot be solved. However, if they belong to any of the standard forms which we are going to discuss, in the subsequent articles they can be solved.

Equations in Which the Variables are Separable

The equation $\frac{dy}{dx} = f(x, y)$ is said to be in variables separable form, if we can express it in the form $f(x)\,dx = g(y)\,dy$. By integrating this, solution of the equation is obtained which is, $\int f(x)\,dx = \int g(y)\,dy + C$

| Example 16 Solve

$$\sec^2 x \tan y\,dx + \sec^2 y \tan x\,dy = 0.$$

Sol. Dividing the given equation by $\tan x \tan y$, we get

$$\frac{\sec^2 x}{\tan x}dx + \frac{\sec^2 y}{\tan y}dy = 0$$

This is variable-separable type

Integrating, $\int \frac{\sec^2 x}{\tan x}dx + \int \frac{\sec^2 y}{\tan y}dy = C'$

$$\ln|\tan x| + \ln|\tan y| = \ln C; \text{ where } C' = \ln C$$

or $\quad \ln|\tan x \cdot \tan y| = \ln C;\ (C > 0)$

$\Rightarrow \quad |\tan x \cdot \tan y| = C$

This is the general solution.

| Example 17 Solve $\frac{dy}{dx} = e^{x-y} + x^2 e^{-y}$.

Sol. Here, $\frac{dy}{dx} = \frac{e^x}{e^y} + \frac{x^2}{e^y} \Rightarrow e^y\,dy = (x^2 + e^x)\,dx$

This is variable-separable form,

$\therefore$ Integrating both the sides,

$$\int e^y\,dy = \int (x^2 + e^x)\,dx \Rightarrow e^y = \frac{x^3}{3} + e^x + C$$

Which is the general solution of the given differential equation, where C is an arbitrary constant.

| Example 18 Solve $\sqrt{1+x^2+y^2+x^2y^2} + xy\dfrac{dy}{dx} = 0.$

Sol. The given differential equation can be written as

$$\sqrt{(1+y^2)(1+x^2)} = -xy\frac{dy}{dx}$$

$$\Rightarrow \quad -\frac{\sqrt{1+x^2}\,dx}{x} = \frac{y\,dy}{\sqrt{1+y^2}}$$

This is the variable-separable form.

$\therefore$ Integrating both the sides, we get

$$-\int \frac{\sqrt{1+x^2}}{x}\,dx = \int \frac{y}{\sqrt{1+y^2}}\,dy$$

$$\Rightarrow -\left[\sqrt{1+x^2} + \frac{1}{2}\log\left(\frac{\sqrt{1+x^2}-1}{\sqrt{1+x^2}+1}\right)\right] = \sqrt{1+y^2} + C$$

This is the general solution to the given differential equation.

| Example 19 Solve $y - x\dfrac{dy}{dx} = a\left(y^2 + \dfrac{dy}{dx}\right).$

Sol. Rewriting the given equation as

$$y - ay^2 = (x+a)\frac{dy}{dx}$$

$$\Rightarrow \quad \frac{dy}{y(1-ay)} = \frac{dx}{(x+a)}$$

This is the variable-separable form.

Integrating both the sides, we get

$$\int \frac{dy}{y(1-ay)} = \int \frac{dx}{(x+a)}$$

$$\Rightarrow \quad \int \left(\frac{1}{y} + \frac{a}{1-ay}\right)dy = \int \frac{dx}{x+a}$$

$$\Rightarrow \quad \ln y - \frac{a}{a}\ln(1-ay) + \ln C = \ln(a+x)$$

$$\Rightarrow \quad \ln\left(\frac{(a+x)(1-ay)}{y}\right) = \ln(C)$$

or $\quad Cy = (a+x)(1-ay)$ is the general solution.

| Example 20 Solve $e^{dy/dx} = x+1$, given that when $x = 0, y = 3.$

Sol. This is an Example of particular solution.

$$e^{(dy/dx)} = x+1$$

$$\therefore \quad \frac{dy}{dx} = \ln(x+1)$$

$$\therefore \quad \int dy = \int \ln(x+1)\,dx \qquad \text{(integration by parts)}$$

i.e. $\quad y = x\ln(x+1) - \int \dfrac{x}{x+1}\,dx$

i.e. $\quad y = x\ln(x+1) - x + \ln(x+1) + C$

This is the general solution.

To find the particular solution, put $x = 0, y = 3$ in the general equation.

$\therefore \quad 3 = 0 - 0 + 0 + C$

$\therefore \quad C = 3$

$\therefore$ The required particular solution is,

$$y = (x+1)\ln(x+1) - x + 3$$

Differential Equations Reducible to the Separable Variable Type

Sometimes differential equation of the first order cannot be solved directly by variable separation but by some substitution we can reduce it to a differential equation with separable variable. "A differential equation of the form $\dfrac{dy}{dx} = f(ax+by+c)$ is solved by writing $ax + by + c = t$."

| Example 21 Solve $\dfrac{dy}{dx} = \sin^2(x+3y) + 5.$

Sol. Let $x + 3y = t$, so that $1 + \dfrac{3dy}{dx} = \dfrac{dt}{dx}$

The given differential equation becomes,

$$\frac{1}{3}\left(\frac{dt}{dx} - 1\right) = \sin^2(t) + 5 \Rightarrow \frac{dt}{dx} = 3\sin^2 t + 16$$

$$\Rightarrow \quad \int \frac{dt}{3\sin^2 t + 16} = \int dx$$

$$\Rightarrow \quad \int \frac{\sec^2 t\,dt}{3\tan^2 t + 16\sec^2 t} = x + C$$

$$\Rightarrow \quad \int \frac{\sec^2 t\,dt}{19\tan^2 t + 16} = x + C$$

$$\Rightarrow \quad \int \frac{du}{19u^2 + 16} = x + C; \text{ where } \tan t = u$$

$$\Rightarrow \quad \sec^2 t\,dt = du$$

$$\Rightarrow \quad \frac{1}{19}\int \frac{du}{u^2 + \frac{16}{19}} \Rightarrow \frac{1}{19}\,\frac{\sqrt{19}}{4}\tan^{-1}\left(\frac{\sqrt{19}u}{4}\right) + C$$

$$\Rightarrow \quad \frac{1}{4\sqrt{19}}\left\{\tan^{-1}\left(\frac{\sqrt{19}}{4}\tan(3y+x)\right)\right\} + C$$

Example 22 Solve $(x+y)^2 \dfrac{dy}{dx} = a^2$.

Sol. $$(x+y)^2 \frac{dy}{dx} = a^2 \quad \text{...(i)}$$

Put $x + y = t \Rightarrow 1 + \dfrac{dy}{dx} = \dfrac{dt}{dx}$ or $\dfrac{dy}{dx} = \left(\dfrac{dt}{dx} - 1\right)$

$\therefore$ Eq. (i) reduces to $t^2 \left\{\dfrac{dt}{dx} - 1\right\} = a^2$

i.e. $t^2 \dfrac{dt}{dx} = a^2 + t^2$, separating the variable and integrating.

$$\int dx = \int \frac{t^2}{a^2+t^2}\,dt = \int \left(1 - \frac{a^2}{a^2+t^2}\right) dt$$

$$\therefore \quad x = t - a\tan^{-1}\left(\frac{t}{a}\right) + C$$

i.e. $$x = x + y - a\tan^{-1}\left(\frac{x+y}{a}\right) + C$$

i.e. $y = a\tan^{-1}\left(\dfrac{x+y}{a}\right) - C$ is the required general solution.

Example 23 Solve $(2x+3y-1)\,dx + (4x+6y-5)\,dy = 0$.

Sol. $(2x+3y-1)\,dx + (4x+6y-5)\,dy = 0 \quad \text{...(i)}$

Substitute $u = 2x + 3y - 1$

$$\therefore \quad \frac{du}{dx} = 2 + \frac{3dy}{dx} \quad \text{or} \quad \frac{dy}{dx} = \frac{1}{3}\left(\frac{du}{dx} - 2\right)$$

$\therefore$ Eq. (i) reduces to

$$u + (2u-3)\frac{1}{3}\left(\frac{du}{dx} - 2\right) = 0$$

i.e. $$u - \frac{2}{3}(2u-3) + \frac{1}{3}(2u-3)\frac{du}{dx} = 0$$

i.e. $$\frac{-u+6}{3} + \frac{1}{3}(2u-3)\frac{du}{dx} = 0$$

Writing this in the variable-separable form

$$\left(\frac{2u-3}{u-6}\right) du = dx$$

$$\therefore \quad \int dx = \int \frac{2u-3}{u-6}\,du$$

$$\therefore \quad x + C = \int \frac{2(u-6)+9}{(u-6)}\,du$$

$$\therefore \quad x + C = 2u + 9\ln|u-6|$$

$$\therefore \quad x + C = 2(2x+3y-1) + 9\ln|2x+3y-7|$$

$3x + 6y - 2 + 9\ln|2x+3y-7| = C$ is the general solution.

Remark

Sometimes transformation to the polar coordinates facilitates separation of varibales. It is convenient to remember the following differentials.

1. $x\,dx + y\,dy = r\,dr$ **2.** $x\,dy - y\,dx = r^2\,d\theta$

3. $dx^2 + dy^2 = dr^2 + r^2 d\,\theta^2$

Example 24 Solve $\dfrac{x\,dx + y\,dy}{x\,dy - y\,dx} = \sqrt{\dfrac{a^2 - x^2 - y^2}{x^2 + y^2}}$.

Sol. Let $x = r\cos\theta,\ y = r\sin\theta$

So that $$x^2 + y^2 = r^2 \quad \text{...(i)}$$

and $$\tan\theta = \frac{y}{x} \quad \text{...(ii)}$$

From Eq. (i), we have $d(x^2+y^2) = d(r^2)$

i.e. $$x\,dx + y\,dy = r\,dr \quad \text{...(iii)}$$

From Eq. (ii), we have $d\left(\dfrac{y}{x}\right) = d(\tan\theta)$

i.e. $$\frac{x\,dy - y\,dx}{x^2} = \sec^2\theta\,d\theta$$

i.e. $x\,dy - y\,dx = x^2\sec^2\theta\,d\theta = r^2\cos^2\theta\sec^2\theta\,d\theta$ ...(iv)

Using Eqs. (iii) and (iv) in the given equation, we get

$$\frac{r\,dr}{r^2\,d\theta} = \sqrt{\frac{a^2-r^2}{r^2}} \text{ i.e. } \frac{dr}{\sqrt{a^2-r^2}} = d\theta$$

i.e. $$\sin^{-1}\left(\frac{r}{a}\right) = \theta + C \quad \text{or} \quad r = a\sin(\theta + C)$$

or $$\sqrt{x^2+y^2} = a\sin\{C + \tan^{-1}(y/x)\}$$

It is advised to remember the results (iii) and (iv).

Homogeneous Differential Equation

By definition, a homogeneouos function $f(x,y)$ of degree n satisfies the property

$$f(\lambda x, \lambda y) = \lambda^n f(x,y)$$

For example, the functions

$$f_1(x,y) = x^3 + y^3$$
$$f_2(x,y) = x^2 + xy + y^2$$
$$f_3(x,y) = x^3 e^{x/y} + xy^2$$

are all homogeneous functions, of degrees three, two and three respectively (verify this assertion).

Observe that any homogeneous function $f(x,y)$ of degree n can be equivalnetly written as follows :

$$f(x,y) = x^n f\left(\frac{y}{x}\right) = y^n f\left(\frac{x}{y}\right)$$

For example, $f(x, y) = x^3 + y^3$

$$= x^3\left(1 + \left(\frac{y}{x}\right)^3\right) = y^3\left(1 + \left(\frac{x}{y}\right)^3\right)$$

Having seen homogeneous functions we define homogeneous DEs as follows :

Any DE of the form $M(x,y)\,dx + N(x,y)dy = 0$

or $\frac{dy}{dx} = -\frac{M(x,y)}{N(x,y)}$ is called homogeneous if $M(x,y)$ and $N(x,y)$ are homogeneous functions of the same degree.

What is so special about homogeneous DEs? Well, it turns out that they areextremely simple to solve. To see how, we express both $M(x,y)$ and $N(x,y)$ as, say $x^n M\left(\frac{y}{x}\right)$ and $x^n N\left(\frac{y}{x}\right)$. This can be done sicne $M(x,y)$ and $N(x,y)$ are both homogeneous function of degree n. Doing this reduces our DE to

$$\frac{dy}{dx} = -\frac{M(x,y)}{N(x,y)} = \frac{x^n M\left(\frac{y}{x}\right)}{x^n N\left(\frac{y}{x}\right)} = -\frac{M\left(\frac{y}{x}\right)}{N\left(\frac{y}{x}\right)} = P\left(\frac{y}{x}\right)$$

(The function $P(t)$ stands for $\frac{-M(t)}{N(t)}$)

Now, the simple substitution $y = vx$ reduces this DE to a VS form

$$y = vx$$

$$\Rightarrow \quad \frac{dy}{dx} = v + x\frac{dy}{dx}$$

Thus, $\frac{dy}{dx} = p\left(\frac{y}{x}\right)$ transforms to

$$v + x\frac{dv}{dx} = P(v)$$

$$\Rightarrow \quad \frac{dv}{P(v) - v} = \frac{dx}{x}$$

This can now be integraed directly since it is in VS form. Let us see some examples of solving homogeneous DEs.

Alogorithm for Solving Homogeneous Differential Equation

***Step* I** Put the differential equation in the form

$$\frac{dy}{dx} = \frac{\phi(x,y)}{\psi(x,y)}$$

***Step* II** Put $y = vx$ and $\frac{dy}{dx} = v + x\frac{dv}{dx}$ in the equation in step I and can out x from the right hand side. The equation reduces to the form $v + x\frac{dv}{dx} = f(v)$.

***Step* III** Shift v on R.H.S and seperate the variables in v and x.

***Step* IV** Integrate both sides to obtain the solution in terms of v and x.

***Step* V** Replace v by $\frac{y}{x}$ in the solution obtained in step IV to obtain the solution in terms of x and y.

Following examples illustrate the procedure.

Example 25 Solve $y\,dx + (2\sqrt{xy} - x)\,dy = 0$.

Sol. $$y\,dx + (2\sqrt{xy} - x)\,dy = 0 \quad \text{...(i)}$$

This is homogeneous type. Substitute $y = ux$

$$\therefore \quad \frac{dy}{dx} = u + \frac{x\,du}{dx}$$

$\therefore$ Equation $ux\,dx + (2\sqrt{x^2u} - x)(u\,dx + x\,du) = 0$

i.e. $x\cdot\{udx + (2\sqrt{u} - 1)\,u\,dx + x\,du\,(2\sqrt{u} - 1)\} = 0$

i.e. $dx\,(2u^{3/2} - u + u) + x\,du\,(2\sqrt{u} - 1) = 0$

Separating the variables, $\frac{dx}{x} + \left(\frac{2\sqrt{u} - 1}{2u^{3/2}}\right)du = 0$

Integrating both the sides $\ln|x| + \ln|u| + \frac{1}{\sqrt{u}} = C$

or $\ln|xu| + \frac{1}{\sqrt{u}} = C$ or $\ln|y| + \sqrt{\frac{x}{y}} = C \quad \left(\because u = \frac{y}{x}\right)$

Which is the general solution.

Example 26 Solve $(x^2 + y^2)\,dx - 2xy\,dy = 0$.

Sol. Here, $\frac{dy}{dx} = \frac{x^2 + y^2}{2xy} = \frac{1}{2}\left[\frac{x}{y} + \frac{y}{x}\right]$

With $y = ux$, $\frac{dy}{dx} = u + x\frac{du}{dx}$, so that the differential equation becomes

$$u + x\frac{du}{dx} = \frac{1}{2}\left(\frac{1}{u} + u\right)$$

$$\Rightarrow \quad \frac{x\,du}{dx} = \frac{1 + u^2}{2u} - u$$

$$\Rightarrow \quad \frac{x\,du}{dx} = \frac{1 - u^2}{2u}$$

$$\Rightarrow \quad \int \frac{2u}{1 - u^2}\,du = \int \frac{dx}{x}$$

$$\Rightarrow \quad -\log|1 - u^2| = \log|x| - \log|C|$$

$$\Rightarrow \quad x(1 - u^2) = C$$

$$\Rightarrow \quad \frac{x^2 - y^2}{x} = C \quad \left(\because u = \frac{y}{x}\right)$$

Hence, $x^2 - y^2 = xC$, is the required solution.

Example 27 Solve $\frac{2\,dy}{dx} = \frac{y}{x} + \frac{y^2}{x^2}$.

Sol. The above equation is homogeneous so that we put $y = ux$.

$$\Rightarrow \quad 2\left[u + x\frac{du}{dx}\right] = u + u^2 \Rightarrow 2u + 2x\frac{du}{dx} = u + u^2$$

$$\Rightarrow \quad 2x\frac{du}{dx} = u^2 - u \Rightarrow \frac{du}{u^2 - u} = \frac{dx}{2x}$$

$$\Rightarrow \quad \int \frac{du}{u(u-1)} = \frac{1}{2}\int \frac{dx}{x}$$

$$\Rightarrow \quad \int \frac{1}{u-1}\,du - \int \frac{1}{u}\,du = \frac{1}{2}\int \frac{dx}{x}$$

$$\Rightarrow \quad \log|u-1| - \log|u| = \frac{1}{2}\log|x| + \log|C|$$

$$\Rightarrow \quad \log\left|\frac{u-1}{u}\right| = \log|C\sqrt{x}|$$

$$\Rightarrow \quad \frac{u-1}{u} = C\sqrt{x} \Rightarrow \frac{y-x}{y} = C\sqrt{x}$$

$$\Rightarrow \quad y - x = C\sqrt{x}\cdot y$$

Which is the required solution.

Example 28 Solve $(1 + 2\,e^{x/y})\,dx + 2\,e^{x/y}\,(1 - x/y)\,dy = 0$.

Sol. The appearance of x/y in the equation suggests the substitution $x = vy$ or $dx = v\,dy + y\,dv$.

$\therefore$ The given equation is

$$(1 + 2e^v)(v\,dy + y\,dv) + 2e^v(1 - v)\,dy = 0$$

i.e. $$y(1 + 2e^v)\,dv + (v + 2e^v)\,dy = 0$$

i.e. $$\frac{1 + 2e^v}{v + 2e^v}\,dv + \frac{dy}{y} = 0$$

Integrating, $$\int \frac{1 + 2e^v}{v + 2e^v}\,dv + \int \frac{dy}{y} = 0$$

$$\log|v + 2e^v| + \log|y| = \log|C|$$

$$\Rightarrow \quad (v + 2e^v)\,y = C \qquad \left(v = \frac{x}{y}\right)$$

$$\Rightarrow \quad \left(\frac{x}{y} + 2e^{x/y}\right)y = C$$

$\Rightarrow$ $(x + 2ye^{x/y}) = C$, is required solution.

Example 29 Show that any equation of the form
$$y\,f(xy)\,dx + x\,g(xy)\,dy = 0$$
can be converted to variable separable form by substituting $xy = v$.

Sol. Since, $xy = v$, $y = \frac{v}{x}$ and $d(xy) = dv$

i.e. $$x\,dy + y\,dx = dv$$

and $$dy = d\left(\frac{v}{x}\right) = \frac{x\,dv - v\,dx}{x^2}$$

i.e. $$x\,dy = dv - \frac{v}{x}\,dx$$

$$\therefore \quad \frac{v}{x}f(v)\,dx + g(v)\left\{dv - \frac{v}{x}\,dx\right\} = 0$$

$$\therefore \quad \frac{v\{f(v) - g(v)\}}{x}\,dx + g(v)\,dv = 0$$

i.e. $$\frac{dx}{x} + \frac{g(v)\,dv}{v\{f(v) - g(v)\}} = 0$$

Which is in variables separable form.

Reducible to Homogeneous Form

Type I

Many a times, the DE specified may not be homogeneous but some suibtale manipulation might reduce it to a homogeneous form. Generally, such equations involve a function of a rational expression whose numerator and denominator are linear functions of the variable, i.e., of the form

$$\frac{dy}{dx} = f\left(\frac{ax + by + c}{dx + cy + f}\right) \qquad \text{...(1)}$$

Note that the presence of the constant c and f causes this DE to be non-homogeneous.

To make it homogeneous, we use the substitutions

$$x \to X + h$$
$$y \to Y + k$$

and select h and k so that

$$\left.\begin{aligned} ah + bk + c &= 0 \\ dh + ek + f &= 0 \end{aligned}\right\} \qquad \text{...(2)}$$

This can always be done $\left(\text{if } \frac{a}{b} \neq \frac{d}{e}\right)$. The RHS of the DE in (1) now reduces to $= f\left(\frac{a(X + h) + b(Y + k) + c}{d(X + h) + e(Y + k + f)}\right)$

$$= f\left(\frac{aX + bY}{dX + eY}\right) \qquad \text{(Using (2))}$$

This expression is clearly homogeneous! The LHS of (1) is $\frac{dy}{dx}$ which equals $\frac{dy}{dY}\cdot\frac{dY}{dX}\cdot\frac{dX}{dx}$. Since $\frac{dy}{dY}\cdot\frac{dx}{dX} = 1$, the LHS $\frac{dy}{dx}$ equals $\frac{dY}{dX}$. Thus, our equation becomes

$$\frac{dY}{dX} = f\left(\frac{aX + bY}{dX + eY}\right) \quad ...(3)$$

We have thus succeeded in transforming the non-homogeneous DE in (1) to the homogeneous DE in (3). This can now be solved as described earlier.

Example 30 Solve the DE $\frac{dy}{dx} = \frac{2y - x - 4}{y - 3x + 3}$.

Sol. We substitute $x \to X + h$ and $y \to Y + k$ where h, k need to be determined

$$\frac{dy}{dx} = \frac{dY}{dX} = \frac{(2Y - X) + (2k - h - 4)}{(Y - 3X) + (k - 3h + 3)}$$

h and k must be chosen so that

$$2k - h - 4 = 0$$
$$k - 3h + 3 = 0$$

This gives $h = 2$ and $k = 3$. Thus,

$$x = X + 2$$
$$y = Y + 3$$

Our DE now reduces to

$$\frac{dY}{dX} = \frac{2Y - X}{Y - 3X}$$

Using the substitution $Y = vX$, and simplifying, we have (verify),

$$\frac{v - 3}{v^2 \;\; 5v + 1} dv = \frac{-dX}{X}$$

We now integrate this DE which is VS; the left-hand side can be integrated by the techniques described in the unit of Indefinite Integration.

Finally, we substitute $v = \frac{Y}{X}$ and

$$X = x - 2$$
$$Y = y - 3$$

to obtain the general solution.

Type II

Suppose our DE is of the form

$$\frac{dy}{dx} = f\left(\frac{ax + by + c}{dx + ey + f}\right)$$

We try to find h, k so that

$$ah + bk + c = 0$$
$$dh + ek + f = 0$$

What if this system does not yield a solution? Recall that this will happen if $\frac{a}{b} = \frac{d}{e}$. How do we reduce the DE to a homogeneous one in such a case?

Let $\frac{a}{d} = \frac{b}{e} = \lambda$ (say).

Thus,

$$\frac{ax + by + c}{dx + ey + f} = \frac{\lambda(dx + ey) + c}{dx + ey + f}$$

This suggests the substitution $dx + ey = v$, which will give

$$d + e\frac{dy}{dx} = \frac{dv}{dx}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{e}\left(\frac{dv}{dx} - d\right)$$

Thus, our DE reduces to

$$\frac{1}{e}\left(\frac{dv}{dx} - d\right) = \frac{\lambda v + c}{v + f}$$

$$\Rightarrow \quad \frac{dv}{dx} = \frac{\lambda ev + ec}{v + f} + d$$

$$= \frac{(\lambda e + d)v + (ec + d)}{v + f}$$

$$\Rightarrow \quad \frac{(v + f)}{(\lambda e + d)\, v + ec + df} dv = dx$$

which is in VS form and hence can be solved.

Example 31 Solve the DE $\frac{dy}{dx} = \frac{x + 2y - 1}{x + 2y + 1}$.

Sol. Note that h, k do not exist in this case which can reduce this DE to homogeneous form. Thus, we use the substitution

$$x + 2y = v$$

$$\Rightarrow \quad 1 + 2\frac{dy}{dx} = \frac{dv}{dx}$$

Thus, our DE becomes

$$\frac{1}{2}\left(\frac{dv}{dx} - 1\right) = \frac{v - 1}{v + 1}$$

$$\Rightarrow \quad \frac{dv}{dx} = \frac{2v - 2}{v + 1} + 1 = \frac{3v - 1}{v + 1}$$

$$\Rightarrow \quad \frac{v + 1}{3v - 1} dv = dx$$

$$\Rightarrow \quad \frac{1}{3}\left(1 + \frac{4}{3v - 1}\right) dv = dx$$

Integrating, we have

$$\frac{1}{3}\left(v + \frac{4}{3}\ln(3v - 1)\right) = x + C_1$$

Substituting $v = x + 2y$, we have

$$x + 2y + \frac{4}{3}\ln(3x + 6y - 1) = 3x + C_2$$

$$\Rightarrow \quad y - x + \frac{2}{3}\ln(3x + 6y - 1) = C$$

Example 32 The solution of the differential equation $\frac{dy}{dx} = \frac{\sin y + x}{\sin 2y - x\cos y}$ is

(a) $\sin^2 y = x\sin y + \frac{x^2}{2} + C$

(b) $\sin^2 y = x\sin y - \frac{x^2}{2} + C$

(c) $\sin^2 y = x + \sin y + \frac{x^2}{2} + C$

(d) $\sin^2 y = x - \sin y + \frac{x^2}{2} + C$

Sol. Here, $\frac{dy}{dx} = \frac{\sin y + x}{\sin 2y - x\cos y}$

$$\Rightarrow \cos y \frac{dy}{dx} = \frac{\sin y + x}{2\sin y - x}, \text{ put } \sin y = t$$

$$\Rightarrow \frac{dt}{dx} = \frac{t + x}{2t - x}, \text{ put } t = vx$$

$$\frac{x\,dv}{dx} + v = \frac{vx + x}{2vx - x} = \frac{v+1}{2v-1}$$

$$\therefore \quad x\frac{dv}{dx} = \frac{v+1}{2v-1} - v = \frac{v + 1 - 2v^2 + v}{2v - 1}$$

or $$\frac{2v^2 - v}{-2v^2 + 2v + 1}\,dv = \frac{dx}{x}$$

On solving, we get

$$\sin^2 y = x\sin y + \frac{x^2}{2} + C$$

Hence, (a) is the correct answer.

Example 33 The equation of curve passing through (1, 0) and satisfying $\left(y\frac{dy}{dx} + 2x\right)^2 = (y^2 + 2x^2)\left(1 + \left(\frac{dy}{dx}\right)^2\right)$, is given by

(a) $\sqrt{2}\,x^{\pm\frac{1}{\sqrt{2}}} = \frac{y + \sqrt{y^2 + 2x^2}}{x}$

(b) $\sqrt{2}x^{\pm\sqrt{2}} = \frac{y + \sqrt{y^2 + \sqrt{2}\,x^2}}{x}$

(c) $\sqrt{2}\,y^{\pm\frac{1}{\sqrt{2}}} = \frac{y + \sqrt{x^2 + 2y^2}}{x}$

(d) None of the above

Sol. The given differential equation can be written as

$$y^2\left(\frac{dy}{dx}\right)^2 + 4x^2 + 4xy\cdot\frac{dy}{dx} = (y^2 + 2x^2)\left(1 + \left(\frac{dy}{dx}\right)^2\right)$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{y}{x} \pm \sqrt{\frac{1}{2}\left(\frac{y}{x}\right)^2 + 1} \quad \text{...(i)}$$

Let $y = vx$

$$\Rightarrow \quad v + x\frac{dv}{dx} = \frac{dy}{dx}$$

$\therefore$ Eq. (i) becomes

$$v + x\frac{dv}{dx} = v \pm \sqrt{\frac{1}{2}v^2 + 1}$$

or $$\int \frac{dv}{\sqrt{\frac{1}{2}v^2 + 1}} = \int \frac{dx}{x}$$

$$\Rightarrow \quad \sqrt{2}\log|v + \sqrt{v^2 + 2}| = \log|xC|$$

$$\Rightarrow \sqrt{2}\log\left|\frac{y + \sqrt{y^2 + 2x^2}}{x}\right| = \log|xC|,$$

putting $x = 1$ and $y = 0$

$$\Rightarrow \quad C = (\sqrt{2})^{\sqrt{2}}$$

$\therefore$ Curves are given by $\frac{y + \sqrt{y^2 + 2x^2}}{x} = \sqrt{2}\,x^{\pm\frac{1}{\sqrt{2}}}$

Hence, (a) is the correct answer.

Exercise for Session 2

1. The solution of $\frac{dy}{dx} = \frac{(x+y)^2}{(x+2)(y-2)}$, is given by

(a) $(x+2)^4 \left(1 + \frac{2y}{x}\right) = ke^{2y/x}$ (b) $(x+2)^4 \left(1 + \frac{2(y-2)}{(x+2)}\right) = k\,e^{\frac{2(y-2)}{(x+2)}}$

(c) $(x+2)^3 \left(1 + 2\frac{(y-2)}{x+2}\right) = ke^{\frac{2(y-2)}{x+2}}$ (d) None of these

2. If $(y^3 - 2x^2y)\,dx + (2xy^2 - x^3)\,dy = 0$, then the value of $xy\sqrt{y^2 - x^2}$, is

(a) $y^2 + x$ (b) xy^2

(c) any constant (d) None of these

3. The solution of $dy/dx = \cos(x+y) + \sin(x+y)$, is given by

(a) $\log\left|1 + \tan\left(\frac{x+y}{2}\right)\right| = x + C$ (b) $\log|1 + \tan(x+y)| = x + C$

(c) $\log|1 - \tan(x+y)| = x + C$ (d) None of these

4. The solution of $\frac{dy}{dx} = (x+y-1) + \frac{x+y}{\log(x+y)}$, is given by

(a) $\{1 + \log(x+y)\} - \log\{1 + \log(x+y)\} = x + C$ (b) $\{1 - \log(x+y)\} - \log\{1 - \log(x+y)\} = x + C$

(c) $\{1 + \log(x+y)\}^2 - \log\{1 + \log(x+y)\} = x + C$ (d) None of these

5. The solution of $(2x^2 + 3y^2 - 7)\,xdx - (3x^2 + 2y^2 - 8)\,ydy = 0$, is given by

(a) $(x^2 + y^2 - 1) = (x^2 + y^2 - 3)^5 C$ (b) $(x^2 + y^2 - 1)^2 = (x^2 + y^2 - 3)^5 C$

(c) $(x^2 + y^2 - 3) = (x^2 + y^2 - 1)^5 C$ (d) None of these

6. The solution of $\frac{dy}{dx} = \frac{(x-1)^2 + (y-2)^2 \tan^{-1}\left(\frac{y-2}{x-1}\right)}{(xy - 2x - y + 2)\tan^{-1}\left(\frac{y-2}{x-1}\right)}$, is equal to

(a) $\{(x-1)^2 + (y-1)^2\}\tan^{-1}\left(\frac{y-2}{x-1}\right) - 2(x-1)(y-2) = 2(x-1)^2 \log C(x-1)$

(b) $\{(x-1)^2 + (y-1)^2\} - 2(x-1)(y-2)\tan^{-1}\left(\frac{y-2}{x-1}\right) = 2(x-1)^2 \log C$

(c) $\{(x-1)^2 + (y-1)^2\}\tan^{-1}\left(\frac{y-2}{x-1}\right) - 2(x-1)(y-2) = \log C(x-1)$

(d) None of the above

7. The solution of $\frac{dy}{dx} = \left(\frac{x+2y-3}{2x+y+3}\right)^2$, is

(a) $(x+3)^3 - (y-3)^3 = C(x-y+6)^4$ (b) $(x+3)^3 - (y-3)^3 = C$

(c) $(x+3)^4 + (y-3)^4 = C$ (d) None of these

8. The solution of $\frac{dy}{dx} = \frac{-\cos x\,(3\cos y - 7\sin x - 3)}{\sin y\,(3\sin x - 7\cos y + 7)}$, is

(a) $(\cos y - \sin x - 1)^2 (\sin x + \cos y - 1)^5 = C$ (b) $(\cos y - \sin x - 1)^2 (\sin x + \cos y - 1)^3 = C$

(c) $(\cos y - \sin x - 1)^2 (\sin x + \cos y - 1)^7 = C$ (d) None of these

9. A curve C has the property that if the tangent drawn at any point P on C meets. The coordinate axes at A and B, then P is the mid point of AB. The curve passes through the point (1, 1). Then the equation of curve is

(a) $xy = 1$ (b) $\frac{x}{y} = 1$

(c) $2x = xy - 1$ (d) None of the above

10. The family of curves whose tangent form an angle $\frac{\pi}{4}$ with the hyperbola $xy = 1$, is

(a) $y = x - 2\tan^{-1}(x) + K$ (b) $y = x + 2\tan^{-1}(x) + K$

(c) $y = 2x - \tan^{-1}(x) + K$ (d) $y = 2x + \tan^{-1}(x) + K$

11. A and B are two separate reservoires of mater capacity of reservoir are filled completely with water their inlets are closed and then the water is released simultaneously from both the reservoirs. The rate of flow of water out of each reservoir at any instant of time is proportional to the quantity of water in the reservoir at that time. One hour after the water is released, the quantity of water in the reservoir A is $1\frac{1}{2}$ times the quantity of the water in reservoir B. The time after which do both the reservoirs have the same quantity of water, is

(a) $\log_{3/4}(2)$ (b) $\log_{3/4}\left(\frac{1}{2}\right)$

(c) $\log_{1/2}\left(\frac{1}{2}\right)$ (d) None of the above

12. A curve passes through (2, 1) and is such that the square of the ordinate is twice the rectangle contained by the abscissa and the intercept of the normal. Then the equation of curve is

(a) $x^2 + y^2 = 9x$ (b) $4x^2 + y^2 = 9x$

(c) $4x^2 + 2y^2 = 9x$ (d) None of the above

13. A normal at $P(x, y)$ on a curve neets the X-axis at Q and N is the foot of the ordinate at P. If $NQ = \frac{x(1+y^2)}{(1+x^2)}$. Then the equation of curve passing through (3, 1) is

(a) $5(1+y^2) = (1+x^2)$ (b) $(1+y^2) = 5(1+x^2)$

(c) $(1+x^2) = (1+y^2).x$ (d) None of the above

14. The curve for which the ratio of the length of the segment intercepted by any tangent on the Y-axis to the length of the radius vector is constant (k), is

(a) $(y + \sqrt{x^2 - y^2})x^{k-1} = c$ (b) $(y + \sqrt{x^2 + y^2})x^{k-1} = c$

(c) $(y - \sqrt{x^2 - y^2})x^{k-1} = c$ (d) $(y - \sqrt{x^2 + y^2})x^{k-1} = c$

15. A point $P(x, y)$ nores on the curve $x^{2/3} + y^{2/3} = a^{2/3}, a > 0$ for each position (x, y) of p, perpendiculars are drawn from origin upon the tangent and normal at P, the length (absolute valve) of them being $P_1(x)$ and $P_2(x)$ brespectively, then

(a) $\frac{dp_1}{dx}.\frac{dp_2}{dx} < 0$ (b) $\frac{dp_1}{dx}.\frac{dp_2}{dx} \le 0$

(c) $\frac{dp_1}{dx}.\frac{dp_2}{dx} > 0$ (d) $\frac{dp_1}{dx}.\frac{dp_2}{dx} \ge 0$

Session 3

Solving of Linear Differential Equations, Bernoulli's Equation, Orthogonal Trajectory

Solving of Linear Differential Equations

First Order Linear Differential Equations

A differential equation is said to be linear if an unknown variable and its derivative occur only in the first degree.

An equation of the form

$$\frac{dy}{dx}+P(x)\cdot y=Q(x).$$

Where $P(x)$ and $Q(x)$ are functions of x only or constant is called a linear equation of the first order.

To get the general solution of the above equation we proceeds as follows. By multiplying both the sides of the above equation by $e^{\int P dx}$, we get

$$e^{\int Pdx}\cdot\frac{dy}{dx}+yP\cdot e^{\int Pdx}=Q\,e^{\int Pdx}$$

i.e. $$e^{\int Pdx}\cdot\frac{dy}{dx}+y\,\frac{d}{dx}(e^{\int Pdx})=Q\,e^{\int Pdx}$$

i.e. $$\frac{d}{dx}(y\,e^{\int Pdx})=Q\cdot e^{\int Pdx}$$

$\therefore$ Integrating, we get $y\,e^{\int Pdx}=\int Q\,e^{\int Pdx}dx+C$

Here, the term $e^{\int Pdx}$ which converts the left hand expression of the equation into a perfect differential is called an Integrating factor. In short it is written as IF. Thus, we remember the solution of the above equation as

$$y\,(\text{IF})=\int Q\,(\text{IF})\,dx+C.$$

Algoritm for Solving A Linear Differential Equation

Step I Write the differential equation in the form $dy/dx+Py=Q$ and obtain P and Q.

Step II Find integrating factor (I.F.) given by I.F. $=e^{\int Pdx}$.

Step III Multiply both sides of equation in Step I by I.F.

Step IV Integrate both sides of the equation obtained in step III. w.r.t x to obtain y (I.F.) $=\int Q.(\text{I. F.})\,dx+C$

This gives the required solution following examples illustrate the procedure.

Example 34 Solve $\dfrac{dy}{dx}+2y=\cos x$.

Sol. It is a linear equation of the form

$$\frac{dy}{dx}+Py=Q(x)$$

where $P=2$ and $Q=\cos x$

Then $$\text{IF}=e^{\int Pdx}=e^{\int 2dx}=e^{2x}$$

Hence, the general solution is $y\,(\text{IF})=\int Q\,(\text{IF})\,dx$

i.e. $$y\cdot e^{2x}=\int e^{2x}\cos x\,dx+C$$

$$y\cdot e^{2x}=\frac{e^{2x}}{5}[2\cos x+\sin x]+C$$

Example 35 Solve $\dfrac{dy}{dx}+\dfrac{y}{x}=\log x$.

