Limits and Derivatives

Chapter 13 Limits and Derivatives Exercise 13.1, 13.2, miscellaneous Solutions

Exercise 13.1 : Solutions of Questions on Page Number : **301**

Q1 :

Evaluate the Given limit: $\lim_{x \to 3} x + 3$

Answer :

$$\lim_{x \to 3} x + 3 = 3 + 3 = 6$$

Q2 :

Evaluate the Given limit: $\lim_{x \to \pi} \left(x - \frac{22}{7} \right)$

Answer :

$$\lim_{x \to \pi} \left(x - \frac{22}{7} \right) = \left(\pi - \frac{22}{7} \right)$$

Q3 :

Evaluate the Given limit: $\lim_{r \to 1} \pi r^2$

Answer :

$$\lim_{r \to 1} \pi r^2 = \pi (1)^2 = \pi$$

Q4 :

Evaluate the Given limit: $\lim_{x \to 4} \frac{4x + 3}{x - 2}$

Answer :

$$\lim_{x\to 4}\frac{4x+3}{x-2}=\frac{4(4)+3}{4-2}=\frac{16+3}{2}=\frac{19}{2}$$

Q5 :

Evaluate the Given limit: $\lim_{x\to -1}\frac{x^{10}+x^5+1}{x-1}$

Answer :

$$\lim_{x\to -1}\frac{x^{10}+x^5+1}{x-1}=\frac{(-1)^{10}+(-1)^5+1}{-1-1}=\frac{1-1+1}{-2}=-\frac{1}{2}$$

Q6 :

Evaluate the Given limit: $\lim_{x\to 0}\frac{(x+1)^5-1}{x}$

Answer :

$$\lim_{x\to 0}\frac{(x+1)^5-1}{x}$$

Put *x* + 1 = *y* so that *y* Ã¢â€ ' 1 as *x* Ã¢â€ ' 0.

$$\text{Accordingly, }\lim_{x\to 0}\frac{(x+1)^5-1}{x}=\lim_{y\to 1}\frac{y^5-1}{y-1}$$

$$=\lim_{y\to 1}\frac{y^5-1^5}{y-1}$$

$$=5.1^{5-1}\qquad\left[\lim_{x\to a}\frac{x^n-a^n}{x-a}=na^{n-1}\right]$$

$$=5$$

$$\therefore \lim_{x\to 0}\frac{(x+5)^5-1}{x}=5$$

Q7 :

Evaluate the Given limit: $\lim_{x\to 2}\frac{3x^2-x-10}{x^2-4}$

Answer :

At x = 2, the value of the given rational function takes the form $\frac{0}{0}$.

$$\therefore \lim_{x\to 2}\frac{3x^2-x-10}{x^2-4}=\lim_{x\to 2}\frac{(x-2)(3x+5)}{(x-2)(x+2)}$$

$$=\lim_{x\to 2}\frac{3x+5}{x+2}$$

$$=\frac{3(2)+5}{2+2}$$

$$=\frac{11}{4}$$

Q8 :

Evaluate the Given limit: $\lim_{x\to 3}\frac{x^4-81}{2x^2-5x-3}$

Answer :

At x = 2, the value of the given rational function takes the form $\frac{0}{0}$.

$$\therefore \lim_{x\to 3}\frac{x^4-81}{2x^2-5x-3}=\lim_{x\to 3}\frac{(x-3)(x+3)(x^2+9)}{(x-3)(2x+1)}$$

$$=\lim_{x\to 3}\frac{(x+3)(x^2+9)}{2x+1}$$

$$=\frac{(3+3)(3^2+9)}{2(3)+1}$$

$$=\frac{6\times 18}{7}$$

$$=\frac{108}{7}$$

Q9 :

Evaluate the Given limit: $\lim_{x\to 0}\frac{ax+b}{cx+1}$

Answer :

$$\lim_{x\to 0}\frac{ax+b}{cx+1}=\frac{a(0)+b}{c(0)+1}=b$$

Q10 :

Evaluate the Given limit: $\lim_{z\to 1}\frac{z^{\frac{1}{3}}-1}{z^{\frac{1}{6}}-1}$

Answer :

$$\lim_{z\to 1}\frac{z^{\frac{1}{3}}-1}{z^{\frac{1}{6}}-1}$$

At *z* = 1, the value of the given function takes the form $\frac{0}{0}$.

Put $z^{\frac{1}{6}}=x$ so that *z* Ã¢â€ ’1 as *x* Ã¢â€ ’ 1.

$$\text{Accordingly, } \lim_{z\to 1}\frac{z^{\frac{1}{3}}-1}{z^{\frac{1}{6}}-1}=\lim_{x\to 1}\frac{x^2-1}{x-1}$$

$$=\lim_{x\to 1}\frac{x^2-1^2}{x-1}$$

$$=2.1^{2-1} \qquad \left[\lim_{x\to a}\frac{x^n-a^n}{x-a}=na^{n-1}\right]$$

$$=2$$

$$\therefore \lim_{z\to 1}\frac{z^{\frac{1}{3}}-1}{z^{\frac{1}{6}}-1}=2$$

Q11 :

Evaluate the Given limit: $\lim_{x\to 1}\frac{ax^2+bx+c}{cx^2+bx+a}, a+b+c\neq 0$

Answer :

$$\lim_{x\to 1}\frac{ax^2+bx+c}{cx^2+bx+a}=\frac{a(1)^2+b(1)+c}{c(1)^2+b(1)+a}$$
$$=\frac{a+b+c}{a+b+c}$$
$$=1 \qquad [a+b+c\neq 0]$$

Q12 :

Evaluate the Given limit: $\lim_{x\to -2}\frac{\frac{1}{x}+\frac{1}{2}}{x+2}$

Answer :

$$\lim_{x\to -2}\frac{\frac{1}{x}+\frac{1}{2}}{x+2}$$

At x = â€“2, the value of the given function takes the form $\frac{0}{0}$.

$$\text{Now, } \lim_{x\to -2}\frac{\frac{1}{x}+\frac{1}{2}}{x+2}=\lim_{x\to -2}\frac{\left(\frac{2+x}{2x}\right)}{x+2}$$
$$=\lim_{x\to -2}\frac{1}{2x}$$
$$=\frac{1}{2(-2)}=\frac{-1}{4}$$

Q13 :

Evaluate the Given limit: $\lim_{x\to 0}\frac{\sin ax}{bx}$

Answer :

$$\lim_{x \to 0} \frac{\sin ax}{bx}$$

At *x* = 0, the value of the given function takes the form $\frac{0}{0}$.

$$\text{Now, } \lim_{x \to 0} \frac{\sin ax}{bx} = \lim_{x \to 0} \frac{\sin ax}{ax} \times \frac{ax}{bx}$$

$$= \lim_{x \to 0} \left(\frac{\sin ax}{ax}\right) \times \left(\frac{a}{b}\right)$$

$$= \frac{a}{b} \lim_{ax \to 0} \left(\frac{\sin ax}{ax}\right) \qquad [x \to 0 \Rightarrow ax \to 0]$$

$$= \frac{a}{b} \times 1 \qquad \left[\lim_{y \to 0} \frac{\sin y}{y} = 1\right]$$

$$= \frac{a}{b}$$

Q14 :

Evaluate the Given limit: $\lim_{x \to 0} \frac{\sin ax}{\sin bx},\ a,\ b \neq 0$

Answer :

$$\lim_{x \to 0} \frac{\sin ax}{\sin bx},\ a,\ b \neq 0$$

At *x* = 0, the value of the given function takes the form $\frac{0}{0}$.

$$\text{Now, } \lim_{x\to 0}\frac{\sin ax}{\sin bx} = \lim_{x\to 0}\frac{\left(\frac{\sin ax}{ax}\right)\times ax}{\left(\frac{\sin bx}{bx}\right)\times bx}$$

$$= \left(\frac{a}{b}\right)\times\frac{\lim\limits_{ax\to 0}\left(\frac{\sin ax}{ax}\right)}{\lim\limits_{bx\to 0}\left(\frac{\sin bx}{bx}\right)} \qquad \begin{bmatrix} x\to 0 \Rightarrow ax\to 0 \\ \text{and } x\to 0\Rightarrow bx\to 0\end{bmatrix}$$

$$= \left(\frac{a}{b}\right)\times\frac{1}{1} \qquad \left[\lim_{y\to 0}\frac{\sin y}{y}=1\right]$$

$$= \frac{a}{b}$$

Q15 :

Evaluate the Given limit: $\lim\limits_{x\to\pi}\frac{\sin(\pi-x)}{\pi(\pi-x)}$

Answer :

$$\lim_{x\to\pi}\frac{\sin(\pi-x)}{\pi(\pi-x)}$$

It is seen that *x* Ã¢â€ ' π ⇒ (π â€“ *x*) Ã¢â€ ' 0

$$\therefore \lim_{x\to\pi}\frac{\sin(\pi-x)}{\pi(\pi-x)} = \frac{1}{\pi}\lim_{(\pi-x)\to 0}\frac{\sin(\pi-x)}{(\pi-x)}$$

$$= \frac{1}{\pi}\times 1 \qquad \left[\lim_{y\to 0}\frac{\sin y}{y}=1\right]$$

$$= \frac{1}{\pi}$$

Q16 :

Evaluate the given limit: $\lim\limits_{x\to 0}\frac{\cos x}{\pi - x}$

Answer :

$$\lim_{x \to 0} \frac{\cos x}{\pi - x} = \frac{\cos 0}{\pi - 0} = \frac{1}{\pi}$$

Q17 :

Evaluate the Given limit: $\lim_{x \to 0} \frac{\cos 2x - 1}{\cos x - 1}$

Answer :

$$\lim_{x \to 0} \frac{\cos 2x - 1}{\cos x - 1}$$

At $x = 0$, the value of the given function takes the form $\frac{0}{0}$.

Now,

$$\lim_{x\to 0}\frac{\cos 2x-1}{\cos x-1}=\lim_{x\to 0}\frac{1-2\sin^2 x-1}{1-2\sin^2\frac{x}{2}-1} \qquad \left[\cos x=1-2\sin^2\frac{x}{2}\right]$$

$$=\lim_{x\to 0}\frac{\sin^2 x}{\sin^2\frac{x}{2}}=\lim_{x\to 0}\frac{\left(\frac{\sin^2 x}{x^2}\right)\times x^2}{\left(\frac{\sin^2\frac{x}{2}}{\left(\frac{x}{2}\right)^2}\right)\times\frac{x^2}{4}}$$

$$=4\frac{\lim_{x\to 0}\left(\frac{\sin^2 x}{x^2}\right)}{\lim_{x\to 0}\left(\frac{\sin^2\frac{x}{2}}{\left(\frac{x}{2}\right)^2}\right)}$$

$$=4\frac{\left(\lim_{x\to 0}\frac{\sin x}{x}\right)^2}{\left(\lim_{\frac{x}{2}\to 0}\frac{\sin\frac{x}{2}}{\frac{x}{2}}\right)^2} \qquad \left[x\to 0\Rightarrow\frac{x}{2}\to 0\right]$$

$$=4\frac{1^2}{1^2} \qquad \left[\lim_{y\to 0}\frac{\sin y}{y}=1\right]$$

$$=4$$

Q18 :

Evaluate the Given limit: $\lim_{x\to 0}\frac{ax+x\cos x}{b\sin x}$

Answer :

$$\lim_{x\to 0}\frac{ax+x\cos x}{b\sin x}$$

At x = 0, the value of the given function takes the form $\frac{0}{0}$.

Now,

$$\lim_{x\to 0}\frac{ax+x\cos x}{b\sin x}=\frac{1}{b}\lim_{x\to 0}\frac{x(a+\cos x)}{\sin x}$$

$$=\frac{1}{b}\lim_{x\to 0}\left(\frac{x}{\sin x}\right)\times\lim_{x\to 0}(a+\cos x)$$

$$=\frac{1}{b}\times\frac{1}{\left(\lim_{x\to 0}\frac{\sin x}{x}\right)}\times\lim_{x\to 0}(a+\cos x)$$

$$=\frac{1}{b}\times(a+\cos 0)\qquad\left[\lim_{x\to 0}\frac{\sin x}{x}=1\right]$$

$$=\frac{a+1}{b}$$

Q19 :

Evaluate the Given limit: $\lim_{x\to 0} x\sec x$

Answer :

$$\lim_{x\to 0} x\sec x=\lim_{x\to 0}\frac{x}{\cos x}=\frac{0}{\cos 0}=\frac{0}{1}=0$$

Q20 :

Evaluate the Given limit: $\lim_{x\to 0}\frac{\sin ax+bx}{ax+\sin bx}\quad a,b,a+b\neq 0$

Answer :

At x = 0, the value of the given function takes the form $\frac{0}{0}$.

Now,

$$\lim_{x\to 0}\frac{\sin ax+bx}{ax+\sin bx}$$

$$=\lim_{x\to 0}\frac{\left(\frac{\sin ax}{ax}\right)ax+bx}{ax+bx\left(\frac{\sin bx}{bx}\right)}$$

$$=\frac{\left(\lim_{ax\to 0}\frac{\sin ax}{ax}\right)\times\lim_{x\to 0}(ax)+\lim_{x\to 0}bx}{\lim_{x\to 0}ax+\lim_{x\to 0}bx\left(\lim_{bx\to 0}\frac{\sin bx}{bx}\right)}\qquad [\text{As } x\to 0\Rightarrow ax\to 0 \text{ and } bx\to 0]$$

$$=\frac{\lim_{x\to 0}(ax)+\lim_{x\to 0}bx}{\lim_{x\to 0}ax+\lim_{x\to 0}bx}\qquad \left[\lim_{x\to 0}\frac{\sin x}{x}=1\right]$$

$$=\frac{\lim_{x\to 0}(ax+bx)}{\lim_{x\to 0}(ax+bx)}$$

$$=\lim_{x\to 0}(1)$$

$$=1$$

Q21 :

Evaluate the Given limit: $\lim_{x\to 0}(\operatorname{cosec} x-\cot x)$

Answer :

At x = 0, the value of the given function takes the form $\infty-\infty$.