Sol. It is a linear differential equation of the form

$$\frac{dy}{dx}+Py=Q(x)$$

Here, $$P=\frac{1}{x},\ Q=\log x$$

Then $$\text{IF}=e^{\int Pdx}=e^{\int 1/x\,dx}=e^{\log x}=x$$

Hence, the general solution is

$$y\,(\text{IF})=\int Q\,(\text{IF})\,dx+C$$

i.e. $$yx=\int(\log x)\,x\,dx+C$$

i.e. $$yx=(\log x)\cdot\frac{x^2}{2}-\int\frac{1}{x}\cdot\frac{x^2}{2}+C$$

i.e. $$yx=\frac{x^2}{2}(\log x)-\frac{x^2}{4}+C$$

| Example 36 Solve $\frac{dy}{dx} = \frac{y}{2y \ln y + y - x}$.

Sol. The equation can be written as

$$\frac{dx}{dy} = \frac{2y \ln y + y - x}{y} = (2 \ln y + 1) - \frac{x}{y}$$

i.e. $$\frac{dx}{dy} + \frac{1}{y} \cdot x = (2 \ln y + 1)$$

In this equation it is clear that $P = \frac{1}{y}$ and $Q = (2 \ln y + 1)$.

Which are function of y only because equation contains derivatives of x with respect to y.

$$\text{IF} = e^{\int P dy} = e^{\int 1/y \, dy} = e^{\ln y} = y$$

$\therefore$ The solution is; $x\,(\text{IF}) = \int (2 \ln y + 1)\,(\text{IF})\, dy$

i.e. $$xy = \int (2 \ln y + 1) \cdot y \, dy = y^2 \ln y + C$$

i.e. $$x = y \ln y + \frac{C}{y}$$

Note In some cases a linear differential equation may be of the form $\frac{dx}{dy} + P_1 x = Q_1$, where P_1 and Q_1 are function of y alone. In such a case the integrating factor is $e^{\int P_1 \, dy}$.

| Example 37 Solve $\cos^2 x \frac{dy}{dx} - y \tan 2x = \cos^4 x$, where $|x| < \frac{\pi}{4}$ and $y\left(\frac{\pi}{6}\right) = \frac{3\sqrt{3}}{8}$.

Sol. The given equation can be written as

$$\frac{dy}{dx} - \sec^2 x \cdot \tan 2x \cdot y = \cos^2 x$$

$$\text{IF} = e^{\int - \tan 2x \cdot \sec^2 x \, dx} = e^{\int + \frac{2 \tan x}{\tan^2 x - 1} \sec^2 x \, dx}$$

$$= e^{\int \frac{dt}{t}}, \text{ where } t = \tan^2 x - 1$$

$$= e^{\ln|t|} = |t| = |\tan^2 x - 1|$$

It is given that $|x| < \frac{\pi}{4}$ and for this region $\tan^2 x < 1$.

$\therefore$ $$\text{IF} = (1 - \tan^2 x)$$

$\therefore$ The solution is

$$y(1 - \tan^2 x) = \int \cos^2 x (1 - \tan^2 x)\, dx$$

$$= \int (\cos^2 x - \sin^2 x)\, dx$$

$$= \int (\cos 2x)\, dx = \frac{\sin 2x}{2} + C$$

Now, when $x = \frac{\pi}{6}, y = \frac{3\sqrt{3}}{8}$

$\therefore$ $$\frac{3\sqrt{3}}{8}\left(1 - \frac{1}{3}\right) = \frac{1}{2} \cdot \frac{\sqrt{3}}{2} + C \Rightarrow C = 0$$

$\therefore$ $$y = \frac{\sin 2x}{2(1 - \tan^2 x)}$$

| Example 38 Solve $\frac{dy}{dx} + y\,\phi'(x) = \phi(x) \cdot \phi'(x)$, where $\phi(x)$ is a given function.

Sol. Here, $P = \phi'(x)$ and $Q = \phi(x)\,\phi'(x)$

$$\text{IF} = e^{\int \phi'(x)\, dx} = e^{\phi(x)}$$

$\therefore$ The solution is

$$y\, e^{\phi(x)} = \int \phi(x) \cdot \phi'(x) \cdot e^{\phi(x)} dx = \int t \cdot e^t \, dt,$$

where $\phi(x) = t$

$$y\, e^{\phi(x)} = e^t (t - 1) + C$$

i.e. $$y e^{\phi(x)} = \{\phi(x) - 1\}\, e^{\phi(x)} + C$$

Bernoulli's Equation

Sometimes a differential equation is not linear but it can be converted into a linear differential equation by a suitable substitution. An equation of the form

$$\frac{dy}{dx} + Py = Q\,y^n. \qquad (n \neq 0, 1)$$

Where P and Q are functions of x only, is known as Bernoulli's equation (for $n = 0$ the equation is linear.)

It is easy to reduce the above equation into linear form as below :

Dividing both the sides by y^n, we get

$$y^{-n} \frac{dy}{dx} + P\,y^{1-n} = Q$$

Putting $y^{1-n} = z$ and hence, $(1-n)\,y^{-n} \frac{dy}{dx} = \frac{dz}{dx}$ the equation becomes $\frac{dz}{dx} + (1-n)\,Pz = (1-n)\,Q$ which is linear in z.

Here, $$\text{IF} = e^{\int (1-n) P\, dx}$$

$\therefore$ The solution is,

$$z\, e^{\int (1-n) P dx} = \int \{(1-n) \cdot Q \cdot e^{\int (1-n) P dx}\}\, dx$$

Example 39 Solve $(y \log x - 1)\, y\, dx = x\, dy$.

Sol. The given differential equation can be written as

$$x \frac{dy}{dx} + y = y^2 \log x \quad \text{...(i)}$$

Dividing by xy^2, hence

$$\frac{1}{y^2} \cdot \frac{dy}{dx} + \frac{1}{xy} = \frac{1}{x} \log x$$

Let $$\frac{1}{y} = v \Rightarrow -\frac{1}{y^2} \cdot \frac{dy}{dx} = \frac{dv}{dx}$$

So that $$\frac{dv}{dx} - \frac{1}{x} v = -\frac{1}{x} \log x$$

Which is the standard linear differential equations, with

$$P = -\frac{1}{x},\ Q = -\frac{1}{x} \log x$$

$$\text{IF} = e^{\int -1/x\, dx} = e^{-\ln x} = e^{\ln x^{-1}} = \frac{1}{x}$$

The solution is given by

$$v \cdot \frac{1}{x} = \int \frac{1}{x}\left(-\frac{1}{x} \log x\right) dx = -\int \frac{\log x}{x^2}\, dx$$

$$= \frac{\log x}{x} - \int \frac{1}{x} \cdot \frac{1}{x}\, dx = \frac{\log x}{x} + \frac{1}{x} + C$$

$\Rightarrow$ $$v = 1 + \log x + Cx = \log(ex) + Cx$$

or $$\frac{1}{y} = \log(ex) + Cx \quad \text{or} \quad y\{\log(ex) + Cx\} = 1$$

Example 40 Solve $\frac{dy}{dx} + xy = xy^2$.

Sol. Dividing by y^2, we get

$$y^{-2} \frac{dy}{dx} + \frac{1}{y} \cdot x = x$$

Let $$\frac{1}{y} = z$$

So that $$-\frac{1}{y^2} \cdot \frac{dy}{dx} = \frac{dz}{dx}$$

$\therefore$ The given equation reduces to

$$\frac{dz}{dx} - xz = -x$$

$$\text{IF} = e^{\int -x\, dx} = e^{-x^2/2}$$

$\therefore$ The solution is

$$z\, e^{-x^2/2} = \int -x \cdot e^{-x^2/2}\, dx = e^{-x^2/2} + C$$

i.e. $$\frac{1}{y} = 1 + Ce^{x^2/2}$$

Example 41 Solve $\frac{dy}{dx} = \frac{y\, \phi'(x) - y^2}{\phi(x)}$, where $\phi(x)$ is a given function.

Sol. The equation can be written as

$$\frac{dy}{dx} - \frac{\phi'(x)}{\phi(x)} y = -\frac{y^2}{\phi(x)}$$

i.e. $$-y^{-2} \frac{dy}{dx} + \frac{\phi'(x)}{\phi(x)} \cdot \frac{1}{y} = \frac{1}{\phi(x)}$$

Let $\frac{1}{y} = z$. So that, $-\frac{1}{y^2} \frac{dy}{dx} = \frac{dz}{dx}$

$\therefore$ $$\frac{dz}{dx} + \frac{\phi'(x)}{\phi(x)} \cdot z = \frac{1}{\phi(x)}$$

$$\text{IF} = e^{\int \frac{\phi'(x)}{\phi(x)} dx} = e^{\ln \phi(x)} = \phi(x)$$

$\therefore$ The solution is $z \cdot \phi(x) = \int \frac{1}{\phi(x)} \cdot \phi(x)\, dx = x + C$

i.e. $$\frac{\phi(x)}{y} = x + C \quad ie, \quad \frac{\phi(x)}{x + C} = y$$

Remark

Another type of equation which is reducible to the linear form is

$$f'(y) \frac{dy}{dx} + P(x) \cdot f(y) = Q(x)$$

An equation of this type can be easily reduced the linear form by taking $z = f(y)$.

Example 42 Solve $\sec^2 y \frac{dy}{dx} + 2x \tan y = x^3$.

Sol. Let $\tan y = z$ so that $\sec^2 y \cdot \frac{dy}{dx} = \frac{dz}{dx}$

Thus, the given equation reduces to

$$\frac{dz}{dx} + 2x \cdot z = x^3$$

$$\text{IF} = e^{\int 2x\, dx} = e^{x^2}$$

$\therefore$ The solution is, $z \cdot e^{x^2} = \int x^3 \cdot e^{x^2} dx$

i.e. $$\tan y \cdot e^{x^2} = \frac{1}{2} \int x^2 e^{x^2} \cdot (2x)\, dx = \frac{1}{2} \int t \cdot e^t\, dt,$$

where $t = x^2$

$$\tan y \cdot e^{x^2} = \frac{1}{2}(t \cdot e^t - e^t) + C$$

i.e. $$\tan y = Ce^{-x^2} + \frac{e^{-x^2}}{2} \cdot e^{x^2}(x^2 - 1)$$

$$\tan y = Ce^{-x^2} + \frac{1}{2}(x^2 - 1)$$

Example 43 Solve $\frac{dy}{dx} + x(x+y) = x^3(x+y)^3 - 1.$

Sol. The given equation can be written as

$$\left(\frac{dy}{dx}+1\right) + x(x+y) = x^3(x+y)^3$$

i.e. $$\frac{d(x+y)}{dx} + x(x+y) = x^3(x+y)^3$$

i.e. $$(x+y)^{-3} \cdot \frac{d(x+y)}{dx} + x(x+y)^{-2} = x^3$$

Let $(x+y)^{-2} = z$ so that $-2(x+y)^{-3}\frac{d(x+y)}{dx} = \frac{dz}{dx}$

The given equation reduces to

$$-\frac{1}{2}\frac{dz}{dx} + xz = x^3$$

i.e. $$\frac{dz}{dx} - 2xz = -2x^3$$

$$\text{IF} = e^{\int -2x\,dx} = e^{-x^2}$$

∴ The solution is

$$z \cdot e^{-x^2} = \int -2x^3 \cdot e^{-x^2}\,dx = (x^2+1)e^{-x^2} + C$$

$$\frac{1}{(x+y)^2} = Ce^{x^2} + x^2 + 1$$

Example 44 Solve $\sin y \cdot \frac{dy}{dx} = \cos y(1 - x\cos y).$

Sol. The given differential equation is

$$\sin y\frac{dy}{dx} = \cos y(1 - x\cos y)$$

or $$\sin y\frac{dy}{dx} - \cos y = -x\cos^2 y$$

Dividing by $\cos^2 y$, we get

$$\tan y \cdot \sec y \cdot \frac{dy}{dx} - \sec y = -x$$

Let $\sec y = v \Rightarrow \sec y \tan y \cdot \frac{dy}{dx} = \frac{dv}{dx}$

So that $$\frac{dv}{dx} - v = -x$$

Which is linear differential equation with $P = -1, Q = -x$

$$\text{IF} = e^{\int P dx} = e^{\int -1\,dx} = e^{-x}$$

The solution is given by

$$v \cdot e^{-x} = \int -x \cdot e^{-x}\,dx = xe^{-x} + e^{-x} + C$$

$$= e^{-x}(x+1) + C$$

or $$v = (1+x) + Ce^x$$

or $$\sec y = (1+x) + Ce^x$$

Orthogonal Trajectory

Any curve, which cuts every member of a given family of curves at right angles, is called an orthogonal trajectory of the family. For example, each straight line passing through the origin, *ie*, $y = kx$ is an orthogonal trajectory of the family of the circles $x^2 + y^2 = a^2$.

Procedure for Finding the Orthogonal Trajectory

(i) Let $f(x, y, c) = 0$ be the equation of the given family of curves, where c is an arbitrary parameter.

(ii) Differentiate $f = 0$; w.r.t. 'x' and eliminate 'c', i.e. form a differential equation.

(iii) Substitute $-\frac{dx}{dy}$ for $\frac{dy}{dx}$ in the above differential equation. This will give the differential equation of the orthogonal trajectories.

(iv) By solving this differential equation, we get the required orthogonal trajectories.

Example 45 Find the orthogonal trajectories of the hyperbola $xy = C$.

Sol. The equation of the given family of curves is $xy = c$ ...(i)

Differentiating Eq. (i) w.r.t. x, we get

$$\frac{x\,dy}{dx} + y = 0 \quad \text{...(ii)}$$

Substitute $-\frac{dx}{dy}$ for $\frac{dy}{dx}$ in Eq. (ii), we get

$$-\frac{x\,dx}{dy} + y = 0 \quad \text{...(iii)}$$

This is the differential equation for the orthogonal trajectory of given family of hyperbola. Eq. (iii) can be rewritten as $x\,dx = y\,dy$, which on integration gives

$$x^2 - y^2 = C.$$

This is the family of required orthogonal trajectories.

Example 46 Find the orthogonal trajectories of the curves $y = cx^2$.

Sol. Here, $$y = cx^2 \quad \text{...(i)}$$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = 2cx \quad \text{...(ii)}$$

Eliminating c from Eqs. (i) and (ii),

$$y = \left(\frac{1}{2x}\frac{dy}{dx}\right)x^2$$

$$\Rightarrow \quad 2y = x\frac{dy}{dx} \quad \text{...(iii)}$$

This is the differential equation of the family of curves given in Eq. (i).

Now, to obtain orthogonal trajectory replace $\frac{dy}{dx}$ by $-\frac{dx}{dy}$ in Eq. (iii).

$$\Rightarrow \quad 2y = -x\frac{dx}{dy}$$

or $\quad 2y\,dy = -x\,dx$

Integrating both the sides, we get

$$y^2 = -\frac{x^2}{2} + C_1$$

$\Rightarrow x^2 + 2y^2 = C_1$, is the required family of orthogonal trajectory.

| Example 47 Find the equation of all possible curves that will cut each member of the family of circles $x^2 + y^2 - 2cx = 0$ at right angle.

Sol. Here, $\quad x^2 + y^2 - 2cx = 0 \quad$ ...(i)

Differentiating w.r.t. x, we get

$$2x + 2yy_1 - 2c = 0$$

$$\Rightarrow \quad c = x + yy_1 \quad \text{...(ii)}$$

From Eqs. (i) and (ii), we eliminate c

$$\Rightarrow \quad x^2 + y^2 - 2(x + yy_1)x = 0$$

or $\quad -x^2 + y^2 - 2xy\frac{dy}{dx} = 0$

This is the differential equation representing the given family of circles. To find differential equation of the orthogonal trajectories, we replace $\frac{dy}{dx}$ by $-\frac{dx}{dy}$.

$$\Rightarrow \quad y^2 - x^2 = -2xy\frac{dx}{dy}$$

$$\Rightarrow \quad y^2\,dy = x^2\,dy - 2xy\,dx$$

$$\Rightarrow \quad -dy = \frac{yd(x^2) - x^2\,dy}{y^2}$$

$$\Rightarrow \quad -dy = d\left(\frac{x^2}{y}\right)$$

Integrating both the sides, we get

$$-y = \frac{x^2}{y} + C \Rightarrow x^2 + y^2 + Cy = 0$$

Represents family of orthogonal trajectory.

| Example 48 Find the orthogonal trajectory of the circles

$$x^2 + y^2 - ay = 0.$$

Sol. Here, $\quad x^2 + y^2 - ay = 0 \quad$ ...(i)

Differentiating, we get

$$2x + 2yy_1 - ay_1 = 0$$

$$\Rightarrow \quad a = \frac{2(x + yy_1)}{y_1} \quad \text{...(ii)}$$

Substituting 'a' in Eq. (i), we get

$$x^2 + y^2 - \frac{2(x + yy_1)}{y_1}y = 0$$

$$\Rightarrow \quad (x^2 - y^2)y_1 - 2xy = 0$$

This is the differential equation of the family of circles given in Eq. (i).

$\therefore$ The differential representing the orthogonal trajectory is obtained by replacing $\frac{dy}{dx}$ by $-\frac{dy}{dx}$.

i.e. $\quad -(x^2 - y^2)\frac{dx}{dy} - 2xy = 0$

$$\Rightarrow \quad 2xy\,dy - y^2\,dx = -x^2\,dx$$

$$\Rightarrow \quad \frac{x\,d(y^2) - y^2\,dx}{x^2} = -dx$$

$$\Rightarrow \quad d\left(\frac{y^2}{x}\right) = -dx$$

Integrating both the sides, we get

$y^2 + x^2 = Cx$, is required family of orthogonal trajectories.

Exercise for Session 3

1. The solution of $(1+x^2)\frac{dy}{dx}+y=e^{\tan^{-1}x}$, is given by

(a) $2ye^{\tan^{-1}x}=e^{2\tan^{-1}x}+C$
(b) $ye^{\tan^{-1}y}=e^{2\tan^{-1}x}+C$
(c) $2ye^{\tan^{-1}y}=e^{2\tan^{-1}y}+C$
(d) None of these

2. The solution of $\frac{dy}{dx}+\frac{x}{1-x^2}y=x\sqrt{y}$, is given by

(a) $3\sqrt{y}+(1-x^2)=C(1-x^2)^{1/4}$
(b) $\frac{3}{2}\sqrt{y}+(1-x^2)=C(1-x^2)^{3/2}$
(c) $3\sqrt{y}-(1-x^2)=C(1-x^2)^{3/2}$
(d) None of these

3. The solution of $\frac{dy}{dx}+x\sin 2y=x^3\cos^2 y$, is

(a) $e^{x^2}=(x^2-1)e^{x^2}\tan y+C$
(b) $e^{x^2}\tan y=\frac{1}{2}(x^2-1)e^{x^2}+C$
(c) $e^{x^2}\tan y=(x^2-1)\tan y+C$
(d) None of these

4. The solution of $3x(1-x^2)y^2\,dy/dx+(2x^2-1)y^3=ax^3$ is

(a) $y^3=ax+C\sqrt{1-x^2}$
(b) $y^3=ax+Cx\sqrt{1-x^2}$
(c) $y^2=ax+C\sqrt{1-x^2}$
(d) None of these

5. The solution of $\frac{dy}{dx}+\frac{y}{x}\log y=\frac{y}{x^2}(\log y)^2$, is

(a) $x=\frac{1}{2x}\log y+C$
(b) $x^2+\log y=C$
(c) $\frac{1}{x\log y}=\frac{1}{2x^2}+C$
(d) None of these

6. The solution of $\frac{dy}{dx}+yf'(x)-f(x)\cdot f'(x)=0, y\neq f(x)$ is

(a) $y=f(x)+1+ce^{-f(x)}$
(b) $y-ce^{-f(x)}$
(c) $y=f(x)-1+ce^{-f(x)}$
(d) None of these

7. The solution of $(x^{2-1})\,dy/dx\cdot\sin y-2x\cdot\cos y=2x-2x^3$, is

(a) $(x^2-1)\cos y=\frac{x^4}{2}-x^2+C$
(b) $(x^2-1)\sin y=\frac{x^4}{2}-x^2+C$
(c) $(x^2-1)\cos y=\frac{x^4}{4}-\frac{x^2}{2}+C$
(d) $(x^2-1)\sin y=\frac{x^4}{4}-\frac{x^2}{2}+C$

8. The Curve possessing the property text the intercept made by the tangent at any point of the curve on the y-axis is equal to square of the abscissa of the point of tangency, is given by

(a) $y^2=x+C$
(b) $y=2x^2+cx$
(c) $y=-x^2+cx$
(d) None of these

9. The tangent at a point P of a curve meets the y-axis at A, and the line parallel to y-axis at A, and the line parallel to y-axis through P meets the x-axis at B. If area of ΔOAB is constant (O being the origin), Then the curve is

(a) $cx^2-xy+k=0$
(b) $x^2+y^2=cx$
(c) $3x^2+4y^2=k$
(d) $xy-x^2y^2+kx=0$

10. The value of k such that the family of parabolas $y=cx^2+k$ is the orthogonal trajectory of the family of ellipse $x^2+2y^2-y=C$, is

(a) y_2
(b) y_3
(c) y_4
(d) y_5

Session 4

Exact Differential Equations

Exact Differential Equations

A differential equation of the form $M(x, y)\,dx + N(x, y)\,dy = 0$ is said to be exact (or total) if its left hand expression is the exact differential of some function $u(x, y)$.

i.e. $$du = M \cdot dx + N \cdot dy$$

Hence, its solution is $u(x, y) = c$ (where c is an arbitrary constant). But then there is a question that how do we confirm whether the above mentioned equation is exact. The answer to this question is the following theorem.

Theorem The necessary and sufficient condition for the differential equation $M\,dx + N\,dy = 0$ to be exact is $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.

The solution of $M\,dx + N\,dy = 0$ is ,

$$\int_{y-\text{constant}} M\,dx + \int (\text{terms of } N \text{ not containing } x)\,dy = C$$

provided $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.

| Example 49 Solve $(x^2 - ay)\,dx + (y^2 - ax)\,dy = 0$.

Sol. Here, we have $M = x^2 - ay$ and $N = y^2 - ax$

$$\therefore \quad \frac{\partial M}{\partial y} = -a = \frac{\partial N}{\partial x}$$

Thus, the equation is exact.

$\therefore$ The solution is, $\int_{y-\text{constant}} (x^2 - ay)\,dx + \int y^2\,dy = C$

$$\Rightarrow \quad \frac{x^3}{3} - axy + \frac{y^3}{3} = C$$

| Example 50 Solve $(2x \log y)\,dx + \left(\frac{x^2}{y} + 3y^2\right)dy = 0$.

Sol. Here, we have $M = 2x \log y$ and $N = \frac{x^2}{y} + 3y^2$

$$\therefore \quad \frac{\partial M}{\partial y} = \frac{2x}{y}$$

and $\frac{\partial N}{\partial x} = \frac{2x}{y}$ and hence the equation is exact.

$\therefore$ The solution is,

$$\int_{y-\text{constant}} (2x \log y)\,dx + \int 3y^2\,dy = C$$

$$\Rightarrow \quad x^2 \log y + y^3 = C$$

Equations Reducible to the Exact Form

Sometimes a differential equation of the form $M\,dx + N\,dy = 0$ which is not exact can be reduced to an exact form by multiplying by a suitable function $f(x, y)$ which is not identically zero. This function $f(x, y)$ which then multiplied to a non-exact differential equation makes it exact is known as integrating factor.

One can find integrating factors by inspection but for that some experience and practice is required.

For finding the integrating factors by inspection, the following identities must be remembered.

1. $x\,dy + y\,dx = d(xy)$
2. $x\,dx + y\,dy = \frac{1}{2} d(x^2 + y^2)$
3. $\frac{x\,dy - y\,dx}{x^2} = d\left(\frac{y}{x}\right)$
4. $\frac{y\,dx - x\,dy}{y^2} = d\left(\frac{x}{y}\right)$
5. $\frac{x\,dy - y\,dx}{xy} = \frac{dy}{y} - \frac{dx}{x} = d\left[\log\left(\frac{y}{x}\right)\right]$
6. $\frac{y\,dx - x\,dy}{xy} = d\left[\log\left(\frac{x}{y}\right)\right]$
7. $\frac{x\,dy - y\,dx}{x^2 + y^2} = \frac{\frac{x\,dy - y\,dx}{x^2}}{1 + \frac{y^2}{x^2}} = \frac{d\left(\frac{y}{x}\right)}{1 + \left(\frac{y}{x}\right)^2} = d\left[\tan^{-1}\left(\frac{y}{x}\right)\right]$
8. $\frac{dx + dy}{x + y} = d[\ln(x + y)]$
9. $d(\ln(xy)) = \frac{x\,dy + y\,dx}{xy}$
10. $d\left(\frac{1}{2}\ln(x^2 + y^2)\right) = \frac{x\,dx + y\,dy}{x^2 + y^2}$

11. $d\left(-\frac{1}{xy}\right)=\frac{x\,dy+y\,dx}{x^2y^2}$

12. $d\left(\frac{e^y}{x}\right)=\frac{x\,e^y\,dy-e^y\,dx}{x^2}$

13. $d\left(\frac{e^x}{y}\right)=\frac{ye^x dx-e^x dy}{y^2}$

14. $d(x^m y^n)=x^{m-1}y^{n-1}(my\,dx+nx\,dy)$

15. $\frac{d[f(x,y)]^{1-n}}{1-n}=\frac{f'(x,y)}{(f(x,y))^n}$

Example 51 Solve $(x^2-ay)\,dx+(y^2-ax)\,dy=0$.

Sol. The given differential equation is

$$x^2dx+y^2dy-a(y\,dx+x\,dy)=0$$

$$\Rightarrow \quad d\left(\frac{x^3}{3}\right)+d\left(\frac{y^3}{3}\right)-ad(xy)=0$$

Integrating, we get $\frac{x^3}{3}+\frac{y^3}{3}-a\,xy=k$

$$\Rightarrow \quad x^3+y^3-3axy=3k=C$$

Example 52 Solve $(2x\log y)\,dx+\left(\frac{x^2}{y}+3y^2\right)dy=0$.

Sol. The given differential equation is

$$(\log y)\,2x\,dx+\frac{x^2}{y}dy+3y^2\,dy=0$$

$$\Rightarrow \quad (\log y)\,d(x^2)+x^2\,d(\log y)+d(y^3)=0$$

$$\Rightarrow \quad d(x^2\log y)+d(y^3)=0$$

$$\Rightarrow \quad x^2\log y+y^3=C$$

(integrating both the sides)

Example 53 Solve $x\,dx+y\,dy=x\,dy-y\,dx$.

Sol. The given equation can be written as

$$\frac{1}{2}d(x^2+y^2)=x^2\,d\left(\frac{y}{x}\right)$$

$$\Rightarrow \quad \frac{d(x^2+y^2)}{x^2+y^2}=\frac{2x^2\,d\left(\frac{y}{x}\right)}{x^2+y^2}$$

$$\Rightarrow \quad \frac{d(x^2+y^2)}{x^2+y^2}=\frac{2d(y/x)}{1+(y/x)^2}$$

$$\Rightarrow \quad \log(x^2+y^2)=2\tan^{-1}\left(\frac{y}{x}\right)+C$$

Example 54 Solve

$$\frac{y+\sin x\cos^2(xy)}{\cos^2(xy)}dx+\left(\frac{x}{\cos^2(xy)}+\sin y\right)dy=0.$$

Sol. The given differential equation can be written as

$$\frac{y\,dx+x\,dy}{\cos^2(xy)}+\sin x\,dx+\sin y\,dy=0$$

$$\Rightarrow \quad \sec^2(xy)\,d(xy)+\sin x\,dx+\sin y\,dy=0$$

$$d(\tan(xy))+d(-\cos x)+d(-\cos y)=0$$

$$\Rightarrow \quad \tan(xy)-\cos x-\cos y=C$$

Example 55 Solve $\dfrac{x+y\dfrac{dy}{dx}}{y-x\dfrac{dy}{dx}}=x^2+2y^2+\dfrac{y^4}{x^2}$.

Sol. The given equation can be written as

$$\frac{x\,dx+y\,dy}{(x^2+y^2)^2}=\frac{y\,dx-x\,dy}{y^2}\cdot\frac{y^2}{x^2}$$

$$\Rightarrow \quad \int\frac{d(x^2+y^2)}{(x^2+y^2)^2}=2\int\frac{1}{x^2/y^2}\,d\left(\frac{x}{y}\right)$$

Integrating both the sides, we get

$$-\frac{1}{(x^2+y^2)}=-\frac{1}{(x/y)}+C$$

$$\Rightarrow \quad \frac{y}{x}-\frac{1}{x^2+y^2}=C$$

Example 56 The solution of

$$e^{\frac{x(y^2-1)}{y}}\{xy^2\,dy+y^3dx\}+\{ydx-xdy\}=0,\text{ is}$$

(a) $e^{xy}+e^{x/y}+C=0$ (b) $e^{xy}-e^{x/y}+C=0$

(c) $e^{xy}+e^{y/x}+C=0$ (d) $e^{xy}-e^{y/x}+C=0$

Sol. Here, $e^{\frac{x(y^2-1)}{y}}\cdot y^2\{xdy+ydx\}+\{ydx-xdy\}=0$

$$\Rightarrow \quad e^{xy}\cdot y^2\cdot\{xdy+ydx\}+e^{x/y}\{ydx-xdy\}=0$$

or $$e^{xy}\cdot\{xdy+ydx\}+e^{x/y}\frac{\{ydx-xdy\}}{y^2}=0$$

or $$e^{xy}\cdot d(xy)+e^{x/y}\cdot d\left(\frac{x}{y}\right)=0$$

or $$d(e^{xy})+d(e^{x/y})=0$$

Integrating both the sides, we get

$$e^{xy}+e^{x/y}+C=0$$

Hence, (a) is the correct answer.

Example 57 The solution of $x^2dy - y^2dx + xy^2(x-y)\,dy = 0$, is

(a) $\log\left|\dfrac{x-y}{xy}\right| = \dfrac{y^2}{2} + C$ (b) $\log\left|\dfrac{xy}{x-y}\right| = \dfrac{x^2}{2} + C$

(c) $\log\left|\dfrac{x-y}{xy}\right| = \dfrac{x^2}{2} + C$ (d) $\log\left|\dfrac{x-y}{xy}\right| = x + C$

Sol. Here, $$x^2y^2\left(\frac{dy}{y^2} - \frac{dx}{x^2}\right) + x^2y^3\left(\frac{1}{y} - \frac{1}{x}\right)dy = 0$$

$$\Rightarrow \quad -d\left(\frac{1}{y} - \frac{1}{x}\right) + y\left(\frac{1}{y} - \frac{1}{x}\right)dy = 0$$

$$\Rightarrow \quad -\frac{d\left(\frac{1}{y} - \frac{1}{x}\right)}{\left(\frac{1}{y} - \frac{1}{x}\right)} + y\,dy = 0 \text{ or } \frac{d\left(\frac{1}{y} - \frac{1}{x}\right)}{\left(\frac{1}{y} - \frac{1}{x}\right)} = d\left(\frac{y^2}{2}\right)$$

Integrating both the sides, we get

$$\log\left|\frac{1}{x} - \frac{1}{y}\right| = \frac{y^2}{2} + C$$

Hence, (a) is the correct answer.

Example 58 The solution of the differential equation $ydx - xdy + xy^2\,dx = 0$, is

(a) $\dfrac{x}{y} + x^2 = \lambda$ (b) $\dfrac{x}{y} + \dfrac{x^2}{2} = \lambda$

(c) $\dfrac{x}{2y^2} + \dfrac{x^2}{4} = \lambda$ (d) None of these

Sol. Given equation is, $ydx - xdy + xy^2\,dx = 0$

Which could be converted into exact form

i.e. $$\frac{ydx - xdy}{y^2} + xdx = 0$$

$$\Rightarrow \quad d\left(\frac{x}{y}\right) + d\left(\frac{x^2}{2}\right) = 0$$

Integrating both the sides, we get

$$\frac{x}{y} + \frac{x^2}{2} = \text{constant}$$

or $$\frac{x}{y} + \frac{x^2}{2} = \lambda$$

Hence, (b) is the correct answer.

Example 59 The solution of differential equation $xdy\,(y^2e^{xy} + e^{x/y}) = ydx\,(e^{x/y} - y^2e^{xy})$, is

(a) $xy = \log(e^x + \lambda)$ (b) $x^2/y = \log(e^{x/y} + \lambda)$

(c) $xy = \log(e^{x/y} + \lambda)$ (d) $xy^2 = \log(e^{x/y} + \lambda)$

Sol. The given equation is

$$(xy^2e^{xy})\,dy + (xe^{x/y})\,dy = (ye^{x/y})\,dx - (y^3e^{xy})\,dx$$

$$\Rightarrow \quad y^2e^{xy}(xdy + ydx) = e^{x/y}(ydx - xdy)$$

$$\Rightarrow \quad e^{xy}(d(xy)) = e^{x/y}\cdot\left(\frac{ydx - xdy}{y^2}\right)$$

$$\Rightarrow \quad e^{xy}(d(xy)) = e^{x/y}\cdot d\left(\frac{x}{y}\right)$$

$$\Rightarrow \quad d(e^{xy}) = d(e^{x/y})$$

Integrating both the sides, we get

$$e^{xy} = e^{x/y} + \lambda$$

$$\Rightarrow \quad xy = \log(e^{x/y} + \lambda)$$

Hence, (c) is the correct answer.

Example 60 The solution of the differential equation $(y + x\sqrt{xy}\,(x+y))\,dx + (y\sqrt{xy}\,(x+y) - x)\,dy = 0$, is

(a) $\dfrac{x^2+y^2}{2} + 2\tan^{-1}\sqrt{\dfrac{x}{2y}} = C$

(b) $\dfrac{x^2+y^2}{2} + 2\tan^{-1}\sqrt{\dfrac{x}{y}} = C$

(c) $\dfrac{x^2+y^2}{\sqrt{2}} + 2\tan^{-1}\sqrt{\dfrac{x}{y}} = C$

(d) None of these

Sol. The given equation can be written as

$$(ydx - xdy) + x\sqrt{xy}\,(x+y)\,dx + y\sqrt{xy}\,(x+y)\,dy = 0$$

$$\Rightarrow \quad (ydx - xdy) + (x+y)\sqrt{xy}\,(xdx + ydy) = 0$$

$$\Rightarrow \quad \frac{ydx - xdy}{y^2} + \left(\frac{x}{y} + 1\right)\cdot\sqrt{\frac{x}{y}}\,d\left(\frac{x^2+y^2}{2}\right) = 0$$

$$\Rightarrow \quad d\left(\frac{x}{y}\right) + d\left(\frac{x^2+y^2}{2}\right)\left(\frac{x}{y} + 1\right)\sqrt{\frac{x}{y}} = 0$$

$$\Rightarrow \quad d\left(\frac{x^2+y^2}{2}\right) + \frac{d\left(\frac{x}{y}\right)}{\left(\frac{x}{y} + 1\right)\cdot\sqrt{\frac{x}{y}}} = 0$$

or $$d\left(\frac{x^2+y^2}{2}\right) + 2d\left(\tan^{-1}\left(\sqrt{\frac{x}{y}}\right)\right) = 0$$

Integrating both the sides, we get

$$\Rightarrow \quad \frac{x^2+y^2}{2} + 2\tan^{-1}\sqrt{\frac{x}{y}} = C$$

Hence, (b) is the correct answer.

Exercise for Session 4

1. The solution of $xdy + ydx + 2x^3dx = 0$, is

(a) $xy + x^4 = C$ (b) $xy + \frac{1}{2}x^4 = C$ (c) $\frac{x^2}{y} + \frac{x^4}{4} = C$ (d) None of these

2. The solution of, $ydx - xdy + (1 + x^2)\,dx + x^2 \sin y\, dy = 0$, is given by

(a) $x + 1 - y^2 + \cos y + C = 0$ (b) $y + 1 - x^2 + x\cos y + C = 0$

(c) $\frac{x}{y} + \frac{1}{y} - y + \cos y + C = 0$ (d) $\frac{y}{x} + \frac{1}{x} - x + \cos y + C = 0$

3. The solution of $(1 + x\sqrt{x^2 + y^2})\,dx + (-1 + \sqrt{x^2 + y^2})\,ydy = 0$, is

(a) $2x - y^2 + \frac{2}{3}(x^2 + y^2)^{3/2} = C$ (b) $2x - y + \frac{2}{3}(x^2 + y^2)^{3/2} = C$

(c) $2y - x^2 + \frac{2}{3}(x^2 + y^2)^{3/2} = C$ (d) None of these

4. The solution of, $\frac{xdy}{x^2 + y^2} = \left(\frac{y}{x^2 + y^2} - 1\right)dx$, is given by

(a) $\tan^{-1}\left(\frac{x}{y}\right) + x = C$ (b) $\tan^{-1}\left(\frac{y}{x}\right) + x = C$ (c) $\tan^{-1}\left(\frac{y}{x}\right) + xy = C$ (d) $\tan^{-1}\left(\frac{y}{x}\right) + x^2 = C$

5. The solution of $ye^{x/y}\,dx = (xe^{x/y} + y^2 \sin y)\,dy$, is given by

(a) $e^{x/y} = -\cos y + C$ (b) $e^{x/y} + 2\cos y = C$ (c) $e^{x/y} = x\cos y + C$ (d) $e^{x/y} = 2\cos y\, e^{x/y} + C$

6. The solution of $x \sin\left(\frac{y}{x}\right)dy = \left\{y \sin\left(\frac{y}{x}\right) - x\right\}dx$, is given by

(a) $\log x - \cos\left(\frac{y}{x}\right) = \log C$ (b) $\log x - \sin\left(\frac{y}{x}\right) = C$ (c) $\log\left(\frac{x}{y}\right) - \cos\left(\frac{y}{x}\right) = \log C$ (d) None of these

7. The solution of $\frac{xdx + ydy}{xdy - ydx} = \sqrt{\frac{a^2 - x^2 - y^2}{x^2 + y^2}}$, is given by

(a) $\sin^{-1}(\sqrt{x^2 + y^2}) = a\tan^{-1}\left(\frac{y}{x}\right) + C$ (b) $\sin^{-1}(\sqrt{x^2 + y^2}) = \frac{1}{a}\tan^{-1}\left(\frac{y}{x}\right) + C$

(c) $\sin^{-1}\left(\frac{\sqrt{x^2 + y^2}}{a}\right) = \tan^{-1}\left(\frac{y}{x}\right) + C$ (d) None of the above

8. The solution of $(1 + e^{x/y})\,dx + e^{x/y}\left(1 - \frac{x}{y}\right)dy = 0$, is given by

(a) $x - ye^{x/y} = C$ (b) $x + ye^{x/y} = C$ (c) $y - \frac{x}{y}e^{x/y} = C$ (d) None of these

9. The solution of $\frac{x + y\,dy/dx}{y - x\,dy/dx} = \frac{x\sin^2(x^2 + y^2)}{y^3}$, is given by

(a) $-\cot(x^2 + y^2) = \left(\frac{x}{y}\right)^2 + C$ (b) $\tan(x^2 + y^2) = x^2y^2 + C$

(c) $\cot(x^2 + y^2) = \frac{x}{y} + C$ (d) None of these

10. The solution of $\frac{dy}{dx} + \frac{y}{x} = \frac{1}{(1 + \log x + \log y)^2}$, Is given by

(a) $xy\,(1 + \log(xy)) = C$ (b) $xy^2\,(1 + \log(xy)) = C$ (c) $xy\,(1 + \log(xy))^2 = C$ (d) $xy\,(1 + (\log xy)^2) = C$

Session 5

Solving of First Order and Higher Degrees, Application of Differential Equations, Application of First Order Differential Equations

Solving of First Order and Higher Degrees

Differential Equation of First Order and Higher Degrees

A differential equation of first order is of the form $f(x, y, P)$ where $P = dy / dx$. If in the equation degree of P is greater than one, then the equation is of first order and higher degree.