Now,

$$\lim_{x\to 0}(\text{cosec } x - \cot x)$$

$$= \lim_{x\to 0}\left(\frac{1}{\sin x} - \frac{\cos x}{\sin x}\right)$$

$$= \lim_{x\to 0}\left(\frac{1-\cos x}{\sin x}\right)$$

$$= \lim_{x\to 0}\frac{\left(\frac{1-\cos x}{x}\right)}{\left(\frac{\sin x}{x}\right)}$$

$$= \frac{\lim_{x\to 0}\frac{1-\cos x}{x}}{\lim_{x\to 0}\frac{\sin x}{x}}$$

$$= \frac{0}{1} \qquad \left[\lim_{x\to 0}\frac{1-\cos x}{x} = 0 \text{ and } \lim_{x\to 0}\frac{\sin x}{x} = 1\right]$$

$$= 0$$

Q22 :

$$\lim_{x\to\frac{\pi}{2}}\frac{\tan 2x}{x-\frac{\pi}{2}}$$

Answer :

$$\lim_{x\to\frac{\pi}{2}}\frac{\tan 2x}{x-\frac{\pi}{2}}$$

At $x = \frac{\pi}{2}$, the value of the given function takes the form $\frac{0}{0}$.

Now, put $x - \frac{\pi}{2} = y$ so that $x \to \frac{\pi}{2},\ y \to 0$.

$$\therefore \lim_{x\to\frac{\pi}{2}} \frac{\tan 2x}{x-\frac{\pi}{2}} = \lim_{y\to 0} \frac{\tan 2\left(y+\frac{\pi}{2}\right)}{y}$$

$$= \lim_{y\to 0} \frac{\tan(\pi+2y)}{y}$$

$$= \lim_{y\to 0} \frac{\tan 2y}{y} \qquad [\tan(\pi+2y) = \tan 2y]$$

$$= \lim_{y\to 0} \frac{\sin 2y}{y\cos 2y}$$

$$= \lim_{y\to 0} \left(\frac{\sin 2y}{2y} \times \frac{2}{\cos 2y}\right)$$

$$= \left(\lim_{2y\to 0} \frac{\sin 2y}{2y}\right) \times \lim_{y\to 0} \left(\frac{2}{\cos 2y}\right) \qquad [y\to 0 \Rightarrow 2y\to 0]$$

$$= 1\times\frac{2}{\cos 0} \qquad \left[\lim_{x\to 0}\frac{\sin x}{x} = 1\right]$$

$$= 1\times\frac{2}{1}$$

$$= 2$$

Q23 :

Find $\lim_{x\to 0} f(x)$ and $\lim_{x\to 1} f(x)$, where $f(x) = \begin{cases} 2x+3, & x\le 0 \\ 3(x+1), & x>0 \end{cases}$

Answer :

The given function is

$f(x) = \begin{cases} 2x+3, & x\le 0 \\ 3(x+1), & x>0 \end{cases}$

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0}[2x+3] = 2(0)+3 = 3$$

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0} 3(x+1) = 3(0+1) = 3$$

$$\therefore \lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) = \lim_{x\to 0} f(x) = 3$$

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1} 3(x+1) = 3(1+1) = 6$$

$$\lim_{x\to1^+} f(x) = \lim_{x\to1} 3(x+1) = 3(1+1) = 6$$
$$\therefore \lim_{x\to1^-} f(x) = \lim_{x\to1^+} f(x) = \lim_{x\to1} f(x) = 6$$

Q24 :

Find $\lim_{x\to1}$ ***f*(*x*), where *f*(*x*) =** $\begin{cases} x^2-1, & x\le1 \\ -x^2-1, & x>1 \end{cases}$

Answer :

The given function is

$$f(x) = \begin{cases} x^2-1, & x\le1 \\ -x^2-1, & x>1 \end{cases}$$

$$\lim_{x\to1^-} f(x) = \lim_{x\to1}\left[x^2-1\right] = 1^2-1 = 1-1 = 0$$
$$\lim_{x\to1^+} f(x) = \lim_{x\to1}\left[-x^2-1\right] = -1^2-1 = -1-1 = -2$$

It is observed that $\lim_{x\to1^-} f(x) \ne \lim_{x\to1^+} f(x)$.

Hence, $\lim_{x\to1} f(x)$ does not exist.

Q25 :

Evaluate $\lim_{x\to0}$ ***f*(*x*), where *f*(*x*) =** $\begin{cases} \frac{|x|}{x}, & x\ne0 \\ 0, & x=0 \end{cases}$

Answer :

The given function is

f(*x*) = $\begin{cases} \frac{|x|}{x}, & x\ne0 \\ 0, & x=0 \end{cases}$

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-}\left[\frac{|x|}{x}\right]$$

$$= \lim_{x\to 0}\left(\frac{-x}{x}\right) \qquad \left[\text{When } x \text{ is negaitve, } |x| = -x\right]$$

$$= \lim_{x\to 0}(-1)$$

$$= -1$$

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+}\left[\frac{|x|}{x}\right]$$

$$= \lim_{x\to 0}\left[\frac{x}{x}\right] \qquad \left[\text{When } x \text{ is positive, } |x| = x\right]$$

$$= \lim_{x\to 0}(1)$$

$$= 1$$

It is observed that $\lim_{x\to 0^-} f(x) \neq \lim_{x\to 0^+} f(x)$.

Hence, $\lim_{x\to 0} f(x)$ does not exist.

Q26 :

Find $\lim_{x\to 0}$ ***f*(*x*), where *f*(*x*) =** $\begin{cases} \dfrac{x}{|x|}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

Answer :

The given function is

$$f(x)=\begin{cases}\dfrac{x}{|x|}, & x\neq 0\\ 0, & x=0\end{cases}$$

$$\lim_{x\to 0^-} f(x)=\lim_{x\to 0^-}\left[\frac{x}{|x|}\right]$$
$$=\lim_{x\to 0}\left[\frac{x}{-x}\right] \qquad \left[\text{When } x<0,\ |x|=-x\right]$$
$$=\lim_{x\to 0}(-1)$$
$$=-1$$

$$\lim_{x\to 0^+} f(x)=\lim_{x\to 0^+}\left[\frac{x}{|x|}\right]$$
$$=\lim_{x\to 0}\left[\frac{x}{x}\right] \qquad \left[\text{When } x>0,\ |x|=x\right]$$
$$=\lim_{x\to 0}(1)$$
$$=1$$

It is observed that $\lim_{x\to 0^-} f(x)\neq \lim_{x\to 0^+} f(x)$.

Hence, $\lim_{x\to 0} f(x)$ does not exist.

Q27 :

Find $\lim_{x\to 5}$ *f*(*x*), where *f*(*x*) = $|x|-5$

Answer :

The given function is *f*(*x*) = $|x|-5$.

$$\lim_{x\to 5^-} f(x) = \lim_{x\to 5^-}\left[|x|-5\right]$$
$$= \lim_{x\to 5}(x-5) \qquad \left[\text{When } x>0,\ |x|=x\right]$$
$$= 5-5$$
$$= 0$$
$$\lim_{x\to 5^+} f(x) = \lim_{x\to 5^+}\left(|x|-5\right)$$
$$= \lim_{x\to 5}(x-5) \qquad \left[\text{When } x>0,\ |x|=x\right]$$
$$= 5-5$$
$$= 0$$
$$\therefore\ \lim_{x\to 5^-} f(x) = \lim_{x\to 5^+} f(x) = 0$$
$$\text{Hence, } \lim_{x\to 5} f(x) = 0$$

Q28 :

Suppose *f*(*x*) = $\begin{cases} a+bx, & x<1 \\ 4, & x=1 \\ b-ax & x>1 \end{cases}$ **and if** $\lim_{x\to 1}$ ***f*(*x*) = *f*(1) what are possible values of *a* and *b*?**

Answer :

The given function is

$$f(x) = \begin{cases} a+bx, & x<1 \\ 4, & x=1 \\ b-ax & x>1 \end{cases}$$

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1}(a+bx) = a+b$$
$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1}(b-ax) = b-a$$
$$f(1) = 4$$
$$\text{It is given that } \lim_{x\to 1} f(x) = f(1).$$
$$\therefore \lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x) = \lim_{x\to 1} f(x) = f(1)$$
$$\Rightarrow a+b=4 \text{ and } b-a=4$$
$$\text{On solving these two equations, we obtain } a=0 \text{ and } b=4.$$

Thus, the respective possible values of *a* and *b* are 0 and 4.

Q29 :

Let $a_1,\ a_2,\ \ldots,\ a_n$ **be fixed real numbers and define a function**

$$f(x) = (x - a_1)(x - a_2)\ldots(x - a_n)$$

What is $\lim_{x \to a_1}$ ***f*(*x*)? For some** $a \neq a_1,\ a_2 \ldots,\ a_n,$ **compute** $\lim_{x \to a}$ ***f*(*x*).**

Answer :

The given function is $f(x) = (x - a_1)(x - a_2)\ldots(x - a_n)$.

$$\lim_{x \to a_1} f(x) = \lim_{x \to a_1}\left[(x - a_1)(x - a_2)\ldots(x - a_n)\right]$$

$$= \left[\lim_{x \to a_1}(x - a_1)\right]\left[\lim_{x \to a_1}(x - a_2)\right]\ldots\left[\lim_{x \to a_1}(x - a_n)\right]$$

$$= (a_1 - a_1)(a_1 - a_2)\ldots(a_1 - a_n) = 0$$

$$\therefore \lim_{x \to a_1} f(x) = 0$$

$$\text{Now, } \lim_{x \to a} f(x) = \lim_{x \to a}\left[(x - a_1)(x - a_2)\ldots(x - a_n)\right]$$

$$= \left[\lim_{x \to a}(x - a_1)\right]\left[\lim_{x \to a}(x - a_2)\right]\ldots\left[\lim_{x \to a}(x - a_n)\right]$$

$$= (a - a_1)(a - a_2)\ldots\ldots(a - a_n)$$

$$\therefore \lim_{x \to a} f(x) = (a - a_1)(a - a_2)\ldots(a - a_n)$$

Q30 :

If ***f*(*x*) =** $\begin{cases} |x| + 1, & x < 0 \\ 0, & x = 0 \\ |x| - 1, & x > 0 \end{cases}$.

For what value (s) of a does $\lim_{x \to a}$ ***f*(*x*) exists?**

Answer :

The given function is

$$f(x)=\begin{cases}|x|+1, & x<0\\ 0, & x=0\\ |x|-1, & x>0\end{cases}$$

When $a=0$,

$$\lim_{x\to 0^-} f(x)=\lim_{x\to 0^-}(|x|+1)$$
$$=\lim_{x\to 0}(-x+1) \qquad [\text{If } x<0,\ |x|=-x]$$
$$=-0+1$$
$$=1$$

$$\lim_{x\to 0^+} f(x)=\lim_{x\to 0^+}(|x|-1)$$
$$=\lim_{x\to 0}(x-1) \qquad [\text{If } x>0,\ |x|=x]$$
$$=0-1$$
$$=-1$$

Here, it is observed that $\lim_{x\to 0^-} f(x)\neq \lim_{x\to 0^+} f(x)$.

$\therefore \lim_{x\to 0} f(x)$ does not exist.

When $a < 0$,

$$\lim_{x\to a^-} f(x)=\lim_{x\to a^-}(|x|+1)$$
$$=\lim_{x\to a}(-x+1) \qquad [x<a<0\Rightarrow\ |x|=-x]$$
$$=-a+1$$

$$\lim_{x\to a^+} f(x)=\lim_{x\to a^+}(|x|+1)$$
$$=\lim_{x\to a}(-x+1) \qquad [a<x<0\Rightarrow\ |x|=-x]$$
$$=-a+1$$

$$\therefore \lim_{x\to a^-} f(x)=\lim_{x\to a^+} f(x)=-a+1$$

Thus, limit of $f(x)$ exists at $x=a$, where $a<0$.

When $a > 0$

$$\lim_{x\to a^-} f(x) = \lim_{x\to a^-} (|x|-1)$$
$$= \lim_{x\to a} (x-1) \qquad [0 < x < a \Rightarrow |x| = x]$$
$$= a-1$$
$$\lim_{x\to a^+} f(x) = \lim_{x\to a^+} (|x|-1)$$
$$= \lim_{x\to a} (x-1) \qquad [0 < a < x \Rightarrow |x| = x]$$
$$= a-1$$
$$\therefore \lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = a-1$$

Thus, limit of $f(x)$ exists at $x = a$, where $a > 0$.

Thus, $\lim_{x\to a} f(x)$ exists for all $a \neq 0$.

Q31 :

If the function *f*(*x*) satisfies $\lim_{x\to 1} \frac{f(x)-2}{x^2-1} = \pi$, evaluate $\lim_{x\to 1} f(x)$.