The differential equation of first order and higher degree can be written in the form

$$P^n + F_1(x, y) P^{n-1} + \ldots + F_{n-1}(x, y) P + F_n(x, y) = 0$$

The differential equations of these category can be solved by one or more of the following methods :

(i) Equations solvable for P.

(ii) Equations solvable for y.

(iii) Equations solvable for x.

(iv) Clairaut's equations.

Now, we shall discuss these cases.

(i) Equations Solvable For P

If the equation

$$P^n + F_1(x, y) P^{n-1} + \ldots + F_{n-1}(x, y) P + F_n(x, y) = 0,$$

is solvable for P, then LHS expression can be resolved into n linear factors and hence can be put in the form

$$(P - f_1(x, y))(P - f_2(x, y)) \ldots (P - f_n(x, y)) = 0.$$

Equating each of these factors to zero, we get n differential equations of the first order and first degree.

$$\frac{dy}{dx} = f_1(x, y), \frac{dy}{dx} = f_2(x, y), \ldots, \frac{dy}{dx} = f_n(x, y)$$

Let the solutions of these obtained equations are

$$\phi_1(x, y, c_1) = 0, \phi_2(x, y, c_2) = 0, \ldots, \phi_n(x, y, c_n) = 0$$

respectively.

Hence, the general solution is given by

$$\phi_1(x, y, c), \phi_2(x, y, c), \ldots, \phi_n(x, y, c) = 0$$

Here, the arbitrary constant $c_1, c_2, \ldots, c_n$ are replaced by a single arbitrary constant c because every first order equation has only one arbitrary constant in its solution.

Example 61 Solve $(p - x)(p - e^x)(p - 1/y) = 0$; where $p = \dfrac{dy}{dx}$.

Sol. The component linear equations are $p = x, p = e^x, p = \dfrac{1}{y}$

If $\dfrac{dy}{dx} = x$, then $dy = x\,dx \Rightarrow y = \dfrac{x^2}{2} + C_1$

If $\dfrac{dy}{dx} = e^x$, then $dy = e^x\,dx \Rightarrow y = e^x + C_2$

If $\dfrac{dy}{dx} = \dfrac{1}{y}$, then $y\,dy = dx \Rightarrow \dfrac{y^2}{2} = x + C_3$

$\therefore$ The required solution is

$$\left(y - \frac{x^2}{2} + C\right)(y - e^x + C)\left(\frac{y^2}{2} - x + C\right) = 0$$

Example 62 Solve $x^2p^2 + xyp - 6y^2 = 0$.

Sol. The given equation is

$$x^2p^2 + xyp - 6y^2 = 0$$

Solving as a quadratic in p, we get

$$p = \frac{(-xy \pm \sqrt{x^2y^2 + 24x^2y^2})}{2x^2} = \frac{2y}{x}, -\frac{3y}{x}$$

If $p = \dfrac{2y}{x}$, then $\dfrac{dy}{dx} = \dfrac{2y}{x} \Rightarrow \dfrac{dy}{y} = \dfrac{2dx}{x}$

$\Rightarrow \quad \log\left|\dfrac{y}{x^2}\right| = k \Rightarrow y = C_1 x^2$

If $p = -\dfrac{3y}{x}$, then $\dfrac{dy}{dx} = -\dfrac{3y}{x} \Rightarrow \dfrac{dy}{y} = -\dfrac{3dx}{x}$

$\Rightarrow \quad x^3y = C_2$

$\therefore$ The required solution is $(y - Cx^2)(x^3y - C) = 0$.

Example 63 Solve $xy^2(p^2+2)=2py^3+x^3$.

Sol. The given equation can be written as

$$(xy^2p^2-x^3)+2(xy^2-py^3)=0$$

$$\Rightarrow \quad x(y^2p^2-x^2)+2y^2(x-py)=0$$

$$\Rightarrow \quad (py-x)\{x(py+x)-2y^2\}=0$$

If $py-x=0$, then $y\,dy-x\,dx=0 \Rightarrow y^2-x^2=C_1$

If $xyp+x^2-2y^2=0$, then $2y\dfrac{dy}{dx}-\dfrac{4y^2}{x}=-2x$

$$\Rightarrow \quad \frac{dt}{dx}-\frac{4}{x}t=-2x,$$

where $t=y^2$ $\quad$ $\text{IF}=e^{-\int \frac{4}{x}dx}=e^{-4\ln x}=\dfrac{1}{x^4}$

Its solution is $t\left(\dfrac{1}{x^4}\right)=\int -2x\cdot\dfrac{1}{x^4}\,dx$

i.e. $\dfrac{t}{x^4}=\dfrac{1}{x^2}+C_2$ i.e. $y^2=x^2+C_2x^4$

Hence, the required solution is

$$(y^2-x^2-C)(y^2-x^2-Cx^4)=0$$

(ii) Equations Solvable For y

Equation that comes under this category, can be expressed in the form

$$y=g(x,p)$$

(i.e. an explicit function y in term of x and p) ...(i)

Differentiating Eq. (i) w.r.t. x, we get

$$\frac{dy}{dx}=p=F\left(x,p,\frac{dp}{dx}\right)$$

Which is a differential equation of the first order containing x and p. Let us suppose that its solution is

$$\phi(x,p,c)=0 \qquad \text{...(ii)}$$

Then, the solution is obtained by eliminating p between $y=g(x,p)$ and $\phi(x,p,c)=0$. However, if eliminating of p is difficult express x and y as a function of the parameter p.

Example 64 Solve $xp^2-2yp+ax=0$.

Sol. The given equation can be written as,

$$y=\frac{xp}{2}+\frac{ax}{2p} \qquad \text{...(i)}$$

$$\frac{dy}{dx}=\frac{p}{2}+\frac{x}{2}\cdot\frac{dp}{dx}+\frac{a}{2p}-\frac{ax}{2p^2}\cdot\frac{dp}{dx}$$

$$\Rightarrow \quad p(p^2-a)=x(p^2-a)\cdot\frac{dp}{dx}$$

$$\Rightarrow \quad \frac{dp}{dx}=\frac{p}{x}\Rightarrow p=Cx$$

(The equation $p^2-a=0$ gives us singular solution in which we are not interested).

The substitute p in Eq. (i), we get the required solution

$$2y=Cx^2+\frac{a}{C}$$

Example 65 Solve $y=2px-p^2$.

Sol. Differentiating the given equation w.r.t. x, we get

$$\frac{dy}{dx}=2p+2x\frac{dp}{dx}-2p\frac{dp}{dx}$$

or $\quad 2(x-p)\dfrac{dp}{dx}+p=0$

or $\quad \dfrac{dx}{dp}+\dfrac{2}{p}x=2$

It is a linear equation in x and p.

$$\text{IF}=e^{\int \frac{2}{p}dx}=e^{2\log p}=p^2$$

$\therefore$ The solution is $xp^2=\int p^2\cdot 2\,dp=\dfrac{2}{3}p^3+C$

Thus, the solution of the given equation is

$$x=\frac{2}{3}p+Cp^{-2}\text{, where } p \text{ is parameter.}$$

(iii) Equations Solvable For x

This type of equation can be put in the form

$$x=g(y,p) \qquad \text{...(i)}$$

Differentiating w.r.t. y, we get

$$\frac{1}{p}=\frac{dx}{dy}=G\left(x,p,\frac{dp}{dx}\right)$$

which is a differential equation of 1st order containing y and p and its solution is

$$\phi(y,p,c)=0$$

Then, the solution is obtained by eliminating p between $x=g(y,p)$ and $\phi(y,p,c)=0$. However, if eliminating of p is difficult express x and y as a function of the parameter p.

Example 66 Solve $y=2px+y^2p^3$.

Sol. Solving for x, we get

$$x=\frac{y}{2p}-\frac{y^2p^2}{2} \qquad \text{...(i)}$$

Differentiating Eq. (i) w.r.t. y, we get

$$\frac{dx}{dy}=\frac{1}{2p}-\frac{y}{2p^2}\cdot\frac{dp}{dy}-yp^2-y^2p\cdot\frac{dp}{dy}$$

or $\frac{1}{p} - \frac{1}{2p} + yp^2 = -y\left(\frac{1}{2p^2} + yp\right) \cdot \frac{dp}{dy}$

or $(1 + 2yp^3)\, p = -y(1 + 2p^3 y) \cdot \frac{dp}{dy}$

or $\frac{dp}{p} + \frac{dy}{y} = 0 \Rightarrow py = C \Rightarrow p = \frac{C}{y}$

Substituting this in the Eq. (i), we get

$$x = \frac{y^2}{2C} - \frac{C^2}{2} \Rightarrow y^2 = 2Cx + C^2$$

(iv) Clairaut's Equation

The differential equation $y = px + f(p)$ is known as Clairaut's equation. The solution of equation of this type is given by $y = cx + f(c)$.

Which is obtained by replacing p by c in the given equation.

Remark

Some equations can be reduced to Clairaut's form by suitable substitution.

| Example 67 Solve $y = px + \frac{p}{\sqrt{1+p^2}}$.

Sol. Its solution is, $y = cx + \frac{c}{\sqrt{1+c^2}}$

| Example 68 Solve $\sqrt{1+p^2} = \tan(px - y)$.

Sol. The given equation is

$$\sqrt{1+p^2} = \tan(px - y)$$

or $px - y = \tan^{-1}(\sqrt{1+p^2})$

or $y = px - \tan^{-1}(\sqrt{1+p^2})$

Its solution is, $y = cx - \tan^{-1}(\sqrt{1+c^2})$

| Example 69 Solve $y^2 \log y = pxy + p^2$.

Sol. Let $\log y = t$. Then $\frac{1}{y}\frac{dy}{dx} = \frac{dt}{dx}$

So, if $\frac{dt}{dx} = p$, then $\frac{p}{y} = p$

Substituting these in the given equation, we have

$$y^2 t = y \cdot p\, xy + p^2 y^2 \quad \text{or} \quad t = px + p^2$$

Which is in Clairaut's form.

Thus, the required solution is

$$t = cx + c^2 \quad \text{or} \quad \log y = cx + c^2$$

(c being an arbitrary constant.)

Application of Differential Equations

Differential Equation of First Order But not of First Degree

1. The most general form of a first order and higher degree differential equation is $p^n + P_1 p^{n-1} + P_2 p^{n-2} + \ldots\ldots + P_n = 0$ where $P_1, P_2, \ldots\ldots, P_n$ are function of x, y and $p = dy/dx$. If a 1st order any degree equation can be resolved into differential equation (involving p) of first degree and 1st order, in such case we say that the equation is solvable for p.

   Let their solution be $g_1(x, y, c_1) \times g_2(x, y, c_2) \times \ldots \times g_n(x, y, c_n) = 0$, (where $c_1, c_2, \ldots\ldots\ldots, c_n$, are arbitrary constant) we take $c_1 = c_2 = \ldots\ldots = c_n = c$ because the differential equation of 1st order 1st degree contain only one arbitrary constant. So solution is

   $$g_1(x, y, c) \times g_2(x, y, c) \times \ldots \times g_n(x, y, c) = 0$$

2. The most general form of a first order and higher degree differential equation is $p^n + P_1 p^{n-1} + P_2 p^{n-2} + \ldots + P_n = 0$, where $P_1, P_2, \ldots\ldots, P_n$ are function of x, y and $p = dy/dx$. If differential equation is expressible in the form $y = f(x, p)$, then

   ***Step* 1** Differentiate w.r.t. x, we get $p\frac{dy}{dx} = f\left(x, p, \frac{dp}{dx}\right)$.

   ***Step* 2** Solving this we obtain $\phi(x, p, c) = 0$.

   ***Step* 3** The solution of differential equation is obtained by eliminating p.

Application of First Order Differential Equations

Growth and Decay Problems

Let $N(t)$ denotes the amount of substance (or population) that is either growing or decaying. If we assume that dN/dt, the time rate of change of this amount of substance, is proportional to the amount of substance present, then

$$\frac{dN}{dt} = kN \quad \text{or} \quad \frac{dN}{dt} - kN = 0 \qquad \ldots\text{(i)}$$

Where k is the constant of proportionality. We are assuming that $N(t)$ is a differentiable, hence continuous, function of time.

Example 70 The population of a certain country is known to increase at a rate proportional to the number of people presently living in the country. If after two years the population has doubled and after three years the population is 20000, estimate the number of people initially living in the country.

Sol. Let N denotes the number of people living in the country at any time t, and let N_0 denote the number of people initially living in the country. Then, from Eq. (i)

$$\frac{dN}{dt} - kN = 0$$

Which has the solution $N = Ce^{kt}$ ...(i)

At $t = 0$, $N = N_0$; hence, it follows from Eq. (i) that $N_0 = Ce^{k(0)}$ or that $C = N_0$.

Thus, $$N = N_0 e^{kt} \quad \text{...(ii)}$$

At $t = 2$, $N = 2N_0$. Substituting these values into Eq. (ii), we have

$$2N_0 = N_0 e^{2k} \text{ from which } k = \frac{1}{2}\ln 2 = 0.347$$

Substituting this value into Eq. (i) gives

$$N = N_0 e^{(0.347)\,t} \quad \text{...(iii)}$$

At $t = 3$, $N = 20000$. Substituting these values into Eq. (iii), we obtain

$$20000 = N_0 e^{(0.347)(3)}$$

Example 71 A certain radioactive material is known to decay at a rate proportional to the amount present. If initially there is 50 mg of the material present and after two hours it is observed that the material has lost 10% of its original mass, find (a) and expression for the mass of the material remaining at any time t, (b) the mass of the material after four hours, and (c) the time at which the material has decayed to one half of its initial mass.

Sol. (a) Let N denotes the amount of material present at time t. Then, from Eq. (i)

$$\frac{dN}{dt} - kN = 0$$

This differential equation is separable and linear, its solution is

$$N = Ce^{kt} \quad \text{...(i)}$$

At $t = 0$, we are given that $N = 50$. Therefore, from Eq. (i), $50 = Ce^{k(0)}$ or $C = 50$.

Thus, $$N = 50\,e^{kt} \quad \text{...(ii)}$$

At $t = 2$, 10% of the original mass of 50 mg or 5 mg has decayed. Hence, at $t = 2$, $N = 50 - 5 = 45$. Substituting these values into Eq. (ii) and solving for k, we have

$$45 = 50e^{2k} \quad \text{or} \quad k = \frac{1}{2}\ln\frac{45}{50} = -0.053$$

Substituting this value into Eq. (ii), we obtain the amount of mass present at any time t as

$$N = 50e^{-0.053\,t} \quad \text{...(iii)}$$

Where t is measured in hours.

(b) We require N at $t = 4$. Substituting $t = 4$ into Eq. (iii) and then solving for N, we find $N = 50e^{(-0.053)(4)}$

(c) We require t when $N = 50/2 = 25$. Substituting $N = 25$ into Eq. (iii) and solving for t, we find $25 = 50e^{-0.053t}$

or $$-0.053\,t = \ln\frac{1}{2} \quad \text{or} \quad t = 13\text{ h}$$

Example 72 Five mice in a stable population of 500 are intentionally infected with a contagious disease to test a theory of epidemic spread that postulates the rate of change in the infected population is proportional to the product of the number of mice who have the disease with the number that are disease free. Assuming the theory is correct, how long will it take half the population to contract the disease?

Sol. Let $N(t)$ denotes the number of mice with the disease at time t. We are given that $N(0) = 5$, and it follows that $500 - N(t)$ is the number of mice without the disease at time t. The theory predicts that

$$\frac{dN}{dt} = kN\,(500 - N) \quad \text{...(i)}$$

Where k is a constant of proportionality. This equation is different from Eq. (i) because the rate of change is no longer proportional to just the number of mice who have the disease. Eq. (i) has the differential form

$$\frac{dN}{N(500 - N)} - k\,dt = 0 \quad \text{...(ii)}$$

Which is separable. Using partial fraction decomposition, we have

$$\frac{1}{N(500 - N)} = \frac{1/500}{N} + \frac{1/500}{500 - N}$$

Hence, Eq. (ii) may be rewritten as

$$\frac{1}{500}\left(\frac{1}{N} + \frac{1}{500 - N}\right)dN - k\,dt = 0$$

It solution is $\frac{1}{500}\int\left(\frac{1}{N} + \frac{1}{500 - N}\right)dN - \int k\,dt = C$

or $$\frac{1}{500}(\ln|N| - \ln|500 - N|) - kt = C$$

Which may be rewritten as

$$\ln\left|\frac{N}{500 - N}\right| = 500\,(C + kt)$$

$$\frac{N}{500-N} = e^{500(C+kt)} \quad \text{...(iii)}$$

But $e^{500(C+kt)} = e^{500} e^{kt}$. Setting $C_1 = e^{500C}$, we can write Eq. (iii) as

$$\frac{N}{500-N} = C_1 e^{500kt} \quad \text{...(iv)}$$

At $t = 0$, $N = 5$. Substituting these values into Eq. (iv), we find

$$\frac{4}{495} = C_1 e^{500k(0)} = C_1$$

So, $C_1 = 1/99$ and Eq. (iv) becomes

$$\frac{N}{500-N} = \frac{1}{99} e^{500kt} \quad \text{...(v)}$$

We could solve Eq. (v) for N, but this is not necessary. We seek a value of t when $N = 250$, one half the population. Substituting $N = 250$ into Eq. (v) and solving for t, we obtain

$$1 = \frac{1}{99} e^{500kt}; \qquad \ln 99 = 500kt$$

or $t = 0.0091/k$ time units. Without additional information, we cannot obtain a numerical value for the constant of proportionality k or be more definitive about t.

Geometrical Applications

Let $P(x_1, y_1)$ be any point on the curve $y = f(x)$, then slope of the tangent at $P(= \tan\psi) = \left(\frac{dy}{dx}\right)_{(x_1, y_1)}$ and hence we find the following facts.

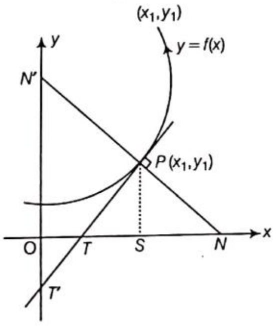


Fig. 4.2

(i) The equation of the tangent at P is,

$y - y_1 = \frac{dy}{dx}(x - x_1)$ when it cuts x-axis, $y = 0$.

$\therefore$ x-intercept of the tangent $= x_1 - y_1 \left(\frac{dx}{dy}\right)$

y-intercept of the tangent $= y_1 - x_1 \frac{dy}{dx}$

(ii) The equation of normal at P is,

$y - y_1 = -\frac{1}{(dy/dx)}(x - x_1)$ x and y-intercepts of normal are; $x_1 + y_1 \frac{dy}{dx}$ and $y_1 + x_1 \frac{dx}{dy}$

(iii) Length of tangent $= PT = |y_1| \sqrt{1 + (dx/dy)^2_{(x_1, y_1)}}$

(iv) Length of normal $= PN = |y_1| \sqrt{1 + (dy/dx)^2_{(x_1, y_1)}}$

(v) Length of subtangent $= ST = \left| y_1 \left(\frac{dx}{dy}\right)_{(x_1, y_1)} \right|$

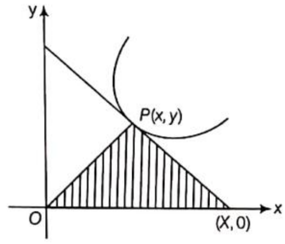
(vi) Length of subnormal $= SN = \left| y_1 \left(\frac{dy}{dx}\right)_{(x_1, y_1)} \right|$

(vii) Length of radius vector $= \sqrt{x_1^2 + y_1^2}$

Example 73 Find the curve for which the area of the triangle formed by the x-axis tangent drawn at any point on the curve and radius vector of the point of tangency is constant equal to a^2.

Sol. Tangent drawn at any point (x, y) is

$$Y - y = \frac{dy}{dx}(X - x)$$



When $\quad Y = 0, X = x - y\frac{dx}{dy}$

Area of $\Delta = 2a^2$ (given)

i.e. $\quad \left| \frac{1}{2} \cdot X \cdot Y \right| = 2a^2$

i.e. $\quad \left| xy - y^2 \frac{dx}{dy} \right| = 2a^2$

i.e. $\quad xy - y^2 \frac{dx}{dy} = \pm 2a^2$

i.e. $\quad \frac{dx}{dy} - \frac{x}{y} = \pm \frac{2a^2}{y^2}$

$$\text{IF} = e^{\int -\frac{1}{y} dy} = \frac{1}{y}$$

$\therefore$ The solution is $x \cdot \frac{1}{y} = \int \pm \frac{2a^2}{y^2} \cdot \frac{1}{y} dy$

$$\frac{x}{y} = \pm \frac{2a^2 \cdot y^{-2}}{-2} + C, \text{ i.e. } x = Cy + \frac{a^2}{y}$$

Example 74 Find the curve for which the intercept cut off by any tangent on y-axis is proportional to the square of the ordinate of the point of tangency.

Sol. The equation of tangent at any point (x, y) is

$$Y - y = \frac{dy}{dx}(X - x)$$

When $X = 0; Y = y - x\frac{dy}{dx} = y$-intercept

It is given $Y \propto y^2$. i.e. $Y = ky^2$

(k being constant of proportionality)

i.e. $y - x\frac{dy}{dx} = ky^2$

i.e. $\frac{dy}{dx} - \frac{1}{x}y = -\frac{ky^2}{x}$

i.e. $-y^{-2}\frac{dy}{dx} + \frac{1}{x} \cdot \frac{1}{y} = \frac{k}{x}$

Let $\frac{1}{y} = z$ so that, $-\frac{1}{y^2} \cdot \frac{dy}{dx} = \frac{dz}{dx}$ $\quad \therefore \frac{dz}{dx} + \frac{z}{x} = \frac{k}{x}$

$$\text{IF} = e^{\int \frac{1}{x} dx} = e^{\log x} = x$$

$\therefore$ The solution is $zx = \int \frac{kx}{x} dx = kx + C$

$\Rightarrow \quad \frac{x}{y} = kx + C$

$\Rightarrow \quad x = kxy + Cy$

$\Rightarrow \quad x - Cy = kxy \Rightarrow \frac{1}{ky} + \frac{-C}{k} \cdot \frac{1}{x} = 1$

$\Rightarrow \quad \frac{C_1}{x} + \frac{C_2}{y} = 1 \quad \left(\text{where } \frac{1}{k} = C_2 \text{ and } -\frac{C}{k} = C_1\right)$

Example 75 For any differential function $y = f(x)$, the value of $\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^3 \cdot \frac{d^2x}{dy^2}$ is equal to

(a) $2y\frac{dy}{dx}$ (b) $y^2\frac{dy}{dx}$

(c) $y\frac{dy}{dx} + \left(\frac{d^2x}{dy^2}\right)^2$ (d) None of these

Sol. We know, $\frac{dy}{dx} = \left(\frac{dx}{dy}\right)^{-1}$, differentiating both the sides

or $\quad \frac{d^2y}{dx^2} = -\left(\frac{dx}{dy}\right)^{-2} \cdot \frac{d}{dy}\left(\frac{dx}{dy}\right) \cdot \frac{dy}{dx}$

$$= -\left(\frac{dx}{dy}\right)^{-2} \cdot \frac{d^2x}{dy^2} \cdot \frac{dy}{dx}$$

$\Rightarrow \quad \frac{d^2y}{dx^2} = -\frac{d^2x}{dy^2} \cdot \left(\frac{dy}{dx}\right)^3$

or $\quad \frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^3 \cdot \frac{d^2x}{dy^2} = 0$

Hence, (d) is the correct answer.

Example 76 The solution of $y = x\frac{dy}{dx} + \frac{dy}{dx} - \left(\frac{dy}{dx}\right)^2$, is

(a) $y = (x - 1)^2$ (b) $4y = (x + 1)^2$

(c) $(y - 1)^2 = 4x$ (d) None of these

Sol. The given equation can be written as

$$y = xp + p - p^2; \quad \text{where } p = \frac{dy}{dx} \quad \text{...(i)}$$

Differentiating both the sides w.r.t. x, we get

$$p = p + \frac{xdp}{dx} + \frac{dp}{dx} - 2p\frac{dp}{dx}$$

$\therefore \quad \frac{dp}{dx}(x + 1 - 2p) = 0$

$\therefore$ Either $\frac{dp}{dx} = 0$ i.e. $p = C$ ...(ii)

or $\quad x + 1 - 2p = 0 \quad$ *ie*, $p = \frac{1}{2}(x + 1)$ ...(iii)

Eliminating p between Eqs. (i) and (ii), we get

$$y = Cx + C - C^2$$

As the complete solution and eliminating p between Eqs. (i) and (iii)

$$y = \frac{1}{2}(x + 1)x + \frac{1}{2}(x + 1) - \frac{1}{4}(x + 1)^2$$

ie, $\quad 4y = (x + 1)^2$ as the singular solution.

Hence, (b) is the correct answer.

Example 77 The solution of $\left(\frac{dy}{dx}\right)^2 + (2x + y)\frac{dy}{dx} + 2xy = 0$, is

(a) $(y + x^2 - C_1)(x + \log y + y^2 + C_2) = 0$

(b) $(y + x^2 - C_1)(x - \log y - C_2) = 0$

(c) $(y + x^2 - C_1)(x + \log y - C_2) = 0$

(d) None of the above

Sol. The given equation can be written as

$$p^2 + (2x + y)p + 2xy = 0, \text{ where } p = \frac{dy}{dx}$$

i.e. $(p + 2x)(p + y) = 0$

$\therefore$ $p + 2x = 0$, otherwise $p + y = 0$

$\Rightarrow$ $\frac{dy}{dx} + 2x = 0$ or $\frac{dy}{dx} + y = 0$

$\Rightarrow$ $\int dy + 2\int x\,dx = C_1$ or $\int \frac{dy}{y} + \int dx = C_2$

$\Rightarrow$ $y + x^2 = C_1$ or $\log y + x = C_2$

$\therefore$ $(y + x^2 - C_1) = 0$

or $(x + \log y - C_2) = 0$

$\Rightarrow$ $(y + x^2 - C_1)(x + \log y - C_2) = 0$

is the required solution.

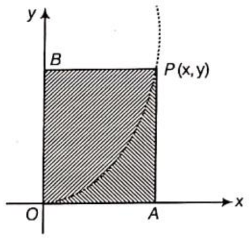
Hence, (c) is the correct answer.

Example 78 A curve $y = f(x)$ passes through the origin. Through any point (x, y) on the curve, lines are drawn parallel to the coordinate axes. If the curve divides the area formed by these lines and coordinate axes in the ratio $m : n$. Then the equation of curve is

(a) $y = Cx^{m/n}$ (b) $my^2 = Cx^{m/n}$

(c) $y^3 = Cx^{m/n}$ (d) None of these

Sol. $\frac{\text{Area of } OBPO}{\text{Area of } OPAO} = \frac{m}{n}$



$\Rightarrow$ $\frac{xy - \int_0^x y\,dx}{\int_0^x y\,dx} = \frac{m}{n} \Rightarrow nxy = (m + n)\int_0^x y\,dx$

Differentiating w.r.t. x, we get

$$n\left(x\frac{dy}{dx} + y\right) = (m + n)y$$

$\Rightarrow$ $nx\frac{dy}{dx} = my$

$\Rightarrow$ $\frac{m}{n}\cdot\frac{dx}{x} = \frac{dy}{y}$, integrating both the sides

$$y = Cx^{m/n}$$

Hence, (a) is the correct answer.

Example 79 The equation of the curve passing through the points $(3a, a)\,(a > 0)$ in the form $x = f(y)$ which satisfy the differential equation; $\frac{a^2}{xy}\cdot\frac{dx}{dy} = \frac{x}{y} + \frac{y}{x} - 2$, is

(a) $x = y + a\left(\frac{1 + e^{y-k}}{1 - 2e^{y-k}}\right)$ (b) $x = y + a\left(\frac{1 + e^{y-k}}{1 - e^{y-k}}\right)$

(c) $y = x + a\left(\frac{1 + e^{y-k}}{1 - e^{y-k}}\right)$ (d) None of these

Sol. Here, $\frac{a^2}{xy}\cdot\frac{dx}{dy} = \frac{x}{y} + \frac{y}{x} - 2$

$\Rightarrow$ $a^2 = \frac{dy}{dx}(x^2 + y^2 - 2xy)$

$\Rightarrow$ $(x - y)^2\cdot\frac{dy}{dx} = a^2$, put $x - y = v$ $\therefore 1 - \frac{dy}{dx} = \frac{dv}{dx}$

$\Rightarrow$ $v^2\left(1 - \frac{dv}{dx}\right) = a^2$

$\Rightarrow$ $v^2 - a^2 = v^2\frac{dv}{dx} \Rightarrow \frac{v^2 dv}{v^2 - a^2} = dx$

$\Rightarrow$ $\left(1 + \frac{a^2}{v^2 - a^2}\right)dv = dx$

Integrating both the sides, we get

$$v + \frac{a}{2}\log\left|\frac{v - a}{v + a}\right| = x + C$$

$\Rightarrow$ $(x - y) + \frac{a}{2}\log\left(\frac{x - y - a}{x - y + a}\right) = x + C$

$\Rightarrow$ $y + C = \frac{a}{2}\log\left(\frac{x - y - a}{x - y + a}\right)$ ...(i)

It passes through $(3a, a)$

$\Rightarrow$ $a + C = \frac{a}{2}\log\left(\frac{1}{3}\right)$

$\Rightarrow$ $C = -a + \frac{a}{2}\log\left(\frac{1}{3}\right)$

$\Rightarrow$ $C = -a\left(\frac{2 + \log 3}{2}\right)$

$\therefore$ $y = \frac{a}{2}(2 + \log 3) + \log\left(\frac{x - y - a}{x - y + a}\right)$

$\Rightarrow$ $\frac{x - y - a}{x - y + a} = e^{y-k}$, where $k = \frac{a}{2}(2 + \log 3)$

$\therefore$ $\frac{x - y}{a} = \frac{1 + e^{y-k}}{1 - e^{y-k}}$

$\Rightarrow$ $x = y + a\left(\frac{1 + e^{y-k}}{1 - e^{y-k}}\right)$, where $k = \frac{a}{2}(2 + \log 3)$

Which is required equation of curve.

Hence, (b) is the correct answer.

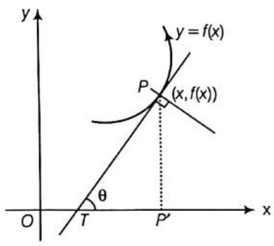
Example 80 The family of curves, the subtangent at any point of which is the arithmetic mean of the coordinates of the point of tangency, is given by

(a) $(x-y)^2 = Cy$ (b) $(y-x)^2 = Cx$

(c) $(x-y)^2 = Cxy$ (d) None of these

Sol. Let the family of curves be $y = f(x)$

$$\tan\theta = \frac{l(PP')}{l(TP')}$$



$$\tan\theta = \frac{l(PP')}{l(TP')}$$

$\therefore$ $$l(\text{subtangent}) = \frac{f(x)}{f'(x)}$$

$\therefore$ $$\frac{y}{y'} = \frac{x+y}{2} \text{ (given)}$$

$\therefore$ $$y' = \frac{2y}{x+y}$$

$\Rightarrow$ $$\frac{dy}{dx} = \frac{2y}{x+y} \quad \therefore \quad \frac{dx}{dy} = \frac{x+y}{2xy} \quad \text{...(i)}$$

It is a homogeneous differential equation.

$\therefore$ Put $x = vy$

Differentiating w.r.t. y, we get

$$\frac{dx}{dy} = v + y\frac{dv}{dy} \quad \text{...(ii)}$$

In Eq. (i) replacing $\frac{dx}{dy}$ by Eq. (ii), we get

$$v + y\frac{dv}{dy} = \frac{vy+y}{2y} = \frac{1+v}{2}$$

$\Rightarrow$ $$y\frac{dv}{dy} = \frac{1+v}{2} - v = \frac{1+v-2v}{2} = \frac{1-v}{2}$$

$\Rightarrow$ $$\frac{2}{1-v}\,dv = \frac{dy}{y}$$

Integrating, $\dfrac{2\log|1-v|}{-1} = \log|y| + \log C_1\,(C_1 > 0)$

$\therefore$ $-2\log|y-x| + 2\log|y| = \log|y| + \log C_1$

$\Rightarrow$ $\log|y-x|^2 = \log|y| - \log C_1$

$\Rightarrow$ $\log|y-x|^2 = \log|y| + \log C,$

where $\log C = -\log C_1$

$\Rightarrow$ $\log|y-x|^2 = \log|yC|$

$\Rightarrow (x-y)^2 = Cy$, is the required equation of family of curves.

Hence, (a) is the correct answer.

Exercise for Session 5

1. The equation of curve for which the normal at every point passes through a fixed point, is

(a) a circle
(b) an ellipse
(c) a hyperbola
(d) None of these

2. If the tangent at any point P of a curve meets the axis of x in T. Then the curve for which $OP = PT$, O being the origin is

(a) $x = Cy^2$
(b) $x = Cy^2$ or $x = C/y^2$
(c) $x = Cy$ or $x = C/y$
(d) None of these

3. According to Newton's law, the rate of cooling is proportional to the difference between the temperature of the body and the temperature of the air. If the temperature of the air is 20°C and body cools for 20 min from 100°C to 60°C, then the time it will take for it temperature to drop to 30°C, is

(a) 30 min
(b) 40 min
(c) 60 min
(d) 80 min

4. Let $f(x, y)$ be a curve in the x-y plane having the property that distance from the origin of any tangent to the curve is equal to distance of point of contact from the y-axis. If $f(1, 2) = 0$, then all such possible curves are

(a) $x^2 + y^2 = 5x$
(b) $x^2 - y^2 = 5x$
(c) $x^2y^2 = 5x$
(d) All of these

5. Given the curves $y = f(x)$ passing through the point (0, 1) and $y = \int_{-\infty}^{x} f(t)$ passing through the point $\left(0, \frac{1}{2}\right)$. The tangents drawn to both the curves at the points with equal abscissae intersect on the x-axis. Then the curve $y = f(x)$, is

(a) $f(x) = x^2 + x + 1$
(b) $f(x) = \frac{x^2}{e^x}$
(c) $f(x) = e^{2x}$
(d) $f(x) = x - e^x$

6. A curve passing through (1, 0) is such that the ratio of the square of the intercept cut by any tangent on the y-axis to the Sub-normal is equal to the ratio of the product of the Coordinates of the point of tangency to the product of square of the slope of the tangent and the subtangent at the same point, is given by

(a) $x = e^{\pm 2\sqrt{y/x}}$
(b) $x = e^{\pm\sqrt{y/x}}$
(c) $y = e^{\pm\sqrt{y/x}} - 1$
(d) $xy + e^{y/x} - 1 = 0$

7. Consider a curve $y = f(x)$ in xy-plane. The curve passes through (0, 0) and has the property that a segment of tangent drawn at any point $P(x, f(x))$ and the line $y = 3$ gets bisected by the line $x + y = 1$ then the equation of curve, is

(a) $y^2 = 9(x - y)$
(b) $(y - 3)^2 = 9(1 - x - y)$
(c) $(y + 3)^2 = 9(1 - x - y)$
(d) $(y - 3)^2 - 9(1 + x + y)$

8. Consider the curved mirror $y = f(x)$ passing through (0, 6) having the property that all light rays emerging from origin, after getting reflected from the mirror becomes parallel to x-axis, then the equation of curve, is

(a) $y^2 = 4(x - y)$ or $y^2 = 36(9 + x)$
(b) $y^2 = 4(1 - x)$ or $y^2 = 36(9 - x)$
(c) $y^2 = 4(1 + x)$ or $y^2 = 36(9 - x)$
(d) None of these

JEE Type Solved Examples : Single Option Correct Type Questions

Ex. 1 *The order of the differential equation of family of curves $y = C_1 \sin^{-1} x + C_2 \cos^{-1} x + C_3 \tan^{-1} x + C_4 \cot^{-1} x$ (where C_1, C_2, C_3 and C_4 are arbitrary constants) is*

(a) 2 (b) 3

(c) 4 (d) None of these

Sol. Here, $y = C_1 \sin^{-1} x + C_2 \cos^{-1} x + C_3 \tan^{-1} x + C_4 \cot^{-1} x$

$$\Rightarrow y = C_1 \sin^{-1} x + C_2\left(\frac{\pi}{2} - \sin^{-1} x\right) + C_3 \tan^{-1} x + C_4\left(\frac{\pi}{2} - \tan^{-1} x\right)$$

$$= (C_1 - C_2)\sin^{-1} x + (C_3 - C_4)\tan^{-1} x + (C_3 - C_4)\frac{\pi}{2}$$

There are only two independent arbitrary constant order of the differential equation is 2.

Hence, (c) is the correct answer.

Ex. 2 *The solution of the differential equation $\dfrac{dy}{dx} = \dfrac{1}{xy(x^2 \sin y^2 + 1)}$ is*

(a) $x^2(\cos y^2 - \sin y^2 - 2ce^{-y^2}) = 2$

(b) $y^2(\sin x^2 - \cos y^2 - 2ce^{-y^2}) = 2$

(c) $x^2(\cos y^2 - \sin y^2 - e^{-y^2}) = 4c$

(d) None of the above

Sol. Here, $\dfrac{dx}{dy} = xy(x^2 \sin y^2 + 1)$

$$\Rightarrow \quad \frac{1}{x^3}\frac{dx}{dy} - \frac{1}{x^2} y = y \sin y^2$$

Let, $$-\frac{1}{x^2} = t \Rightarrow \frac{2}{x^3}\frac{dx}{dy} = \frac{dt}{dy}$$

$$\Rightarrow \quad \frac{dt}{dy} + 2t \cdot y = y \cdot \sin y^2, \; I.F. = e^{\int +2y\,dy} = e^{+y^2}$$

So, required solution is

$$t \cdot e^{y^2} = \int 2y \sin y^2 \times e^{y^2} dy = \frac{1}{2} e^{y^2}(\sin y^2 - \cos y^2) + C$$

$$\Rightarrow \quad 2t = (\sin y^2 - \cos y^2) + 2C e^{-y^2}$$

$$\Rightarrow \quad 2 = -x^2(\sin y^2 - \cos y^2 + 2ce^{-y^2})$$

$$\Rightarrow \quad x^2(\cos y^2 - \sin y^2 - 2ce^{-y^2}) = 2$$

Ex. 3 *The curve satisfying the differential equation $\dfrac{dy}{dx} = \dfrac{y(x+y^3)}{x(y^3 - x)}$ and passing through $(4, -2)$ is*

(a) $y^2 = -2x$ (b) $y = -2x$

(c) $y^3 = -2x$ (d) None of these

Sol. Here, $(xy^3 - x^2)dy = (xy + y^4)dx$

$$\Rightarrow \quad y^3(x\,dy - y\,dx) - x(x\,dy + y\,dx = 0)$$

$$\Rightarrow \quad x^2y^3 d\left(\frac{y}{x}\right) - x\, d(xy) = 0$$

dividing by x^3y^2, we get

$$\Rightarrow \quad \frac{y}{x} d\left(\frac{y}{x}\right) - \frac{1}{x^2y^2} \cdot d\left(\frac{x}{y}\right) = 0$$

$$\Rightarrow \quad \frac{1}{2} d\left(\frac{y}{x}\right)^2 + d\left(\frac{1}{xy}\right) = 0$$

Now integrating, we get

$$\frac{1}{2}\left(\frac{y}{x}\right)^2 + \frac{1}{xy} = c$$

It passes through $(4, -2)$

$$\Rightarrow \quad \frac{1}{8} - \frac{1}{8} = c \Rightarrow C = 0$$

$\therefore y^3 = -2x$ is required curve.

Ex. 4 *Spherical rain drop evaporates at a rate proportional to its surface area. The differential equation corresponding to the rate of change of the radius of the rain drop, if the constant of proportionality is $K > 0$, is*

(a) $\dfrac{dr}{dt} + K = 0$ (b) $\dfrac{dr}{dt} - K = 0$

(c) $\dfrac{dr}{dt} = Kr$ (d) None of these

Sol. $$\frac{dV}{dt} = -k4\pi r^2 \quad \text{...(i)}$$

But $$V = \frac{4}{3}\pi r^3$$

$$\Rightarrow \quad \frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt} \quad \text{...(ii)}$$

Therefore, $$\frac{dr}{dt} = -K$$

Hence, (a) is the correct answer.

Ex. 5 *A function $y = f(x)$ satisfies the differential equation $f(x) \cdot \sin 2x - \cos x + (1 + \sin^2 x) f'(x) = 0$ with initial condition $y(0) = 0$. The value of $f(\pi/6)$ is equal to*

(a) 1/5 (b) 3/5

(c) 4/5 (d) 2/5

Sol. $y \sin 2x - \cos x + (1 + \sin^2 x)\dfrac{dy}{dx} = 0$ where $y = f(x)$

$$\frac{dy}{dx} + \left(\frac{\sin 2x}{1 + \sin^2 x}\right) y = \frac{\cos x}{1 + \sin^2 x}$$

$$\text{IF} = e^{\int \frac{\sin 2x}{1+\sin^2 x} dx} = e^{\int \frac{dt}{t}} = e^{\ln(1+\sin^2 x)}$$

$$= 1 + \sin^2 x \text{ (by putting } 1 + \sin^2 x = t)$$

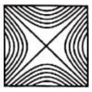
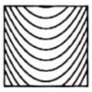
$$y(1 + \sin^2 x) = \int \cos x \, dx$$

$$y(1 + \sin^2 x) = \sin x + C; \; y(0) = 0 \Rightarrow C = 0$$

Therefore, $$y = \frac{\sin x}{1 + \sin^2 x}; \; y\left(\frac{\pi}{6}\right) = \frac{2}{5}$$

Hence, (d) is the correct answer.

● ***Ex. 6*** *The general solution of the differential equation $\frac{dy}{dx} = \frac{1-x}{y}$ is a family of curves which looks most like which of the following?*

(a)  (b)  (c)  (d) 

Sol. $$\int y\,dy = \int (1 - x)\, dx$$

$$\frac{y^2}{2} = x - \frac{x^2}{2} + C$$

$$x^2 + y^2 - 2x = C$$

$$(x-1)^2 + y^2 = C + 1 = C$$

Hence, (b) is the correct answer.

Remark

Family of concentric circles with (1, 0) as the centre and variable radius.

● ***Ex. 7*** *Water is drained from a vertical cylindrical tank by opening a valve at the base of the tank. It is known that the rate at which the water level drops is proportional to the square root of water depth y, where the constant of proportionality $k > 0$ depends on the acceleration due to gravity and the geometry of the hole. If t is measured in minutes and $k = \frac{1}{15}$, then the time to drain the tank, if the water is 4 m deep to start with is*

(a) 30 min (b) 45 min (c) 60 min (d) 80 min

Sol. $\frac{dy}{dt} = -k\sqrt{y}$; when $t = 0$; $y = 4$

$$\int_4^0 \frac{dy}{\sqrt{y}} = -k\int_0^t dt$$

$$[2\sqrt{y}]_4^0 = -kt = -\frac{t}{15}$$

$$0 - 4 = -\frac{t}{15}$$

$\Rightarrow$ $$t = 60 \text{ min}$$

Hence, (c) is the correct answer.

● ***Ex. 8*** *Number of straight lines which satisfy the differential equation $\frac{dy}{dx} + x\left(\frac{dy}{dx}\right)^2 - y = 0$ is*

(a) 1 (b) 2 (c) 3 (d) 4

Sol. $$y = kx + b; \; \frac{dy}{dx} = k$$

$\Rightarrow$ $$kx + b \equiv k + xk^2 \quad \Rightarrow k = k^2 \text{ and } b = k$$

$$k = 0 \text{ or } k = 1$$

Hence, (b) is the correct answer.

● ***Ex. 9*** *Consider the two statements :*

Statement I *$y = \sin kt$ satisfy the differential equation $y'' + 9y = 0$.*

Statement II *$y = e^{kt}$ satisfy the differential equation $y'' + y' - 6y = 0$.*

The value of k for which both the statements are correct is

(a) -3 (b) 0 (c) 2 (d) 3

Sol. **Statement I** $y = \sin kt$, $y' = k\cos kt$; $y'' = -k^2 \sin kt$

$\therefore$ $$-k^2 \sin kt + 9 \sin kt = 0$$

$$\sin kt\,[9 - k^2] = 0 \Rightarrow k = 0, k = 3, k = -3$$

Statement II $y = e^{kt}$, $y' = ke^{kt}$; $y'' = k^2 e^{kt}$

$\therefore$ $$k^2 e^{kt} + ke^{kt} - 6e^{kt} = 0$$

$$e^{kt}[k^2 + k - 6] = 0$$

$$(k+3)(k-2) = 0$$

$$k = -3 \text{ or } 2$$

Common value is $k = -3$.

Hence, (a) is the correct answer.

● ***Ex. 10*** *If $y = \frac{x}{\ln|cx|}$ (where c is an arbitrary constant) is the general solution of the differential equation $\frac{dy}{dx} = \frac{y}{x} + \phi\left(\frac{x}{y}\right)$, then the function $\phi\left(\frac{x}{y}\right)$ is*

(a) $\frac{x^2}{y^2}$ (b) $-\frac{x^2}{y^2}$ (c) $\frac{y^2}{x^2}$ (d) $-\frac{y^2}{x^2}$

Sol. $\ln c + \ln|x| = \frac{x}{y}$

Differentiating w.r.t. x, $\frac{1}{x} = \frac{y - xy_1}{y^2}$

$$\frac{y^2}{x} = y - x\frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{y}{x} - \frac{y^2}{x^2} \Rightarrow \phi\left(\frac{x}{y}\right) = -\frac{y^2}{x^2}$$

Hence, (d) is the correct answer.

• **Ex. 11** *If* $\int_a^x ty(t)\,dt = x^2 + y(x)$, *then* y *as a function of* x *is*

(a) $y = 2-(2+a^2)e^{\frac{x^2-a^2}{2}}$ (b) $y = 1-(2+a^2)e^{\frac{x^2-a^2}{2}}$

(c) $y = 2-(1+a^2)e^{\frac{x^2-a^2}{2}}$ (d) None of these

Sol. Differentiating both the sides, we get

$$xy(x) = 2x - y'(x)$$

Hence, $\dfrac{dy}{dx} - xy = -2x$ $\left[y'(x) = \dfrac{dy}{dx}; y(x) = y\right]$

$$\text{IF} = e^{\int -x\,dx} = e^{\frac{-x^2}{2}}$$

$$ye^{\frac{-x^2}{2}} = \int -2xe^{\frac{-x^2}{2}}\,dx$$

Let $e^{-\frac{x^2}{2}} = t \Rightarrow -xe^{-\frac{x^2}{2}}\,dx = dt$

$$I = \int 2dt$$

$$ye^{-\frac{x^2}{2}} = 2e^{-\frac{x^2}{2}} + C$$

$$y = 2 + Ce^{\frac{x^2}{2}}$$

If $x = a \Rightarrow a^2 + y = 0 \Rightarrow y = -a^2$ (from the given equation)

Hence, $-a^2 = 2 + Ce^{\frac{a^2}{2}}; Ce^{\frac{a^2}{2}} = -(2+a^2)$

$$C = -(2+a^2)e^{-\frac{a^2}{2}}; y = 2-(2+a^2)e^{\frac{x^2-a^2}{2}}$$

Hence, (a) is the correct answer.

• **Ex. 12** *The differential equation* $\dfrac{dy}{dx} = \dfrac{\sqrt{1-y^2}}{y}$ *determines a family of circles with* **[IIT JEE 2007]**

(a) variable radii and a fixed centre at (0, 1)

(b) variable radii and fixed centre at (0, −1)

(c) fixed radius 1 and variable centres along the x-axis

(d) fixed radius 1 and variable centres along the y-axis

Sol. $\because$ $\dfrac{dy}{dx} = \dfrac{\sqrt{1-y^2}}{y}$

$\Rightarrow$ $\int \dfrac{y}{\sqrt{1-y^2}}\,dy = \int dx \Rightarrow -\dfrac{1}{2}\cdot 2\sqrt{1-y^2} = x + C$

$\Rightarrow$ $-\sqrt{1-y^2} = x + C \Rightarrow 1 - y^2 = (x+C)^2$

$\Rightarrow$ $(x+C)^2 + y^2 = 1$

Therefore, the differential equation represents a circle of fixed radius 1 and variable centres along the x-axis. Hence, (c) is the correct answer.

JEE Type Solved Examples :
More than One Correct Option Type Questions

• **Ex. 13** *A curve* $y = f(x)$ *has the property that the perpendicular distance of the origin from the normal at any point P of the curve is equal to the distance of the point P from the x-axis. Then the differential equation of the curve*

(a) is homogeneous

(b) can be converted into linear differential equation with some suitable substitution

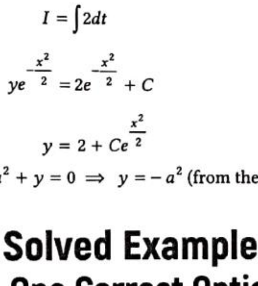
(c) is the family of circles touching the x-axis at the origin

(d) the family of circles touching the y-axis at the origin

Sol. Equation of normal

$$Y - y = -\frac{1}{m}(X - x) \Rightarrow -my + my = X - x$$

$$X + my - (x + my) = 0$$



Perpendicular from (0, 0) $= \left|\dfrac{x+my}{\sqrt{1+m^2}}\right| = y \Rightarrow x^2 + 2xym = y^2$

$$\frac{dy}{dx} = \frac{y^2 - x^2}{2xy} \Rightarrow \text{homogeneous}$$

Also, $x \cdot 2y \cdot \dfrac{dy}{dx} - x^2 = y^2$

Put $y^2 = t;$ $2y\dfrac{dy}{dx} = \dfrac{dt}{dx}; x \cdot \dfrac{dt}{dx} + x^2 = t$

$\dfrac{dt}{dx} - \dfrac{1}{x}t = -x$ which is linear differential equation.

Hence, (a), (b) and (d) are the correct answers.

• **Ex. 14** *A differentiable function satisfies* $f(x) = \int_0^x \{f(t)\cos t - \cos(t-x)\}\,dt$. *Which is of the following hold good?*

(a) $f(x)$ has a minimum value $1-e$

(b) $f(x)$ has a maximum value $1-e^{-1}$

(c) $f''\left(\dfrac{\pi}{2}\right) = e$ (d) $f'(0) = 1$

Sol. $f(x) = \int_0^x \{f(t)\cos t - \cos(t-x)\}\,dx$

$$= \int_0^x f(t)\cos t\,dt - \int_0^x \cos(-t)\,dt \left[\int_0^a f(x)dx = \int_0^a f(a-x)\,dx\right]$$

$$f(x) = \int_0^x f(t) \cos t \, dt - \sin x$$

Differentiating both the sides, we get

$$f'(x) = f(x) \cos x - \cos x$$

Let $\quad f(x) = y;\ f'(x) = \dfrac{dy}{dx}$

$$\frac{dy}{dx} - y \cos x = -\cos x \qquad \text{(L.D.E.)}$$

$$\text{IF} = e^{-\int \cos x \, dx} = e^{-\sin x}$$

Therefore, $y \cdot e^{-\sin x} = -\int e^{-\sin x} \cos x \, dx;$

$$y \cdot e^{-\sin x} = C + e^{-\sin x};\ y = Ce^{\sin x} + 1$$

If $x = 0;$ $\quad y = 0$ (from the given relation)

$\Rightarrow \quad C = -1$

Therefore, $\quad f(x) = 1 - e^{\sin x}$

Now, minimum value $= 1 - e$ (when $x = \pi/2$)

Maximum value $= 1 - e^{-1}$ (when $x = -\pi/2$)

$$f'(x) = -e^{\sin x} \cos x$$

Therefore, $\quad f'(0) = -1$

$$f''(x) = -[\cos^2 x \cdot e^{\sin x} - e^{\sin x} \cdot \sin x]$$

$$f''\left(\frac{\pi}{2}\right) = e$$

Hence, (a), (b) and (c) are the correct answers.

● **Ex. 15** *Let $\dfrac{dy}{dx} + y = f(x)$ where y is a continuous function of x with $y(0) = 1$ and $f(x) = \begin{cases} e^{-x}, & \text{if } 0 \le x \le 2 \\ e^{-2}, & \text{if } x > 2 \end{cases}$. Which is of the following hold(s) good?*

(a) $y(1) = 2e^{-1}$ (b) $y'(1) = -e^{-1}$

(c) $y(3) = -2e^{-3}$ (d) $y'(3) = -2e^{-3}$

Sol. $\dfrac{dy}{dx} + y = f(x) \Rightarrow \text{IF} = e^x$

$$ye^x = \int e^x f(x) \, dx + C$$

Now, if $0 \le x \le 2$, then $ye^x = \int e^x e^{-x} \, dx + C$

$\Rightarrow \quad ye^x = x + C$

$x = 0, y(0) = 1, C = 1$

$\therefore \quad ye^x = x + 1 \qquad \text{...(i)}$

$$y = \frac{x+1}{e^x};\ y(1) = \frac{2}{e} \Rightarrow y' = \frac{e^x - (x+1)e^x}{e^{2x}}$$

$$y'(1) = \frac{e - 2e}{e^2} = \frac{-e}{e^2} = -\frac{1}{e}$$

If $x > 2$, $\quad ye^x = \int e^{x-2} \, dx$

$$ye^x = e^{x-2} + C$$

$$y = e^{-2} + Ce^{-x}$$

As y is continuous.

$$\therefore \quad \lim_{x \to 2} \frac{x+1}{e^x} = \lim_{x \to 2} (e^{-2} + Ce^{-x})$$

$$3e^{-2} = e^{-2} + Ce^{-2} \Rightarrow C = 2$$

$\therefore$ for $x > 2$

$$y = e^{-2} + 2e^{-x}$$

Hence, $\quad y(3) = 2e^{-3} + e^{-2} = e^{-2}(2e^{-1} + 1)$

$$y' = -2e^{-x}$$

$$y'(3) = -2e^{-3}$$

Hence, (a), (b) and (d) are the correct answers.

● **Ex. 16** *A curve $y = f(x)$ passes through (1, 1) and tangent at $P(x, y)$ cuts the x-axis and y-axis at A and B respectively such that $BP : AP = 3 : 1$, then* **[IIT JEE 2006]**

(a) equation of curve is $xy' - 3y = 0$

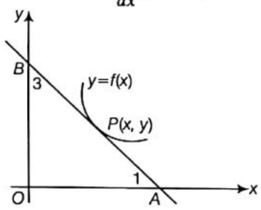
(b) normal at (1, 1) is $x + 3y = 4$

(c) curve passes through $\left(2, \dfrac{1}{8}\right)$

(d) equation of curve is $xy' + 3y = 0$

Sol. Equation of the tangent to the curve $y = f(x)$ at (x, y) is

$$Y - y = \frac{dy}{dx}(X - x)$$



$\because$ Thus, cuts the x-axis at A and y-axis at B.

$$\therefore \quad A\left(\frac{x\dfrac{dy}{dx} - y}{\dfrac{dy}{dx}}, 0\right) \text{ and } B\left(0, -x\frac{dy}{dx} + y\right)$$

$\because \quad BP : PA = 3 : 1$

$$\Rightarrow \quad x = \frac{\dfrac{3\left(x\dfrac{dy}{dx} - y\right)}{(dy/dx)} + 1 \times 0}{4}$$

$$\Rightarrow \quad x\frac{dy}{dx} + 3y = 0$$

$$\Rightarrow \quad \int \frac{dy}{y} = \int -3\frac{dx}{x}$$

$$\Rightarrow \quad \log y = -3 \log x + \log C$$

$$\Rightarrow \quad y = \frac{C}{x^3}$$

$\because$ Curve passes through (1, 1) $\therefore$ $C = 1$

$\therefore$ Curve is $x^3 y = 1$ which also passes through $\left(2, \dfrac{1}{8}\right)$.

Hence, (c) and (d) are the correct answers.

JEE Type Solved Examples : Statement I and II Type Questions

Ex. 17 *Let a solution $y = y(x)$ of the differential equation $x\sqrt{x^2-1}\,dy - y\sqrt{y^2-1}\,dx = 0$ satisfy $y(2) = \frac{2}{\sqrt{3}}$.* *[IIT JEE 2008]*

Statement I $y(x) = \sec\left(\sec^{-1} x - \frac{\pi}{6}\right)$.

Statement II $y(x)$ *is given by* $\frac{1}{y} = \frac{2\sqrt{3}}{x} - \sqrt{1-\frac{1}{x^2}}$.

(a) Statement I is true, Statement II is also true; Statement II is the correct explanation of Statement I.
(b) Statement I is true, Statement II is also true; Statement II is not the correct explanation of Statement I.
(c) Statement I is true, Statement II is false.
(d) Statement I is false, Statement II is true.

Sol. $\because\ x\sqrt{x^2-1}\,dy - y\sqrt{y^2-1}\,dx = 0$

Which can be rewritten as $\frac{dx}{x\sqrt{x^2-1}} = \frac{dy}{y\sqrt{y^2-1}}$

Integration yields, $\int \frac{dx}{x\sqrt{x^2-1}} = \int \frac{dy}{y\sqrt{y^2-1}}$

$\Rightarrow \quad \sec^{-1} x = \sec^{-1} y + C$

$\Rightarrow \quad \sec^{-1}(2) = \sec^{-1}(2/\sqrt{3}) + C$

$\Rightarrow \quad \frac{\pi}{3} = \frac{\pi}{6} + C \Rightarrow C = \frac{\pi}{6}$

Thus, $\sec^{-1} x = \sec^{-1} y + \frac{\pi}{6}$

$$\Rightarrow \quad y = \sec\left(\sec^{-1} x - \frac{\pi}{6}\right) = \frac{1}{\cos\left(\cos^{-1}\frac{1}{x} - \frac{\pi}{6}\right)} = \frac{1}{\frac{1}{x}\cdot\frac{\sqrt{3}}{2} + \sqrt{1-\frac{1}{x^2}}\cdot\frac{1}{2}}$$

$$\Rightarrow \quad \frac{1}{y} = \frac{\sqrt{3}}{2x} + \frac{\sqrt{1-\frac{1}{x^2}}}{2}$$

Hence, (c) is the correct answer.

JEE Type Solved Examples : Passage Based Questions

Passage
(Q. Nos. 18 to 20)

A curve $y = f(x)$ satisfies the differential equation $(1+x^2)\frac{dy}{dx} + 2yx = 4x^2$ and passes through the origin.

18 The function $y = f(x)$
(a) is strictly increasing, $\forall\, x \in R$
(b) is such that it has a minima but no maxima
(c) is such that it has a maxima but no minima
(d) has no inflection point

19 The area enclosed by $y = f^{-1}(x)$, the x-axis and the ordinate at $x = 2/3$ is
(a) $2\ln 2$ (b) $\frac{4}{3}\ln 2$ (c) $\frac{2}{3}\ln 2$ (d) $\frac{1}{3}\ln 2$

20 For the function $y = f(x)$ which one of the following does not hold good?
(a) $f(x)$ is a rational function
(b) $f(x)$ has the same domain and same rage
(c) $f(x)$ is a transcendental function
(d) $y = f(x)$ is a bijective mapping

Sol. (Q. Nos. 18 to 20)

$$\frac{dy}{dx} + \left(\frac{2x}{1+x^2}\right)y = \frac{4x^2}{1+x^2}$$

$$\text{IF} = e^{\int \frac{2x}{1+x^2}dx} = e^{\ln(1+x^2)} = (1+x^2)$$

$\therefore \quad y(1+x^2) = \int 4x^2\,dx = \frac{4x^3}{3} + C$

Passing through $(0, 0) \Rightarrow C = 0$

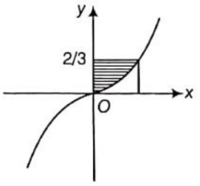
$\therefore \quad y = \frac{4x^3}{3(1+x^2)}$

$$\frac{dy}{dx} = \frac{4}{3}\left[\frac{(1+x^2)3x^2 - x^3\cdot 2x}{(1+x^2)^2}\right] = \frac{4}{3}\left[\frac{3x^2+x^4}{(1+x^2)^2}\right] = \frac{4x^2(3+x^2)}{3(1+x^2)^2}$$

Hence, $\frac{dy}{dx} > 0, \forall\, x \neq 0; \frac{dy}{dx} = 0$ at $x = 0$

and it does not change sign $\Rightarrow x = 0$ is the point of inflection $y = f(x)$ is increasing for all $x \in R$.

$x \to \infty;\ y \to \infty;\ x \to -\infty;\ y \to -\infty$



Area enclosed by $y = f^{-1}(x)$, x-axis and ordinate at $x = \frac{2}{3}$

$$A = \frac{2}{3} - \frac{4}{3}\int_0^1 \frac{x^3}{1+x^2}\,dx$$

Put $\quad 1 + x^2 = t \Rightarrow 2x\,dx = dt$

$$A = \frac{2}{3} - \frac{2}{3}\int_1^2 \frac{(t-1)}{t}\,dt$$

$$= \frac{2}{3} - \frac{2}{3}\int_1^2 \left(1 - \frac{1}{t}\right) dt$$

$$= \frac{2}{3} - \frac{2}{3}[t - \ln t]_1^2 = \frac{2}{3} - \frac{2}{3}[(2 - \ln 2) - 1]$$

$$= \frac{2}{3} - \frac{2}{3}[1 - \ln 2] = \frac{2}{3}\ln 2$$

JEE Type Solved Examples : Single Integer Answer Type Questions

● ***Ex. 21*** *Let $y = f(x)$ be a curve passing through (4, 3) such that slope of normal at any point lying in the first quadrant is negative and the normal and tangent at any point P cuts the Y-axis at A and B respectively such that the mid-point of AB is origin, then the number of solutions of $y = f(x)$ and $y = |5 - |x||$, is*

Sol. Equation of tangent at any point (x_1, y_1) of curve $y = f(x)$ is $(y - y_1) = f'(x_1)(x - x_1)$, so $B(0, y_1 - x_1 f'(x_1))$

Equation of normal at (x_1, y_1) is $(y, y_1) = -\frac{1}{f'(x_1)}(x, x_1)$ so,

$$A = \left(0, y_1 + \frac{x_1}{f'(x_1)}\right) \text{ mid point of } AB \text{ is origin, so}$$

$$2y_1 - x_1\left(f'(x_1) - \frac{1}{f'(x_1)}\right) = 0$$

Thus differential equation of curve $y = f(x)$, is

$$x\left(\frac{dy}{dx}\right)^2 - 2y\frac{dy}{dx} - x = 0$$

Thus, $$\frac{dy}{dx} = \frac{y \pm \sqrt{x^2 + y^2}}{x}$$

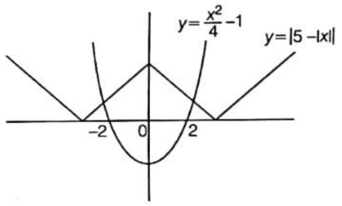
In first quadrant, $x > 0, y > 0, \frac{dy}{dx} > 0,$

So $$\frac{dy}{dx} = \frac{y + \sqrt{x^2 + y^2}}{x}, \text{ put } y = vx$$

$$\Rightarrow \quad v + x\frac{dv}{dx} = \frac{vx + x\sqrt{1 + v^2}}{x}$$

$$\Rightarrow \quad \int \frac{dv}{\sqrt{1 + v^2}} = \int \frac{dx}{x}$$

on solving we get $\quad y = \frac{x^2}{4} - 1$



$\therefore$ number of solutions for $y = f(x)$ and $y = |5 - |x||$

$\therefore$ number of solutions are 2.

● ***Ex. 22*** *A real valued function, $f(x)$, $f: \left(0, \frac{\pi}{2}\right) \to R^+$ satisfies the differential equation $xf'(x) = 1 + f(x)\{x^2 f(x)^{-1}\}$ and $f\left(\frac{\pi}{4}\right) = \frac{4}{\pi}$, then $\lim_{x \to 0} f(x)$, is*

Sol. Here, $\quad xf'(x) = 1 + x^2 f^2(x) - f(x)$

$$\Rightarrow \quad \frac{x f'(x) + f(x)}{1 + x^2 f^2(x)} = 1$$

Integrating both sides

$$\int \frac{(x f'(x) + f(x))dx}{1 + (x f(x))^2} = x + C$$

$$\Rightarrow \quad \tan^{-1}(xf(x)) = x + C, \text{ as } f\left(\frac{\pi}{4}\right) = \frac{4}{\pi}.$$

$$\Rightarrow \quad \tan^{-1} 1 = \frac{\pi}{4} + C$$

$$\Rightarrow \quad C = 0$$

$$\therefore \quad x f(x) = \tan x$$

$$\Rightarrow \quad f(x) = \frac{\tan x}{x}$$

and $$\lim_{x \to 0} f(x) = \lim_{x \to 0} \frac{\tan x}{x} = 1$$

Ex. 23 *If the area bounded by $y = f(x), x = \frac{1}{2}, x = \frac{\sqrt{3}}{2}$ and the X-axis is A sq units where $f(x) = x + \frac{2}{3}x^3 + \frac{2}{3}\cdot\frac{4}{5}x^5 + \frac{2}{3}\cdot\frac{4}{5}\cdot\frac{6}{7}x^7 + \ldots \infty, |x| < 1$, Then the value of $[4A]$ is (where $[\cdot]$ is G.I.F)*

Sol. Here, $f'(x) = 1 + 2x^2 + \frac{2}{3}\cdot 4x^4 + \frac{2}{3}\cdot\frac{4}{5}\cdot 6x^6 + \ldots \infty$

$$= 1 + x\left(\frac{d}{dx}(xf(x))\right)$$

$$\Rightarrow \quad f'(x) = 1 + x\,|xf'(x) + f(x)|$$

$$\Rightarrow (1 - x^2)f'(x) = 1 + xf(x)$$

$$\Rightarrow \quad \frac{dy}{dx} - \frac{x}{1-x^2}\cdot y = \frac{1}{x^2}, \text{ I.F.} = e^{\int \frac{-x}{1-x^2}dx} = e^{\frac{1}{2}\log|1-x^2|} = \sqrt{1-x^2}$$

$$\therefore \quad y\cdot\sqrt{1-x^2} = \int \frac{1}{1-x^2}\cdot\sqrt{1-x^2} + C$$

$$\Rightarrow \quad y\sqrt{1-x^2} = \sin^{-1}x + C, \text{ as } f(0) = 0 \Rightarrow C = 0$$

$$\Rightarrow \quad y = \frac{\sin^{-1}x}{\sqrt{1-x^2}}$$

$$\Rightarrow A = \int_{1/2}^{\sqrt{3}/2} \frac{\sin^{-1}x}{\sqrt{1-x^2}}dx = \int_{\pi/6}^{\pi/3} t\,dt = \left(\frac{t^2}{2}\right)_{\frac{\pi}{6}}^{\frac{\pi}{2}} = \frac{1}{2}\left[\frac{\pi^2}{4} - \frac{\pi^2}{36}\right]$$

$$\therefore \quad [4A] = 1$$

Subjective Type Questions

Ex. 24 *For a certain curve $y = f(x)$ satisfying $\frac{d^2y}{dx^2} = 6x - 4$; $f(x)$ has a local minimum value 5 when $x = 1$. Find the equation of the curve and also the global maximum and global minimum values of $f(x)$ given that $0 \le x \le 2$.*

Sol. Integrating, $\frac{d^2y}{dx^2} = 6x - 4$, we get $\frac{dy}{dx} = 3x^2 - 4x + C$

when $x = 1, \frac{dy}{dx} = 0$. So that $C = 1$

Hence, $\quad \frac{dy}{dx} = 3x^2 - 4x + 1 \quad$ ...(i)

Integrating, we get

$y = x^3 - 2x^2 + x + C_1$, when $x = 1, y = 5$,

so that $C_1 = 5$

Thus, we have $\quad y = x^3 - 2x^2 + x + 5$

Form Eq. (i), we get the critical points $x = \frac{1}{3}, x = 1$

At the critical point $x = \frac{1}{3}, \frac{d^2y}{dx^2}$ is (–ve).

Therefore, at $x = \frac{1}{3}$, y has a local maximum.

At $x = 1, \frac{d^2y}{dx^2}$ is (+ ve).

Therefore, at $x = 1$, y has a local minimum.

Also, $\quad f(1) = 5$

$$\Rightarrow \quad f\left(\frac{1}{3}\right) = \frac{139}{27}$$

$$f(0) = 5, f(2) = 7$$

Hence, the global maximum value = 7, the global minimum value = 5.

Ex. 25 *If $\phi(x)$ is a differentiable real-valued function satisfying $\phi'(x) + 2\phi(x) \le 1$, prove that $\phi(x) - \frac{1}{2}$ is a non-increasing function of x.*

Sol. $\quad \phi'(x) + 2\phi(x) \le 1$

$$\Rightarrow \quad e^{2x}\phi'(x) + 2\phi(x)e^{2x} \le e^{2x}$$

$$\Rightarrow \quad \frac{d}{dx}\left(e^{2x}\phi(x) - \frac{1}{2}e^{2x}\right) \le 0$$

$\Rightarrow e^{2x}\left(\phi(x) - \frac{1}{2}\right)$ is a non-increasing function of x.

$\Rightarrow \phi(x) - \frac{1}{2}$ is a non-increasing function of x.

Ex. 26 *Determine all curve for which the ratios of the length of the segment intercepted by any tangent on the y-axis to the length of the radius vector is a constant.*

Sol. Let $y = f(x)$ be the equation of the required curve.

Given that $\quad \frac{\left|y - x\frac{dy}{dx}\right|}{\sqrt{x^2 + y^2}} = k$ (a constant)

$$\Rightarrow \quad \frac{dy}{dx} = \frac{y}{x} \pm k\sqrt{1 + \left(\frac{y}{x}\right)^2}$$

Let $y = vx$, then $v + x\frac{dv}{dx} = v \pm k\sqrt{1 + v^2}$

$$\Rightarrow \quad \frac{dv}{\sqrt{1+v^2}} = \pm k\frac{dx}{x}, \text{ integrating we get}$$

$$\Rightarrow \quad \log\left|v + \sqrt{1+v^2}\right| = \pm k\ln x + C$$

$$\Rightarrow \quad \log\left|\frac{xy + \sqrt{x^2+y^2}}{x}\right| = \pm k\ln x + C$$

Which are the equations of the required curves.

Ex. 27 *Let $u(x)$ and $v(x)$ satisfy the differential equations $\frac{du}{dx}+p(x)\,u=f(x)$ and $\frac{dv}{dx}+p(x)\,v=g(x)$, where $p(x)$, $f(x)$ and $g(x)$ are continuous functions. If $u(x_1)>v(x_1)$ for some x_1 and $f(x)>g(x)$, for all $x>x_1$, prove that any point (x, y), where $x>x_1$, does not satisfy the equation $y=u(x)$ and $y=v(x)$.* **[IIT JEE 1997]**

Sol. Given that

$$\frac{du}{dx}+p(x)\cdot u=f(x) \text{ and } \frac{dv}{dx}+p(x)\cdot v=g(x)$$

Subtracting, we get

$$\frac{d(u-v)}{dx}+p(x)\cdot(u-v)=f(x)-g(x)$$

Multiplying by $e^{\int p(x)\,dx}$, we get

$$e^{\int p(x)\,dx}\cdot\frac{d(u-v)}{dx}+(u-v)\cdot p(x)\cdot e^{\int p(x)\,dx}$$

$$=\{f(x)-g(x)\}\cdot e^{\int p(x)\,dx}$$

i.e. $$\frac{d}{dx}\{(u-v)\,e^{\int p dx}\}=\{f(x)-g(x)\}\,e^{\int p(x)\,dx}$$

Since, exponential function takes only positive values and

$f(x)>g(x)$ for all $x>x_1$, RHS is + ve; $x>x_1$

$\therefore$ $$\frac{d}{dx}\{(u-v)\cdot e^{\int p(x)\,dx}\}=0$$

ie, $(u-v)\cdot e^{\int p(x)\,dx}$ is increasing function.

Hence, if $e^{\int p(x)\,dx}=\phi(x)$, then for $x>x_1$

We have, $\{u(x)-v(x)\}\,\phi(x)>\{u(x_1)-v(x_1)\}\,\phi(x_1)$

i.e. $$u(x)-v(x)>\frac{\{u(x_1)-v(x_1)\}\cdot\phi(x_1)}{\phi(x)}>0\ [\because u(x_1)>v(x_1)]$$

Thus, $u(x)>v(x),\ \forall\ x>x_1$

i.e. $u(x)\neq v(x),\ \forall\ x>x_1$

Hence, no point (x, y) such that $x>x_1$ can satisfy the equations $y=u(x)$ and $y=v(x)$.

Ex. 28 *A normal is drawn at a point $P(x, y)$ of a curve. It meets the x-axis at Q. If PQ is of constant length k, then show that the differential equation describing such curves is, $y\frac{dy}{dx}=\pm\sqrt{k^2-y^2}$ and the equation of such a curve passing through $(0, k)$.* **[IIT JEE 1994]**

Sol. Let $y=f(x)$ be the curve such that the normal at $P(x, y)$ to this curve meets x-axis at Q. Then,

PQ = length of the normal at P

$$=y\sqrt{1+\left(\frac{dy}{dx}\right)^2}$$

But $PQ=k$

$\therefore$ $$y\sqrt{1+\left(\frac{dy}{dx}\right)^2}=k$$

$\Rightarrow$ $$y^2+y^2\left(\frac{dy}{dx}\right)^2=k^2 \text{ or } y\frac{dy}{dx}=\pm\sqrt{k^2-y^2}$$

$\Rightarrow$ $$\frac{y\,dy}{\sqrt{k^2-y^2}}=\pm\,dx$$

Integrating both the sides, we get

$-\sqrt{k^2-y^2}=\pm\,x+C$, since it passes through $(0, k)\to C=0$.

$\therefore$ $-\sqrt{k^2-y^2}=\pm\,x$

or $k^2-y^2=x^2$

$\Rightarrow$ $x^2+y^2=k^2$, is required equation of the curve.

Ex. 29 *A curve passing through the point (1, 1) has the property that the perpendicular distance of the normal at any point P on the curve from the origin is equal to the distance of P from x-axis. Determine the equation of the curve.* **[IIT JEE 1999]**

Sol. Let $P(x, y)$ be any point on the curve $y=f(x)$. Then, the equation of the normal at P is,

$$Y-y=-\frac{1}{(dy/dx)}(X-x)$$

or $$X+Y\frac{dy}{dx}-\left(y\frac{dy}{dx}+x\right)=0 \quad \text{...(i)}$$

It is given that distance of Eq. (i) from origin = Distance from x-axis (i.e. y)

i.e. $$\left|\frac{0-\left(y\frac{dy}{dx}+x\right)}{\sqrt{1+\left(\frac{dy}{dx}\right)^2}}\right|=y$$

$\Rightarrow$ $$\left(y\frac{dy}{dx}+x\right)^2=y^2\left[1+\left(\frac{dy}{dx}\right)^2\right]$$

$\Rightarrow$ $$x^2+2xy\frac{dy}{dx}=y^2$$

or $$\frac{dy}{dx}=\frac{y^2-x^2}{2xy}$$

which is homogeneous differential equation and we can solve by homogeneous or by total differential.