Answer :

$$\lim_{x\to 1} \frac{f(x)-2}{x^2-1} = \pi$$
$$\Rightarrow \frac{\lim_{x\to 1}(f(x)-2)}{\lim_{x\to 1}(x^2-1)} = \pi$$
$$\Rightarrow \lim_{x\to 1}(f(x)-2) = \pi \lim_{x\to 1}(x^2-1)$$
$$\Rightarrow \lim_{x\to 1}(f(x)-2) = \pi(1^2-1)$$
$$\Rightarrow \lim_{x\to 1}(f(x)-2) = 0$$
$$\Rightarrow \lim_{x\to 1} f(x) - \lim_{x\to 1} 2 = 0$$
$$\Rightarrow \lim_{x\to 1} f(x) - 2 = 0$$
$$\therefore \lim_{x\to 1} f(x) = 2$$

Q32 :

If $f(x)=\begin{cases} mx^2+n, & x<0 \\ nx+m, & 0\le x\le 1 \\ nx^3+m, & x>1 \end{cases}$. **For what integers *m* and *n* does** $\lim_{x\to 0} f(x)$ **and** $\lim_{x\to 1} f(x)$ **exist?**

Answer :

The given function is

$$f(x)=\begin{cases} mx^2+n, & x<0 \\ nx+m, & 0\le x\le 1 \\ nx^3+m, & x>1 \end{cases}$$

$$\lim_{x\to 0^-} f(x)=\lim_{x\to 0}\left(mx^2+n\right)$$
$$=m(0)^2+n$$
$$=n$$

$$\lim_{x\to 0^+} f(x)=\lim_{x\to 0}\left(nx+m\right)$$
$$=n(0)+m$$
$$=m.$$

Thus, $\lim_{x\to 0} f(x)$ exists if *m* = *n*.

$$\lim_{x\to 1^-} f(x)=\lim_{x\to 1}\left(nx+m\right)$$
$$=n(1)+m$$
$$=m+n$$

$$\lim_{x\to 1^+} f(x)=\lim_{x\to 1}\left(nx^3+m\right)$$
$$=n(1)^3+m$$
$$=m+n$$

$$\therefore \lim_{x\to 1^-} f(x)=\lim_{x\to 1^+} f(x)=\lim_{x\to 1} f(x).$$

Thus, $\lim_{x\to 1} f(x)$ exists for any integral value of *m* and *n*.

Exercise 13.2 : Solutions of Questions on Page Number : **312**

Q1 :

Find the derivative of x^2 - 2 at x = 10.

Answer :

Let $f(x) = x^2$ â€“ 2. Accordingly,

$$f'(10) = \lim_{h\to 0}\frac{f(10+h)-f(10)}{h}$$
$$= \lim_{h\to 0}\frac{\left[(10+h)^2-2\right]-(10^2-2)}{h}$$
$$= \lim_{h\to 0}\frac{10^2+2.10.h+h^2-2-10^2+2}{h}$$
$$= \lim_{h\to 0}\frac{20h+h^2}{h}$$
$$= \lim_{h\to 0}(20+h) = (20+0) = 20$$

Thus, the derivative of x^2 â€“ 2 at x = 10 is 20.

Q2 :

Find the derivative of 99*x* at *x* = 100.

Answer :

Let $f(x) = 99x$. Accordingly,

$$f'(100) = \lim_{h\to 0}\frac{f(100+h)-f(100)}{h}$$
$$= \lim_{h\to 0}\frac{99(100+h)-99(100)}{h}$$
$$= \lim_{h\to 0}\frac{99\times 100+99h-99\times 100}{h}$$
$$= \lim_{h\to 0}\frac{99h}{h}$$
$$= \lim_{h\to 0}(99) = 99$$

Thus, the derivative of 99x at x = 100 is 99.

Q3 :

Find the derivative of *x* at *x* = 1.

Answer :

Let$f(x) = x$. Accordingly,

$$
\begin{aligned}
f'(1) &= \lim_{h\to 0}\frac{f(1+h)-f(1)}{h} \\
&= \lim_{h\to 0}\frac{(1+h)-1}{h} \\
&= \lim_{h\to 0}\frac{h}{h} \\
&= \lim_{h\to 0}(1) \\
&= 1
\end{aligned}
$$

Thus, the derivative of *x* at *x* = 1 is 1.

Q4 :

Find the derivative of the following functions from first principle.

(i) x^3 â€“ 27 (ii) (*x* â€“ 1) (*x* â€“ 2)

(ii) $\frac{1}{x^2}$ **(iv)** $\frac{x+1}{x-1}$

Answer :

(i) Let *f*(*x*) = x^3 â€“ 27. Accordingly, from the first principle,

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0}\frac{\left[(x+h)^3-27\right]-\left(x^3-27\right)}{h} \\
&= \lim_{h\to 0}\frac{x^3+h^3+3x^2h+3xh^2-x^3}{h} \\
&= \lim_{h\to 0}\frac{h^3+3x^2h+3xh^2}{h} \\
&= \lim_{h\to 0}\left(h^2+3x^2+3xh\right) \\
&= 0+3x^2+0=3x^2
\end{aligned}
$$

(ii) Let *f*(*x*) = (*x* â€“ 1) (*x* â€“ 2). Accordingly, from the first principle,

$$\begin{aligned}
f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\\
&= \lim_{h\to 0}\frac{(x+h-1)(x+h-2)-(x-1)(x-2)}{h}\\
&= \lim_{h\to 0}\frac{\left(x^2+hx-2x+hx+h^2-2h-x-h+2\right)-\left(x^2-2x-x+2\right)}{h}\\
&= \lim_{h\to 0}\frac{\left(hx+hx+h^2-2h-h\right)}{h}\\
&= \lim_{h\to 0}\frac{2hx+h^2-3h}{h}\\
&= \lim_{h\to 0}(2x+h-3)\\
&= (2x+0-3)\\
&= 2x-3
\end{aligned}$$

(iii) Let $f(x)=\frac{1}{x^2}$. Accordingly, from the first principle,

$$\begin{aligned}
f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\\
&= \lim_{h\to 0}\frac{\frac{1}{(x+h)^2}-\frac{1}{x^2}}{h}\\
&= \lim_{h\to 0}\frac{1}{h}\left[\frac{x^2-(x+h)^2}{x^2(x+h)^2}\right]\\
&= \lim_{h\to 0}\frac{1}{h}\left[\frac{x^2-x^2-h^2-2hx}{x^2(x+h)^2}\right]\\
&= \lim_{h\to 0}\frac{1}{h}\left[\frac{-h^2-2hx}{x^2(x+h)^2}\right]\\
&= \lim_{h\to 0}\left[\frac{-h-2x}{x^2(x+h)^2}\right]\\
&= \frac{0-2x}{x^2(x+0)^2}=\frac{-2}{x^3}
\end{aligned}$$

(iv) Let $f(x)=\frac{x+1}{x-1}$. Accordingly, from the first principle,

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$=\lim_{h\to 0}\frac{\left(\frac{x+h+1}{x+h-1}-\frac{x+1}{x-1}\right)}{h}$$

$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{(x-1)(x+h+1)-(x+1)(x+h-1)}{(x-1)(x+h-1)}\right]$$

$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{(x^2+hx+x-x-h-1)-(x^2+hx-x+x+h-1)}{(x-1)(x+h-1)}\right]$$

$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{-2h}{(x-1)(x+h-1)}\right]$$

$$=\lim_{h\to 0}\left[\frac{-2}{(x-1)(x+h-1)}\right]$$

$$=\frac{-2}{(x-1)(x-1)}=\frac{-2}{(x-1)^2}$$

Q5 :

For the function

$$f(x)=\frac{x^{100}}{100}+\frac{x^{99}}{99}+\ldots+\frac{x^2}{2}+x+1$$

Prove that $f'(1)=100f'(0)$

Answer :

The given function is

$$f(x)=\frac{x^{100}}{100}+\frac{x^{99}}{99}+...+\frac{x^2}{2}+x+1$$

$$\frac{d}{dx}f(x)=\frac{d}{dx}\left[\frac{x^{100}}{100}+\frac{x^{99}}{99}+...+\frac{x^2}{2}+x+1\right]$$

$$\frac{d}{dx}f(x)=\frac{d}{dx}\left(\frac{x^{100}}{100}\right)+\frac{d}{dx}\left(\frac{x^{99}}{99}\right)+...+\frac{d}{dx}\left(\frac{x^2}{2}\right)+\frac{d}{dx}(x)+\frac{d}{dx}(1)$$

On using theorem $\frac{d}{dx}(x^n)=nx^{n-1}$, we obtain

$$\frac{d}{dx}f(x)=\frac{100x^{99}}{100}+\frac{99x^{98}}{99}+...+\frac{2x}{2}+1+0$$

$$=x^{99}+x^{98}+...+x+1$$

$$\therefore f'(x)=x^{99}+x^{98}+...+x+1$$

At $x=0$,

$$f'(0)=1$$

At $x=1$,

$$f'(1)=1^{99}+1^{98}+...+1+1=[1+1+...+1+1]_{100\text{ terms}}=1\times100=100$$

Thus, $f'(1)=100\times f^1(0)$

Q6 :

Find the derivative of $x^n+ax^{n-1}+a^2x^{n-2}+...+a^{n-1}x+a^n$ **for some fixed real number *a*.**

Answer :

Let $f(x)=x^n+ax^{n-1}+a^2x^{n-2}+...+a^{n-1}x+a^n$

$$\therefore f'(x)=\frac{d}{dx}\left(x^n+ax^{n-1}+a^2x^{n-2}+...+a^{n-1}x+a^n\right)$$

$$=\frac{d}{dx}(x^n)+a\frac{d}{dx}(x^{n-1})+a^2\frac{d}{dx}(x^{n-2})+...+a^{n-1}\frac{d}{dx}(x)+a^n\frac{d}{dx}(1)$$

On using theorem $\frac{d}{dx}x^n=nx^{n-1}$, we obtain

$$f'(x)=nx^{n-1}+a(n-1)x^{n-2}+a^2(n-2)x^{n-3}+...+a^{n-1}+a^n(0)$$

$$=nx^{n-1}+a(n-1)x^{n-2}+a^2(n-2)x^{n-3}+...+a^{n-1}$$

Q7 :

For some constants *a* and *b*, find the derivative of

(i) (*x* â€“ *a*) (*x* â€“ *b*) (ii) (*ax*² + *b*)² (iii) $\frac{x-a}{x-b}$

Answer :

(i) Let *f* (*x*) = (*x* â€“ *a*) (*x* â€“ *b*)

$$\Rightarrow f(x) = x^2 - (a+b)x + ab$$

$$\therefore f'(x) = \frac{d}{dx}\left(x^2 - (a+b)x + ab\right)$$

$$= \frac{d}{dx}(x^2) - (a+b)\frac{d}{dx}(x) + \frac{d}{dx}(ab)$$

On using theorem $\frac{d}{dx}(x^n) = nx^{n-1}$, we obtain

$$f'(x) = 2x - (a+b) + 0 = 2x - a - b$$

(ii) Let $f(x) = (ax^2 + b)^2$

$$\Rightarrow f(x) = a^2x^4 + 2abx^2 + b^2$$

$$\therefore f'(x) = \frac{d}{dx}(a^2x^4 + 2abx^2 + b^2) = a^2\frac{d}{dx}(x^4) + 2ab\frac{d}{dx}(x^2) + \frac{d}{dx}(b^2)$$

On using theorem $\frac{d}{dx}x^n = nx^{n-1}$, we obtain

$$f'(x) = a^2(4x^3) + 2ab(2x) + b^2(0)$$

$$= 4a^2x^3 + 4abx$$

$$= 4ax(ax^2 + b)$$

(iii) Let $f(x) = \frac{(x-a)}{(x-b)}$

$$\Rightarrow f'(x) = \frac{d}{dx}\left(\frac{x-a}{x-b}\right)$$

By quotient rule,

$$f'(x)=\frac{(x-b)\frac{d}{dx}(x-a)-(x-a)\frac{d}{dx}(x-b)}{(x-b)^2}$$

$$=\frac{(x-b)(1)-(x-a)(1)}{(x-b)^2}$$

$$=\frac{x-b-x+a}{(x-b)^2}$$

$$=\frac{a-b}{(x-b)^2}$$

Q8 :

Find the derivative of $\frac{x^n-a^n}{x-a}$ **for some constant *a*.**

Answer :

$$\text{Let } f(x)=\frac{x^n-a^n}{x-a}$$

$$\Rightarrow f'(x)=\frac{d}{dx}\left(\frac{x^n-a^n}{x-a}\right)$$

By quotient rule,

$$f'(x)=\frac{(x-a)\frac{d}{dx}(x^n-a^n)-(x^n-a^n)\frac{d}{dx}(x-a)}{(x-a)^2}$$

$$=\frac{(x-a)(nx^{n-1}-0)-(x^n-a^n)}{(x-a)^2}$$

$$=\frac{nx^n-anx^{n-1}-x^n+a^n}{(x-a)^2}$$

Q9 :

Find the derivative of

(i) $2x-\frac{3}{4}$ **(ii) (5*x*^{3} + 3*x* â€“ 1) (*x* â€“ 1)**

(iii) $x^{\text{â€“3}}$ **(5 + 3*x*) (iv)** x^5 **(3 â€“ 6**$x^{\text{â€“9}}$**)**

(v) $x^{\text{â€“4}}$ **(3 â€“ 4**$x^{\text{â€“5}}$**) (vi)** $\frac{2}{x+1}-\frac{x^2}{3x-1}$

Answer :

(i) Let $f(x)=2x-\frac{3}{4}$

$$
\begin{aligned}
f'(x) &= \frac{d}{dx}\left(2x-\frac{3}{4}\right) \\
&= 2\frac{d}{dx}(x)-\frac{d}{dx}\left(\frac{3}{4}\right) \\
&= 2-0 \\
&= 2
\end{aligned}
$$

(ii) Let *f* (*x*) = (5x^3 + 3*x* â€“ 1) (*x* â€“ 1)

By Leibnitz product rule,

$$
\begin{aligned}
f'(x) &= \left(5x^3+3x-1\right)\frac{d}{dx}(x-1)+(x-1)\frac{d}{dx}\left(5x^3+3x-1\right) \\
&= \left(5x^3+3x-1\right)(1)+(x-1)\left(5.3x^2+3-0\right) \\
&= \left(5x^3+3x-1\right)+(x-1)\left(15x^2+3\right) \\
&= 5x^3+3x-1+15x^3+3x-15x^2-3 \\
&= 20x^3-15x^2+6x-4
\end{aligned}
$$

(iii) Let*f* (*x*) = $x^{\text{â€“ 3}}$ (5 + 3*x*)

By Leibnitz product rule,

$$
\begin{aligned}
f'(x) &= x^{-3}\frac{d}{dx}(5+3x)+(5+3x)\frac{d}{dx}\left(x^{-3}\right)\\
&= x^{-3}(0+3)+(5+3x)\left(-3x^{-3-1}\right)\\
&= x^{-3}(3)+(5+3x)\left(-3x^{-4}\right)\\
&= 3x^{-3}-15x^{-4}-9x^{-3}\\
&= -6x^{-3}-15x^{-4}\\
&= -3x^{-3}\left(2+\frac{5}{x}\right)\\
&= \frac{-3x^{-3}}{x}(2x+5)\\
&= \frac{-3}{x^{4}}(5+2x)
\end{aligned}
$$

(iv) Let *f* (*x*) = x^{5} (3 â€“ 6*x*$^{\text{â€9}}$)

By Leibnitz product rule,

$$
\begin{aligned}
f'(x) &= x^{5}\frac{d}{dx}\left(3-6x^{-9}\right)+\left(3-6x^{-9}\right)\frac{d}{dx}\left(x^{5}\right)\\
&= x^{5}\left\{0-6(-9)x^{-9-1}\right\}+\left(3-6x^{-9}\right)\left(5x^{4}\right)\\
&= x^{5}\left(54x^{-10}\right)+15x^{4}-30x^{-5}\\
&= 54x^{-5}+15x^{4}-30x^{-5}\\
&= 24x^{-5}+15x^{4}\\
&= 15x^{4}+\frac{24}{x^{5}}
\end{aligned}
$$

(v) Let *f* (*x*) = *x*$^{\text{â€4}}$ (3 â€“ 4*x*$^{\text{â€5}}$)

By Leibnitz product rule,

$$
\begin{aligned}
f'(x) &= x^{-4}\frac{d}{dx}\left(3-4x^{-5}\right)+\left(3-4x^{-5}\right)\frac{d}{dx}\left(x^{-4}\right)\\
&= x^{-4}\left\{0-4(-5)x^{-5-1}\right\}+\left(3-4x^{-5}\right)(-4)x^{-4-1}\\
&= x^{-4}\left(20x^{-6}\right)+\left(3-4x^{-5}\right)\left(-4x^{-5}\right)\\
&= 20x^{-10}-12x^{-5}+16x^{-10}\\
&= 36x^{-10}-12x^{-5}\\
&= -\frac{12}{x^{5}}+\frac{36}{x^{10}}
\end{aligned}
$$

(vi) Let *f* (*x*) = $\frac{2}{x+1}-\frac{x^{2}}{3x-1}$

$$f'(x) = \frac{d}{dx}\left(\frac{2}{x+1}\right) - \frac{d}{dx}\left(\frac{x^2}{3x-1}\right)$$

By quotient rule,

$$f'(x) = \left[\frac{(x+1)\frac{d}{dx}(2) - 2\frac{d}{dx}(x+1)}{(x+1)^2}\right] - \left[\frac{(3x-1)\frac{d}{dx}(x^2) - x^2\frac{d}{dx}(3x-1)}{(3x-1)^2}\right]$$

$$= \left[\frac{(x+1)(0) - 2(1)}{(x+1)^2}\right] - \left[\frac{(3x-1)(2x) - (x^2)(3)}{(3x-1)^2}\right]$$

$$= \frac{-2}{(x+1)^2} - \left[\frac{6x^2 - 2x - 3x^2}{(3x-1)^2}\right]$$

$$= \frac{-2}{(x+1)^2} - \left[\frac{3x^2 - 2x^2}{(3x-1)^2}\right]$$

$$= \frac{-2}{(x+1)^2} - \frac{x(3x-2)}{(3x-1)^2}$$

Q10 :

Find the derivative of cos *x* from first principle.

Answer :

Let *f* (*x*) = cos *x*. Accordingly, from the first principle,

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\cos(x+h) - \cos x}{h}$$

$$= \lim_{h \to 0} \left[\frac{\cos x \cos h - \sin x \sin h - \cos x}{h} \right]$$

$$= \lim_{h \to 0} \left[\frac{-\cos x (1 - \cos h) - \sin x \sin h}{h} \right]$$

$$= \lim_{h \to 0} \left[\frac{-\cos x (1 - \cos h)}{h} - \frac{\sin x \sin h}{h} \right]$$

$$= -\cos x \left(\lim_{h \to 0} \frac{1 - \cos h}{h} \right) - \sin x \lim_{h \to 0} \left(\frac{\sin h}{h} \right)$$

$$= -\cos x (0) - \sin x (1) \qquad \left[\lim_{h \to 0} \frac{1 - \cos h}{h} = 0 \text{ and } \lim_{h \to 0} \frac{\sin h}{h} = 1 \right]$$

$$= -\sin x$$

$$\therefore f'(x) = -\sin x$$

Q11 :

Find the derivative of the following functions:

(i) sin *x* cos *x* (ii) sec *x* (iii) 5 sec *x* + 4 cos *x*

(iv) cosec *x* (v) 3cot *x* + 5cosec *x*

(vi) 5sin *x* - 6cos *x* + 7 (vii) 2tan *x* - 7sec *x*

Answer :

(i) Let*f* (*x*) = sin *x* cos *x*. Accordingly, from the first principle,

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0}\frac{\sin(x+h)\cos(x+h)-\sin x\cos x}{h} \\
&= \lim_{h\to 0}\frac{1}{2h}\left[2\sin(x+h)\cos(x+h)-2\sin x\cos x\right] \\
&= \lim_{h\to 0}\frac{1}{2h}\left[\sin 2(x+h)-\sin 2x\right] \\
&= \lim_{h\to 0}\frac{1}{2h}\left[2\cos\frac{2x+2h+2x}{2}\cdot\sin\frac{2x+2h-2x}{2}\right] \\
&= \lim_{h\to 0}\frac{1}{h}\left[\cos\frac{4x+2h}{2}\sin\frac{2h}{2}\right] \\
&= \lim_{h\to 0}\frac{1}{h}\left[\cos(2x+h)\sin h\right] \\
&= \lim_{h\to 0}\cos(2x+h).\lim_{h\to 0}\frac{\sin h}{h} \\
&= \cos(2x+0).1 \\
&= \cos 2x
\end{aligned}
$$

(ii) Let*f* (*x*) = sec *x*. Accordingly, from the first principle,

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$
$$=\lim_{h\to 0}\frac{\sec(x+h)-\sec x}{h}$$
$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{1}{\cos(x+h)}-\frac{1}{\cos x}\right]$$
$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{\cos x-\cos(x+h)}{\cos x\cos(x+h)}\right]$$
$$=\frac{1}{\cos x}.\lim_{h\to 0}\frac{1}{h}\left[\frac{-2\sin\left(\frac{x+x+h}{2}\right)\sin\left(\frac{x-x-h}{2}\right)}{\cos(x+h)}\right]$$
$$=\frac{1}{\cos x}.\lim_{h\to 0}\frac{1}{h}\left[\frac{-2\sin\left(\frac{2x+h}{2}\right)\sin\left(-\frac{h}{2}\right)}{\cos(x+h)}\right]$$
$$=\frac{1}{\cos x}.\lim_{h\to 0}\frac{\left[\sin\left(\frac{2x+h}{2}\right)\frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)}\right]}{\cos(x+h)}$$
$$=\frac{1}{\cos x}.\lim_{\frac{h}{2}\to 0}\frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)}.\lim_{h\to 0}\frac{\sin\left(\frac{2x+h}{2}\right)}{\cos(x+h)}$$
$$=\frac{1}{\cos x}.1.\frac{\sin x}{\cos x}$$
$$=\sec x\tan x$$

(iii) Let*f* (*x*) = 5 sec *x* + 4 cos *x*. Accordingly, from the first principle,

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$=\lim_{h\to 0}\frac{5\sec(x+h)+4\cos(x+h)-[5\sec x+4\cos x]}{h}$$

$$=5\lim_{h\to 0}\frac{[\sec(x+h)-\sec x]}{h}+4\lim_{h\to 0}\frac{[\cos(x+h)-\cos x]}{h}$$

$$=5\lim_{h\to 0}\frac{1}{h}\left[\frac{1}{\cos(x+h)}-\frac{1}{\cos x}\right]+4\lim_{h\to 0}\frac{1}{h}\left[\cos(x+h)-\cos x\right]$$

$$=5\lim_{h\to 0}\frac{1}{h}\left[\frac{\cos x-\cos(x+h)}{\cos x\cos(x+h)}\right]+4\lim_{h\to 0}\frac{1}{h}[\cos x\cos h-\sin x\sin h-\cos x]$$

$$=\frac{5}{\cos x}\lim_{h\to 0}\frac{1}{h}\left[\frac{-2\sin\left(\frac{x+x+h}{2}\right)\sin\left(\frac{x-x-h}{2}\right)}{\cos(x+h)}\right]+4\lim_{h\to 0}\frac{1}{h}\left[-\cos x(1-\cos h)-\sin x\sin h\right]$$

$$=\frac{5}{\cos x}.\lim_{h\to 0}\frac{1}{h}\left[\frac{-2\sin\left(\frac{2x+h}{2}\right)\sin\left(-\frac{h}{2}\right)}{\cos(x+h)}\right]+4\left[-\cos x\lim_{h\to 0}\frac{(1-\cos h)}{h}-\sin x\lim_{h\to 0}\frac{\sin h}{h}\right]$$

$$=\frac{5}{\cos x}.\lim_{h\to 0}\left[\frac{\sin\left(\frac{2x+h}{2}\right).\frac{\sin\left(\frac{h}{2}\right)}{\frac{h}{2}}}{\cos(x+h)}\right]+4\left[(-\cos x).(0)-(\sin x).1\right]$$

$$=\frac{5}{\cos x}.\left[\lim_{h\to 0}\frac{\sin\left(\frac{2x+h}{2}\right)}{\cos(x+h)}.\lim_{h\to 0}\frac{\sin\left(\frac{h}{2}\right)}{\frac{h}{2}}\right]-4\sin x$$

$$=\frac{5}{\cos x}.\frac{\sin x}{\cos x}.1-4\sin x$$

$$=5\sec x\tan x.-4\sin x$$

(iv) Let *f* (*x*) = cosec *x*. Accordingly, from the first principle,

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$f'(x)=\lim_{h\to 0}\frac{1}{h}\left[\operatorname{cosec}(x+h)-\operatorname{cosec}x\right]$$

$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{1}{\sin(x+h)}-\frac{1}{\sin x}\right]$$

$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{\sin x-\sin(x+h)}{\sin(x+h)\sin x}\right]$$

$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{2\cos\left(\frac{x+x+h}{2}\right)\cdot\sin\left(\frac{x-x-h}{2}\right)}{\sin(x+h)\sin x}\right]$$

$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{2\cos\left(\frac{2x+h}{2}\right)\sin\left(-\frac{h}{2}\right)}{\sin(x+h)\sin x}\right]$$

$$=\lim_{h\to 0}\frac{-\cos\left(\frac{2x+h}{2}\right)\cdot\frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)}}{\sin(x+h)\sin x}$$

$$=\lim_{h\to 0}\left(\frac{-\cos\left(\frac{2x+h}{2}\right)}{\sin(x+h)\sin x}\right).\lim_{\frac{h}{2}\to 0}\frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)}$$

$$=\left(\frac{-\cos x}{\sin x\sin x}\right).1$$

$$=-\operatorname{cosec}x\cot x$$

(v) Let $f(x)=3\cot x+5\operatorname{cosec} x$. Accordingly, from the first principle,

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$=\lim_{h\to 0}\frac{3\cot(x+h)+5\operatorname{cosec}(x+h)-3\cot x-5\operatorname{cosec} x}{h}$$

$$=3\lim_{h\to 0}\frac{1}{h}\left[\cot(x+h)-\cot x\right]+5\lim_{h\to 0}\frac{1}{h}\left[\operatorname{cosec}(x+h)-\operatorname{cosec} x\right] \quad ...(1)$$

Now, $\lim_{h\to 0}\frac{1}{h}\left[\cot(x+h)-\cot x\right]$

$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{\cos(x+h)}{\sin(x+h)}-\frac{\cos x}{\sin x}\right]$$

$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{\cos(x+h)\sin x-\cos x\sin(x+h)}{\sin x\sin(x+h)}\right]$$

$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{\sin(x-x-h)}{\sin x\sin(x+h)}\right]$$