Here, using total differential,

$$2xy\,dy-y^2\,dx=-x^2\,dx$$

$\Rightarrow$ $$\frac{x\,d(y^2)-y^2\,dx}{x^2}=-dx$$

$\Rightarrow$ $$d\left(\frac{y^2}{x}\right)=-dx$$

Integrating both the sides, we get

$\Rightarrow$ $$\frac{y^2}{x}=-x+C \quad \text{...(ii)}$$

It passes through (1, 1) $\Rightarrow C=2$

$\therefore$ $\frac{y^2}{x}=-x+2$ or $y^2=-x^2+2x$

$\Rightarrow$ $x^2+y^2-2x=0$, is required equation of curve.

● **Ex. 30** *A country has a food deficit of 10%. Its population grows continuously at a rate of 3% per year. Its annual food production every year is 4% more than that of the last year. Assuming that the average food requirement per person remains constant, prove that the country will become self-sufficient in food after n years, where n is the smallest integer bigger than or equal to* $\dfrac{\log_e 10 - \log_e 9}{(\log_e 1.04) - 0.03}$ **[IIT JEE 2000]**

Sol. Let P_0 be the initial population, Q_0 be its initial food production.

Let P be the population of the country in year t and Q be its food production in year t.

$$\Rightarrow \quad \frac{dP}{dt} = \frac{3P}{100} \quad \text{or} \quad \frac{dP}{P} = \frac{3}{100}\,dt$$

Integrating, we get

$$\log P = \frac{3}{100}t + C$$

At $t = 0$, we have $P = P_0$

$$\Rightarrow \quad C = \log P_0$$

$$\Rightarrow \quad P = P_0\, e^{0.03t} \quad \text{...(i)}$$

It is given that the annual food production every year is 4% more than that of last year.

$$\Rightarrow \quad Q = Q_0\left(1 + \frac{4}{100}\right)^t$$

Let the average consumption per person be k units.

$$\Rightarrow \quad Q_0 = kP_0\left(\frac{90}{100}\right) = 0.9\,kP_0$$

$$\therefore \quad Q = 0.9\,kP_0\,(1.04)^t \quad \text{...(ii)}$$

This gives quantity of food available in year t. The population in year t is,

$$P = P_0\, e^{0.03t} \quad \text{[from Eq. (i)]}$$

$\therefore$ Consumption in year, $t = kP_0e^{0.03t}$ ...(iii)

The country will be self sufficient, if

$$Q \ge P$$

$$\Rightarrow \quad 0.9k\,P_0\,(1.04)^t \ge kP_0e^{0.03t}$$

$$\Rightarrow \quad \frac{9}{10}(1.04)^t \ge e^{0.03t}$$

$$\Rightarrow \quad (1.04)^t\, e^{-0.03t} \ge \frac{10}{9}$$

$$\Rightarrow \quad t\log(1.04) - 0.03\,t \ge \log\left(\frac{10}{9}\right)$$

$$\Rightarrow \quad t\,\{\log(1.04) - 0.03\} \ge \log\left(\frac{10}{9}\right)$$

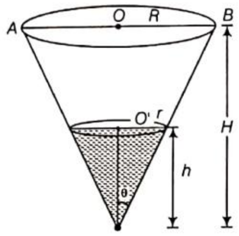
$$\Rightarrow \quad t \ge \frac{\log 10 - \log 9}{\log(1.04) - 0.03}$$

Thus, the least number of year in which country becomes self sufficient.

$$\Rightarrow \quad t = \frac{\log 10 - \log 9}{\log(1.04) - 0.03}$$

● **Ex. 31** *A right circular cone with radius R and height H contains a liquid which evaporates at a rate proportional to its surface area in contact with air (proportionality constant = k > 0), find the time after which the cone is empty.* **[IIT JEE 2003]**

Sol. Let the semi-vertical angle of the cone be θ and let the height of the liquid at time 't' be 'h' from the vertex V and radius of the liquid cone be r. Let V be the volume at time t. Then,



$$V = \frac{1}{3}\pi r^2 h$$

$$\Rightarrow \quad V = \frac{1}{3}\pi r^3 \cot\theta \quad \left(\because \tan\theta = \frac{r}{h}\right)$$

Let S be the surface area of the liquid in contact with air at time t.

Then, $S = \pi r^2$

$$\Rightarrow \quad \frac{dV}{dt} \propto S$$

$\Rightarrow \quad \dfrac{dV}{dt} = -kS$, k is constant of proportionality.

$$\Rightarrow \quad \frac{d}{dt}\left(\frac{1}{3}\pi r^3 \cot\theta\right) = -k\pi r^2$$

$$\Rightarrow \quad \pi r^2\frac{dr}{dt}\cot\theta = -k\pi r^2 \Rightarrow \cot\theta\, dr = -k\, dt$$

On integrating, we get $\cot\theta\int_R^0 dr = -k\int_0^T dt$

$\Rightarrow$ $R\cot\theta = +kT$, where T is required time.

$$\Rightarrow \quad T = H/k \quad (\text{as } \tan\theta = R/H)$$

● **Ex. 32** *Solve the equation*

$$x\int_0^x y(t)\,dt = (x+1)\int_0^x t\,y(t)\,dt,\ x > 0.$$

Sol. Differentiating the equation w.r.t. x, we get

$$xy(x) + 1\cdot\int_0^x y(t)\,dt = (x+1)\,xy(x) + 1\cdot\int_0^x t\,y(t)\,dt$$

i.e. $$\int_0^x y(t)\,dt = x^2\,y(x) + \int_0^x t\,y(t)\,dt$$

Again, differentiating w.r.t. x, we get

$$y(x) = x^2\,y'(x) + 2xy(x) + xy(x)$$

i.e. $$(1-3x)\,y(x) = \frac{x^2\,dy(x)}{dx}$$

i.e. $$\frac{(1-3x)\,dx}{x^2} = \frac{dy(x)}{y(x)}$$

Integrating, we get $y = \dfrac{C}{x^3}e^{-1/x}$

• Ex. 33 *If (y_1, y_2) are two solutions of the differential equation*
$$\frac{dy}{dx} + P(x)\cdot y = Q(x)$$
Then prove that $y = y_1 + C(y_1 - y_2)$ is the general solution of the equation where C is any constant. For what relation between the constant α, β will the linear combination $\alpha y_1 + \beta y_2$ also be a solution.

Sol. As y_1, y_2 are the solutions of the differential equation;
$$\frac{dy}{dx} + P(x)\cdot y = Q(x) \quad \text{...(i)}$$
$$\therefore \quad \frac{dy_1}{dx} + P(x)\cdot y_1 = Q(x) \quad \text{...(ii)}$$
$$\text{and} \quad \frac{dy_2}{dx} + P(x)\cdot y_2 = Q(x) \quad \text{...(iii)}$$
From Eqs. (i) and (ii), $\left(\frac{dy}{dx} - \frac{dy_1}{dx}\right) + P(x)(y - y_1) = 0$
$$\therefore \quad \frac{d}{dx}(y - y_1) + P(x)\cdot(y - y_1) = 0 \quad \text{...(iv)}$$
From Eqs. (ii) and (iii), $\frac{d}{dx}(y_1 - y_2) + P(x)(y_1 - y_2) = 0$ ...(v)

From Eqs. (iv) and (v), $\frac{\frac{d}{dx}(y - y_1)}{\frac{d}{dx}(y_1 - y_2)} = \frac{y - y_1}{y_1 - y_2}$
$$\Rightarrow \quad \frac{\frac{d}{dx}(y - y_1)}{y - y_1} = \frac{\frac{d}{dx}(y_1 - y_2)}{y_1 - y_2}$$
Integrating both the sides, we get
$$\log(y - y_1) = \log(y_1 - y_2)$$
$$\therefore \quad y = y_1 + C(y_1 - y_2)$$
Now, $y = \alpha y_1 + \beta y_2$ will be a solutions, if
$$\frac{d}{dx}(\alpha y_1 + \beta y_2) + P(x)(\alpha y_1 + \beta y_2) = Q(x)$$
$$\text{or} \quad \alpha\left(\frac{dy_1}{dx} + P(x)\, y_1\right) + \beta\left(\frac{dy_2}{dx} + P(x)\, y_2\right) = Q(x)$$
$$\text{or} \quad \alpha\, Q(x) + \beta\, Q(x) = Q(x) \quad \text{[using Eqs. (ii) and (iii)]}$$
$$\therefore \quad (\alpha + \beta)\, Q(x) = Q(x)$$
$$\text{Hence,} \quad \alpha + \beta = 1$$

• Ex. 34 *Find a pair of curves such that*

(a) the tangents drawn at points with equal abscissae intersect on the y-axis.
(b) the normal drawn at points with equal abscissae intersect on x-axis.
(c) one curve passes through (1, 1) and other passes through (2, 3).

Sol. Let the curve be $y = f_1(x)$ and $y = f_2(x)$ equation of tangents with equal abscissa, x are
$$(y - f_1(x)) = f'_1(x)(X - x)$$
$$\text{and} \quad Y - f_2(x) = f'_2(x)(X - x)$$
These tangent intersect at y-axis,
$$\Rightarrow \quad -x f'_1(x) + f_1(x) = -x f'_2(x) + f_2(x)$$
$$\Rightarrow \quad f_1(x) - f_2(x) = x(f'_1(x) - f'_2(x))$$
Integrating both the sides, we get
$$\Rightarrow \quad \ln|f_1(x) - f_2(x)| = \ln|x| + C$$
$$\Rightarrow \quad f_1(x) - f_2(x) = \pm C_1 x \quad \text{...(i)}$$
Now, equations of normal with equal abscissa x, are
$$y - f_1(x) = -\frac{1}{f'_1(x)}(X - x)$$
$$\text{and} \quad (y - f_2(x)) = -\frac{1}{f'_2(x)}(X - x)$$
As these normal intersect on the x-axis,
$$x + f_1(x)\cdot f'_1(x) = x + f_2(x)\cdot f'_2(x)$$
$$\Rightarrow \quad f_1(x)\cdot f_1'(x) = f_2(x)\cdot f_2'(x) \text{ Integrating}$$
$$\Rightarrow \quad f'^2_1(x) - f_2^2(x) = C_2$$
$$\Rightarrow \quad f_1(x) + f_2(x) = \frac{C_2}{f_1(x) - f_2(x)} = \pm\frac{C_2}{C_1 x} = \pm\frac{\lambda_2}{x}$$
[using Eq. (i)] ...(ii)

From Eqs. (i) and (ii), we get
$$2f_1(x) = \pm\left(\frac{\lambda_2}{x} + C_1 x\right), 2f_2(x) = \pm\left(\frac{\lambda_2}{x} - C_1 x\right)$$
We have, $f_1(1) = 1$ and $f_2(2) = 3$
$$\Rightarrow \quad f_1(x) = \frac{2}{x} - x \quad \text{and} \quad f_2(x) = \frac{2}{x} + x$$

• Ex. 35 *Given two curves $y = f(x)$ passing through (0, 1) and $y = \int_{-\infty}^{x} f(t)\, dt$ passing through (0, / n). The tangents drawn to both the curves at the points with equal abscissae intersect on the x-axis find the curve $y = f(x)$.*

Sol. Equation of the tangent to the curve; $y = f(x)$ is
$$(Y - y) = f'(x)(X - x)$$
Equation of tangent to the curve $g(x) = y_1 = \int_{-\infty}^{x} f(t)\, dt$ is
$$(Y - y_1) = g'(x)(X - x) = f(x)(X - x)$$
Given that tangent with equal abscissas intersect on the x-axis.
$$\Rightarrow \quad x - \frac{y}{f'(x)} = x - \frac{y_1}{f(x)}$$
$$\Rightarrow \quad \frac{f(x)}{f'(x)} = \frac{y_1}{f(x)} \quad [\because y = f(x)]$$
$$\Rightarrow \quad \frac{f(x)}{y_1} = \frac{f'(x)}{f(x)} \Rightarrow \frac{g'(x)}{g(x)} = \frac{f'(x)}{f(x)}$$
$$\Rightarrow \quad \frac{g'(x)}{g(x)} = k \quad \Rightarrow g(x) = Ce^{kx}$$
$$\Rightarrow \quad g'(x) = k\,Ce^{kx} \Rightarrow f(x) = k\,Ce^{kx}$$
$y = f(x)$ passes through $(0, 1) \Rightarrow kC = 1$

$y_1 = g(x)$ passes through

$(0, 1/n) \Rightarrow C = \frac{1}{n} \Rightarrow k = n$
$$\Rightarrow \quad f(x) = e^{nx}$$

● **Ex. 36** *A normal is drawn at a point P(x, y) of a curve. It meets the x-axis and the y-axis in point A and B, respectively, such that $\frac{1}{OA}+\frac{1}{OB}=1$, where O is the origin, find the equation of such a curve passing through (5, 4).*

Sol. The equation of the normal at (x, y) is

$$(X-x)+(Y-y)\frac{dy}{dx}=0$$

$$\Rightarrow \quad \frac{X}{x+y\frac{dy}{dx}}+\frac{Y}{\frac{(x+y\,dy/dx)}{dy/dx}}=1$$

$$\Rightarrow \quad OA=x+y\frac{dy}{dx},\ OB=\frac{\left(x+y\frac{dy}{dx}\right)}{\frac{dy}{dx}}$$

Also, $\frac{1}{OA}+\frac{1}{OB}=1 \Rightarrow 1+\frac{dy}{dx}=x+y\frac{dy}{dx}$

$\Rightarrow (y-1)\frac{dy}{dx}+(x-1)=0$

Integrating, we get

$$(y-1)^2+(x-1)^2=C$$

Since, the curve passes through (5, 4), $C=25$.

Hence, the curve is $(x-1)^2+(y-1)^2=25$.

● **Ex. 37** *A line is drawn from a point P (x, y) on curve y = f(x), making an angle with the x-axis which is supplementary to the one made by the tangent to the curve at P(x, y). The line meets the x-axis at A. Another line perpendicular to the first, is drawn from P(x, y) meeting the y-axis at B. If OA = OB, where O is origin, find all curve which passes through (1, 1).*

Sol. The equation of the line through $P(x, y)$ making an angle with the x-axis which is supplementary to the angle made by the tangent at $P(x, y)$ is

$$Y-y=-\frac{dy}{dx}(X-x) \quad \text{...(i)}$$

where it meets the x-axis.

$$Y=0,\quad X=x+\frac{y}{dy/dx} \Rightarrow OA=x+y\frac{dx}{dy} \quad \text{...(ii)}$$

The line through $P(x, y)$ and perpendicular to Eq. (i) is

$$Y-y=\frac{dx}{dy}(X-x)$$

where it meets the y-axis.

$$X=0,\ Y=y-x\frac{dx}{dy} \Rightarrow OB=y-x\frac{dx}{dy} \quad \text{...(iii)}$$

Since, $OA=OB$

$$\Rightarrow \quad x+y\frac{dx}{dy}=y-x\frac{dx}{dy}$$

$$\text{or} \quad (y-x)=(y+x)\frac{dx}{dy}$$

$$\text{or} \quad \frac{dy}{dx}=\frac{y+x}{y-x},\ \text{put } y=vx$$

$$\Rightarrow \quad x\frac{dv}{dx}=\frac{1+2v-v^2}{v-1}$$

$$\Rightarrow \quad \frac{(1-v)\,dv}{1+2v-v^2}+\frac{dx}{x}=0$$

$$\Rightarrow \quad \log(1+2v-v^2)+\log x=C_1$$

$$\Rightarrow \quad x^2+2xy-y^2=C$$

where $C_1=\log\sqrt{C}$

Since, curve passes through $(1, 1) \rightarrow C=2$

$\therefore$ Required curve, $\quad x^2-y^2+2xy=2$

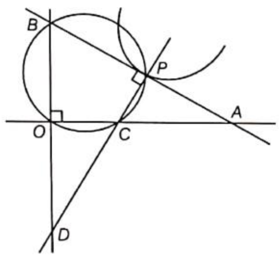
● **Ex. 38** *The tangent and a normal to a curve at any point P meet the x and y axes at A, B, C and D respectively. Find the equation of the curve passing through (1, 0) if the centre of circle through O, C, P and B lies on the line y = x (where O is origin).*

Sol. Let $P(x, y)$ be a point on the curve.

$$\Rightarrow \quad C\equiv\left(x+y\frac{dy}{dx},0\right)$$

$$B\equiv\left(0,y-x\frac{dy}{dx}\right)$$

Circle passing through O, C, P and B has its centre at mid-point of BC.



Let the centre of the circle be (α, β).

$$\Rightarrow \quad 2\alpha=x+y\frac{dy}{dx}$$

$$\text{and} \quad 2\beta=y-x\frac{dy}{dx}$$

and since $\beta=\alpha$, $y-x\frac{dy}{dx}=x+y\frac{dy}{dx}$

$$\Rightarrow \quad \frac{dy}{dx}=\frac{y-x}{y+x}$$

Let $\quad y=vx \Rightarrow x\frac{dv}{dx}=-\frac{(1+v^2)}{1+v}$

$$\Rightarrow \quad \frac{1+v}{v^2+1}dv=-\frac{dx}{x}$$

Integrating both the sides, we get

$$\int \frac{1+v}{v^2+1}\, dv = -\int \frac{dx}{x}$$

$$\Rightarrow \quad \frac{1}{2}\int \frac{2v}{v^2+1}\, dv + \int \frac{dv}{v^2+1} = -\int \frac{dx}{x}$$

$$\Rightarrow \quad \frac{1}{2}\log |v^2+1| + \tan^{-1}|v| = -\log x + C$$

$$\Rightarrow \quad \log \{(\sqrt{v^2+1})\, x\} + \tan^{-1} v = C$$

$$\Rightarrow \quad \log \sqrt{x^2+y^2} + \tan^{-1}\frac{y}{x} = C$$

As $x = 1$ and $y = 0$,

$$\log 1 + \tan^{-1} 0 = C \Rightarrow C = 0$$

$\therefore$ Required curve, $(\log \sqrt{x^2+y^2}) + \tan^{-1}\left(\frac{y}{x}\right) = 0$

• **Ex. 39** *If $f(x)$ be a positive, continuous and differentiable on the interval (a, b). If $\lim_{x \to a^+} f(x) = 1$ and $\lim_{x \to b^-} f(x) = 3^{1/4}$. Also, $f'(x) \ge f^3(x) + \frac{1}{f(x)}$, then*

(a) $b - a \ge \pi/4$ (b) $b - a \le \pi/4$
(c) $b - a \le \pi/24$ (d) None of these

Sol. Since,

$$f'(x) \ge f^3(x) + \frac{1}{f(x)}$$

$$\Rightarrow \quad f'(x)\, f(x) \ge 1 + f^4(x)$$

$$\Rightarrow \quad \frac{f(x)\, f'(x)}{1 + f^4(x)} \ge 1$$

On integrating w.r.t. 'x' from $x = a$ to $x = b$.

$$\frac{1}{2}(\tan^{-1}(f^2(x)))_a^b \ge (b-a)$$

or $$(b-a) \le \frac{1}{2}\left\{\lim_{x \to b^-}(\tan^{-1}(f^2(x))) - \lim_{x \to a^+}(\tan^{-1}(f^2(x)))\right\}$$

or $$(b-a) \le \pi/24$$

Hence, (c) is the correct answer.

Differential Equations Exercise 1 :
Single Option Correct Type Questions

1. If the differential equation of the family of curve given by $y = Ax + Be^{2x}$, where A and B are arbitrary constants, is of the form $(1-2x)\frac{d}{dx}\left(\frac{dy}{dx} + ly\right) + k\left(\frac{dy}{dx} + ly\right) = 0$, then the ordered pair (k, l) is

(a) $(2, -2)$ (b) $(-2, 2)$
(c) $(2, 2)$ (d) $(-2, -2)$

2. A curve passes through the point $\left(1, \frac{\pi}{4}\right)$ and its slope at any point is given by $\frac{y}{x} - \cos^2\left(\frac{y}{x}\right)$. Then, the curve has the equation

(a) $y = x\tan^{-1}\left(\ln \frac{e}{x}\right)$ (b) $y = x\tan^{-1}(\ln 2)$
(c) $y = \frac{1}{x}\tan^{-1}\left(\ln \frac{e}{x}\right)$ (d) None of these

3. The x-intercept of the tangent to a curve is equal to the ordinate of the point of contact. The equation of the curve through the point (1, 1) is

(a) $ye^{\frac{x}{y}} = e$ (b) $xe^{\frac{x}{y}} = e$
(c) $xe^{\frac{y}{x}} = e$ (d) $ye^{\frac{y}{x}} = e$

4. A function $y = f(x)$ satisfies the condition $f'(x)\sin x + f(x)\cos x = 1$, $f(x)$ being bounded when $x \to 0$. If $I = \int_0^{\pi/2} f(x)\,dx$, then

(a) $\frac{\pi}{2} < I < \frac{\pi^2}{4}$ (b) $\frac{\pi}{4} < I < \frac{\pi^2}{2}$
(c) $1 < I < \frac{\pi}{2}$ (d) $0 < I < 1$

5. A curve is such that the area of the region bounded by the coordinate axes, the curve and the ordinate of any point on it is equal to the cube of that ordinate. The curve represents

(a) a pair of straight lines (b) a circle
(c) a parabola (d) an ellipse

6. The value of the constant 'm' and 'c' for which $y = mx + c$ is a solution of the differential equation $D^2y - 3Dy - 4y = -4x$.

(a) is $m = -1; c = 3/4$ (b) is $m = 1; c = -3/4$
(c) no such real m, c (d) is $m = 1; c = 3/4$

7. The real value of m for which the substitution, $y = u^m$ will transform the differential equation, $2x^4y\frac{dy}{dx} + y^4 = 4x^6$ into a homogeneous equation is

(a) $m = 0$ (b) $m = 1$
(c) $m = 3/2$ (d) No value of m

8. The solution of the differential equation,

$$x^2\frac{dy}{dx}\cdot\cos\frac{1}{x} - y\sin\frac{1}{x} = -1,$$

where $y \to -1$ as $x \to \infty$ is

(a) $y = \sin\frac{1}{x} - \cos\frac{1}{x}$ (b) $y = \frac{x+1}{x\sin\frac{1}{x}}$
(c) $y = \sin\frac{1}{x} + \cos\frac{1}{x}$ (d) $y = \frac{x+1}{x\cos\frac{1}{x}}$

9. A wet porous substance in the open air loses its moisture at a rate proportional to the moisture content. If a sheet hung in the wind loses half its moisture during the first hour, then the time when it would have lost 99.9% of its moisture is (weather conditions remaining same)

(a) more than 100 h
(b) more than 10 h
(c) approximately 10 h
(d) approximately 9 h

10. A curve C passes through origin and has the property that at each point (x, y) on it, the normal line at that point passes through (1, 0). The equation of a common tangent to the curve C and the parabola $y^2 = 4x$ is

(a) $x = 0$ (b) $y = 0$
(c) $y = x + 1$ (d) $x + y + 1 = 0$

11. A function $y = f(x)$ satisfies

$$(x+1)\cdot f'(x) - 2(x^2 + x)\,f(x) = \frac{e^{x^2}}{(x+1)}, \forall\, x > -1.$$

If $f(0) = 5$, then $f(x)$ is

(a) $\left(\frac{3x+5}{x+1}\right)\cdot e^{x^2}$ (b) $\left(\frac{6x+5}{x+1}\right)\cdot e^{x^2}$
(c) $\left(\frac{6x+5}{(x+1)^2}\right)\cdot e^{x^2}$ (d) $\left(\frac{5-6x}{x+1}\right)\cdot e^{x^2}$

12. The curve, with the property that the projection of the ordinate on the normal is constant and has a length equal to 'a', is

(a) $x - a\ln(\sqrt{y^2 - a^2} + y) = C$

(b) $x + \sqrt{a^2 - y^2} = C$

(c) $(y - a)^2 = Cx$

(d) $ay = \tan^{-1}(x + C)$

13. The differential equation corresponding to the family of curves $y = e^x(ax + b)$ is

(a) $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - y = 0$ (b) $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = 0$

(c) $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = 0$ (d) $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} - y = 0$

14. The equation to the orthogonal trajectories of the system of parabolas $y = ax^2$ is

(a) $\frac{x^2}{2} + y^2 = C$ (b) $x^2 + \frac{y^2}{2} = C$

(c) $\frac{x^2}{2} - y^2 = C$ (d) $x^2 - \frac{y^2}{2} = C$

15. A function $f(x)$ satisfying $\int_0^1 f(tx)\,dt = n\,f(x)$, where $x > 0$, is

(a) $f(x) = C \cdot x^{\frac{1-n}{n}}$ (b) $f(x) = C \cdot x^{\frac{n}{n-1}}$

(c) $f(x) = C \cdot x^{\frac{1}{n}}$ (d) $f(x) = C \cdot x^{(1-n)}$

16. The substitution $y = z^\alpha$ transforms the differential equation $(x^2y^2 - 1)\,dy + 2xy^3\,dx = 0$ into a homogeneous differential equation for

(a) $\alpha = -1$ (b) 0

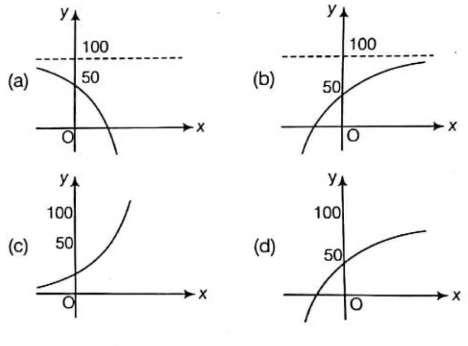
(c) $\alpha = 1$ (d) No value of α

17. A curve passing through (2, 3) and satisfying the differential equation $\int_0^x ty(t)\,dt = x^2y(x), (x > 0)$ is

(a) $x^2 + y^2 = 13$ (b) $y^2 = \frac{9}{2}x$

(c) $\frac{x^2}{8} + \frac{y^2}{18} = 1$ (d) $xy = 6$

18. Which one of the following curves represents the solution of the initial value problem $Dy = 100 - y$, where $y(0) = 50$?



Differential Equations Exercise 2 : More than One Option Correct Type Questions

19. The differential equation $x\frac{dy}{dx} + \frac{3}{\frac{dy}{dx}} = y^2$

(a) is of order 1 (b) is of degree 2

(c) is linear (d) is non-linear

20. The function $f(x)$ satisfying the equation $f^2(x) + 4f'(x)\cdot f(x) + [f'(x)]^2 = 0$.

(a) $f(x) = C \cdot e^{(2-\sqrt{3})x}$

(b) $f(x) = C \cdot e^{(2+\sqrt{3})x}$

(c) $f(x) = C \cdot e^{(\sqrt{3}-2)x}$

(d) $f(x) = C \cdot e^{-(2+\sqrt{3})x}$

where C is an arbitrary constant.

21. Which of the following pair(s) is/are orthogonal?

(a) $16x^2 + y^2 = C$ and $y^{16} = kx$

(b) $y = x + Ce^{-x}$ and $x + 2 = y + ke^{-y}$

(c) $y = Cx^2$ and $x^2 + 2y^2 = k$

(d) $x^2 - y^2 = C$ and $xy = k$

22. Family of curves whose tangent at a point with its intersection with the curve $xy = c^2$ form an angle of $\frac{\pi}{4}$ is

(a) $y^2 - 2xy - x^2 = k$

(b) $y^2 + 2xy - x^2 = k$

(c) $y = x - 2c\tan^{-1}\left(\frac{x}{c}\right) + k$

(d) $y = c\ln\left|\frac{c + x}{c - x}\right| - x + k$

23. The general solution of the differential equation, $x\left(\frac{dy}{dx}\right)=y\cdot\log\left(\frac{y}{x}\right)$ is

(a) $y=xe^{1-Cx}$

(b) $y=xe^{1+Cx}$

(c) $y=ex\cdot e^{Cx}$

(d) $y=xe^{Cx}$

where C is an arbitrary constant.

24. Which of the following equation(s) is/are linear?

(a) $\frac{dy}{dx}+\frac{y}{x}=\ln x$

(b) $y\left(\frac{dy}{dx}\right)+4x=0$

(c) $dx+dy=0$

(d) $\frac{d^2y}{dx^2}=\cos x$

25. The equation of the curve passing through (3, 4) and satisfying the differential equation,

$$y\left(\frac{dy}{dx}\right)^2+(x-y)\frac{dy}{dx}-x=0$$

can be

(a) $x-y+1=0$

(b) $x^2+y^2=25$

(c) $x^2+y^2-5x-10=0$

(d) $x+y-7=0$

26. Identify the statement(s) which is/are true?

(a) $f(x,y)=e^{y/x}+\tan\frac{y}{x}$ is homogeneous of degree zero.

(b) $x\cdot\log\frac{y}{x}\,dx+\frac{y^2}{x}\sin^{-1}\frac{y}{x}\,dy=0$ is homogeneous differential equation.

(c) $f(x,y)=x^2+\sin x\cdot\cos y$ is not homogeneous.

(d) $(x^2+y^2)\,dx-(xy^2-y^3)\,dy=0$ is a homogeneous differential equation.

27. The graph of the function $y=f(x)$ passing through the point $(0,-1)$ and satisfying the differential equation $\frac{dy}{dx}+y\cos x=\cos x$ is such that

(a) it is a constant function

(b) it is periodic

(c) it is neither an even nor an odd function

(d) it is continuous and differentiable for all x

28. A function $y=f(x)$ satisfying the differential equation $\frac{dy}{dx}\cdot\sin x-y\cos x+\frac{\sin^2 x}{x^2}=0$ is such that, $y\to 0$ as $x\to\infty$, then the statement which is correct?

(a) $\lim_{x\to 0} f(x)=1$

(b) $\int_0^{\pi/2} f(x)\,dx$ is less than $\frac{\pi}{2}$

(c) $\int_0^{\pi/2} f(x)\,dx$ is greater than unity

(d) $f(x)$ is an odd function

29. Identify the statement(s) which is/are true?

(a) The order of differential equation $\sqrt{1+\frac{d^2y}{dx^2}}=x$ is 1.

(b) Solution of the differential equation $xdy-ydx=\sqrt{x^2+y^2}\,dx$ is $y+\sqrt{x^2+y^2}=Cx^2$.

(c) $\frac{d^2y}{dx^2}=2\left(\frac{dy}{dx}-y\right)$ is differential equation of family of curves $y=e^x(A\cos x+B\sin x)$.

(d) The solution of differential equation $(1+y^2)+(x-2e^{\tan^{-1}y})\frac{dy}{dx}=0$ is $xe^{\tan^{-1}y}=e^{2\tan^{-1}y}+k$.

30. Let $y=(A+Bx)e^{3x}$ is a solution of the differential equation $\frac{d^2y}{dx^2}+m\frac{dy}{dx}+ny=0$, $m,n\in I$, then

(a) $m=-6$

(b) $n=-6$

(c) $m=9$

(d) $n=9$

Differential Equations Exercise 3 : Statement I and II Type Questions

Directions

(Q. Nos. 31 to 40)

For the following questions, choose the correct answers from the codes (a), (b), (c) and (d) defined as follows :

(a) Statement I is true, Statement II is true and Statement II is the correct explanation for Statement I.
(b) Statement I is true, Statement II is true and Statement II is not the correct explanation for Statement I.
(c) Statement I is true, Statement II is false.
(d) Statement I is false, Statement II is true.

31. A curve C has the property that its initial ordinate of any tangent drawn is less than the abscissa of the point of tangency by unity.

Statement I Differential equation satisfying the curve is linear.

Statement II Degree of differential equation is one.

32. Statement I Differential equation corresponding to all lines, $ax + by + c = 0$ has the order 3.

Statement II General solution of a differential equation of nth order contains n independent arbitrary constants.

33. Statement I Integral curves denoted by the first order linear differential equation $\frac{dy}{dx} - \frac{1}{x} y = -x$ are family of parabolas passing through the origin.

Statement II Every differential equation geometrically represents a family of curve having some common property.

34. Statement I The solution of $(y\,dx - x\,dy)\cot\left(\frac{x}{y}\right) = ny^2\,dx$ is $\sin\left(\frac{x}{y}\right) = Ce^{nx}$

Statement II Such type of differential equations can only be solved by the substitution $x = vy$.

35. Statement I The order of the differential equation whose general solution is
$y = c_1 \cos 2x + c_2 \sin^2 x + c_4 e^{2x} + c_5 e^{2x + c_6}$ is 3.

Statement II Total number of arbitrary parameters in the given general solution in the Statement I is 6.

36. Consider differential equation $(x^2 + 1)\cdot\frac{d^2y}{dx^2} = 2x\cdot\frac{dy}{dx}$

Statement I For any member of this family $y \to \infty$ as $x \to \infty$.

Statement II Any solution of this differential equation is a polynomial of odd degree with positive coefficient of maximum power.

37. Statement I Order of differential equation of family of parallel whose axis is parallel to Y-axis and latusrectum is fixed is 2.

Statement II Order of first equation is same as actual number of arbitrary constant present in differential equation.

38. Statement I The differential equation of all non-vertical lines in a plane is $\frac{d^2x}{dy^2} = 0$.

Statement II The general equation of all non-vertical lines in a plane is $ax + by = 1$, where $b \neq 0$.

39. Statement I The order of differential equation of all conics whose centre lies at origin is, 2.

Statement II The order of differential equation is same as number of arbitrary unknowns in the given curve.

40. Statement I $y = a\sin x + b\cos x$ is general solution of $y'' + y = 0$.

Statement II $y = a\sin x + b\cos x$ is a trigonometric function.

Differential Equations Exercise 4 : Passage Based Questions

Passage I

(Q. Nos. 41 to 43)

Let $y = f(x)$ satisfies the equation

$f(x) = (e^{-x} + e^{x})\cos x - 2x - \int_0^x (x - t)\, f'(t) dt.$

41. y satisfies the differential equation

(a) $\frac{dy}{dx} + y = e^x(\cos x - \sin x) - e^{-x}(\cos x + \sin x)$

(b) $\frac{dy}{dx} - y = e^x(\cos x - \sin x) + e^{-x}(\cos x + \sin x)$

(c) $\frac{dy}{dx} + y = e^x(\cos x + \sin x) - e^{-x}(\cos x - \sin x)$

(d) $\frac{dy}{dx} - y = e^x(\cos x - \sin x) + e^{-x}(\cos x - \sin x)$

42. The value of $f'(0) + f''(0)$ equals to

(a) -1 (b) 2
(c) 1 (d) 0

43. $f(x)$ as a function of x equals to

(a) $e^{-x}(\cos x - \sin x) + \frac{e^x}{5}(3\cos x + \sin x) + \frac{2}{5}e^{-x}$

(b) $e^{-x}(\cos x + \sin x) + \frac{e^x}{5}(3\cos x - \sin x) - \frac{2}{5}e^{-x}$

(c) $e^{-x}(\cos x - \sin x) + \frac{e^x}{5}(3\cos x - \sin x) + \frac{2}{5}e^{-x}$

(d) $e^{-x}(\cos x + \sin x) + \frac{e^x}{5}(3\cos x - \sin x) - \frac{2}{5}e^{-x}$

Passage II

(Q. Nos. 44 to 46)

For certain curves $y = f(x)$ satisfying $\frac{d^2y}{dx^2} = 6x - 4$, $f(x)$ has local minimum value 5 when $x = 1$.

44. Number of critical point for $y = f(x)$ for $x \in [0, 2]$

(a) 0 (b) 1
(c) 2 (d) 3

45. Global minimum value of $y = f(x)$ for $x \in [0, 2]$ is

(a) 5 (b) 7
(c) 8 (d) 9

46. Global maximum value of $y = f(x)$ for $x \in [0, 2]$ is

(a) 5 (b) 7
(c) 8 (d) 9

Passage III

(Q. Nos. 47 to 49)

If any differential equation in the form

$f(f_1(x, y))\, d(f_1(x, y)) + \phi(f_2(x, y))\, d(f_2(x, y)) + \ldots = 0$

then each term can be integrated separately.

For example,

$$\int \sin xy\, d(xy) + \int \left(\frac{x}{y}\right) d\left(\frac{x}{y}\right) = -\cos xy + \frac{1}{2}\left(\frac{x}{y}\right)^2 + C$$

47. The solution of the differential equation $xdy - y\,dx = \sqrt{x^2 - y^2}\, dx$ is

(a) $Cx = e^{\sin^{-1}\frac{y}{x}}$ (b) $xe^{\sin^{-1}\frac{y}{x}} = C$

(c) $x + e^{\sin^{-1}\frac{y}{x}} = C$ (d) None of these

48. The solution of the differential equation $(xy^4 + y)\,dx - xdy = 0$ is

(a) $\frac{x^3}{4} + \frac{1}{2}\left(\frac{x}{y}\right)^2 = C$ (b) $\frac{x^4}{4} + \frac{1}{3}\left(\frac{x}{y}\right)^3 = C$

(c) $\frac{x^4}{4} - \frac{1}{2}\left(\frac{x}{y}\right)^3 = C$ (d) $\frac{x^3}{4} - \frac{1}{2}\left(\frac{x}{y}\right)^2 = C$

49. Solution of differential equation

$(2x\cos y + y^2\cos x)\,dx + (2y\sin x - x^2\sin y)\,dy = 0$ is

(a) $x^2\cos y + y^2\sin x = C$
(b) $x\cos y - y\sin x = C$
(c) $x^2\cos^2 y + y^2\sin^2 x = C$
(d) None of the above

Passage IV

(Q. Nos. 50 to 52)

Differential equation $\frac{dy}{dx} = f(x)\,g(x)$ can be solved by separating variable $\frac{dy}{g(y)} = f(x)\,dx$.

50. The equation of the curve to the point (1, 0) which satisfies the differential equation $(1 + y^2)\,dx - xy\,dy = 0$ is

(a) $x^2 + y^2 = 1$ (b) $x^2 - y^2 = 1$
(c) $x^2 + y^2 = 2$ (d) $x^2 - y^2 = 2$

51. Solution of the differential equation $\frac{dy}{dx}+\frac{1+y^2}{\sqrt{1-x^2}}=0$ is

(a) $\tan^{-1} y+\sin^{-1} x=C$
(b) $\tan^{-1} x+\sin^{-1} y=C$
(c) $\tan^{-1} y\cdot\sin^{-1} x=C$
(d) $\tan^{-1} y-\sin^{-1} x=C$

52. If $\frac{dy}{dx}=1+x+y+xy$ and $y(-1)=0$, then y is equal to

(a) $e^{\frac{(1-x)^2}{2}}$
(b) $e^{\frac{(1+x)^2}{2}}-1$
(c) $\ln(1+x)-1$
(d) $1+x$

Passage V

(Q. Nos. 53 to 55)

Let C be the set of curves having the property that the point of intersection of tangent with y-axis is equidistant from the point of tangency and origin (0, 0)

53. If $C_1, C_2 \in C$

C_1: Curve is passing through (1, 0)

C_2: Curve is passing through (−1, 0)

The number of common tangents for C_1 and C_2 is

(a) 1
(b) 2
(c) 3
(d) None of these

54. If $C_3 \in C$

C_3: is passing through (2, 4). If $\frac{x}{a}+\frac{y}{b}=1$. is tangent to C_3, then

(a) $25a+10b^2-ab^2=0$
(b) $25a+10b-13ab=0$
(c) $13a+25b-16ab=0$
(d) $29a+b+13ab=0$

55. If common tangents of C_1 and C_2 form an equilateral triangle, where $C_1, C_2 \in C$ and C_1: Curve passes through (2, 0), then C_2 may passes through

(a) $(-1/3, 1/3)$
(b) $(-1/3, 1)$
(c) $(-2/3, 4)$
(d) $(-2/3, 2)$

Differential Equations Exercise 5 : Matching Type Questions

56. Match the following :

	Column I		Column II
(A)	$\frac{xdx+ydy}{xdy-ydx}=\sqrt{\frac{a^2-x^2-y^2}{x^2+y^2}}$	(p)	$y=\frac{C}{x^3}e^{-1/x}$
(B)	Solution of $\cos^2 x\frac{dy}{dx}-\tan 2x\cdot y=\cos^4 x$, where $\lvert x\rvert<\frac{\pi}{4}$ and $y\left(\frac{\pi}{6}\right)=\frac{3\sqrt{3}}{8}$	(q)	$\sqrt{x^2+y^2}=a\sin\left\{C+\tan^{-1}\left(\frac{y}{x}\right)\right\}$
(C)	The equation of all possible curves that will cut each member of the family of circles $x^2+y^2-2cx=0$ at right angle	(r)	$x^2+y^2+Cy=0$
(D)	Solution of the equation $x\int_0^x y(t)\,dt=(x+1)\int_0^x ty(t)\,dt,\ x>0$ is	(s)	$y=\frac{\sin 2x}{2(1-\tan^2 x)}$

57. Match the following :

	Column I		Column II
(A)	Circular plate is expanded by heat from radius 5 cm to 5.06 cm. Approximate increase in area is	(p)	4
(B)	Side of cube increasing by 1%, then percentage increase in volume is	(q)	0.6π
(C)	If the rate of decrease of $\frac{x^2}{2}-2x+5$ is twice the rate of decrease of x, then x is equal to	(r)	3
(D)	Rate of increase in area of equilateral triangle of side 15 cm, when each side is increasing at the rate of 0.1 cm/s; is	(s)	$\frac{3\sqrt{3}}{4}$

58. Match the following :

	Column I		Column II
(A)	The differential equation of the family of curves $y = e^x (A\cos x + B\sin x)$, where A, B are arbitrary constants, has the degree n and order m. Then, the values of n and m are, respectively	(p)	2, 1
(B)	The degree and order of the differential equation of the family of all parabolas whose axis is the x-axis, are respectively	(q)	1, 1
(C)	The order and degree of the differential equations of the family of circles touching the x-axis at the origin, are respectively	(r)	2, 2
(D)	The degree and order of the differential equation of the family of ellipse having the same foci, are respectively.	(s)	1, 2

Differential Equations Exercise 6 :
Single Integer Answer Type Questions

59. Find the constant of integration by the general solution of the differential equation $(2x^2y - 2y^4)\,dx + (2x^3 + 3xy^3)\,dy = 0$ if curve passes through (1, 1).

60. A tank initially contains 50 gallons of fresh water. Brine contains 2 pounds per gallon of salt, flows into the tank at the rate of 2 gallons per minutes and the mixture kept uniform by stirring, runs out at the same rate. If it will take for the quantity of salt in the tank to increase from 40 to 80 pounds (in seconds) is 206λ, then find λ.

61. If $f : R - \{-1\} \rightarrow$ and f is differentiable function which satisfies : $f\{x + f(y) + xf(y)\} = y + f(x) + yf(x)\ \forall\ x, y \in R - \{-1\}$, then find the value of $2010\,[1 + f(2009)]$.

62. If $\phi(x)$ is a differential real-valued function satisfying $\phi'(x) + 2\phi(x) \leq 1$, then the value of $2\phi(x)$ is always less than or equal to ....

63. The degree of the differential equation satisfied by the curves $\sqrt{1+x} - a\sqrt{1+y} = 1$, is ....

64. Let $f(x)$ be a twice differentiable bounded function satisfayi $2f^5(x)\cdot f'(x) + 2(f'(x))^3 \cdot f^5(x) = f''(x)$. If $f(x)$ is bounded in between $y = k$, and $y = k_2$, Then the number of integers between k_1 and k_2 is/are (where $f(0) = f'(0) = 0$).

65. Let $y(x)$ be a function satisfying $d^2y/dx^2 - dy/dx + e^{2x} = 0$, $y(0) = 2$ and $y'(0) = 1$. If maximum value of $y(x)$ is $y(\alpha)$, Then Integral part of (2α) is ....

Differential Equations Exercise 7 :
Subjective Type Questions

66. Find the time required for a cylindrical tank of radius r and height H to empty through a round hole of area 'a' at the bottom. The flow through a hole is according to the law $U(t) = u\sqrt{2gh(t)}$ where $v(t)$ and $h(t)$ are respectively the velocity of flow through the hole and the height of the water level above the hole at time t and g is the acceleration due to gravity.

67. The hemispherical tank of radius 2 m is initially full of water and has an outlet of 12 cm^2 cross-sectional area at the bottom. The outlet is opened at some instant. The flow through the outlet is according to the law $v(t) = 0\cdot6\sqrt{2gh(t)}$, where $v(t)$ and $h(t)$ are respectively velocity of the flow through the outlet and the height of water level above the outlet at time t and g is the acceleration due to gravity. Find the time it takes to empty the tank.

68. Let $f : R^+ \to R$ satisfies the functional equation $f(xy) = e^{xy-x-y}(e^y f(x) + e^x f(y)) \forall x, y \in R^+$. If $f'(1) = e$, determine $f(x)$.

69. Let $y = f(x)$ be curve passing through $(1, \sqrt{3})$ such that tangent at any point P on the curve lying in the first quadrant has positive slope and the tangent and the normal at the point P cut the x-axis at A and B respectively so that the mid-point of AB is origin. Find the differential equation of the curve and hence determine $f(x)$.

70. If y_1 and y_2 are the solution of differential equation $dy/dx + Py = Q$, where P and Q are function of x alone and $y_2 = y_1 z$, then prove that $z = 1 + c \cdot e^{-\int \frac{Q}{y_1} dx}$, where c is an arbitrary constant.

71. Let $y = f(x)$ be a differentiable function $\forall x \in R$ and satisfies :
$$f(x) = x + \int_0^1 x^2 z f(z)\,dz + \int_0^1 x z^2 f(z)\,dz.$$
Determine the function.

72. If $f : R - \{-1\} \to R$ and f is differentiable function satisfies : $f((x) + f(y) + x f(y)) = y + f(x) + yf(x) \forall x, y \in R - \{-1\}$ Find $f(x)$.

Differential Equations Exercise 8 : Questions Asked in Previous 10 Years' Exams

(i) JEE Advanced & IIT-JEE

73. A solution curve of the differential equation $(x^2 + xy + 4x + 2y + 4)\frac{dy}{dx} - y^2 = 0, x > 0$, passes through the point $(1, 3)$. Then, the solution curve **[More than One Correct 2016]**

(a) intersects $y = x + 2$ exactly at one point
(b) intersects $y = x + 2$ exactly at two points
(c) intersects $y = (x + 2)^2$
(d) does not intersect $y = (x + 3)^2$

74. Let $f : (0, \infty) \to R$ be a differentiable function such that $f'(x) = 2 - \frac{f(x)}{x}$ for all $x \in (0, \infty)$ and $f(1) \neq 1$. Then **[More than One Correct 2016]**

(a) $\lim_{x \to 0+} f'\left(\frac{1}{x}\right) = 1$
(b) $\lim_{x \to 0+} x f\left(\frac{1}{x}\right) = 2$
(c) $\lim_{x \to 0+} x^2 f'(x) = 0$
(d) $|f(x)| \le 2$ for all $x \in (0, 2)$

75. Let $y(x)$ be a solution of the differential equation $(1 + e^x)y' + ye^x = 1$. If $y(0) = 2$, then which of the following statement(s) is/are true? **[More than One Correct 2015]**

(a) $y(-4) = 0$
(b) $y(-2) = 0$
(c) $y(x)$ has a critical point in the interval $(-1, 0)$
(d) $y(x)$ has no critical point in the interval $(-1, 0)$

76. Consider the family of all circles whose centres lie on the straight line $y = x$. If this family of circles is represented by the differential equation $Py'' + Qy' + 1 = 0$, where P, Q are the functions of x, y and y' (here, $y' = \frac{dy}{dx}, y'' = \frac{d^2y}{dx^2}$), then which of the following statement(s) is/are true? **[More than One correct 2015]**

(a) $P = y + x$
(b) $P = y - x$
(c) $P + Q = 1 - x + y + y' + (y')^2$
(d) $P - Q = x + y - y' - (y')^2$

77. The function $y = f(x)$ is the solution of the differential equation $\frac{dy}{dx} + \frac{xy}{x^2 - 1} = \frac{x^4 + 2x}{\sqrt{1 - x^2}}$ in $(-1, 1)$ satisfying $f(0) = 0$. Then, $\int_{-\frac{\sqrt{3}}{2}}^{\sqrt{3}/2} f(x)\,dx$ is **[Only One correct 2014]**

(a) $\frac{\pi}{3} - \frac{\sqrt{3}}{2}$ (b) $\frac{\pi}{3} - \frac{\sqrt{3}}{4}$ (c) $\frac{\pi}{6} - \frac{\sqrt{3}}{4}$ (d) $\frac{\pi}{6} - \frac{\sqrt{3}}{2}$

78. Let $f : [1/2, 1] \to R$ (the set of all real numbers) be a positive, non-constant and differentiable function such that $f'(x) < 2f(x)$ and $f(1/2) = 1$. Then, the value of $\int_{1/2}^1 f(x)\,dx$ lies in the interval **[Only One Correct 2013]**

(a) $(2e - 1, 2e)$ (b) $(e - 1, 2e - 1)$
(c) $\left(\frac{e-1}{2}, e - 1\right)$ (d) $\left(0, \frac{e-1}{2}\right)$

79. A curve passes through the point $\left(1, \frac{\pi}{6}\right)$. Let the slope of the curve at each point (x, y) be $\frac{y}{x} + \sec\left(\frac{y}{x}\right), x > 0$. Then, the equation of the curve is **[Only One Correct 2013]**

(a) $\sin\left(\frac{y}{x}\right) = \log x + \frac{1}{2}$ (b) $\operatorname{cosec}\left(\frac{y}{x}\right) = \log x + 2$

(c) $\sec\left(\frac{2y}{x}\right) = \log x + 2$ (d) $\cos\left(\frac{2y}{x}\right) = \log x + \frac{1}{2}$

- **Directions** (Q. Nos. 80 to 83) Let $f:[0, 1] \to R$ (the set of all real numbers) be a function. Suppose the function f is twice differentiable, $f(0) = f(1) = 0$ and satisfies

$$f''(x) - 2f'(x) + f(x) \geq e^x, x \in [0, 1]$$

[Passage Based Questions 2013]

80. If the function $e^{-x} f(x)$ assumes its minimum in interval $[0, 1]$ at $x = 1/4$, then which of the following is true?

(a) $f'(x) < f(x), \frac{1}{4} < x < \frac{3}{4}$ (b) $f'(x) > f(x), 0 < x < \frac{1}{4}$

(c) $f'(x) < f(x), 0 < x < \frac{1}{4}$ (d) $f'(x) < f(x), \frac{3}{4} < x < 1$

81. Which of the following is true?

(a) $0 < f(x) < \infty$ (b) $-\frac{1}{2} < f(x) < \frac{1}{2}$

(c) $-\frac{1}{4} < f(x) < 1$ (d) $-\infty < f(x) < 0$

82. Which of the following is true?

(a) g is increasing on $(1, \infty)$

(b) g is decreasing on $(1, \infty)$

(c) g is increasing on $(1, 2)$ and decreasing on $(2, \infty)$

(d) g is decreasing on $(1, 2)$ and increasing on $(2, \infty)$

83. Consider the statements.

I. There exists some $x \in R$ such that, $f(x) + 2x = 2(1 + x^2)$

II. There exists some $x \in R$ such that, $2f(x) + 1 = 2x(1 + x)$

(a) Both I and II are true (b) I is true and II is false

(c) I is false and II is true (d) Both I and II are false

84. If $y(x)$ satisfies the differential equation $y' - y\tan x = 2x\sec x$ and $y(0)$, then **[More than One Correct 2016]**

(a) $y\left(\frac{\pi}{4}\right) = \frac{\pi^2}{8\sqrt{2}}$ (b) $y'\left(\frac{\pi}{4}\right) = \frac{\pi^2}{18}$

(c) $y\left(\frac{\pi}{3}\right) = \frac{\pi^2}{9}$ (d) $y'\left(\frac{\pi}{3}\right) = \frac{4\pi}{3} + \frac{2\pi^2}{3\sqrt{3}}$

85. Let $y'(x) + y(x)g'(x) = g(x)g'(x), y(0) = 0, x \in R$, where $f'(x)$ denotes $\frac{d f(x)}{dx}$ and $g(x)$ is a given non-constant differentiable function on R with $g(0) = g(2) = 0$. Then, the value of $y(2)$ is ...... **[Integer Type 2011]**

86. Let $f: R \to R$ be a continuous function, which satisfies $f(x) = \int_0^x f(t)\, dt$. Then, the value of $f(\ln 5)$ is .... **[Integer Type 2009]**

- **Direction** For the following question, choose the correct answer from the codes (a), (b), (c) and (d) defined as follows

(a) Statement I is true, Statement II is also true; Statement II is the correct explanation of Statement I.

(b) Statement I is true, Statement II is also true; Statement II is not the correct explanation of Statement I.

(c) Statement I is true; Statement II is false.

(d) Statement I is false; Statement II is true.

87. Let a solution $y = y(x)$ of the differential equation $x\sqrt{x^2 - 1}\, dy - y\sqrt{y^2 - 1}\, dx = 0$ satisfy $y(2) = \frac{2}{\sqrt{3}}$

Statement I $y(x) = \sec\left(\sec^{-1} x - \frac{\pi}{6}\right)$ and

Statement II $y(x)$ is given by $\frac{1}{y} = \frac{2\sqrt{3}}{x} - \sqrt{1 - \frac{1}{x^2}}$

[Statement Based Questions 2008]

(ii) JEE Main & AIEEE

88. If a curve $y = f(x)$ passes through the point $(1, -1)$ and satisfies the differential equation, $y(1 + xy)dx = x\, dy$, then $f\left(-\frac{1}{2}\right)$ is equal to **[2016 JEE Main]**

(a) $-\frac{2}{5}$ (b) $-\frac{4}{5}$ (c) $\frac{2}{5}$ (d) $\frac{4}{5}$

89. Let $y(x)$ be the solution of the differential equation $(x\log x)\frac{dy}{dx} + y = 2x\log x, (x \geq 1)$. Then, $y(e)$ is equal to **[2015 JEE Main]**

(a) e (b) 0 (c) 2 (d) $2e$

90. Let the population of rabbits surviving at a time t be governed by the differential equation $\frac{dp(t)}{dt} = \frac{1}{2}p(t) - 200$. If $p(0) = 100$, then $p(t)$ is equal to **[2014 JEE Main]**

(a) $400 - 300e^{\frac{t}{2}}$

(b) $300 - 200e^{-\frac{t}{2}}$

(c) $600 - 500e^{\frac{t}{2}}$

(d) $400 - 300e^{-\frac{t}{2}}$

91. At present, a firm is manufacturing 2000 items. It is estimated that the rate of change of production P with respect to additional number of workers x is given by $\frac{dP}{dx} = 100 - 12\sqrt{x}$. If the firm employees 25 more workers, then the new level of production of items is **[2013 JEE Main]**

(a) 2500 (b) 3000
(c) 3500 (d) 4500

92. The population $p(t)$ at time t of a certain mouse species satisfies the differential equation $\frac{dp(t)}{dt} = 0.5(t) - 450$. If $p(0) = 850$, then the time at which the population becomes zero is **[2012 AIEEE]**

(a) 2 log 18 (b) log 9
(c) $\frac{1}{2}$ log 18 (d) log 18

93. If $\frac{dy}{dx} = y + 3 > 0$ and $y(0) = 2$, then $y(\log 2)$ is equal to **[2011 JEE Main]**

(a) 5 (b) 13
(c) −2 (d) 7

94. Let I be the purchase value of an equipment and $V(t)$ be the value after it has been used for t years. The value $V(t)$ depreciates at a rate given by differential equation $\frac{dV(t)}{dt} = -k(T-t)$, where $k > 0$ is a constant and T is the total life in years of the equipment. Then, the scrap value $V(T)$ of the equipment is **[2010 AIEEE]**

(a) $I - \frac{kT^2}{2}$ (b) $I - \frac{k(T-t)^2}{2}$
(c) e^{-kT} (d) $T^2 - \frac{1}{k}$

95. Solution of the differential equation
$$\cos x\, dy = y(\sin x - y)\, dx,\ 0 < x < \frac{\pi}{2},\text{ is}$$ **[2010 AIEEE]**

(a) $\sec x = (\tan x + C)y$
(b) $y \sec x = \tan x + C$
(c) $y \tan x = \sec x + C$
(d) $\tan x = (\sec x + C)y$

96. The differential equation which represents the family of curves $y = c_1 e^{c_2 x}$, where c_1 and c_2 are arbitrary constants, is **[2009 AIEEE]**

(a) $y' = y^2$ (b) $y'' = y'y$
(c) $yy'' = y'$ (d) $yy'' = (y')^2$

97. The differential equation of the family of circles with fixed radius 5 units and centre on the line $y = 2$ is **[AIEEE 2008]**

(a) $(x-2)y'^2 = 25 - (y-2)^2$
(b) $(y-2)y'^2 = 25 - (y-2)^2$
(c) $(y-2)^2 y'^2 = 25 - (y-2)^2$
(d) $(x-2)^2 y'^2 = 25 - (y-2)^2$

Answers

Exercise for Session 1

1. (a) **2.** (a) **3.** (d) **4.** (a) **5.** (d)
6. (a) **7.** (b) **8.** (a) **9.** (a) **10.** (b)

Exercise for Session 2

1. (b) **2.** (c) **3.** (a) **4.** (a) **5.** (c)
6. (a) **7.** (a) **8.** (a) **9.** (a) **10.** (a)
11. (b) **12.** (c) **13.** (a) **14.** (b) **15.** (b)

Exercise for Session 3

1. (a) **2.** (a) **3.** (b) **4.** (b) **5.** (c)
6. (c) **7.** (a) **8.** (c) **9.** (a) **10.** (c)

Exercise for Session 4

1. (b) **2.** (d) **3.** (a) **4.** (b) **5.** (a)
6. (a) **7.** (c) **8.** (b) **9.** (a) **10.** (d)

Exercise for Session 5

1. (a) **2.** (c) **3.** (c) **4.** (a) **5.** (c)
6. (a) **7.** (b) **8.** (c)

Chapter Exercises

1. (a) **2.** (a) **3.** (a) **4.** (a) **5.** (c)
6. (b) **7.** (c) **8.** (a) **9.** (c) **10.** (a)
11. (b) **12.** (a) **13.** (b) **14.** (a) **15.** (a)
16. (a) **17.** (d) **18.** (b) **19.** (a,b,d)
20. (c,d) **21.** (a,b,c,d) **22.** (a,b,c,d)
23. (a,b,c) **24.** (a,c,d) **25.** (a,b)
26. (a,b,c) **27.** (a,b,d) **28.** (a,b,c)
29. (b,c,d) **30.** (a,d) **31.** (b)
32. (d) **33.** (d) **34.** (c) **35.** (c) **36.** (a)
37. (b) **38.** (d) **39.** (d) **40.** (b) **41.** (a)
42. (d) **43.** (c) **44.** (c) **45.** (a) **46.** (b)
47. (a) **48.** (b) **49.** (a) **50.** (b) **51.** (a)
52. (b) **53.** (c) **54.** (a) **55.** (a)
56. (A) → (q), (B) → (s), (C) → (r), (D) → (p)
57. (A) → (q), (B) → (r), (C) → (p), (D) → (s)
58. (A) → (s), (B) → (s), (C) → (q), (D) → (p)
59. (1) **60.** (8) **61.** (1) **62.** (1) **63.** (1)
64. (3) **65.** (1)
66. $t = \frac{\pi r^2}{\mu a}\sqrt{\frac{2H}{g}}$
67. $t = \frac{7\pi \times 10^5}{135\sqrt{g}}$
68. $f(x) = e^x \log x$
69. $x + f(x)f'(x) = \sqrt{x^2 + f^2(x)}$ and $f^2(x) = 1 + 2x$
71. $f(x) = \frac{20x}{119}(4 + 9x)$
72. $f(x) = \frac{-x}{1+x}$
73. (a, d) **74.** (a) **75.** (a, c) **76.** (b, c) **77.** (b)
78. (d) **79.** (a) **80.** (c) **81.** (d) **82.** (b)
83. (c) **84.** (d) **85.** (0) **86.** (0) **87.** (c)
88. (d) **89.** (c) **90.** (a) **91.** (c) **92.** (a)
93. (d) **94.** (a) **95.** (a) **96.** (d) **97.** (c)

Solutions

1. $y \cdot e^{-2x} = Ax\,e^{-2x} + B$

$e^{-2x} \cdot y_1 - 2ye^{-2x} = A(e^{-2x} - 2x\,e^{-2x})$

Cancelling e^{-2x} throughout

$$y_1 - 2y = A(1-2x) \qquad \text{...(i)}$$

Differentiating again $y_2 - 2y_1 = -2A$

$\Rightarrow$ $$A = \frac{2y_1 - y_2}{2}$$

On substituting A in Eq. (i)

$$2(y_2 - 2y) = (2y_1 - y_2)(1-2x)$$

$$2y_1 - 4y = 2y_1(1-2x) - (1-2x)y_2$$

$$(1-2x)\frac{d}{dx}\left(\frac{dy}{dx} - 2y\right) + 2\left(\frac{dy}{dx} - 2y\right) = 0$$

Hence, $k = 2$ and $l = -2$

$\Rightarrow$ Ordered pair $(k, l) \equiv (2, -2)$

2. $\frac{dy}{dx} = \frac{y}{x} - \cos^2\frac{y}{x}$ $y = vx$

$$v + x\frac{dv}{dx} = v - \cos^2 v$$

$$\int \frac{dv}{\cos^2 v} + \int \frac{dx}{x} = C \Rightarrow \tan v + \ln x = C$$

$$\tan\frac{y}{x} + \ln x = C$$

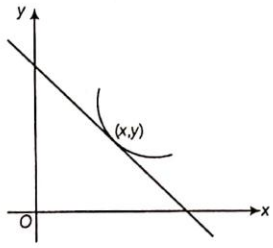
If $x = 1, y = \frac{\pi}{4} \Rightarrow C = 1 \Rightarrow \tan\frac{y}{x} = 1 - \ln x = \ln\frac{e}{x}$

$$y = x\tan^{-1}\left(\ln\frac{e}{x}\right)$$

3. $Y - y = m(X - x)$. For X-intercept $Y = 0$

$$X = x - \frac{y}{m}$$

Therefore, $x - \frac{y}{m} = y$



or $$\frac{dy}{dx} = \frac{y}{x-y}$$

Put $y = vx$ $\quad v + x\frac{dv}{dx} = \frac{v}{1-v}$

$$x\frac{dv}{dx} = \frac{v}{1-v} - v = \frac{v - v + v^2}{1-v}$$

$$\int \frac{1-v}{v^2}\,dv = \int \frac{dx}{x}$$

$$-\frac{1}{v} - \ln v = \ln x + C$$

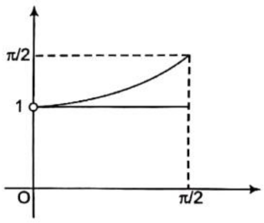
$$-\frac{x}{y} - \ln\frac{y}{x} = \ln x + C \Rightarrow -\frac{x}{y} = \ln y + C$$

$x = 1, y = 1 \Rightarrow C = -1$

$$1 - \frac{x}{y} = \ln y \Rightarrow y = e \cdot e^{-x/y}$$

$$e^{-x/y} = \frac{e}{y} \Rightarrow ye^{x/y} = e$$

4. $\sin x\frac{dy}{dx} + y\cos x = 1$



$$\frac{dy}{dx} + y\cot x = \operatorname{cosec} x$$

$$\text{IF} = e^{\int \cot x\,dx} = e^{\ln(\sin x)} = \sin x$$

$$y\sin x = \int \operatorname{cosec} x \cdot \sin x\,dx$$

$$y\sin x = x + C$$

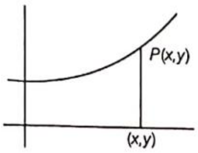
If $x = 0$, y is finite $\therefore$ $C = 0$

$$y = x(\operatorname{cosec} x) = \frac{x}{\sin x}$$

Now, $I < \frac{\pi^2}{4}$ and $I > \frac{\pi}{2}$

Hence, $\frac{\pi}{2} < I < \frac{\pi^2}{4}$

5. $\int_0^x f(x)\,dx = y^3$. Differentiating $f(x) = 3y^2 \cdot \frac{dy}{dx}$



$$y = 3y^2\frac{dy}{dx} \Rightarrow y = 0 \qquad \text{(rejected)}$$

or $$3y\,dy = dx$$

$$\frac{3y^2}{2} = x + C \Rightarrow \text{Parabola}$$

6. $y = mx + c; \frac{dy}{dx} = m; \frac{d^2y}{dx^2} = 0$

Substituting in $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} - 4y = -4x$

$$0 - 3m - 4(mx + c) = -4x$$
$$-3m - 4c - 4mx = -4x$$
$$-(3m + 4c) = 4x(m - 1) \quad \text{...(i)}$$

Eq. (i) is true for all real x, if $m = +1$ and $c = -3/4$.

7. $y = u^m \Rightarrow \frac{dy}{dx} = mu^{m-1}\frac{dy}{dx}$

Since, $2x^4 \cdot u^m \cdot mu^{m-1} \cdot \frac{du}{dx} + u^{4m} = 4x^6$

$$\frac{du}{dx} = \frac{4x^6 - u^{4m}}{2m\, x^4\, u^{2m-1}}$$

$\Rightarrow \quad 4m = 6 \Rightarrow m = \frac{3}{2}$

and $\quad 2m - 1 = 2 \Rightarrow m = \frac{3}{2}$

8. $\frac{dy}{dx} - \frac{y}{x^2}\tan\frac{1}{x} = -\sec\frac{1}{x}\cdot\frac{1}{x^2}$

$\text{IF} = e^{-\int \frac{1}{x^2}\tan\frac{1}{x}dx} = \sec\frac{1}{x} \Rightarrow y \cdot \sec\frac{1}{x}$

$$= -\int \sec^2\left(\frac{1}{x}\right)\frac{1}{x^2}\,dx = \tan\frac{1}{x} + C$$

If $y \to -1$, then $x \to \infty$

$\Rightarrow \quad C = -1 \Rightarrow y = \sin\frac{1}{x} - \cos\frac{1}{x}$

9. $\frac{dM}{dt} = -KM \Rightarrow M = Ce^{-kt}$ when $t = 0$; $M = M_0$

$\Rightarrow \quad C = M_0$

$\Rightarrow \quad M = M_0e^{-kt}$

when $t = 1$, $M = \frac{M_0}{2}$

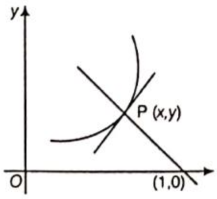
$\Rightarrow \quad k = \ln 2$

Therefore, $M = M_0 e^{-t\ln 2}$

when $\quad M = \frac{M_0}{1000}$, then $t = \log_2 1000 = 9.98$

$= 10\,\text{h}$ approximately

10. Slope of the normal $= \frac{y}{x-1}$



$\therefore \quad \frac{dy}{dx} = \frac{1-x}{y}$

$$\frac{y^2}{2} = x - \frac{x^2}{2} + C \quad \text{...(ii)}$$

Eq. (ii) passes through (0, 0).

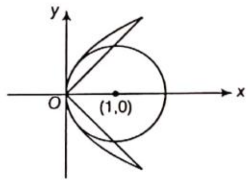
Thus, $C = 0$

$$x^2 + y^2 - 2x = 0$$

Now, tangent to $y^2 = 4x$

$$y = mx + \frac{1}{m} \quad \text{...(iii)}$$

If it touches the circle



$$x^2 + y^2 - 2x = 0$$

Then, $\quad \left|\frac{m + (1/m)}{\sqrt{1 + m^2}}\right| = 1$

$\Rightarrow \quad 1 + m^2 = m^2$

$\Rightarrow \quad m \to \infty$

Hence, tangent is y-axis *ie*, $x = 0$.

11. $f'(x) - \frac{2x(x+1)}{x+1}f(x) = \frac{e^{x^2}}{(x+1)^2}$

$$\text{IF} = e^{\int -2x\,dx} = e^{-x^2}$$

$\therefore \quad f(x)\cdot e^{-x^2} = \int \frac{dx}{(x+1)^2}$

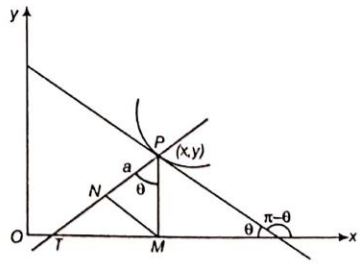
$\Rightarrow \quad f(x)\cdot e^{-x^2} = -\frac{1}{x+1} + C$

At $x = 0$, $f(0) = 5 \Rightarrow C = 6$

$\therefore \quad f(x) = \left(\frac{6x+5}{x+1}\right)\cdot e^{x^2}$

12. Ordinate $= PM$. Let $P \equiv (x, y)$

Projection of ordinate on normal $= PN$



$\therefore$ $PN = PN\cos\theta = a$ (given)

$\therefore$ $\dfrac{y}{\sqrt{1+\tan^2\theta}} = a$

$\Rightarrow$ $y = a\sqrt{1+(y_1)^2}$

$\Rightarrow$ $\dfrac{dy}{dx} = \dfrac{\sqrt{y^2-a^2}}{a}$

$\Rightarrow$ $\int \dfrac{a\,dy}{\sqrt{y^2-a^2}} = \int dx$

$\Rightarrow$ $a\ln|y+\sqrt{y^2-a^2}| = x + C$

13. $y = e^x(ax+b)$ ...(i)

Differentiating w.r.t. x, we get

$\dfrac{dy}{dx} = e^x(ax+b) + e^x\cdot a$ or $\dfrac{dy}{dx} = y + ae^x$ ...(ii)

Again, differentiating both the sides

$\dfrac{d^2y}{dx^2} = \dfrac{dy}{dx} + ae^x$ ...(iii)

From Eq. (iii) – Eq. (ii)

$$\frac{d^2y}{dx^2} - \frac{2dy}{dx} + y = 0$$

is required differential equation.

14. $\dfrac{dy}{dx} = 2ax = 2x\cdot\dfrac{y}{x^2}$; $\dfrac{dy}{dx} = \dfrac{2y}{x}$;

Now, $m\dfrac{dy}{dx} = -1 \Rightarrow m = -\dfrac{x}{2y} \Rightarrow \dfrac{dy}{dx} = -\dfrac{x}{2y}$

$$y^2 = -\frac{x^2}{2} + C$$

15. $\int_0^1 f(tx)\,dt = n\cdot f(x)$

Put, $x = y \Rightarrow dt = \dfrac{1}{x}dy$

$\therefore$ $\dfrac{1}{x}\int_0^x f(y)\,dy = nf(x)$

$\therefore$ $\int_0^x f(y)\,dy = x\cdot n\cdot f(x)$

Differentiating, $f(x) = n[f(x) + xf'(x)]$

$f(x)(1-n) = nx\,f'(x)$

$\therefore$ $\dfrac{f'(x)}{f(x)} = \dfrac{1-n}{nx}$

Integrating, $\ln f(x) = \left(\dfrac{1-n}{n}\right)\ln Cx = \ln(Cx)^{\frac{1-n}{n}}$;

$\therefore$ $f(x) = Cx^{\frac{1-n}{n}}$

16. $(x^2z^{2\alpha}-1)\,\alpha z^{\alpha-1}\,dz + 2x\,z^{3\alpha}\,dx = 0$

or $\alpha(x^2\,z^{3\alpha-1} - z^{\alpha-1})\,dz + 2x\,z^{3\alpha}\,dx = 0$

for homogeneous every term must be of the same degree,

$$3\alpha + 1 = \alpha - 1 \Rightarrow \alpha = -1$$

17. Differentiate, $xy(x) = x^2y'(x) + 2\,xy(x)$

or $xy(x) + x^2\,y'(x) = 0$

$$x\frac{dy}{dx} + y = 0$$

$\ln y + \ln x = \ln C$

$xy = C$

At point (2, 3),

$2\times 3 = C \Rightarrow C = 6$

$\therefore$ $xy = 6$

18. $\int \dfrac{dy}{100-y} = \int dx$

$-\ln(100-y) = x + C$

$\ln(100-y) = -x + C$

Also $x = 0,\ y = 50$

$\therefore$ $C = \ln 50$

$x = \ln 50 - \ln(100-y)$

$\Rightarrow$ $\ln\dfrac{50}{100-y} = x$

$\Rightarrow$ $\dfrac{50}{100-y} = e^x$

$\Rightarrow$ $100 - y = 50e^{-x}$

$\Rightarrow$ $y = 100 - 50e^{-x}$

19. Here, $x\left(\dfrac{dy}{dx}\right)^2 - 3y^2\left(\dfrac{dy}{dx}\right) + 3 = 0$

has order 1, degree 2 and non-linear.

20. Here, $(f'(x))^2 + 4f'(x)\cdot f(x) + (f(x))^2 = 0$

$\Rightarrow$ $\left(\dfrac{f'(x)}{f(x)}\right)^2 + 4\left(\dfrac{f'(x)}{f(x)}\right) + 1 = 0$

$\Rightarrow$ $\dfrac{f'(x)}{f(x)} = \dfrac{-4\pm\sqrt{16-4}}{2}$

$\Rightarrow$ $\dfrac{f'(x)}{f(x)} = -2\pm\sqrt{3}$

Integrating both the sides

$\log|f(x)| = (-2\pm\sqrt{3})\,x + C_1$

$\Rightarrow$ $f(x) = e^{(-2\pm\sqrt{3})x}\cdot C$

21. (a) $32x + 2y\dfrac{dy}{dx} = 0$

$\Rightarrow$ $\dfrac{dy}{dx} = m_1 = -\dfrac{16x}{y}$

and $16y^{15}\dfrac{dy}{dx} = k$

$\Rightarrow$ $\dfrac{dy}{dx} = m_2 = \dfrac{k}{16y^{15}}$

$m_1m_2 = -\dfrac{16x}{y}\cdot\dfrac{k}{16y^{15}} = -\dfrac{x}{y^{16}}\cdot k$

$= -\dfrac{x}{y^{16}}\cdot\dfrac{y^{16}}{x} = -1$

(b) $\dfrac{dy}{dx} = 1 - ce^{-x} = 1 - (y-x) = -(y-x-1)$ [using $ce^{-x} = y - x$]

and $\dfrac{dy}{dx} - k\cdot\dfrac{dy}{dx}e^{-y} = 1$

$\dfrac{dy}{dx}[1 - ke^{-y}] = 1$

or $[1-(x+2-y)]\dfrac{dy}{dx}=1$ [using $ke^{-y}=x-y+2$]

$$\frac{dy}{dx}=m_2=\frac{1}{y-x-1}$$

$\Rightarrow$ $m_1m_2=-1$

(c) $\dfrac{dy}{dx}=2cx=2x\cdot\dfrac{y}{x^2}=\dfrac{2y}{x}=m_1$

Also, $2x+4y\dfrac{dy}{dx}=0 \Rightarrow \dfrac{dy}{dx}=-\dfrac{x}{2y}=m_2$

Hence, $m_1m_2=-1$

(d) $x^2-y^2=c$

$$2x-2y\frac{dy}{dx}=0 \Rightarrow \frac{dy}{dx}=\frac{x}{y}=m_1$$

$$xy=k$$

$$x\frac{dy}{dx}+y=0 \Rightarrow \frac{dy}{dx}=-\frac{y}{x}=m_2$$

$\therefore$ $m_1m_2=-1$

Hence, (a), (b), (c) and (d) are all correct.