$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{\sin(-h)}{\sin x\sin(x+h)}\right]$$

$$=-\left(\lim_{h\to 0}\frac{\sin h}{h}\right).\left(\lim_{h\to 0}\frac{1}{\sin x\cdot\sin(x+h)}\right)$$

$$=-1.\frac{1}{\sin x\cdot\sin(x+0)}=\frac{-1}{\sin^2 x}=-\operatorname{cosec}^2 x \quad ...(2)$$

$$\lim_{h\to0}\frac{1}{h}\left[\operatorname{cosec}(x+h)-\operatorname{cosec}x\right]$$

$$=\lim_{h\to0}\frac{1}{h}\left[\frac{1}{\sin(x+h)}-\frac{1}{\sin x}\right]$$

$$=\lim_{h\to0}\frac{1}{h}\left[\frac{\sin x-\sin(x+h)}{\sin(x+h)\sin x}\right]$$

$$=\lim_{h\to0}\frac{1}{h}\left[\frac{2\cos\left(\frac{x+x+h}{2}\right)\cdot\sin\left(\frac{x-x-h}{2}\right)}{\sin(x+h)\sin x}\right]$$

$$=\lim_{h\to0}\frac{1}{h}\left[\frac{2\cos\left(\frac{2x+h}{2}\right)\sin\left(-\frac{h}{2}\right)}{\sin(x+h)\sin x}\right]$$

$$=\lim_{h\to0}\frac{-\cos\left(\frac{2x+h}{2}\right)\cdot\frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)}}{\sin(x+h)\sin x}$$

$$=\lim_{h\to0}\left(\frac{-\cos\left(\frac{2x+h}{2}\right)}{\sin(x+h)\sin x}\right).\lim_{\frac{h}{2}\to0}\frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)}$$

$$=\left(\frac{-\cos x}{\sin x\sin x}\right).1$$

$$=-\operatorname{cosec}x\cot x \qquad \ldots(3)$$

From (1), (2), and (3), we obtain

$$f'(x)=-3\operatorname{cosec}^2x-5\operatorname{cosec}x\cot x$$

(vi) Let *f* (*x*) = 5sin *x* â€“ 6cos *x* + 7. Accordingly, from the first principle,

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$=\lim_{h\to 0}\frac{1}{h}\left[5\sin(x+h)-6\cos(x+h)+7-5\sin x+6\cos x-7\right]$$

$$=\lim_{h\to 0}\frac{1}{h}\left[5\{\sin(x+h)-\sin x\}-6\{\cos(x+h)-\cos x\}\right]$$

$$=5\lim_{h\to 0}\frac{1}{h}\left[\sin(x+h)-\sin x\right]-6\lim_{h\to 0}\frac{1}{h}\left[\cos(x+h)-\cos x\right]$$

$$=5\lim_{h\to 0}\frac{1}{h}\left[2\cos\left(\frac{x+h+x}{2}\right)\sin\left(\frac{x+h-x}{2}\right)\right]-6\lim_{h\to 0}\frac{\cos x\cos h-\sin x\sin h-\cos x}{h}$$

$$=5\lim_{h\to 0}\frac{1}{h}\left[2\cos\left(\frac{2x+h}{2}\right)\sin\frac{h}{2}\right]-6\lim_{h\to 0}\left[\frac{-\cos x(1-\cos h)-\sin x\sin h}{h}\right]$$

$$=5\lim_{h\to 0}\left(\cos\left(\frac{2x+h}{2}\right)\frac{\sin\frac{h}{2}}{\frac{h}{2}}\right)-6\lim_{h\to 0}\left[\frac{-\cos x(1-\cos h)}{h}-\frac{\sin x\sin h}{h}\right]$$

$$=5\left[\lim_{h\to 0}\cos\left(\frac{2x+h}{2}\right)\right]\left[\lim_{\frac{h}{2}\to 0}\frac{\sin\frac{h}{2}}{\frac{h}{2}}\right]-6\left[(-\cos x)\left(\lim_{h\to 0}\frac{1-\cos h}{h}\right)-\sin x\lim_{h\to 0}\left(\frac{\sin h}{h}\right)\right]$$

$$=5\cos x.1-6\left[(-\cos x).(0)-\sin x.1\right]$$

$$=5\cos x+6\sin x$$

(vii) Let *f* (*x*) = 2 tan *x* â€“ 7 sec *x*. Accordingly, from the first principle,

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$=\lim_{h\to 0}\frac{1}{h}\left[2\tan(x+h)-7\sec(x+h)-2\tan x+7\sec x\right]$$

$$=\lim_{h\to 0}\frac{1}{h}\left[2\left\{\tan(x+h)-\tan x\right\}-7\left\{\sec(x+h)-\sec x\right\}\right]$$

$$=2\lim_{h\to 0}\frac{1}{h}\left[\tan(x+h)-\tan x\right]-7\lim_{h\to 0}\frac{1}{h}\left[\sec(x+h)-\sec x\right]$$

$$=2\lim_{h\to 0}\frac{1}{h}\left[\frac{\sin(x+h)}{\cos(x+h)}-\frac{\sin x}{\cos x}\right]-7\lim_{h\to 0}\frac{1}{h}\left[\frac{1}{\cos(x+h)}-\frac{1}{\cos x}\right]$$

$$=2\lim_{h\to 0}\frac{1}{h}\left[\frac{\sin(x+h)\cos x-\sin x\cos(x+h)}{\cos x\cos(x+h)}\right]-7\lim_{h\to 0}\frac{1}{h}\left[\frac{\cos x-\cos(x+h)}{\cos x\cos(x+h)}\right]$$

$$=2\lim_{h\to 0}\frac{1}{h}\left[\frac{\sin(x+h-x)}{\cos x\cos(x+h)}\right]-7\lim_{h\to 0}\frac{1}{h}\left[\frac{-2\sin\left(\frac{x+x+h}{2}\right)\sin\left(\frac{x-x-h}{2}\right)}{\cos x\cos(x+h)}\right]$$

$$=2\lim_{h\to 0}\left[\left(\frac{\sin h}{h}\right)\frac{1}{\cos x\cos(x+h)}\right]-7\lim_{h\to 0}\frac{1}{h}\left[\frac{-2\sin\left(\frac{2x+h}{2}\right)\sin\left(-\frac{h}{2}\right)}{\cos x\cos(x+h)}\right]$$

$$=2\left(\lim_{h\to 0}\frac{\sin h}{h}\right)\left(\lim_{h\to 0}\frac{1}{\cos x\cos(x+h)}\right)-7\left(\lim_{\frac{h}{2}\to 0}\frac{\sin\frac{h}{2}}{\frac{h}{2}}\right)\left(\lim_{h\to 0}\frac{\sin\left(\frac{2x+h}{2}\right)}{\cos x\cos(x+h)}\right)$$

$$=2.1.\frac{1}{\cos x\cos x}-7.1\left(\frac{\sin x}{\cos x\cos x}\right)$$

$$=2\sec^2 x-7\sec x\tan x$$

Exercise Miscellaneous : Solutions of Questions on Page Number : **317**

Q1 :

Find the derivative of the following functions from first principle:

(i) â€“*x* (ii) (â€“*x*)â€“1 (iii) sin (*x* + 1)

(iv) $\cos\left(x-\frac{\pi}{8}\right)$

Answer :

(i) Let $f(x)$ = â€“x. Accordingly, $f(x+h) = -(x+h)$

By first principle,

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0} \frac{-(x+h)-(-x)}{h} \\ &= \lim_{h\to 0} \frac{-x-h+x}{h} \\ &= \lim_{h\to 0} \frac{-h}{h} \\ &= \lim_{h\to 0}(-1) = -1 \end{aligned}$$

(ii) Let $f(x) = (-x)^{-1} = \frac{1}{-x} = \frac{-1}{x}$. Accordingly, $f(x+h) = \frac{-1}{(x+h)}$

By first principle,

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0} \frac{1}{h}\left[\frac{-1}{x+h} - \left(\frac{-1}{x}\right)\right] \\ &= \lim_{h\to 0} \frac{1}{h}\left[\frac{-1}{x+h} + \frac{1}{x}\right] \\ &= \lim_{h\to 0} \frac{1}{h}\left[\frac{-x+(x+h)}{x(x+h)}\right] \\ &= \lim_{h\to 0} \frac{1}{h}\left[\frac{-x+x+h}{x(x+h)}\right] \end{aligned}$$

$$\begin{aligned} &= \lim_{h\to 0} \frac{1}{h}\left[\frac{h}{x(x+h)}\right] \\ &= \lim_{h\to 0} \frac{1}{x(x+h)} \\ &= \frac{1}{x\cdot x} = \frac{1}{x^2} \end{aligned}$$

(iii) Let $f(x) = \sin(x + 1)$. Accordingly, $f(x+h) = \sin(x+h+1)$

By first principle,

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\\
&= \lim_{h\to 0}\frac{1}{h}\left[\sin(x+h+1)-\sin(x+1)\right]\\
&= \lim_{h\to 0}\frac{1}{h}\left[2\cos\left(\frac{x+h+1+x+1}{2}\right)\sin\left(\frac{x+h+1-x-1}{2}\right)\right]\\
&= \lim_{h\to 0}\frac{1}{h}\left[2\cos\left(\frac{2x+h+2}{2}\right)\sin\left(\frac{h}{2}\right)\right]\\
&= \lim_{h\to 0}\left[\cos\left(\frac{2x+h+2}{2}\right)\cdot\frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)}\right]\\
&= \lim_{h\to 0}\cos\left(\frac{2x+h+2}{2}\right)\cdot\lim_{\frac{h}{2}\to 0}\frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)} \qquad \left[\text{As } h\to 0 \Rightarrow \frac{h}{2}\to 0\right]\\
&= \cos\left(\frac{2x+0+2}{2}\right)\cdot 1 \qquad \left[\lim_{x\to 0}\frac{\sin x}{x}=1\right]\\
&= \cos(x+1)
\end{aligned}
$$

(iv) Let $f(x)=\cos\left(x-\frac{\pi}{8}\right)$. Accordingly, $f(x+h)=\cos\left(x+h-\frac{\pi}{8}\right)$

By first principle,

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\\
&= \lim_{h\to 0}\frac{1}{h}\left[\cos\left(x+h-\frac{\pi}{8}\right)-\cos\left(x-\frac{\pi}{8}\right)\right]
\end{aligned}
$$

$$=\lim_{h\to 0}\frac{1}{h}\left[-2\sin\frac{\left(x+h-\frac{\pi}{8}+x-\frac{\pi}{8}\right)}{2}\sin\left(\frac{x+h-\frac{\pi}{8}-x+\frac{\pi}{8}}{2}\right)\right]$$

$$=\lim_{h\to 0}\frac{1}{h}\left[-2\sin\left(\frac{2x+h-\frac{\pi}{4}}{2}\right)\sin\frac{h}{2}\right]$$

$$=\lim_{h\to 0}\left[-\sin\left(\frac{2x+h-\frac{\pi}{4}}{2}\right)\frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)}\right]$$

$$=\lim_{h\to 0}\left[-\sin\left(\frac{2x+h-\frac{\pi}{4}}{2}\right)\right].\lim_{\frac{h}{2}\to 0}\frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)} \qquad \left[\text{As } h\to 0\Rightarrow\frac{h}{2}\to 0\right]$$

$$=-\sin\left(\frac{2x+0-\frac{\pi}{4}}{2}\right).1$$

$$=-\sin\left(x-\frac{\pi}{8}\right)$$

Q2 :

Find the derivative of the following functions (it is to be understood that *a*, *b*, *c*, *d*, *p*, q, *r* and *s* are fixed non-zero constants and *m* and *n* are integers): (*x* + *a*)

Answer :

Let $f(x) = x + a$. Accordingly, $f(x+h)=x+h+a$

By first principle,

$$\begin{aligned} f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0}\frac{x+h+a-x-a}{h} \\ &= \lim_{h\to 0}\left(\frac{h}{h}\right) \\ &= \lim_{h\to 0}(1) \\ &= 1 \end{aligned}$$

Q3 :

Find the derivative of the following functions (it is to be understood that *a*, *b*, *c*, *d*, *p*, q, *r* and *s* are fixed non-zero constants and *m* and *n* are integers): $(px+q)\left(\frac{r}{x}+s\right)$

Answer :

$$\text{Let } f(x) = (px+q)\left(\frac{r}{x}+s\right)$$

By Leibnitz product rule,

$$\begin{aligned} f'(x) &= (px+q)\left(\frac{r}{x}+s\right)' + \left(\frac{r}{x}+s\right)(px+q)' \\ &= (px+q)\left(rx^{-1}+s\right)' + \left(\frac{r}{x}+s\right)(p) \\ &= (px+q)\left(-rx^{-2}\right) + \left(\frac{r}{x}+s\right)p \\ &= (px+q)\left(\frac{-r}{x^2}\right) + \left(\frac{r}{x}+s\right)p \\ &= \frac{-pr}{x} - \frac{qr}{x^2} + \frac{pr}{x} + ps \\ &= ps - \frac{qr}{x^2} \end{aligned}$$

Q4 :

Find the derivative of the following functions (it is to be understood that *a*, *b*, *c*, *d*, *p*, q, *r* and *s* are fixed non-zero constants and *m* and *n* are integers): $(ax+b)(cx+d)^2$

Answer :

Let $f(x)=(ax+b)(cx+d)^2$

By Leibnitz product rule,

$$\begin{aligned}f'(x)&=(ax+b)\frac{d}{dx}(cx+d)^2+(cx+d)^2\frac{d}{dx}(ax+b)\\&=(ax+b)\frac{d}{dx}\left(c^2x^2+2cdx+d^2\right)+(cx+d)^2\frac{d}{dx}(ax+b)\\&=(ax+b)\left[\frac{d}{dx}\left(c^2x^2\right)+\frac{d}{dx}(2cdx)+\frac{d}{dx}d^2\right]+(cx+d)^2\left[\frac{d}{dx}ax+\frac{d}{dx}b\right]\\&=(ax+b)\left(2c^2x+2cd\right)+\left(cx+d^2\right)a\\&=2c(ax+b)(cx+d)+a(cx+d)^2\end{aligned}$$

Q5 :

Find the derivative of the following functions (it is to be understood that *a*, *b*, *c*, *d*, *p*, q, *r* and *s* are fixed non-zero constants and *m* and *n* are integers): $\frac{ax+b}{cx+d}$

Answer :

Let $f(x)=\frac{ax+b}{cx+d}$

By quotient rule,

$$\begin{aligned}f'(x)&=\frac{(cx+d)\frac{d}{dx}(ax+b)-(ax+b)\frac{d}{dx}(cx+d)}{(cx+d)^2}\\&=\frac{(cx+d)(a)-(ax+b)(c)}{(cx+d)^2}\\&=\frac{acx+ad-acx-bc}{(cx+d)^2}\\&=\frac{ad-bc}{(cx+d)^2}\end{aligned}$$

Q6 :

Find the derivative of the following functions (it is to be understood that *a*, *b*, *c*, *d*, *p*, q, *r* and *s* are fixed non-zero constants and *m* and *n* are integers):

$$\frac{1+\frac{1}{x}}{1-\frac{1}{x}}$$

Answer :

$$\text{Let } f(x)=\frac{1+\frac{1}{x}}{1-\frac{1}{x}}=\frac{\frac{x+1}{x}}{\frac{x-1}{x}}=\frac{x+1}{x-1},\ \text{where } x\neq 0$$

By quotient rule,

$$f'(x)=\frac{(x-1)\frac{d}{dx}(x+1)-(x+1)\frac{d}{dx}(x-1)}{(x-1)^2},\ x\neq 0,\ 1$$

$$=\frac{(x-1)(1)-(x+1)(1)}{(x-1)^2},\ x\neq 0,\ 1$$

$$=\frac{x-1-x-1}{(x-1)^2},\ x\neq 0,\ 1$$

$$=\frac{-2}{(x-1)^2},\ x\neq 0,\ 1$$

Q7 :

Find the derivative of the following functions (it is to be understood that *a*, *b*, *c*, *d*, *p*, q, *r* and *s* are fixed non-zero constants and *m* and *n* are integers): $\frac{1}{ax^2+bx+c}$

Answer :

Let $f(x)=\frac{1}{ax^2+bx+c}$

By quotient rule,

$$f'(x)=\frac{\left(ax^2+bx+c\right)\frac{d}{dx}(1)-\frac{d}{dx}\left(ax^2+bx+c\right)}{\left(ax^2+bx+c\right)^2}$$

$$=\frac{\left(ax^2+bx+c\right)(0)-(2ax+b)}{\left(ax^2+bx+c\right)^2}$$

$$=\frac{-(2ax+b)}{\left(ax^2+bx+c\right)^2}$$

Q8 :

Find the derivative of the following functions (it is to be understood that *a*, *b*, *c*, *d*, *p*, q, *r* and *s* are fixed non-zero constants and *m* and *n* are integers): $\frac{ax+b}{px^2+qx+r}$

Answer :

$$\text{Let } f(x)=\frac{ax+b}{px^2+qx+r}$$

By quotient rule,

$$f'(x)=\frac{\left(px^2+qx+r\right)\frac{d}{dx}(ax+b)-(ax+b)\frac{d}{dx}\left(px^2+qx+r\right)}{\left(px^2+qx+r\right)^2}$$

$$=\frac{\left(px^2+qx+r\right)(a)-(ax+b)(2px+q)}{\left(px^2+qx+r\right)^2}$$

$$=\frac{apx^2+aqx+ar-2apx^2-aqx-2bpx-bq}{\left(px^2+qx+r\right)^2}$$

$$=\frac{-apx^2-2bpx+ar-bq}{\left(px^2+qx+r\right)^2}$$

Q9 :

Find the derivative of the following functions (it is to be understood that *a*, *b*, *c*, *d*, *p*, q, *r* and *s* are fixed non-zero constants and *m* and *n* are integers): $\frac{px^2+qx+r}{ax+b}$

Answer :

$$\text{Let } f(x)=\frac{px^2+qx+r}{ax+b}$$

By quotient rule,

$$f'(x)=\frac{(ax+b)\frac{d}{dx}\left(px^2+qx+r\right)-\left(px^2+qx+r\right)\frac{d}{dx}(ax+b)}{(ax+b)^2}$$

$$=\frac{(ax+b)(2px+q)-\left(px^2+qx+r\right)(a)}{(ax+b)^2}$$

$$=\frac{2apx^2+aqx+2bpx+bq-apx^2-aqx-ar}{(ax+b)^2}$$

$$=\frac{apx^2+2bpx+bq-ar}{(ax+b)^2}$$

Q10 :

Find the derivative of the following functions (it is to be understood that *a*, *b*, *c*, *d*, *p*, q, *r* and *s* are fixed non-zero constants and *m* and *n* are integers): $\frac{a}{x^4}-\frac{b}{x^2}+\cos x$

Answer :

$$\text{Let } f(x)=\frac{a}{x^4}-\frac{b}{x^2}+\cos x$$

$$f'(x)=\frac{d}{dx}\left(\frac{a}{x^4}\right)-\frac{d}{dx}\left(\frac{b}{x^2}\right)+\frac{d}{dx}(\cos x)$$

$$=a\frac{d}{dx}\left(x^{-4}\right)-b\frac{d}{dx}\left(x^{-2}\right)+\frac{d}{dx}(\cos x)$$

$$=a\left(-4x^{-5}\right)-b\left(-2x^{-3}\right)+(-\sin x) \qquad \left[\frac{d}{dx}\left(x^n\right)=nx^{n-1}\text{and}\frac{d}{dx}(\cos x)=-\sin x\right]$$

$$=\frac{-4a}{x^5}+\frac{2b}{x^3}-\sin x$$

Q11 :

Find the derivative of the following functions (it is to be understood that *a*, *b*, *c*, *d*, *p*, q, *r* and *s* are fixed non-zero constants and *m* and *n* are integers): $4\sqrt{x}-2$

Answer :

$$\text{Let } f(x)=4\sqrt{x}-2$$

$$f'(x)=\frac{d}{dx}\left(4\sqrt{x}-2\right)=\frac{d}{dx}\left(4\sqrt{x}\right)-\frac{d}{dx}(2)$$

$$=4\frac{d}{dx}\left(x^{\frac{1}{2}}\right)-0=4\left(\frac{1}{2}x^{\frac{1}{2}-1}\right)$$

$$=\left(2x^{-\frac{1}{2}}\right)=\frac{2}{\sqrt{x}}$$

Q12 :

Find the derivative of the following functions (it is to be understood that *a*, *b*, *c*, *d*, *p*, q, *r* and *s* are fixed non-zero constants and *m* and *n* are integers): $(ax+b)^n$

Answer :

$$\text{Let } f(x)=(ax+b)^n\text{. Accordingly, } f(x+h)=\{a(x+h)+b\}^n=(ax+ah+b)^n$$

By first principle,

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$=\lim_{h\to 0}\frac{(ax+ah+b)^n-(ax+b)^n}{h}$$

$$=\lim_{h\to 0}\frac{(ax+b)^n\left(1+\frac{ah}{ax+b}\right)^n-(ax+b)^n}{h}$$

$$=(ax+b)^n\lim_{h\to 0}\frac{\left(1+\frac{ah}{ax+b}\right)^n-1}{h}$$

$$=(ax+b)^n\lim_{h\to 0}\frac{1}{n}\left[\left\{1+n\left(\frac{ah}{ax+b}\right)+\frac{n(n-1)}{\underline{|2}}\left(\frac{ah}{ax+b}\right)^2+\ldots\right\}-1\right]$$

(Using binomial theorem)

$$=(ax+b)^n\lim_{h\to 0}\frac{1}{h}\left[n\left(\frac{ah}{ax+b}\right)+\frac{n(n-1)a^2h^2}{\underline{|2}(ax+b)^2}+\ldots\text{(Terms containing higher degrees of }h)\right]$$

$$=(ax+b)^n\lim_{h\to 0}\left[\frac{na}{(ax+b)}+\frac{n(n-1)a^2h}{\underline{|2}(ax+b)^2}+\ldots\right]$$

$$=(ax+b)^n\left[\frac{na}{(ax+b)}+0\right]$$

$$=na\frac{(ax+b)^n}{(ax+b)}$$

$$=na(ax+b)^{n-1}$$

Q13 :

Find the derivative of the following functions (it is to be understood that *a*, *b*, *c*, *d*, *p*, q, *r* and *s* are fixed non-zero constants and *m* and *n* are integers): (*ax* + *b*)^n (*cx* + *d*)^m

Answer :

Let $f(x)=(ax+b)^n(cx+d)^m$

By Leibnitz product rule,

$$f'(x)=(ax+b)^n\frac{d}{dx}(cx+d)^m+(cx+d)^m\frac{d}{dx}(ax+b)^n \qquad ...(1)$$

Now, let $f_1(x)=(cx+d)^m$

$$f_1(x+h)=(cx+ch+d)^m$$

$$\begin{aligned}
f_1'(x)&=\lim_{h\to0}\frac{f_1(x+h)-f_1(x)}{h}\\
&=\lim_{h\to0}\frac{(cx+ch+d)^m-(cx+d)^m}{h}\\
&=(cx+d)^m\lim_{h\to0}\frac{1}{h}\left[\left(1+\frac{ch}{cx+d}\right)^m-1\right]\\
&=(cx+d)^m\lim_{h\to0}\frac{1}{h}\left[\left(1+\frac{mch}{(cx+d)}+\frac{m(m-1)}{2}\frac{(c^2h^2)}{(cx+d)^2}+...\right)-1\right]\\
&=(cx+d)^m\lim_{h\to0}\frac{1}{h}\left[\frac{mch}{(cx+d)}+\frac{m(m-1)c^2h^2}{2(cx+d)^2}+...(\text{Terms containing higher degrees of }h)\right]\\
&=(cx+d)^m\lim_{h\to0}\left[\frac{mc}{(cx+d)}+\frac{m(m-1)c^2h}{2(cx+d)^2}+...\right]\\
&=(cx+d)^m\left[\frac{mc}{cx+d}+0\right]\\
&=\frac{mc(cx+d)^m}{(cx+d)}\\
&=mc(cx+d)^{m-1}
\end{aligned}$$

$$\frac{d}{dx}(cx+d)^m=mc(cx+d)^{m-1} \qquad ...(2)$$

Similarly, $\frac{d}{dx}(ax+b)^n=na(ax+b)^{n-1}$ $\qquad ...(3)$

Therefore, from (1), (2), and (3), we obtain

$$\begin{aligned}
f'(x)&=(ax+b)^n\left\{mc(cx+d)^{m-1}\right\}+(cx+d)^m\left\{na(ax+b)^{n-1}\right\}\\
&=(ax+b)^{n-1}(cx+d)^{m-1}\left[mc(ax+b)+na(cx+d)\right]
\end{aligned}$$

Q14 :

Find the derivative of the following functions (it is to be understood that *a*, *b*, *c*, *d*, *p*, q, *r* and *s* are fixed non-zero constants and *m* and *n* are integers): sin (*x* + *a*)

Answer :

Let $f(x) = \sin(x+a)$

$f(x+h) = \sin(x+h+a)$

By first principle,

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0} \frac{\sin(x+h+a)-\sin(x+a)}{h} \\
&= \lim_{h\to 0} \frac{1}{h}\left[2\cos\left(\frac{x+h+a+x+a}{2}\right)\sin\left(\frac{x+h+a-x-a}{2}\right)\right] \\
&= \lim_{h\to 0} \frac{1}{h}\left[2\cos\left(\frac{2x+2a+h}{2}\right)\sin\left(\frac{h}{2}\right)\right] \\
&= \lim_{h\to 0}\left[\cos\left(\frac{2x+2a+h}{2}\right)\left\{\frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)}\right\}\right] \\
&= \lim_{h\to 0}\cos\left(\frac{2x+2a+h}{2}\right)\lim_{\frac{h}{2}\to 0}\left\{\frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)}\right\} && \left[\text{As } h\to 0 \Rightarrow \frac{h}{2}\to 0\right] \\
&= \cos\left(\frac{2x+2a}{2}\right)\times 1 && \left[\lim_{x\to 0}\frac{\sin x}{x}=1\right] \\
&= \cos(x+a)
\end{aligned}
$$

Q15 :

Find the derivative of the following functions (it is to be understood that *a*, *b*, *c*, *d*, *p*, q, *r* and *s* are fixed non-zero constants and *m* and *n* are integers): cosec *x* cot *x*

Answer :

Let $f(x) = \operatorname{cosec} x \cot x$

By Leibnitz product rule,

$$f'(x) = \operatorname{cosec} x (\cot x)' + \cot x (\operatorname{cosec} x)' \qquad ...(1)$$