22. Let $m=\dfrac{dy}{dx}$ be the slope of tangent (x, y) to the required curve.

m_1 = slope of the tangent at $xy=c^2$

$$=-\frac{c^2}{x^2}=-\frac{y}{x}$$

Hence, $$\frac{m+\dfrac{c^2}{x^2}}{1-\dfrac{c^2}{x^2}m}=\pm 1 \quad \text{or} \quad \frac{m+\dfrac{y}{x}}{1-\dfrac{y}{x}m}=\pm 1 \quad \text{...(i)}$$

Consider $y^2-2xy-x^2=k$

$\Rightarrow$ $2y\dfrac{dy}{dx}-2\left(y+x\dfrac{dy}{dx}\right)-2x=0$

$\Rightarrow$ $\dfrac{dy}{dx}(y-x)=x+y$

$\Rightarrow$ $\dfrac{dy}{dx}=\dfrac{x+y}{y-x}=m$ (say)

From Eq. (i)

$$\text{LHS}=\frac{\dfrac{x+y}{y-x}+\dfrac{y}{x}}{1-\dfrac{y}{x}\left(\dfrac{x+y}{y-x}\right)}$$

$$=\frac{x^2+y^2}{x^2+y^2}=1 \text{ RHS}$$

Similarly option, (b), (c) and (d) safisfy

23. $x\dfrac{dy}{dx}=y\log\left(\dfrac{y}{x}\right)\dfrac{dy}{dx}=\dfrac{y}{x}\log\left(\dfrac{y}{x}\right)$, put $y=vx$

$\Rightarrow$ $\dfrac{dy}{dx}=v+x\dfrac{dv}{dx}$

$\therefore$ $v+x\dfrac{dv}{dx}=v\log v \Rightarrow x\dfrac{dv}{dx}=v(\log v-1)$

$\Rightarrow$ $\displaystyle\int\frac{dv}{v(\log v-1)}=\int\frac{dx}{x} \Rightarrow \int\frac{dt}{t-1}=\int\frac{dx}{x}$,

let $\log v=t$

$\Rightarrow$ $\log(t-1)=\log(x)+\log C$

$\Rightarrow$ $\log(t-1)=\log(xC)$

$\Rightarrow$ $t=1+xC$

$\Rightarrow$ $\log\dfrac{y}{x}=1+xC$

$\Rightarrow$ $y=x\cdot e^{1+Cx}$

or $y=x\cdot e^{1-Cx}$

and $\log\dfrac{y}{x}=\log e+Cx$

$\Rightarrow$ $y=ex\cdot e^{Cx}$

24. Clearly, (a) and (c) are of the form $\dfrac{dy}{dx}+Py=Q$, which is linear in y.

Also, (d) is $\dfrac{d^2y}{dx^2}=\cos x$, on integrating $\dfrac{dy}{dx}=\sin x+C$ which is also linear in y.

25. $\dfrac{dy}{dx}=\dfrac{(y-x)\pm\sqrt{(x-y)^2+4xy}}{2y}$

$\Rightarrow$ $\dfrac{dy}{dx}=1$ or $\dfrac{dy}{dx}=-\dfrac{x}{y}$

$\Rightarrow$ $x-y+1=0$ and $x^2+y^2=25$

26. (a) $f(x, tx)=e^t+\tan^{-1}(t)$, independent of x.

$\Rightarrow$ Homogeneous differential equation.

(b) $\log\left(\dfrac{y}{x}\right)dx+\dfrac{y^2}{x^2}\sin^{-1}\dfrac{y}{x}\cdot dy=0$

$$\Rightarrow \frac{dy}{dx}=\frac{\log\left(\dfrac{y}{x}\right)}{\dfrac{y^2}{x^2}\sin^{-1}\left(\dfrac{y}{x}\right)}$$

$$f(x, y)=\frac{\log\left(\dfrac{y}{x}\right)}{\dfrac{y^2}{x^2}\sin^{-1}\left(\dfrac{y}{x}\right)}$$

$\therefore$ $f(x, tx)=\dfrac{\log(t)}{t^2\sin^{-1}(t)}$, independent of x

$\Rightarrow$ Homogeneous differential equation.

(c) $f(x, y)=x^2+\sin x\cdot\cos y$

$f(x, tx)=x^2+\sin x\cdot\cos(tx)$, not independent of x.

$\Rightarrow$ Not homogeneous differential equation.

(d) $f(x, y)=\dfrac{x^2+y^2}{xy^2-y^3}$

$\Rightarrow f(x, tx)$ is not independent of x.

$\Rightarrow$ Not homogeneous differential equation.

27. Integrating Factor $= e^{\int \cos x\, dx} = e^{\sin x}$

or $\quad y \cdot e^{\sin x} = \int e^{+\sin x} \cos x = -e^{+\sin x} + C$

At point (0, −1),

$$-1\, e^0 = -e^0 + C \Rightarrow 0$$

$\therefore \quad ye^{\sin x} = -e^{\sin x}$

$$y = -1$$

28. $\dfrac{dy}{dx} - y \cot x = -\dfrac{\sin x}{x^2}$

Integrating Factor $= e^{\int -\cot x\, dx} = e^{-\log \sin x} = \dfrac{1}{\sin x}$

$\therefore$ Required solution is $y \cdot \dfrac{1}{\sin x} = \int -\dfrac{\sin x}{x^2} \cdot \dfrac{1}{\sin x}\, dx + C$

$$y \cdot \frac{1}{\sin x} = \frac{1}{x} + C$$

As $x \to \infty, y \to 0 \Rightarrow C = 0$

$\therefore \quad y = \dfrac{\sin x}{x} \Rightarrow \lim_{x \to 0} f(x) = 1 \quad$ ...(i)

$\therefore \quad I = \int_0^{\pi/2} \dfrac{\sin x}{x}\, dx$

Since, $\dfrac{\sin x}{x}$ is decreasing, when $x > 0$

$\Rightarrow \quad f(x) < f(0)$

$\Rightarrow \quad \int_0^{\pi/2} f(x) < \dfrac{\pi}{2}$ and $x < \dfrac{\pi}{2}$

$\Rightarrow \quad f(x) > \int \left(\dfrac{\pi}{2}\right) \Rightarrow \int_0^{\pi/2} f(x)\, dx > 1$

29. (a) Order of the differential equation is 2.

(b) $\quad \dfrac{xdy - ydx}{\sqrt{x^2 + y^2}} = dx \Rightarrow \dfrac{\dfrac{xdy - ydx}{x^2}}{\sqrt{\left(1 + \dfrac{y^2}{x^2}\right)}} = \dfrac{dx}{x}$

$\therefore \quad \ln \left| \dfrac{y}{x} + \sqrt{1 + \dfrac{y^2}{x^2}} \right| = \ln |Cx|$

$\therefore \quad \dfrac{y}{x} + \dfrac{\sqrt{x^2 + y^2}}{x} = Cx$

i.e. $\quad y + \sqrt{x^2 + y^2} = Cx^2$

(c) $y = e^x (A \cos x + B \sin x)$

$\dfrac{dy}{dx} = e^x (A \cos x + B \sin x) + e^x(-A \sin x + B \cos x)$

$= y + e^x (-A \sin x + B \cos x)$

$\therefore \quad \dfrac{d^2y}{dx^2} = \dfrac{dy}{dx} + e^x (-A \sin x + B \cos x)$

$+ e^x (-A \cos x - B \sin x)$

$\therefore \quad \dfrac{d^2y}{dx^2} = \dfrac{dy}{dx} + e^x (-A \sin x + B \cos x) - y$

$\dfrac{dy}{dx} + \dfrac{dy}{dx} - y - y = 2\left(\dfrac{dy}{dx} - y\right)$

(d) $(1 + y^2) \dfrac{dy}{dx} + x = 2e^{\tan^{-1} y}$

$\Rightarrow \quad \dfrac{dx}{dy} + \dfrac{1}{1 + y^2} x = \dfrac{e^{\tan^{-1} y}}{1 + y^2}$

$\text{IF} = e^{\int \frac{1}{1 + y^2}} = e^{\tan^{-1} y}$

$\Rightarrow \quad xe^{\tan^{-1} y} = 2 \int e^{\tan^{-1} y} \cdot \dfrac{e^{\tan^{1} y}}{1 + y^2}\, dy$

$\Rightarrow \quad xe^{\tan^{-1} y} = e^{2 \tan^{-1} y} + k$

30. $\dfrac{dy}{dx} = 3(A + Bx)\, e^{3x} = Be^{3x}$

$\Rightarrow \quad \dfrac{dy}{dx} + my = (3 + m)(A + Bx)\, e^{3x} + Be^{3x}$

$\Rightarrow \quad \dfrac{d^2y}{dx^2} + m\dfrac{dy}{dx} + ny = (9 + 3m + n)(A + Bx)\, e^{3x}$

$+ B(6 + m)\, e^{3x} = 0$

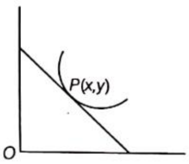
$\Rightarrow \quad 3m + n + 9 = 0$ and $m + 6 = 0$

$\Rightarrow \quad m = -6$ and $n = 9$

31. Equation of tangent

$$Y - y = m(X - x)$$

Put $X = 0, \quad Y = y - mx$



Since, initial ordinate is

$$y - mx = x - 1$$

$\Rightarrow \quad mx - y = 1 - x$

$\dfrac{dy}{dx} - \dfrac{1}{x} y = \dfrac{1 - x}{x}$ which is a linear differential equation.

Hence, Statement I is correct and its degree is 1.

$\Rightarrow$ Statement II is also correct. Since, every 1st degree differential equation need not be linear, hence Statement II is not the correct explanation of Statement I.

32. **Statement I** The order of differential equation is 2.

$\therefore$ Statement I is false.

33. Integral curves are $y = cx - x^2$

The differential equation does not represent all the parabolas passing through origin but it represents all parabolas through origin with axis of symmetry parallel to y-axis and coefficient of x^2 as −1, hence Statement I is false.

Statement II is universally true.

34. $\quad \dfrac{ydx - xdy}{y^2} \cdot \cot \dfrac{x}{y} = x\, dx$

or $\quad \int \cot \dfrac{x}{y} \cdot d\left(\dfrac{x}{y}\right) = \int x\, dx$

or $$\int \cot t \, dx = nx + C$$

$$\ln(\sin t) = nx + C; \sin\frac{x}{y} = Ce^{nx}$$

35. $$y = c_1 \cos 2x + c_2 \sin^2 x + c_3 \cos^2 x + c_4 e^{2x} + c_5$$

$$e^{2x+c_6} = c_1 \cos 2x + c_2\left[\frac{1-\cos 2x}{2}\right] + c_3\left[\frac{\cos 2x + 1}{2}\right] + c_4 e^{2x} + c_5 e^{2x} \cdot e^{c_6}$$

$$= \left(c_1 - \frac{c_2}{2} + \frac{c_3}{2}\right)\cos 2x + \left(\frac{c_2}{2} + \frac{c_3}{2}\right) + (c_4 + c'_5)\, e^{2x}$$

$$= \lambda_1 \cos 2x + \lambda_2 e^{2x} + \lambda_3$$

$\Rightarrow$ Total number of independent parameters in the given general solution is 3.

36. The given differential equation is

$$\frac{d\left(\frac{dy}{dx}\right)}{\frac{dy}{dx}} = \frac{2x}{x^2+1}\, dx$$

$$\Rightarrow \quad \ln\left(\frac{dy}{dx}\right) = \ln(x^2+1) + \ln C, C > 0$$

$$\Rightarrow \quad \frac{dy}{dx} = C(x^2+1)$$

$$\Rightarrow \quad y = C\left(\frac{x^3}{3} + x\right) + C', C' \in R$$

Obviously $y \to \infty$, as $x \to \infty$; as $C > 0$

37. $(x-h)^2 = 4b(y-k)$ here b is constant and h, k are parameters.

38. The general equation of all non-vertical lines in a plane is $ax + by = 1$, where $b \neq 0$.

Now, $$ax + by = 1$$

$$\Rightarrow \quad a + b\frac{dy}{dx} = 0 \quad \text{(differentiating w.r.t. } x\text{)}$$

$$\Rightarrow \quad b\frac{d^2y}{dx^2} = 0 \quad \text{(differentiating w.r.t. } x\text{)}$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = 0 \quad (\text{as } b \neq 0)$$

Hence, the differential equation is $\frac{d^2y}{dx^2} = 0$.

39. The equation $ax^2 + 2hxy + by^2 = 1$ represents the family of all conics whose centre lies at the origin for different values of a, h, b.

$\therefore$ Order is 3.

Thus, Statement I is false and Statement II is true.

Hence, option (d) is the correct answer.

40. $y = a\sin x + b\cos x$

$$\frac{dy}{dx} = a\cos x - b\sin x$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = -a\cos x - b\sin x = -y$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} + y = 0$$

But Statement II is not the correct explanation of the Statement I.

41. $f(0) = 2$

$$f(x) = (e^x + e^{-x})\cos x - 2x - \left[x\int_0^x f'(t)\,dt - \int_0^x \underbrace{t}_{\text{I}}\, \underbrace{f'(t)}_{\text{II}}\,dt\right]$$

$$f(x) = (e^x + e^{-x})\cos x - 2x - [x(f(x) - f(0)) - \{t \cdot f(t)|_0^x - \int_0^x f(t)dt\}]$$

$$f(x) = (e^x + e^{-x})\cos x - 2x - xf(x) + 2x + \left[x\, f(x) - \int_0^x f(t)\,dt\right]$$

$$f(x) = (e^x + e^{-x})\cos x - \int_0^x f(t)\,dt \quad \text{...(i)}$$

On differentiating Eq. (i)

$$f'(x) + f(x) = \cos x\,(e^x - e^{-x}) - (e^x + e^{-x})\sin x \quad \text{...(ii)}$$

Hence, $$\frac{dy}{dx} + y = e^x(\cos x - \sin x) - e^x(\cos x + \sin x)$$

42. $f'(0) + f(0) = 0 - 2 \cdot 0 = 0$

43. IF of Eq. (i) is e^x

$$y \cdot e^x = \int e^{2x}(\cos x - \sin x)\,dx - \int(\cos x + \sin x)\,dx$$

$$y \cdot e^x = \int e^{2x}(\cos x - \sin x)\,dx - (\sin x - \cos x) + C$$

Let $I = \int e^{2x}(\cos x - \sin x)\,dx = e^{2x}(A\cos x + B\sin x)$

Solving, $A = 3/5$ and $B = -1/5$ and $C = 2/5$

$$\therefore \quad y = e^x\left(\frac{3}{5}\cos x - \frac{1}{5}\sin x\right) - (\sin x - \cos x)\,e^{-x} + \frac{2}{5}e^{-x}$$

Solutions (Q. Nos. 44 to 46)

Integrating, $\frac{d^2y}{dx^2} = 6x - 4$,

we get $\frac{dy}{dx} = 3x^2 - 4x + A$

When $x = 1$, $\frac{dy}{dx} = 0$ so that $A = 1$.

Hence, $$\frac{dy}{dx} = 3x^2 - 4x + 1 \quad \text{...(i)}$$

Integrating, we get $y = x^3 - 2x^2 + x + B$.

When $x = 1, y = 5$, so that $B = 5$

Thus, we have $y = x^3 - 2x^2 + x + 5$

From Eq. (i), we get the critical points $x = \frac{1}{3}, x = 1$

At the critical point $x = \frac{1}{3}, \frac{d^2y}{dx^2}$ is negative.

Therefore, at $x = \frac{1}{3}$, y has a local maximum.

At $x = 1$, $\frac{d^2y}{dx^2}$ is positive. Therefore, at $x = 1$, y has a local minimum.

Also, $f(1) = 5$, $f\left(\frac{1}{3}\right) = \frac{157}{27}$, $f(0) = 5$, $f(2) = 7$

Hence, the global maximum value = 7.
and the global minimum value = 5.

47. $xdy - ydx = \sqrt{x^2 - y^2}\, dx$

$\Rightarrow d\left(\frac{y}{x}\right) = \frac{\sqrt{1-(y/x)^2}}{x} dx \Rightarrow \int \frac{d(y/x)}{\sqrt{1-(y/x)^2}} = \int \frac{dx}{x}$

$\Rightarrow \sin^{-1}\frac{y}{x} = \ln x = \ln C$ or $Cx = e^{\sin^{-1} y/x}$

48. $(xy^4 + y)\, dx - x\, dy = 0$

$\Rightarrow \quad xy^4 dx + y\, dx - x\, dy = 0$

$\Rightarrow \quad \int x^3\, dx - \int \left(\frac{x}{y}\right)^2 d\left(\frac{x}{y}\right) = 0$

$\Rightarrow \quad \frac{x^4}{4} + \frac{1}{3}\left(\frac{x}{y}\right)^3 = C$

49. $2x \cos y\, dx - x^2 \sin y\, dy + y^2 \cos x\, dx + 2y \sin x\, dy = 0$

$\Rightarrow \quad \int d(x^2 \cos y) + \int d(y^2 \sin x) = 0$

$\Rightarrow \quad x^2 \cos y + y^2 \sin x = C$

50. $\frac{dx}{x} = \frac{ydy}{1+y^2}$

$\Rightarrow \quad \ln x = \frac{1}{2} \cdot \ln(1+y^2) + C$

From the given condition, $C = 0$

$\therefore \quad x^2 - y^2 = 1$

51. $\frac{dy}{dx} + \frac{1+y^2}{\sqrt{1-x^2}} = 0$

$\Rightarrow \quad \frac{dy}{1+y^2} + \frac{dx}{\sqrt{1-x^2}} = 0$

$\Rightarrow \quad \tan^{-1} y + \sin^{-1} x = C$

52. $\frac{dy}{dx} = (1+x) \cdot (1+y)$ gives $y = e^{\frac{(1+x)^2}{2}} - 1$

53. Tangent at point $P(x,y)$, is $y - y = f'(x)(x - x)$

$Q:(0, y - x f'(x))$

Then, $\quad PQ = OQ$

$\Rightarrow x^2 + x^2(f'(x))^2 = y^2 + x^2(f'(x))^2 - 2xy(f'(x))$

$\Rightarrow x^2 = y^2 - 2xy \cdot f'(x)$

or $dy/dx = \frac{y^2 - x^2}{2xy}$, put $y = tx$

$\Rightarrow \quad t + x\frac{dt}{dx} = \frac{dy}{dx}$

$\therefore \quad t + x\frac{dt}{dx} = \frac{t^2 - 1}{2t}$

$\Rightarrow \quad x\frac{dt}{dx} = \frac{-(1+t^2)}{2t} \Rightarrow \int \frac{2t}{1+t^2} dt = -\int \frac{dx}{x}$

$\Rightarrow \quad \frac{c}{x} = 1 + \frac{y^2}{x^2} \Rightarrow x^2 + y^2 - cx = 0$

$\therefore \quad C_1: x^2 + y^2 - x = 0$

$C_2: x^2 + y^2 + x = 0$

C_1 and C_2 touch externally $\Rightarrow$ number of common tangents = 3.

54. $C_3:(x-5)^2 + y^2 = 5^2$

Tangent, $bx + ay - ab = 0$, length of perpendicular to tangent from centre = radius.

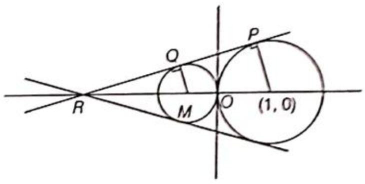
$\Rightarrow \quad |5b - ab| = 5\sqrt{a^2 + b^2}$

$\Rightarrow \quad a^2b^2 - 10ab^2 = 25a^2$

$\Rightarrow \quad ab^2 - 10b^2 = 25a$

55. $C_1: x^2 + y^2 - 2x = 0$

$C_2: x^2 + y^2 - Cx = 0$



$\angle QRM = 30^\circ$

$\sin 30^\circ = \frac{1}{1+OR} \Rightarrow OR = 1$

$\sin 30^\circ = \frac{QM}{1-QM} \Rightarrow QM = 1/3$

radius of $C_2 = 1/3 \Rightarrow C = -2/3, C_2: x^2 + y^2 + \frac{2}{3}x = 0$

Point $(-1/3, 1/3)$ will satisfy C_2.

56. (A) Let $x = r\cos\theta$, $y = r\sin\theta$

$\therefore \quad x^2 + y^2 = r^2(\sin^2\theta + \cos^2\theta) = r^2 \quad$ ...(i)

and $\quad \tan\theta = \frac{y}{x} \quad$ ...(ii)

$\therefore \quad d(x^2 + y^2) = d(r^2)$

From Eq. (i), $x\, dx + y\, dy = rdr \quad$ ...(iii)

From Eq. (ii),

$d\left(\frac{y}{x}\right) = d(\tan\theta)$

$\Rightarrow \quad \frac{xdy - ydx}{x^2} = \sec^2\theta\, d\theta$

$\therefore \quad x\, dy - y\, dx = x^2 \sec^2\theta\, d\theta$

$= r^2\cos^2\theta \sec^2\theta\, d\theta = r^2 d\theta \quad$ ...(iv)

From Eqs. (iii) and (iv)

$\frac{rdr}{r^2 d\theta} = \sqrt{\frac{a^2 - r^2}{r^2}}$

or $\quad \frac{dr}{\sqrt{a^2 - r^2}} = d\theta$

$\sin^{-1}\left(\frac{r}{a}\right) = \theta + C$

$\Rightarrow \quad r = a\sin(\theta + C)$

$\Rightarrow \quad \sqrt{x^2 + y^2} = a\sin\left\{C + \tan^{-1}\left(\frac{y}{x}\right)\right\}$

(B) $\frac{dy}{dx} - \sec^2 x \tan 2xy = \cos^2 x$

$$IF = e^{\int \tan 2x \sec^2 x\, dx} = e^{\int \frac{2\tan x}{\tan^2 x - 1} \times \sec^2 x\, dx}$$

$$= e^{\int \frac{dt}{t}}, \text{ where } t = \tan^2 x - 1$$

$$= e^{\ln|t|} = |t| = |\tan^2 x - 1|$$

Given that, $|x| < \frac{\pi}{4}$

$\therefore \quad \tan^2 x < 1$

$\therefore \quad IF = 1 - \tan^2 x$

Solution is $y(1 - \tan^2 x) = \int \cos^2 x (1 - \tan^2 x)\, dx$

$$= \int (\cos^2 x - \sin^2 x)\, dx$$

$$= \int \cos 2x\, dx = \frac{\sin 2x}{2} + C$$

when $x = \frac{\pi}{6}, y = \frac{3\sqrt{3}}{8}$

$$\therefore \quad \frac{3\sqrt{3}}{8}\left(1 - \frac{1}{3}\right) = \frac{1}{2} \times \frac{\sqrt{3}}{2} + C$$

$$\therefore \quad C = 0 \Rightarrow y = \frac{\sin 2x}{2(1 - \tan^2 x)}$$

(C) $$x^2 + y^2 - 2cx = 0 \quad \text{...(i)}$$

$$\therefore \quad 2x + 2y\frac{dy}{dx} - 2c = 0$$

or $$C = x + \frac{ydy}{dx} \quad \text{...(ii)}$$

From Eqs. (i) and (ii), $x^2 + y^2 - 2\left(x + \frac{ydy}{dx}\right)x = 0$

or $-x^2 + y^2 - 2xy\frac{dy}{dx} = 0$ is the differential equation representing the given family of circles. To find the orthogonal trajectories.

$$y^2 - x^2 = -2xy\frac{dx}{dy}$$

or $$y^2\, dy = x^2\, dy - 2xy\, dx$$

or $$-dy = \frac{yd(x^2) - x^2\, dy}{y^2}$$

or $$-dy = d\left(\frac{x^2}{y}\right)$$

or $$-y = \frac{x^2}{y} + C$$

$\Rightarrow \quad x^2 + y^2 + Cy = 0$

$\Rightarrow$ Orthogonal trajectory.

(D) Differentiating the equation w.r.t. x, we get

$$xy(x) + 1\int_0^x y(t)\, dt = (x+1)\, xy(x) + 1\int_0^x ty(t)\, dt$$

Again, differentiating, w.r.t. x, we get

$$y(x) = x^2 y'(x) + 2xy(x) + xy(x)$$

or $$(1 - 3x)\, y(x) = \frac{x^2 dy(x)}{dx}$$

or $$\frac{(1 - 3x)\, dx}{x^2} = \frac{dy(x)}{y(x)} \text{ or } y = \frac{C}{x^3} e^{-1/x}$$

57. (A) $r = 5$ cm, $\delta r = 0.06$, $A = \pi r^2$

$\delta A = 2\pi r\, \delta r = 10\pi \times 0.06 = 0.6\pi$

(B) $v = x^3$

$\delta v = 3x^2\, \delta x$

$$\frac{\delta v}{v} \times 100 = 3\frac{\delta x}{x} \times 100 = 3 \times 1 = 3$$

(C) $(x - 2)\frac{dx}{dt} = 2\frac{dx}{dt} \Rightarrow x = 4$

(D) $$A = \frac{\sqrt{3}}{4}x^2$$

$$\frac{dA}{dt} = \frac{\sqrt{3}}{2}x\frac{dx}{dt} = \frac{\sqrt{3}}{2}\cdot 15 \cdot \frac{1}{10} = \frac{3\sqrt{3}}{4}$$

58. (A) $\frac{dy}{dx} = e^x(A\cos x + B\sin x) + e^x(-A\sin x + B\cos x)$

$$= y + e^x(-A\sin x + B\cos x)$$

$$\therefore \frac{d^2y}{dx^2} = \frac{dy}{dx} + e^x(-A\sin x + B\cos x) + e^x(-A\cos x - B\sin x)$$

$$= \frac{dy}{dx} + \left(\frac{dy}{dx} - y\right) - y = 2\left(\frac{dy}{dx} - y\right)$$

So, degree = 1 and order = 2

(B) The equation of the family is $y^2 = 4a(x - b)$ where a, b are arbitrary constants.

$$\therefore \quad 2y\frac{dy}{dx} = 4a \text{ or } \left(\frac{dy}{dx}\right)^2 + y\frac{d^2y}{dx^2} = 0$$

So, degree = 1 and order = 2

(C) The equation of the family is

$$(x - c)^2 + y^2 = c^2$$

or $$x^2 + y^2 - 2cx = 0 \text{ or } \frac{x^2 + y^2}{x} = 2c$$

$$\Rightarrow \quad \frac{\left(2x + 2y\frac{dy}{dx}\right)x - (x^2 + y^2)\cdot 1}{x^2} = 0$$

So, degree = 1 and order = 1

(D) The equation of the family is $\frac{x^2}{a^2 + \lambda} + \frac{y^2}{b^2 + \lambda} = 1$ because they have the same foci $(\pm\sqrt{a^2 - b^2}, 0)$.

On differentiating, $\frac{2x}{d^2 + \lambda} + \frac{2y}{b^2 + \lambda}\cdot\frac{dy}{dx} = 0$

or $$\frac{x}{a^2 + \lambda} + \frac{yp}{b^2 + \lambda} = 0 \qquad \left(\text{Let } p = \frac{dy}{dx}\right)$$

or $x(b^2 + \lambda) + yp(a^2 + \lambda) = 0$

$$\Rightarrow \qquad \lambda = \frac{-b^2x - a^2yp}{x + yp}$$

$\therefore$ The differential equation is

$$\frac{x^2}{a^2 + \frac{-b^2x - a^2yp}{x + yp}} + \frac{y^2}{b^2 + \frac{-b^2x - a^2yp}{x + yp}} = 1$$

or $$\frac{x(x + yp)}{a^2 - b^2} + \frac{y(x + yp)}{(b^2 - a^2)p} = 1$$

So, order = 1 and degree = 2

59. $2x^2y\,dx - 2y^4\,dx + 2x^3dy + 3xy^3\,dy = 0$

Dividing throughout by x^3y, we get

$$2\frac{dx}{x} - \frac{2y^3}{x^3}dx + 2\cdot\frac{dy}{y} + \frac{3y^2}{x^2}dy = 0$$

$$\Rightarrow \qquad 2\frac{dx}{x} + 2\frac{dy}{y} + \frac{3y^2x^4\,dy - 2y^3x^3dx}{x^6} = 0$$

$$\Rightarrow \qquad 2\frac{dx}{x} + 2\frac{dy}{y} + \frac{3y^2x^2\,dy - 2y^3x\,dx}{x^4} = 0$$

$$\Rightarrow \qquad 2d(\ln x) + 2d(\ln y) + d\left(\frac{y^3}{x^2}\right) = 0$$

$$\Rightarrow \qquad 2\ln|x| + 2\ln|y| + \frac{y^3}{x^2} = C$$

when $x = 1, y = 1$. So, $C = 1$

60. Let the salt content at time 't' be u lb, so that its rate of change is du/dt.

$$= 2 \text{ gal} \times 2 \text{ lb} = 4 \text{ lb / min}$$

If c be the concentration of the brine at time t, the rate at which the salt content decreases due to the out flow

$$= 2 \text{ gal} \times c \text{ lb / min}$$
$$= 2c \text{ lb / min}$$

$$\therefore \qquad \frac{du}{dt} = 4 - 2c \qquad \text{...(i)}$$

Also, since there is no increase in the volume of the liquid, the concentrations $c = \frac{u}{50}$

$\therefore$ Eq. (i) becomes $\frac{du}{dt} = 4 - \frac{2u}{50}$

Separating the variables and integrating, we have

$$\int dt = 25\int \frac{du}{100 - u}$$

or $$t = -25\ln(100 - u) + K \qquad \text{...(ii)}$$

Initially when $t = 0, u = 0$

$$0 = -25\ln 100 + K \qquad \text{...(iii)}$$

Eliminating 'K' from Eqs. (ii) and (iii), we get

$$t = 25\ln\left(\frac{100}{100 - u}\right)$$

Taking $t = t_1$ when $u = 40$ and $t = t_2$ when $u = 80$, we have

$$t_1 = 25\ln\left(\frac{100}{60}\right) \text{ and } t_2 = 25\ln\left(\frac{100}{20}\right)$$

$\therefore$ The required time $(t_2 - t_1) = 25\left(\ln 5 - \ln\frac{5}{3}\right) = 25\ln 3$

$$= 25 \times 1.0986 = 26 \text{ min } 28 \text{ s}$$
$$= 1648 \text{ s} = 206 \times 8 = 206 \times \lambda$$

$$\therefore \qquad \lambda = 8$$

61. $f(x + f(y) + xf(y)) = y + f(x) + yf(x)$ ...(i)

Differentiating w.r.t. x and y is constant

$$f'(x + f(y) + xf(y))(1 + f(y)) = f'(x) + yf'(x) \qquad \text{...(ii)}$$

From Eq. (i) again differentiating w.r.t. y as x is constant

$$f'(x + f(y) + xf(y))(1 + x)f'(y) = 1 + f(x) \qquad \text{...(iii)}$$

From Eqs. (ii) and (iii)

$$\frac{1 + f(y)}{(1 + x)f'(y)} = \frac{(1 + y)f'(x)}{1 + f(x)}$$

$$\frac{(1 + y)f'(y)}{1 + f(y)} = \frac{1 + f(x)}{(1 + x)f'(x)} = \lambda$$

$$\therefore \quad f'(y) = \frac{1 + f(y)}{1 + y}\lambda \text{ and } f'(x) = \frac{1 + f(x)}{\lambda(1 + x)}$$

$$\therefore \quad \lambda = \frac{1}{\lambda} \Rightarrow \lambda = \pm 1 \;\therefore\; \frac{f'(x)}{1 + f(x)} = \pm\frac{1}{1 + x}$$

Integrating both the sides $f(x) = C(1 + x)^{\pm 1} - 1$

From Eq. (i) put $x = y = 0$

$$f(f(0)) = f(0)$$

From Eq. (ii), $f(0) = C - 1$

So, $$f(C - 1) = C - 1$$

$$\therefore \qquad C = 0, 1 \qquad \text{(taking +ve sign)}$$

So, $f(x) = -1$ and $f(x) = (1 + x) - 1 = x$ and $C = 1$

$$\therefore \qquad f(x) = (1 + x)^{-1} - 1 = \frac{-x}{1 + x}$$

$$1 + f(x) = \frac{1}{1 + x}$$

$$\therefore \qquad 1 + f(2009) = \frac{1}{2010} \;\therefore\; (2010)(1 + f(2009)) = 1$$

62. $$\phi'(x) + 2\phi(x) \le 1$$

$$e^{2x}\phi'(x) + 2\phi(x)e^{2x} \le e^{2x}$$

$$\frac{d}{dx}\left(e^{2x}\phi(x) - \frac{1}{2}e^{2x}\right) \le 0$$

$\therefore \left(e^{2x}\phi(x) - \frac{1}{2}\right)$ is a non-increasing function of x.

$\Rightarrow \phi(x) - \frac{1}{2}$ is a non-increasing function of x.

$\therefore 2\phi(x)$ is always less than or equal to 1.

63. $\frac{1}{2}(1 + x)^{-1/2} - \frac{a}{2}(1 + y)^{-1/2}\frac{dy}{dx} = 0$

$$\Rightarrow \quad \frac{1}{\sqrt{1 + x}} = \frac{a}{\sqrt{1 + y}}\cdot\frac{dy}{dx} \Rightarrow a = \frac{\sqrt{1 + y}}{\sqrt{1 + x}}\cdot\frac{1}{dy/dx}$$

Putting this value in the given equation,

$$\frac{dy}{dx}\sqrt{1 + x} - \frac{1 + y}{\sqrt{1 + x}} = \frac{dy}{dx}$$

$$\Rightarrow \quad (1+x)\frac{dy}{dx} = 1 + y + \sqrt{1+x}\,\frac{dy}{dx}$$

$$\Rightarrow \quad (1 + x - \sqrt{1+x})\frac{dy}{dx} = 1 + y$$

$\therefore$ Degree of the given equation is 1.

64. Here, $2f^5(x)\cdot f'(x)\cdot\left|1+(f'(x))^2\right| = f''(x)$

$$\Rightarrow \quad f^4(x)d(f^2(x)) = \frac{f''(x)}{1+(f'(x))^2}$$

$$\Rightarrow \quad \left|f^4(x)d(f^2(x))\right| = \int d(\tan^{-1}(f'(x)))$$

$$\Rightarrow \quad \frac{f^6(x)}{3} + C = \tan^{-1}(f'(x)), \text{ as } f(0) = f'(0) = 0 \Rightarrow c = 0$$

$$\Rightarrow \quad \frac{f^6(x)}{3} = \tan^{-1}(f'(x))$$

$$\Rightarrow \quad -\frac{\pi}{2} < \frac{f^6(x)}{3} < \frac{\pi}{2}$$

$$\therefore \quad -\left(\frac{3\pi}{2}\right)^{1/6} < f(x) < \left(\frac{3\pi}{2}\right)^{1/6}$$

$\Rightarrow$ Number of integers between k_1 and k_2 are 3.

65. Put, $dy / dx = t$

$\therefore \quad dt/dx - t + e^{2x} = 0$, I.F. $= e^{\int -dx = e^{-x}}$

$\therefore$ Solution is, $t\cdot e^{-x} = \int -e^{2x}\cdot e^{-x}\,dx + C$

$\Rightarrow \quad te^{-x} = -e^x + C\; y'(0) = 1 \Rightarrow C = 2$

$\therefore \quad dy/dx = \dfrac{2-e^x}{e-x}\int dy = \int (2e^x - e^{2x})dx$

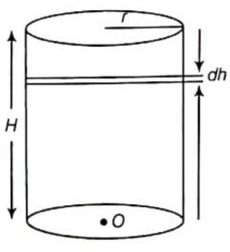
$\Rightarrow \quad y = 2^{ex} - \dfrac{e^{2x}}{2} + C^1, y(0) = 2 \Rightarrow C^1 = 1/2$

$\therefore \quad y(x) = 2e^x - \dfrac{e^{2x}}{2} + \dfrac{1}{2} \Rightarrow \quad y(x) \le \dfrac{5}{2}$ for $x = \log 2$

$\therefore \quad [2x] = [2\log 2] = [\log 4] = 1$

66. Let in time dt the decrease in water level in the tank is dh, then amount of water flown out in time $dt = \pi r^2 \cdot dh$

Now, through the hole the amount of water flown = (Volume of cylinder of cross sectional area 'a' and length vdt).



$$= a\cdot v\cdot dt = a\,\mu\,\sqrt{2gh}\,dt$$

Hence, $\quad \mu a\sqrt{2gh}\,dt = -\pi r^2 dh$

$$\Rightarrow \quad dt = \frac{-\pi r^2}{\mu a\sqrt{2g}}\cdot h^{-1/2}\,dh$$

Now, when $t = 0$, $h = H$ and when $t = t$, $h = 0$

Thus, $\quad \displaystyle\int_0^t dt = \frac{-\pi r^2}{\mu a\sqrt{2g}}\int_H^0 h^{-1/2}\,dh$

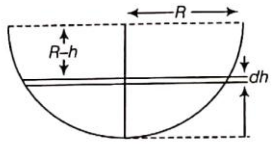
$$\Rightarrow \quad t = \frac{-\pi r^2}{\mu a\sqrt{2g}}\cdot 2(\sqrt{h})_H^0$$

$$\Rightarrow \quad t = \frac{\pi r^2}{\mu a}\cdot\sqrt{\frac{2H}{g}}$$

67. Let at time t the depth of water is h; the radius of water surface is r.

Then, $\quad r^2 = R^2 - (R-h)^2$

$\Rightarrow \quad r^2 = 2Rh - h^2$



Now, if in time dt the decrease in water level is dh, then

$$-\pi r^2\,dh = 0.6\sqrt{2gh}\cdot a\,dt$$

(a is cross-sectional area of the outlet)

$$\Rightarrow \quad \frac{-\pi}{(0.6)\,a\sqrt{2g}}(2Rh - h^2)\frac{dh}{\sqrt{h}} = dt$$

$$\Rightarrow \quad \frac{\pi}{(0.6)\,a\sqrt{2g}}\int_R^0 (h^{3/2} - 2R\,h^{1/2})\,dh = \int_0^t dt$$

$$\Rightarrow \quad \frac{\pi}{(0.6)\,a\sqrt{2g}}\left[\frac{2}{5}h^{5/2} - \frac{4}{3}Rh^{3/2}\right]_R^0 = t$$

$$\Rightarrow \quad t = \frac{\pi}{(0.6)\,a\sqrt{2g}}\left[0 - R^{5/2}\left(\frac{2}{5} - \frac{4}{3}\right)\right] = \frac{7\pi\times 10^5}{135\sqrt{g}}$$

68. We have been given,

$f(xy) = e^{xy - x - y}\{e^y f(x) + e^x f(y)\}, \forall\, x, y \in R^+$ ...(i)

Replacing $x = 1$, $y = 1$ in Eq. (i), we get

$$f(1) = e^{1-1-1}\{e\,f(1) + e\,f(1)\} \Rightarrow f(1) = 0$$

Now, $\quad f'(x) = \displaystyle\lim_{h\to 0}\frac{f(x+h) - f(x)}{h}$

$$= \lim_{h\to 0}\frac{f\left(x\left(1+\frac{h}{x}\right)\right) - f(x)}{h}$$

$$= \lim_{h\to 0}\frac{e^{x+h-x-1-\frac{h}{x}}(e^{1+h/x}\cdot f(x) + e^x\cdot f(1+h/x)) - f(x)}{h}$$

$$= \lim_{h\to 0}\frac{e^h f(x) + e^{h-1-\frac{h}{x}+x}\cdot f\left(1+\frac{h}{x}\right) - f(x)}{h}$$

$$= \lim_{h\to 0}\frac{f(x)(e^h - 1) + e^{h-1-\frac{h}{x}+x}\cdot\left(f\left(1+\frac{h}{x}\right) - f(1)\right)}{h}$$

$$= \lim_{h\to 0} \frac{f(x)(e^h-1)}{h} + \lim_{h\to 0} \frac{e^{h-1-\frac{h}{x}+x}\left(f\left(1+\frac{h}{x}\right)-f(1)\right)}{\frac{h}{x}\cdot x}$$

$$= f(x) + \frac{e^{x-1}\cdot f'(1)}{x} = f(x) + \frac{e^x}{x}$$

$$\Rightarrow \quad f'(x) - f(x) = \frac{e^x}{x}$$

$$\Rightarrow \quad \frac{e^x f'(x) - f(x)e^x}{e^{2x}} = \frac{1}{x}$$

$$\Rightarrow \quad \frac{d}{dx}\left(\frac{f(x)}{e^x}\right) = \frac{1}{x}$$

Integrating both the sides, we get

$$\frac{f(x)}{e^x} = \ln|x| + C$$

Since, $f(1) = 0 \Rightarrow C = 0$

Thus, $f(x) = e^x \cdot \ln(x)$

69. Let $P(x, f(x))$ be a point lying on the curve in first quadrant. Equation of normal and tangent at P are

$$(Y - f(x)) = -\frac{1}{f'(x)}(X - x) \text{ and}$$

$(Y - f(x)) = f'(x)(X - x)$ respectively.

$$\Rightarrow A \equiv \left(x - \frac{f(x)}{f'(x)}, 0\right), B \equiv (x + f(x)f'(x), 0)$$

Since, the mid-point of segment AB is origin.

$$\Rightarrow \quad 2x - \frac{f(x)}{f'(x)} + f(x)\cdot f'(x) = 0$$

$$\Rightarrow \quad f(x)(f'(x))^2 + 2x f'(x) - f(x) = 0$$

$$\Rightarrow \quad f'(x) = \frac{-2x \pm \sqrt{4x^2 + 4f^2(x)}}{2f(x)} = \frac{-x + \sqrt{x^2 + f^2(x)}}{f(x)}$$

Negative sign been neglected as $f'(x) > 0$

Thus, we have, $x + f(x)\cdot f'(x) = \sqrt{x^2 + f^2(x)}$

$$\Rightarrow \quad \frac{d}{2dx}(x^2 + f^2(x)) = \sqrt{x^2 + f^2(x)}$$

$$\Rightarrow \quad \frac{d(x^2 + f^2(x))}{2\sqrt{x^2 + f^2(x)}} = dx$$

Integrating both the sides, we get $\sqrt{x^2 + f^2(x)} = x + \lambda$

It passes through $(1, \sqrt{3})$.