Let $f_1(x) = \cot x$. Accordingly, $f_1(x+h) = \cot(x+h)$

By first principle,

$$f_1'(x) = \lim_{h \to 0} \frac{f_1(x+h) - f_1(x)}{h}$$

$$= \lim_{h \to 0} \frac{\cot(x+h) - \cot x}{h}$$

$$= \lim_{h \to 0} \frac{1}{h}\left(\frac{\cos(x+h)}{\sin(x+h)} - \frac{\cos x}{\sin x}\right)$$

$$= \lim_{h \to 0} \frac{1}{h}\left[\frac{\sin x \cos(x+h) - \cos x \sin(x+h)}{\sin x \sin(x+h)}\right]$$

$$= \lim_{h \to 0} \frac{1}{h}\left[\frac{\sin(x-x-h)}{\sin x \sin(x+h)}\right]$$

$$= \frac{1}{\sin x} . \lim_{h \to 0} \frac{1}{h}\left[\frac{\sin(-h)}{\sin(x+h)}\right]$$

$$= \frac{-1}{\sin x} . \left(\lim_{h \to 0} \frac{\sin h}{h}\right)\left(\lim_{h \to 0} \frac{1}{\sin(x+h)}\right)$$

$$= \frac{-1}{\sin x} . 1 . \left(\frac{1}{\sin(x+0)}\right)$$

$$= \frac{-1}{\sin^2 x}$$

$$= -\operatorname{cosec}^2 x$$

$$\therefore (\cot x)' = -\operatorname{cosec}^2 x \qquad ...(2)$$

Now, let $f_2(x) = \operatorname{cosec} x$. Accordingly, $f_2(x+h) = \operatorname{cosec}(x+h)$

By first principle,

$$f_2'(x) = \lim_{h \to 0} \frac{f_2(x+h) - f_2(x)}{h}$$

$$= \lim_{h \to 0} \frac{1}{h}\left[\operatorname{cosec}(x+h) - \operatorname{cosec} x\right]$$

$$= \lim_{h\to 0} \frac{1}{h}\left[\frac{1}{\sin(x+h)} - \frac{1}{\sin x}\right]$$

$$= \lim_{h\to 0} \frac{1}{h}\left[\frac{\sin x - \sin(x+h)}{\sin x \sin(x+h)}\right]$$

$$= \frac{1}{\sin x}.\lim_{h\to 0} \frac{1}{h}\left[\frac{2\cos\left(\frac{x+x+h}{2}\right)\sin\left(\frac{x-x-h}{2}\right)}{\sin(x+h)}\right]$$

$$= \frac{1}{\sin x}.\lim_{h\to 0} \frac{1}{h}\left[\frac{2\cos\left(\frac{2x+h}{2}\right)\sin\left(\frac{-h}{2}\right)}{\sin(x+h)}\right]$$

$$= \frac{1}{\sin x}.\lim_{h\to 0}\left[\frac{-\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)}.\frac{\cos\left(\frac{2x+h}{2}\right)}{\sin(x+h)}\right]$$

$$= \frac{-1}{\sin x}.\lim_{h\to 0}\frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)}.\lim_{h\to 0}\frac{\cos\left(\frac{2x+h}{2}\right)}{\sin(x+h)}$$

$$= \frac{-1}{\sin x}.1.\frac{\cos\left(\frac{2x+0}{2}\right)}{\sin(x+0)}$$

$$= \frac{-1}{\sin x}.\frac{\cos x}{\sin x}$$

$$= -\operatorname{cosec} x.\cot x$$

$$\therefore (\operatorname{cosec} x)' = -\operatorname{cosec} x.\cot x \qquad ...(3)$$

From (1), (2), and (3), we obtain

$$f'(x) = \operatorname{cosec} x\left(-\operatorname{cosec}^2 x\right) + \cot x\left(-\operatorname{cosec} x \cot x\right)$$

$$= -\operatorname{cosec}^3 x - \cot^2 x \operatorname{cosec} x$$

Q16 :

Find the derivative of the following functions (it is to be understood that *a*, *b*, *c*, *d*, *p*, q, *r* and *s* are fixed non-zero constants and *m* and *n* are integers): $\frac{\cos x}{1+\sin x}$

Answer :

Let $f(x)=\frac{\cos x}{1+\sin x}$

By quotient rule,

$$f'(x)=\frac{(1+\sin x)\frac{d}{dx}(\cos x)-(\cos x)\frac{d}{dx}(1+\sin x)}{(1+\sin x)^2}$$

$$=\frac{(1+\sin x)(-\sin x)-(\cos x)(\cos x)}{(1+\sin x)^2}$$

$$=\frac{-\sin x-\sin^2 x-\cos^2 x}{(1+\sin x)^2}$$

$$=\frac{-\sin x-\left(\sin^2 x+\cos^2 x\right)}{(1+\sin x)^2}$$

$$=\frac{-\sin x-1}{(1+\sin x)^2}$$

$$=\frac{-(1+\sin x)}{(1+\sin x)^2}$$

$$=\frac{-1}{(1+\sin x)}$$

Q17 :

Find the derivative of the following functions (it is to be understood that *a*, *b*, *c*, *d*, *p*, q, *r* and *s* are fixed non-zero constants and *m* and *n* are integers): $\frac{\sin x+\cos x}{\sin x-\cos x}$

Answer :

Let $f(x)=\frac{\sin x+\cos x}{\sin x-\cos x}$

By quotient rule,

$$f'(x)=\frac{(\sin x-\cos x)\frac{d}{dx}(\sin x+\cos x)-(\sin x+\cos x)\frac{d}{dx}(\sin x-\cos x)}{(\sin x-\cos x)^2}$$

$$=\frac{(\sin x-\cos x)(\cos x-\sin x)-(\sin x+\cos x)(\cos x+\sin x)}{(\sin x-\cos x)^2}$$

$$=\frac{-(\sin x-\cos x)^2-(\sin x+\cos x)^2}{(\sin x-\cos x)^2}$$

$$=\frac{-\left[\sin^2 x+\cos^2 x-2\sin x\cos x+\sin^2 x+\cos^2 x+2\sin x\cos x\right]}{(\sin x-\cos x)^2}$$

$$=\frac{-[1+1]}{(\sin x-\cos x)^2}$$

$$=\frac{-2}{(\sin x-\cos x)^2}$$

Q18 :

Find the derivative of the following functions (it is to be understood that *a*, *b*, *c*, *d*, *p*, q, *r* and *s* are fixed non-zero constants and *m* and *n* are integers): $\frac{\sec x-1}{\sec x+1}$

Answer :

Let $f(x)=\frac{\sec x-1}{\sec x+1}$

$$f(x)=\frac{\frac{1}{\cos x}-1}{\frac{1}{\cos x}+1}=\frac{1-\cos x}{1+\cos x}$$

By quotient rule,

$$f'(x) = \frac{(1+\cos x)\frac{d}{dx}(1-\cos x) - (1-\cos x)\frac{d}{dx}(1+\cos x)}{(1+\cos x)^2}$$

$$= \frac{(1+\cos x)(\sin x) - (1-\cos x)(-\sin x)}{(1+\cos x)^2}$$

$$= \frac{\sin x + \cos x \sin x + \sin x - \sin x \cos x}{(1+\cos x)^2}$$

$$= \frac{2\sin x}{(1+\cos x)^2}$$

$$= \frac{2\sin x}{\left(1+\frac{1}{\sec x}\right)^2} = \frac{2\sin x}{\frac{(\sec x+1)^2}{\sec^2 x}}$$

$$= \frac{2\sin x \sec^2 x}{(\sec x+1)^2}$$

$$= \frac{\frac{2\sin x}{\cos x}\sec x}{(\sec x+1)^2}$$

$$= \frac{2\sec x \tan x}{(\sec x+1)^2}$$

Q19 :

Find the derivative of the following functions (it is to be understood that *a*, *b*, *c*, *d*, *p*, q, *r* and *s* are fixed non-zero constants and *m* and *n* are integers): $\sin^n x$

Answer :

Let $y = \sin^n x$.

Accordingly, for $n = 1$, $y = \sin x$.

$$\therefore \frac{dy}{dx} = \cos x, \text{ i.e., } \frac{d}{dx}\sin x = \cos x$$

For $n = 2$, $y = \sin^2 x$.

$$\therefore \frac{dy}{dx} = \frac{d}{dx}(\sin x \sin x)$$

$$= (\sin x)' \sin x + \sin x (\sin x)' \quad \text{[By Leibnitz product rule]}$$

$$= \cos x \sin x + \sin x \cos x$$

$$= 2 \sin x \cos x \quad ...(1)$$

For $n = 3$, $y = \sin^3 x$.

$$\therefore \frac{dy}{dx} = \frac{d}{dx}(\sin x \sin^2 x)$$

$$= (\sin x)' \sin^2 x + \sin x (\sin^2 x)' \quad \text{[By Leibnitz product rule]}$$

$$= \cos x \sin^2 x + \sin x (2 \sin x \cos x) \quad \text{[Using (1)]}$$

$$= \cos x \sin^2 x + 2 \sin^2 x \cos x$$

$$= 3 \sin^2 x \cos x$$

We assert that $\frac{d}{dx}(\sin^n x) = n \sin^{(n-1)} x \cos x$

Let our assertion be true for $n = k$.

i.e., $\frac{d}{dx}(\sin^k x) = k \sin^{(k-1)} x \cos x \quad ...(2)$

Consider

$$\frac{d}{dx}(\sin^{k+1} x) = \frac{d}{dx}(\sin x \sin^k x)$$

$$= (\sin x)' \sin^k x + \sin x (\sin^k x)' \quad \text{[By Leibnitz product rule]}$$

$$= \cos x \sin^k x + \sin x \left(k \sin^{(k-1)} x \cos x\right) \quad \text{[Using (2)]}$$

$$= \cos x \sin^k x + k \sin^k x \cos x$$

$$= (k+1) \sin^k x \cos x$$

Thus, our assertion is true for $n = k + 1$.

Hence, by mathematical induction, $\frac{d}{dx}(\sin^n x) = n \sin^{(n-1)} x \cos x$

Q20 :

Find the derivative of the following functions (it is to be understood that *a*, *b*, *c*, *d*, *p*, q, *r* and *s* are fixed non-zero constants and *m* and *n* are integers): $\frac{a + b \sin x}{c + d \cos x}$

Answer :

$$\text{Let } f(x) = \frac{a + b\sin x}{c + d\cos x}$$

By quotient rule,

$$f'(x) = \frac{(c + d\cos x)\frac{d}{dx}(a + b\sin x) - (a + b\sin x)\frac{d}{dx}(c + d\cos x)}{(c + d\cos x)^2}$$

$$= \frac{(c + d\cos x)(b\cos x) - (a + b\sin x)(-d\sin x)}{(c + d\cos x)^2}$$

$$= \frac{cb\cos x + bd\cos^2 x + ad\sin x + bd\sin^2 x}{(c + d\cos x)^2}$$

$$= \frac{bc\cos x + ad\sin x + bd\left(\cos^2 x + \sin^2 x\right)}{(c + d\cos x)^2}$$

$$= \frac{bc\cos x + ad\sin x + bd}{(c + d\cos x)^2}$$

Q21 :

Find the derivative of the following functions (it is to be understood that *a*, *b*, *c*, *d*, *p*, q, *r* and *s* are fixed non-zero constants and *m* and *n* are integers): $\frac{\sin(x + a)}{\cos x}$

Answer :

Let $f(x) = \frac{\sin(x + a)}{\cos x}$

By quotient rule,

$$f'(x) = \frac{\cos x \frac{d}{dx}\left[\sin(x + a)\right] - \sin(x + a)\frac{d}{dx}\cos x}{\cos^2 x}$$

$$f'(x) = \frac{\cos x \frac{d}{dx}\left[\sin(x + a)\right] - \sin(x + a)(-\sin x)}{\cos^2 x} \qquad \text{... (i)}$$

Let $g(x) = \sin(x + a)$. Accordingly, $g(x + h) = \sin(x + h + a)$

By first principle,

$$g'(x)=\lim_{h\to0}\frac{g(x+h)-g(x)}{h}$$

$$=\lim_{h\to0}\frac{1}{h}\left[\sin(x+h+a)-\sin(x+a)\right]$$

$$=\lim_{h\to0}\frac{1}{h}\left[2\cos\left(\frac{x+h+a+x+a}{2}\right)\sin\left(\frac{x+h+a-x-a}{2}\right)\right]$$

$$=\lim_{h\to0}\frac{1}{h}\left[2\cos\left(\frac{2x+2a+h}{2}\right)\sin\left(\frac{h}{2}\right)\right]$$

$$=\lim_{h\to0}\left[\cos\left(\frac{2x+2a+h}{2}\right)\left\{\frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)}\right\}\right]$$

$$=\lim_{h\to0}\cos\left(\frac{2x+2a+h}{2}\right).\lim_{\frac{h}{2}\to0}\left\{\frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)}\right\}\qquad\left[\text{As } h\to0\Rightarrow\frac{h}{2}\to0\right]$$

$$=\left(\cos\frac{2x+2a}{2}\right)\times1\qquad\left[\lim_{h\to0}\frac{\sin h}{h}=1\right]$$

$$=\cos(x+a)\qquad\ldots\text{(ii)}$$

From (i) and (ii), we obtain

$$f'(x)=\frac{\cos x\cdot\cos(x+a)+\sin x\sin(x+a)}{\cos^2 x}$$

$$=\frac{\cos(x+a-x)}{\cos^2 x}$$

$$=\frac{\cos a}{\cos^2 x}$$

Q22 :

Find the derivative of the following functions (it is to be understood that *a*, *b*, *c*, *d*, *p*, q, *r* and *s* are fixed non-zero constants and *m* and *n* are integers): x^4 (5 sin *x* - 3 cos *x*)

Answer :

Let $f(x)=x^4(5\sin x-3\cos x)$

By product rule,

$$f'(x) = x^4 \frac{d}{dx}(5\sin x - 3\cos x) + (5\sin x - 3\cos x)\frac{d}{dx}(x^4)$$
$$= x^4\left[5\frac{d}{dx}(\sin x) - 3\frac{d}{dx}(\cos x)\right] + (5\sin x - 3\cos x)\frac{d}{dx}(x^4)$$
$$= x^4\left[5\cos x - 3(-\sin x)\right] + (5\sin x - 3\cos x)(4x^3)$$
$$= x^3\left[5x\cos x + 3x\sin x + 20\sin x - 12\cos x\right]$$