Hence, $\lambda + 1 = 2 \Rightarrow \lambda = 1$

$\therefore$ Curve is $x^2 + f^2(x) = (x+1)^2$ or $y^2 = 1 + 2x$

70. We have been given,

$$\frac{dy_1}{dx} + Py_1 = Q, \frac{dy_2}{dx} + Py_2 = Q$$

Now, $y_2 = y_1 z \Rightarrow \frac{dy_2}{dx} = y_1\frac{dz}{dx} + z\frac{dy_1}{dx}$

$$\Rightarrow y_1\frac{dz}{dx} + z\frac{dy_1}{dx} + Py_1 z = Q$$

$$\Rightarrow y_1\cdot\frac{dz}{dx} + z\left(\frac{dy_1}{dx} + py_1\right) = Q$$

$$\Rightarrow \quad y_1\cdot\frac{dz}{dx} + zQ = Q$$

$$\Rightarrow \quad y_1\frac{dz}{dx} = Q(1-z)$$

$$\Rightarrow \quad \frac{dz}{1-z} = \frac{Q}{y_1}dx$$

$\Rightarrow \ln|z-1| = -\int \frac{Q}{y_1}dx + \lambda$, '$\lambda$' being constant of integration

$$\Rightarrow \quad z = 1 + ce^{\int -\frac{Q}{y_1}dx}$$

71. We have, $f(x) = x + x^2\int_0^1 z f(z)\,dz + x\int_0^1 z^2 f(z)\,dz$

$$f(x) = x + x^2\lambda_1 + x\lambda_2 \qquad \text{(say)}$$

Now, $\lambda_1 = \int_0^1 z f(z)\,dz = \int_0^1 ((1+\lambda_2)z + z^2\lambda_1)z\,dz$

$$= \frac{1+\lambda_2}{3} + \frac{\lambda_1}{4}$$

$$\Rightarrow \quad 9\lambda_1 - 4\lambda_2 = 4 \qquad \text{...(i)}$$

Also, $\lambda_2 = \int_0^1 z^2 f(z)\,dz$

$$= \int_0^1 ((1+\lambda_2)z^3 + z^4\lambda_1)\,dz$$

$$= \frac{(1+\lambda_2)}{4} + \frac{\lambda_1}{5}$$

$$\Rightarrow \quad 15\lambda_2 - 4\lambda_1 = 5 \qquad \text{...(ii)}$$

From Eqs. (i) and (ii),

$$\lambda_1 = \frac{80}{119} \text{ and } \lambda_2 = \frac{61}{119}$$

$$\Rightarrow \quad f(x) = x + \frac{80}{119}x^2 + \frac{61}{119}x = \frac{20x}{119}(4 + 9x)$$

72. $f(x + f(y) + x f(y)) = y + f(x) + y f(x)$

Keep y constant and differentiating the expression w.r.t. 'x', we get

$$f'(x + f(y) + x(y))(1 + f(y)) = f'(x)(1+y) \qquad \text{...(i)}$$

Similarly, differentiate the expression w.r.t. 'y' and keep x constant, we get

$$f'(x + f(y) + xf(y))(f'(y)(1+x)) = (1 + f(x)) \qquad \text{...(ii)}$$

Dividing Eq. (i) by Eq. (ii), we get

$$\frac{1 + f(y)}{f'(y)(1+x)} = \frac{f'(x)(1+y)}{(1+f(x))}$$

$$\Rightarrow \quad \frac{1 + f(y)}{f'(y)(1+y)} = \frac{f'(x)(1+x)}{(1+f(x))} = C$$

$$\Rightarrow f'(y) = \frac{1}{C}\left[\frac{1+f(y)}{1+y}\right] \text{ and } f'(x) = C\frac{1+f(x)}{(1+x)}$$

$$\Rightarrow \quad C = \frac{1}{C} \Rightarrow C = \pm 1$$

$$\therefore \quad \frac{f'(x)}{1+f(x)} = \pm\frac{1}{1+x}$$

$$\Rightarrow \quad f(x) = \lambda_1(1+x)^{\pm 1} - 1$$

Replacing $x, y \to 0$ in the Eq. (i), we get

$$f(f(0)) = f(0)$$

Now, $\quad f(x) = \lambda_1 (1+x)^{\pm 1} - 1 \Rightarrow f(0) = \lambda_1 - 1$

and $\quad f(f(0)) = f(\lambda_1 - 1) = \lambda_1 \lambda_1^{\pm 1} - 1$

Since, $\quad f(f(0)) = f(0), \lambda_1 \lambda_1^{\pm 1} - 1 = \lambda_1 - 1$

$\Rightarrow \quad \lambda_1^{1 \pm 1} = \lambda_1$

By taking +ve sign, we get $\lambda_1 = 0, 1$

$\Rightarrow \quad f(x) = -1 \;\text{ or }\; f(x) = x$

By taking – ve sign, we get $\lambda_1 = 1$

$\Rightarrow \quad f(x) = \dfrac{1}{1+x} - 1 = -\dfrac{x}{1+x}$

73. Given, $\quad (x^2 + xy + 4x + 2y + 4)\dfrac{dy}{dx} - y^2 = 0$

$\Rightarrow \quad [(x^2 + 4x + 4) + y(x+2)]\dfrac{dy}{dx} - y^2 = 0$

$\Rightarrow \quad [(x+2)^2 + y(x+2)]\dfrac{dy}{dx} - y^2 = 0$

Put $x + 2 = X$ and $y = Y$, then

$$(X^2 + XY)\frac{dY}{dX} - Y^2 = 0$$

$\Rightarrow \quad X^2 dY + XY dY - Y^2 dX = 0$

$\Rightarrow \quad X^2 dY + Y(X dY - Y dX) = 0$

$\Rightarrow \quad -\dfrac{dY}{Y} = \dfrac{X dY - Y dX}{X^2}$

$\Rightarrow \quad -d(\log|Y|) = d\left(\dfrac{Y}{X}\right)$

On integrating both sides, we get

$$-\log|Y| = \frac{Y}{X} + C, \text{ where } x + 2 = X \text{ and } y = Y$$

$\Rightarrow \quad -\log|y| = \dfrac{y}{x+2} + C \quad$ ...(i)

Since, it passes through the point (1, 3).

$\therefore \quad -\log 3 = 1 + C$

$\Rightarrow \quad C = -1 - \log 3 = -(\log e + \log 3)$

$\quad = -\log 3e$

$\therefore$ Eq. (i) becomes

$$\log|y| + \frac{y}{x+2} - \log(3e) = 0$$

$\Rightarrow \quad \log\left(\dfrac{|y|}{3e}\right) + \dfrac{y}{x+2} = 0 \quad$ ...(ii)

Now, to check option (a), $y = x + 2$ intersects the curve.

$\Rightarrow \quad \log\left(\dfrac{|x+2|}{3e}\right) + \dfrac{x+2}{x+2} = 0$

$\Rightarrow \quad \log\left(\dfrac{|x+2|}{3e}\right) = -1$

$\Rightarrow \quad \dfrac{|x+2|}{3e} = e^{-1} = \dfrac{1}{e}$

$\Rightarrow \quad |x+2| = 3 \;\text{ or }\; x + 2 = \pm 3$

$\therefore \quad x = 1, -5$ (rejected), as $x > 0$ [given]

$\therefore$ $x = 1$ only one solution.

Thus, (a) is the correct answer.

To check option (c), we have $\quad y = (x+2)^2$

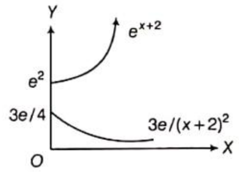
and $\quad \log\left(\dfrac{|y|}{3e}\right) + \dfrac{y}{x+2} = 0$

$\Rightarrow \quad \log\left[\dfrac{|x+2|^2}{3e}\right] + \dfrac{(x+2)^2}{x+2} = 0$

$\Rightarrow \quad \log\left[\dfrac{|x+2|^2}{3e}\right] = -(x+2)$

$\Rightarrow \quad \dfrac{(x+2)^2}{3e} = e^{-(x+2)}$ or $(x+2)^2 \cdot e^{x+2} = 3e$

$\Rightarrow \quad e^{x+2} = \dfrac{3e}{(x+2)^2}$



Clearly, they have no solution.

To check option (d), $y = (x+3)^2$

i.e. $\quad \log\left[\dfrac{|x+3|^2}{3e}\right] + \dfrac{(x+3)^2}{(x+2)} = 0$

To check the number of solutions.

Let $\quad g(x) = 2\log(x+3) + \dfrac{(x+3)^2}{(x+2)} - \log(3e)$

$\therefore \quad g'(x) = \dfrac{2}{x+3} + \left(\dfrac{(x+2)\cdot 2(x+3) - (x+3)^2 \cdot 1}{(x+2)^2}\right) - 0$

$\quad = \dfrac{2}{x+3} + \dfrac{(x+3)(x+1)}{(x+2)^2}$

Clearly, when $x > 0$, then, $g'(x) > 0$

$\therefore \quad g(x)$ is increasing, when $x > 0$.

Thus, when $x > 0$, then $g(x) > g(0)$

$$g(x) > \log\left(\frac{3}{e}\right) + \frac{9}{4} > 0$$

Hence, there is no solution.

Thus, option (d) is true.

74. Here, $\quad f'(x) = 2 - \dfrac{f(x)}{x}$

or $\quad \dfrac{dy}{dx} + \dfrac{y}{x} = 2 \quad$ [i.e. linear differential equation in y]

Integrating Factor, IF $= e^{\int \frac{1}{x} dx} = e^{\log x} = x$

$\therefore$ Required solution is $y \cdot (\text{IF}) = \int Q(\text{IF}) dx + C$

$\Rightarrow \quad y(x) = \int 2(x)\, dx + C$

$\Rightarrow \quad yx = x^2 + C$

$\therefore \quad y = x + \dfrac{C}{x} \quad$ [$\because C \neq 0$, as $f(1) \neq 1$]

(a) $\lim_{x \to 0^+} f'\left(\frac{1}{x}\right) = \lim_{x \to 0^+} (1 - Cx^2) = 1$

$\therefore$ Option (a) is correct.

(b) $\lim_{x \to 0^+} x f\left(\frac{1}{x}\right) = \lim_{x \to 0^+} (1 + Cx^2) = 1$

$\therefore$ Option (b) is incorrect.

(c) $\lim_{x \to 0^+} x^2 f'(x) = \lim_{x \to 0^+} (x^2 - C) = -C \neq 0$

$\therefore$ Option (c) is incorrect.

(d) $f(x) = x + \frac{C}{x}, C \neq 0$

For $C > 0$, $\lim_{x \to 0^+} f(x) = \infty$

$\therefore$ Function is not bounded in (0, 2).

$\therefore$ Option (d) is incorrect.

75. Here, $(1 + e^x)y' + ye^x = 1$

$\Rightarrow \frac{dy}{dx} + e^x \cdot \frac{dy}{dx} + ye^x = 1$

$\Rightarrow dy + e^x dy + ye^x\, dx = dx$

$\Rightarrow dy + d(e^x y) = dx$

On integrating both sides, we get

$$y + e^x y = x + C$$

Given, $y(0) = 2$

$\Rightarrow 2 + e^0 \cdot 2 = 0 + C$

$\Rightarrow C = 4$

$\therefore y(1 + e^x) = x + 4$

$\Rightarrow y = \frac{x + 4}{1 + e^x}$

Now at $x = -4, y = \frac{-4 + 4}{1 + e^{-4}} = 0$

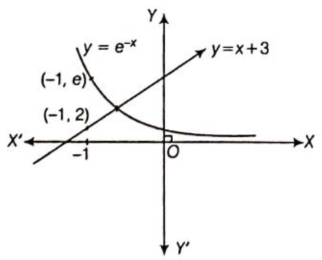
$\therefore y(-4) = 0$ ...(i)

For critical points, $\frac{dy}{dx} = 0$

i.e. $\frac{dy}{dx} = \frac{(1 + e^x) \cdot 1 - (x + 4)e^x}{(1 + e^x)^2} = 0$

$\Rightarrow e^x (x + 3) - 1 = 0$

or $e^{-x} = (x + 3)$



Clearly, the intersection point lies between (− 1, 0).

$\therefore y(x)$ has a critical point in the interval (− 1, 0).

76. Since, centre lies on $y = x$.

$\therefore$ Equation of circle is $x^2 + y^2 - 2ax - 2ay + c = 0$

On differentiating, we get

$$2x + 2yy' - 2a - 2ay' = 0$$

$\Rightarrow x + yy' - a - ay' = 0$

$\Rightarrow a = \frac{x + yy'}{1 + y'}$

Again differentiating, we get

$$0 = \frac{(1 + y')[1 + yy'' + (y')^2] - (x + yy') \cdot (y'')}{(1 + y')^2}$$

$\Rightarrow (1 + y')[1 + (y')^2 + yy''] - (x + yy')(y'') = 0$

$\Rightarrow 1 + y'[(y')^2 + y' + 1] + y''(y - x) = 0$

On comparing with $Py'' + Qy' + 1 = 0$, we get

$$P = y - x$$

and $Q = (y')^2 + y' + 1$

77. (i) Solution of the differential equation $\frac{dy}{dx} + Py = Q$ is

$$y \cdot (\text{IF}) = \int Q \cdot (\text{IF})\, dx + c$$

where, $\text{IF} = e^{\int P\, dx}$

(ii) $\int_{-a}^{a} f(x)\, dx = 2\int_0^a f(x)\, dx$, if $f(-x) = f(x)$

Given differential equation

$$\frac{dy}{dx} + \frac{x}{x^2 - 1} y = \frac{x^4 + 2x}{\sqrt{1 - x^2}}$$

This is a linear differential equation.

$$\text{IF} = e^{\int \frac{x}{x^2 - 1} dx} = e^{\frac{1}{2} \ln|x^2 - 1|} = \sqrt{1 - x^2}$$

$\Rightarrow$ Solution is

$$y\sqrt{1 - x^2} = \int \frac{x(x^3 + 2)}{\sqrt{1 - x^2}} \cdot \sqrt{1 - x^2}\, dx$$

or $y\sqrt{1 - x^2} = \int (x^4 + 2x)\, dx = \frac{x^5}{5} + x^2 + c$

$$f(0) = 0 \Rightarrow c = 0$$

$\Rightarrow f(x)\sqrt{1 - x^2} = \frac{x^5}{5} + x^2$

Now, $\int_{-\sqrt{3}/2}^{\sqrt{3}/2} f(x) dx = \int_{-\sqrt{3}/2}^{\sqrt{3}/2} \frac{x^2}{\sqrt{1 - x^2}} dx$ [using property]

$= 2\int_0^{\sqrt{3}/2} \frac{x^2}{\sqrt{1 - x^2}} dx$

$= 2\int_0^{\pi/3} \frac{\sin^2 \theta}{\cos \theta} \cos \theta\, d\theta$ [taking $x = \sin \theta$]

$= 2\int_0^{\pi/3} \sin^2 \theta\, d\theta = \int_0^{\pi/3} (1 - \cos 2\theta)\, d\theta$

$= \left(\theta - \frac{\sin 2\theta}{2}\right)_0^{\pi/3} = \frac{\pi}{3} - \frac{\sin 2\pi/3}{2} = \frac{\pi}{3} - \frac{\sqrt{3}}{4}$

78. Whenever we have linear differential equation containing inequality, we should always check for increasing or decreasing,

i.e. for $\frac{dy}{dx} + Py < 0 \Rightarrow \frac{dy}{dx} + Py > 0$

Multiply by integrating factor, i.e. $e^{\int P dx}$ and convert into total differential equation.

Here, $f'(x) < 2f(x)$, multiplying by $e^{-\int 2dx}$

$$f'(x)\cdot e^{-2x} - 2e^{-2x}f(x) < 0 \Rightarrow \frac{d}{dx}(f(x)\cdot e^{-2x}) < 0$$

$\therefore\ \phi(x) = f(x)e^{-2x}$ is decreasing for $x \in \left[\frac{1}{2}, 1\right]$

Thus, when $x > \frac{1}{2}$

$$\phi(x) < \phi\left(\frac{1}{2}\right) \Rightarrow e^{-2x}f(x) < e^{-1}\cdot f\left(\frac{1}{2}\right)$$

$$\Rightarrow f(x) < e^{2x-1}\cdot 1,\ \text{given } f\left(\frac{1}{2}\right) = 1$$

$$\Rightarrow 0 < \int_{1/2}^{1} f(x)dx < \int_{1/2}^{1} e^{2x-1}dx$$

$$\Rightarrow 0 < \int_{1/2}^{1} f(x)dx < \left(\frac{e^{2x-1}}{2}\right)_{1/2}^{1}$$

$$\Rightarrow 0 < \int_{1/2}^{1} f(x)dx < \frac{e-1}{2}$$

79. To solve homogeneous differential equation, i.e. substitute $\frac{y}{x} = v$

$$\therefore\ y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$$

Here, slope of the curve at (x, y) is

$$\frac{dy}{dx} = \frac{y}{x} + \sec\left(\frac{y}{x}\right)$$

Put $\frac{y}{x} = v$

$$\therefore\ v + x\frac{dv}{dx} = v + \sec(v)$$

$$\Rightarrow x\frac{dv}{dx} = \sec(v) \Rightarrow \int\frac{dv}{\sec v} = \int\frac{dx}{x}$$

$$\Rightarrow \int\cos v\,dv = \int\frac{dx}{x} \Rightarrow \sin v = \log x + \log c$$

$$\Rightarrow \sin\left(\frac{y}{x}\right) = \log(cx)$$

As it passes through $\left(1, \frac{\pi}{6}\right) \Rightarrow \sin\left(\frac{\pi}{6}\right) = \log c$

$$\Rightarrow \log c = \frac{1}{2}$$

$$\therefore\ \sin\left(\frac{y}{x}\right) = \log x + \frac{1}{2}$$

80. Let $\phi(x) = e^{-x}f(x)$

Here, $\phi'(x) < 0,\ x \in \left(0, \frac{1}{4}\right)$

and $\phi'(x) > 0,\ x \in \left(\frac{1}{4}, 1\right)$

$$\Rightarrow e^{-x}f'(x) - e^{-x}f(x) < 0,\ x \in \left(0, \frac{1}{4}\right)$$

$$\Rightarrow f'(x) < f(x), 0 < x < \frac{1}{4}$$

81. Here, $f''(x) - 2f'(x) + f(x) \ge e^x$

$$\Rightarrow f''(x)e^{-x} - f'(x)e^{-x} - f'(x)e^{-x} + f(x)e^{-x} \ge 0$$

$$\Rightarrow \frac{d}{dx}\{f'(x)e^{-x}\} - \frac{d}{dx}\{f(x)e^{-x}\} \ge 1$$

$$\Rightarrow \frac{d}{dx}\{f'(x)e^{-x} - f(x)e^{-x}\} \ge 1$$

$$\Rightarrow \frac{d^2}{dx^2}\{e^{-x}f(x)\} \ge 1,\ \forall x \in [0, 1]$$

$\therefore\ \phi(x) = e^{-x}f(x)$ is concave function.

$$f(0) = f(1) = 0$$

$$\Rightarrow \phi(0) = 0 = f(1)$$

$$\Rightarrow \phi(x) < 0$$

$$\Rightarrow e^{-x}f(x) < 0$$

$$\therefore\ f(x) < 0$$

82. Here, $f(x) = (1-x)^2\cdot\sin^2 x + x^2 \ge 0,\ \forall\ x$

and $g(x) = \int_1^x\left(\frac{2(t-1)}{t+1} - \log t\right)f(t)dt$

$$\Rightarrow g'(x) = \left\{\frac{2(x-1)}{(x+1)} - \log x\right\}\cdot \underbrace{f(x)}_{+\text{ve}} \quad \text{...(i)}$$

For $g'(x)$ to be increasing or decreasing.

Let $\phi(x) = \frac{2(x-1)}{x+1} - \log x$

$$\phi'(x) = \frac{4}{(x+1)^2} - \frac{1}{x} = \frac{-(x-1)}{x(x+1)^2}$$

$$\phi'(x) < 0,\ \forall\ x > 1$$

$$\Rightarrow \phi(x) < \phi(1) \Rightarrow \phi(x) < 0 \quad \text{...(ii)}$$

From Eqs. (i) and (ii), $g'(x) < 0,\ x \in (1, \infty)$

$\therefore g(x)$ is decreasing on $x \in (1, \infty)$.

83. Here, $f(x) + 2x = (1-x)^2\cdot\sin^2 x + x^2 + 2x$ ...(i)

where, I: $f(x) + 2x = 2(1+x)^2$ ...(ii)

$$\therefore\ 2(1+x^2) = (1-x)^2\sin^2 x + x^2 + 2x$$

$$\Rightarrow (1-x)^2\sin^2 x = x^2 - 2x + 2$$

$$\Rightarrow (1-x)^2\sin^2 x = (1-x)^2 + 1$$

$$\Rightarrow (1-x)^2\cos^2 x = -1$$

which is never possible.

$\therefore$ I is false.

Again, let $h(x) = 2f(x) + 1 - 2x(1+x)$

where, $h(0) = 2f(0) + 1 - 0 = 1$

$h(1) = 2(1) + 1 - 4 = -3$ as $[h(0)h(1) < 0]$

$\Rightarrow h(x)$ must have a solution.

$\therefore$ II is true.

84. Linear differential equation under one variable.

$$\frac{dy}{dx}+Py=Q;\ \text{IF}=e^{\int P\,dx}$$

$\therefore$ Solution is, $y\,(\text{IF})=\int Q\cdot(\text{IF})dx+C$

$$y'-y\tan x=2x\sec x \text{ and } y(0)=0$$

$\Rightarrow$ $$\frac{dy}{dx}-y\tan x=2x\sec x$$

$\therefore$ $$\text{IF}=\int e^{-\tan x}\,dx=e^{\log|\cos x|}=\cos x$$

Solution is $y\cdot\cos x=\int 2x\sec x\cdot\cos x\,dx+C$

$\Rightarrow$ $y\cdot\cos x=x^2+C$

As $y(0)=0$

$\Rightarrow$ $C=0$

$\therefore$ $y=x^2\sec x$

Now, $y\left(\frac{\pi}{4}\right)=\frac{\pi^2}{8\sqrt{2}}$

$\Rightarrow$ $y'\left(\frac{\pi}{4}\right)=\frac{\pi}{\sqrt{2}}+\frac{\pi^2}{8\sqrt{2}} \Rightarrow y\left(\frac{\pi}{3}\right)=\frac{2\pi^2}{9}$

$\Rightarrow$ $y'\left(\frac{\pi}{3}\right)=\frac{4\pi}{3}+\frac{2\pi^2}{3\sqrt{3}}$

85. $\frac{dy}{dx}+y\cdot g'(x)=g(x)\,g'(x)$

$$\text{IF}=e^{\int g'(x)\,dx}=e^{g(x)}$$

$\therefore$ Solution is $y\,(e^{g(x)})=\int g(x)\cdot g'(x)\cdot e^{g(x)}\,dx+C$

Put $g(x)=t,\ g'(x)\,dx=dt$

$$y(e^{g(x)})=\int t\cdot e^t\,dt+C=t\cdot e^t-\int 1\cdot e^t\,dt+C$$

$$=t\cdot e^t-e^t+C$$

$$y\,e^{g(x)}=(g(x)-1)\,e^{g(x)}+C \quad \text{...(i)}$$

Given, $y(0)=0,\ g(0)=g(2)=0$

$\therefore$ Eq. (i) becomes,

$$y(0)\cdot e^{g(0)}=(g(0)-1)\cdot e^{g(0)}+C$$

$\Rightarrow$ $0=(-1)\cdot 1+C \Rightarrow C=1$

$\therefore$ $y(x)\cdot e^{g(x)}=(g(x)-1)\,e^{g(x)}+1$

$\Rightarrow$ $y(2)\cdot e^{g(2)}=(g(2)-1)\,e^{g(2)}+1$, where $g(2)=0$

$\Rightarrow$ $y(2)\cdot 1=(-1)\cdot 1+1$

$y(2)=0$

86. From given integral equation, $f(0)=0$.

Also, differentiating the given integral equation w.r.t. x

$$f'(x)=f(x)$$

If $f(x)\neq 0$

$\Rightarrow$ $\frac{f'(x)}{f(x)}=1 \Rightarrow \log f(x)=x+c$

$\Rightarrow$ $f(x)=e^c e^x$

$\because$ $f(0)=0 \Rightarrow e^c=0$, a contradiction

$\therefore$ $f(x)=0,\ \forall\, x\in R \Rightarrow f(\ln 5)=0$

Alter

Given, $f(x)=\int_0^x f(t)\,dt$

$\Rightarrow$ $f(0)=0$ and $f'(x)=f(x)$

If $f(x)\neq 0$

$\Rightarrow$ $\frac{f'(x)}{f(x)}=1 \Rightarrow \ln f(x)=x+c$

$\Rightarrow$ $f(x)=e^c\cdot e^x$

$\because$ $f(0)=0$

$\Rightarrow e^c=0$, a contradiction

$\therefore$ $f(x)=0,\forall\, x\in R \Rightarrow f(\ln 5)=0$

87. Given, $\frac{dy}{dx}=\frac{y\sqrt{y^2-1}}{x\sqrt{x^2-1}}$

$$\int\frac{dy}{y\sqrt{y^2-1}}=\int\frac{dx}{x\sqrt{x^2-1}}$$

$\Rightarrow$ $\sec^{-1}y=\sec^{-1}x+c$

At $x=2,\ y=\frac{2}{\sqrt{3}};\ \frac{\pi}{6}=\frac{\pi}{3}+c \Rightarrow c=-\frac{\pi}{6}$

Now, $y=\sec\left(\sec^{-1}x-\frac{\pi}{6}\right)=\cos\left[\cos^{-1}\frac{1}{x}-\cos^{-1}\frac{\sqrt{3}}{2}\right]$

$$=\cos\left[\cos^{-1}\left(\frac{\sqrt{3}}{2x}+\sqrt{1-\frac{1}{x^2}}\sqrt{1-\frac{3}{4}}\right)\right]$$

$$y=\frac{\sqrt{3}}{2x}+\frac{1}{2}\sqrt{1-\frac{1}{x^2}}$$

88. Given differential equation is

$$y(1+xy)\,dx=x\,dy$$

$\Rightarrow$ $y\,dx+xy^2\,dx=x\,dy$

$\Rightarrow$ $\frac{x\,dy-y\,dx}{y^2}=x\,dx$

$\Rightarrow$ $-\frac{(y\,dx-x\,dy)}{y^2}=x\,dx \Rightarrow -d\left(\frac{x}{y}\right)=x\,dx$

On integrating both sides, we get

$$-\frac{x}{y}=\frac{x^2}{2}+C \quad \text{...(i)}$$

$\because$ It passes through $(1,-1)$.

$\therefore$ $1=\frac{1}{2}+C \Rightarrow C=\frac{1}{2}$

Now, from Eq. (i) $-\frac{x}{y}=\frac{x^2}{2}+\frac{1}{2}$

$\Rightarrow$ $x^2+1=-\frac{2x}{y} \Rightarrow y=-\frac{2x}{x^2+1}$

$\therefore$ $f\left(-\frac{1}{2}\right)=\frac{4}{5}$

89. Given differential equation is

$$(x\log x)\frac{dy}{dx}+y=2x\log x, \quad (x\geq 1)$$

$\Rightarrow$ $\frac{dy}{dx}+\frac{y}{x\log x}=2$

This is a linear differential equation.

$\therefore \quad IF = e^{\int \frac{1}{x \log x} dx} = e^{\log(\log x)} = \log x$

Now, the solution of given differential equation is given by

$$y \cdot \log x = \int \log x \cdot 2dx$$

$\Rightarrow \quad y \cdot \log x = 2\int \log x \, dx$

$\Rightarrow \quad y \cdot \log x = 2[x \log x - x] + c$

At $\quad x = 1, c = 2$

$\Rightarrow \quad y \cdot \log x = 2[x \log x - x] + 2$

At $\quad x = e,$

$y = 2(e - e) + 2$

$\Rightarrow \quad y = 2$

90. Given differential equation $\frac{dp}{dt} - \frac{1}{2}p(t) = -200$ is a linear differential equation.

Here, $\quad p(t) = \frac{-1}{2}, Q(t) = -200$

$$IF = e^{\int -\left(\frac{1}{2}\right)dt} = e^{-\frac{t}{2}}$$

Hence, solution is

$$p(t).IF = \int Q(t)\, IF\, dt$$

$$p(t) \cdot e^{-\frac{t}{2}} = \int -200 \cdot e^{-\frac{t}{2}} dt$$

$$p(t) \cdot e^{-\frac{t}{2}} = 400 e^{-\frac{t}{2}} + K$$

$\Rightarrow \quad p(t) = 400 + ke^{-1/2}$

If $p(0) = 100$, then $k = -300$

$\Rightarrow \quad p(t) = 400 - 300e^{\frac{t}{2}}$

91. Given, $\frac{dP}{dx} = (100 - 12\sqrt{x})$

$\Rightarrow \quad dP = (100 - 12\sqrt{x})\, dx$

On integrating both sides, we get

$$\int dP = \int (100 - 12\sqrt{x})\, dx$$

$$P = 100x - 8x^{3/2} + C$$

When $x = 0$, then $P = 2000$

$\Rightarrow \quad C = 2000$

Now, when $x = 25$, then

$$P = 100 \times 25 - 8 \times (25)^{3/2} + 2000$$

$$= 2500 - 8 \times 125 + 2000$$

$$= 4500 - 1000 = 3500$$

92. Given

(i) The population of mouse at time 't' satisfies the differential equation $p'(t) = \frac{dp(t)}{dt} = 0.5p(t) - 450$

(ii) Population of mouse at time $t = 0$ is

$$p(0) = 850$$

To find The time at which the population of the mouse will become zero, i.e. to find the value of 't' at which $p(t) = 0$.

Let's solve the differential equation first

$$p'(t) = \frac{dp(t)}{dt} = 0.5\, p(t) - 450$$

$\Rightarrow \quad \frac{2dp(t)}{p(t) - 900} = dt$

$\Rightarrow \quad \int \frac{2dp(t)}{p(t) - 900} = \int dt$

$\Rightarrow 2\log |\, p(t) - 900| = t + C$, where C is the constant of integration.

To find the value of 'C', let's substitute $t = 0$.

$\Rightarrow \; 2 \log |\, p(0) - 900 \,| = 0 + C$

$\Rightarrow \quad C = 2 \log |\, 850 - 900 \,|$

$\Rightarrow \quad C = 2 \log 50$

Now, substituting the value of C back in the solution, we get

$$2 \log |\, p(t) - 900 \,| = t + 2 \log 50$$

Here, since we want to find the value of t at which $p(t) = 0$, hence substituting $p(t) = 0$, we get

$$2 \log |\, 0 - 900 \,| = t + 2 \log 50$$

$\Rightarrow \quad t = 2 \log \left| \frac{900}{50} \right|$

$\Rightarrow \quad t = 2 \log 18$

93. Here, $\frac{dy}{dx} = y + 3 > 0$ and $y(0) = 2$

$\Rightarrow \quad \int \frac{dy}{y+3} = \int dx$

$\Rightarrow \quad \log |y + 3| = x + C$

Since, $\quad y(0) = 2$

$\Rightarrow \quad \log_e |2 + 3| = 0 + C$

$\therefore \quad C = \log_e 5$

$\Rightarrow \quad \log_e |y + 3| = x + \log_e 5$

When $\quad x = \log_e 2$

$\Rightarrow \quad \log_e |y + 3| = \log_e 2 + \log_e 5 = \log_e 10$

$\Rightarrow \quad y + 3 = 10$

$\Rightarrow \quad y = 7$

94. Given, $\quad \frac{d\{V(t)\}}{dt} = -k(T - t)$

$\therefore \quad d\{V(t)\} = -k(T - t)\, dt$

On integrating both sides, we get

$$V(t) = -k\frac{(T-t)^2}{2(-1)} + C$$

$\Rightarrow \quad V(t) = \frac{k}{2}(T - t)^2 + C$

$\because \quad$ At $t = 0$, $V(t) = I$

$\therefore \quad I = \frac{k}{2}(T - 0)^2 + C$

$\Rightarrow \quad C = I - \frac{k}{2}T^2$

$\therefore \quad V(t) = \frac{k}{2}(T-t)^2 + I - \frac{k}{2}T^2$

Now, $\quad V(T) = I - \frac{k}{2}T^2$

95. Since, $\cos x\, dy = y \sin x\, dx - y^2 dx$

$\Rightarrow \quad \frac{1}{y^2}\frac{dy}{dx} - \frac{1}{y}\tan x = -\sec x$

Put $-\frac{1}{y} = z$

$\Rightarrow \quad \frac{1}{y^2}\frac{dy}{dx} = \frac{dz}{dx}$

$\Rightarrow \quad \frac{dz}{dx} + (\tan x)z = -\sec x$

This is a linear differential equation.

Therefore,

$$\text{IF} = e^{\int \tan x\, dx} = e^{\log \sec x} = \sec x$$

Hence, the solution is

$$z \cdot (\sec x) = \int -\sec x \cdot \sec x\, dx + C_1$$

$\Rightarrow \quad -\frac{1}{y}\sec x = -\tan x + C_1$

$\Rightarrow \quad \sec x = y(\tan x + C)$

96. Given, $\quad y = c_1 e^{c_2 x}$

$\Rightarrow \quad y' = c_2 c_1 e^{c_2 x}$

$\Rightarrow \quad y' = c_2 y \quad$ ...(i)

$\Rightarrow \quad y'' = c_2 y' \quad$ ...(ii)

$\Rightarrow \quad y'' = \frac{(y')^2}{y} \quad \left[\text{from Eq. (i), } c_2 = \frac{y'}{y}\right]$

$\Rightarrow \quad yy'' = (y')^2$

97. Equation of circle having centre (h, k) and radius a is $(x-h)^2 + (y-k)^2 = a^2$.

The equation of family of circles with centre on $y = 2$ and of radius 5 is

$$(x-\alpha)^2 + (y-2)^2 = 5^2 \quad \text{...(i)}$$

$\Rightarrow \quad x^2 + \alpha^2 - 2\alpha x + y^2 + 4 - 4y = 25$

On differentiating w.r.t. x, we get

$$2x - 2\alpha + 2y\frac{dy}{dx} - 4\frac{dy}{dx} = 0$$

$\Rightarrow \quad \alpha = x + \frac{dy}{dx}(y-2)$

On putting the value of α in Eq. (i), we get

$$\left[x - x - \frac{dy}{dx}(y-2)\right]^2 + (y-2)^2 = 5^2$$

$\Rightarrow \quad \left(\frac{dy}{dx}\right)^2 (y-2)^2 = 25 - (y-2)^2$

$\Rightarrow \quad y'^2 (y-2)^2 = 25 - (y-2)^2$