Q23 :

Find the derivative of the following functions (it is to be understood that *a*, *b*, *c*, *d*, *p*, q, *r* and *s* are fixed non-zero constants and *m* and *n* are integers): (x^2 + 1) cos *x*

Answer :

Let $f(x) = (x^2 + 1)\cos x$

By product rule,

$$f'(x) = (x^2 + 1)\frac{d}{dx}(\cos x) + \cos x \frac{d}{dx}(x^2 + 1)$$
$$= (x^2 + 1)(-\sin x) + \cos x(2x)$$
$$= -x^2 \sin x - \sin x + 2x\cos x$$

Q24 :

Find the derivative of the following functions (it is to be understood that *a*, *b*, *c*, *d*, *p*, q, *r* and *s* are fixed non-zero constants and *m* and *n* are integers): (ax^2 + sin *x*) (*p* + *q* cos *x*)

Answer :

Let $f(x) = (ax^2 + \sin x)(p + q\cos x)$

By product rule,

$$f'(x) = (ax^2 + \sin x)\frac{d}{dx}(p + q\cos x) + (p + q\cos x)\frac{d}{dx}(ax^2 + \sin x)$$
$$= (ax^2 + \sin x)(-q\sin x) + (p + q\cos x)(2ax + \cos x)$$
$$= -q\sin x(ax^2 + \sin x) + (p + q\cos x)(2ax + \cos x)$$

Q25 :

Find the derivative of the following functions (it is to be understood that *a*, *b*, *c*, *d*, *p*, q, *r* and *s* are fixed non-zero constants and *m* and *n* are integers): $(x+\cos x)(x-\tan x)$

Answer :

Let $f(x)=(x+\cos x)(x-\tan x)$

By product rule,

$$f'(x)=(x+\cos x)\frac{d}{dx}(x-\tan x)+(x-\tan x)\frac{d}{dx}(x+\cos x)$$
$$=(x+\cos x)\left[\frac{d}{dx}(x)-\frac{d}{dx}(\tan x)\right]+(x-\tan x)(1-\sin x)$$
$$=(x+\cos x)\left[1-\frac{d}{dx}\tan x\right]+(x-\tan x)(1-\sin x) \qquad \text{... (i)}$$

Let $g(x)=\tan x$. Accordingly, $g(x+h)=\tan(x+h)$

By first principle,

$$g'(x) = \lim_{h\to 0}\frac{g(x+h)-g(x)}{h}$$
$$= \lim_{h\to 0}\left(\frac{\tan(x+h)-\tan x}{h}\right)$$
$$= \lim_{h\to 0}\frac{1}{h}\left[\frac{\sin(x+h)}{\cos(x+h)}-\frac{\sin x}{\cos x}\right]$$
$$= \lim_{h\to 0}\frac{1}{h}\left[\frac{\sin(x+h)\cos x-\sin x\cos(x+h)}{\cos(x+h)\cos x}\right]$$
$$= \frac{1}{\cos x}.\lim_{h\to 0}\frac{1}{h}\left[\frac{\sin(x+h-x)}{\cos(x+h)}\right]$$
$$= \frac{1}{\cos x}.\lim_{h\to 0}\frac{1}{h}\left[\frac{\sin h}{\cos(x+h)}\right]$$
$$= \frac{1}{\cos x}.\left(\lim_{h\to 0}\frac{\sin h}{h}\right).\left(\lim_{h\to 0}\frac{1}{\cos(x+h)}\right)$$
$$= \frac{1}{\cos x}.1.\frac{1}{\cos(x+0)}$$
$$= \frac{1}{\cos^2 x}$$
$$= \sec^2 x \qquad \dots \text{(ii)}$$

Therefore, from (i) and (ii), we obtain

$$f'(x) = (x+\cos x)(1-\sec^2 x)+(x-\tan x)(1-\sin x)$$
$$= (x+\cos x)(-\tan^2 x)+(x-\tan x)(1-\sin x)$$
$$= -\tan^2 x(x+\cos x)+(x-\tan x)(1-\sin x)$$

Q26 :

Find the derivative of the following functions (it is to be understood that *a*, *b*, *c*, *d*, *p*, q, *r* and *s* are fixed non-zero constants and *m* and *n* are integers): $\frac{4x+5\sin x}{3x+7\cos x}$

Answer :

Let $f(x) = \frac{4x+5\sin x}{3x+7\cos x}$

By quotient rule,

$$f'(x)=\frac{(3x+7\cos x)\frac{d}{dx}(4x+5\sin x)-(4x+5\sin x)\frac{d}{dx}(3x+7\cos x)}{(3x+7\cos x)^2}$$

$$=\frac{(3x+7\cos x)\left[4\frac{d}{dx}(x)+5\frac{d}{dx}(\sin x)\right]-(4x+5\sin x)\left[3\frac{d}{dx}x+7\frac{d}{dx}\cos x\right]}{(3x+7\cos x)^2}$$

$$=\frac{(3x+7\cos x)(4+5\cos x)-(4x+5\sin x)(3-7\sin x)}{(3x+7\cos x)^2}$$

$$=\frac{12x+15x\cos x+28\cos x+35\cos^2 x-12x+28x\sin x-15\sin x+35\sin^2 x}{(3x+7\cos x)^2}$$

$$=\frac{15x\cos x+28\cos x+28x\sin x-15\sin x+35(\cos^2 x+\sin^2 x)}{(3x+7\cos x)^2}$$

$$=\frac{35+15x\cos x+28\cos x+28x\sin x-15\sin x}{(3x+7\cos x)^2}$$

Q27 :

Find the derivative of the following functions (it is to be understood that *a*, *b*, *c*, *d*, *p*, q, *r* and *s* are fixed non-zero constants and *m* and *n* are integers):

$$\frac{x^2\cos\left(\frac{\pi}{4}\right)}{\sin x}$$

Answer :

Let $f(x)=\frac{x^2\cos\left(\frac{\pi}{4}\right)}{\sin x}$

By quotient rule,

$$f'(x) = \cos\frac{\pi}{4}.\left[\frac{\sin x \frac{d}{dx}(x^2) - x^2 \frac{d}{dx}(\sin x)}{\sin^2 x}\right]$$

$$= \cos\frac{\pi}{4}.\left[\frac{\sin x \cdot 2x - x^2 \cos x}{\sin^2 x}\right]$$

$$= \frac{x\cos\frac{\pi}{4}[2\sin x - x\cos x]}{\sin^2 x}$$

Q28 :

Find the derivative of the following functions (it is to be understood that *a*, *b*, *c*, *d*, *p*, q, *r* and *s* are fixed non-zero constants and *m* and *n* are integers): $\frac{x}{1+\tan x}$

Answer :

Let $f(x) = \frac{x}{1+\tan x}$

$$f'(x) = \frac{(1+\tan x)\frac{d}{dx}(x) - x\frac{d}{dx}(1+\tan x)}{(1+\tan x)^2}$$

$$f'(x) = \frac{(1+\tan x) - x \cdot \frac{d}{dx}(1+\tan x)}{(1+\tan x)^2} \qquad \text{... (i)}$$

Let $g(x) = 1 + \tan x$. Accordingly, $g(x+h) = 1 + \tan(x+h)$.

By first principle,

$$g'(x)=\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}$$

$$=\lim_{h\to 0}\left[\frac{1+\tan(x+h)-1-\tan x}{h}\right]$$

$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{\sin(x+h)}{\cos(x+h)}-\frac{\sin x}{\cos x}\right]$$

$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{\sin(x+h)\cos x-\sin x\cos(x+h)}{\cos(x+h)\cos x}\right]$$

$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{\sin(x+h-x)}{\cos(x+h)\cos x}\right]$$

$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{\sin h}{\cos(x+h)\cos x}\right]$$

$$=\left(\lim_{h\to 0}\frac{\sin h}{h}\right)\cdot\left(\lim_{h\to 0}\frac{1}{\cos(x+h)\cos x}\right)$$

$$=1\times\frac{1}{\cos^2 x}=\sec^2 x$$

$$\Rightarrow\frac{d}{dx}(1+\tan x)=\sec^2 x \qquad \text{... (ii)}$$

From (i) and (ii), we obtain

$$f'(x)=\frac{1+\tan x-x\sec^2 x}{(1+\tan x)^2}$$

Q29 :

Find the derivative of the following functions (it is to be understood that *a*, *b*, *c*, *d*, *p*, q, *r* and *s* are fixed non-zero constants and *m* and *n* are integers): (*x* + sec *x*) (*x* - tan *x*)

Answer :

Let $f(x)=(x+\sec x)(x-\tan x)$

By product rule,

$$f'(x)=(x+\sec x)\frac{d}{dx}(x-\tan x)+(x-\tan x)\frac{d}{dx}(x+\sec x)$$
$$=(x+\sec x)\left[\frac{d}{dx}(x)-\frac{d}{dx}\tan x\right]+(x-\tan x)\left[\frac{d}{dx}(x)+\frac{d}{dx}\sec x\right]$$
$$=(x+\sec x)\left[1-\frac{d}{dx}\tan x\right]+(x-\tan x)\left[1+\frac{d}{dx}\sec x\right] \qquad \text{... (i)}$$

Let $f_1(x)=\tan x,\ f_2(x)=\sec x$

Accordingly, $f_1(x+h)=\tan(x+h)$ and $f_2(x+h)=\sec(x+h)$

$$f_1'(x)=\lim_{h\to0}\left(\frac{f_1(x+h)-f_1(x)}{h}\right)$$
$$=\lim_{h\to0}\left(\frac{\tan(x+h)-\tan x}{h}\right)$$
$$=\lim_{h\to0}\left[\frac{\tan(x+h)-\tan x}{h}\right]$$
$$=\lim_{h\to0}\frac{1}{h}\left[\frac{\sin(x+h)}{\cos(x+h)}-\frac{\sin x}{\cos x}\right]$$
$$=\lim_{h\to0}\frac{1}{h}\left[\frac{\sin(x+h)\cos x-\sin x\cos(x+h)}{\cos(x+h)\cos x}\right]$$
$$=\lim_{h\to0}\frac{1}{h}\left[\frac{\sin(x+h-x)}{\cos(x+h)\cos x}\right]$$
$$=\lim_{h\to0}\frac{1}{h}\left[\frac{\sin h}{\cos(x+h)\cos x}\right]$$
$$=\left(\lim_{h\to0}\frac{\sin h}{h}\right).\left(\lim_{h\to0}\frac{1}{\cos(x+h)\cos x}\right)$$
$$=1\times\frac{1}{\cos^2 x}=\sec^2 x$$
$$\Rightarrow \frac{d}{dx}\tan x=\sec^2 x \qquad \text{... (ii)}$$

$$f_2'(x) = \lim_{h\to 0}\left(\frac{f_2(x+h)-f_2(x)}{h}\right)$$

$$= \lim_{h\to 0}\left(\frac{\sec(x+h)-\sec x}{h}\right)$$

$$= \lim_{h\to 0}\frac{1}{h}\left[\frac{1}{\cos(x+h)}-\frac{1}{\cos x}\right]$$

$$= \lim_{h\to 0}\frac{1}{h}\left[\frac{\cos x-\cos(x+h)}{\cos(x+h)\cos x}\right]$$

$$= \frac{1}{\cos x}.\lim_{h\to 0}\frac{1}{h}\left[\frac{-2\sin\left(\frac{x+x+h}{2}\right)\cdot\sin\left(\frac{x-x-h}{2}\right)}{\cos(x+h)}\right]$$

$$= \frac{1}{\cos x}.\lim_{h\to 0}\frac{1}{h}\left[\frac{-2\sin\left(\frac{2x+h}{2}\right)\cdot\sin\left(\frac{-h}{2}\right)}{\cos(x+h)}\right]$$

$$= \frac{1}{\cos x}.\lim_{h\to 0}\left[\frac{\sin\left(\frac{2x+h}{2}\right)\left\{\frac{\sin\left(\frac{h}{2}\right)}{\frac{h}{2}}\right\}}{\cos(x+h)}\right]$$

$$= \sec x.\frac{\left\{\lim_{h\to 0}\sin\left(\frac{2x+h}{2}\right)\right\}\left\{\lim_{\frac{h}{2}\to 0}\frac{\sin\left(\frac{h}{2}\right)}{\frac{h}{2}}\right\}}{\lim_{h\to 0}\cos(x+h)}$$

$$= \sec x.\frac{\sin x.1}{\cos x}$$

$$\Rightarrow \frac{d}{dx}\sec x = \sec x\tan x \qquad \text{.. (iii)}$$

From (i), (ii), and (iii), we obtain

$$f'(x) = (x+\sec x)\left(1-\sec^2 x\right)+(x-\tan x)(1+\sec x\tan x)$$

Q30 :

Find the derivative of the following functions (it is to be understood that *a*, *b*, *c*, *d*, *p*, q, *r* and *s* are fixed non-zero constants and *m* and *n* are integers): $\frac{x}{\sin^n x}$

Answer :

Let $f(x)=\frac{x}{\sin^n x}$

By quotient rule,

$$f'(x)=\frac{\sin^n x\frac{d}{dx}x-x\frac{d}{dx}\sin^n x}{\sin^{2n} x}$$

It can be easily shown that $\frac{d}{dx}\sin^n x=n\sin^{n-1}x\cos x$

Therefore,

$$f'(x)=\frac{\sin^n x\frac{d}{dx}x-x\frac{d}{dx}\sin^n x}{\sin^{2n} x}$$

$$=\frac{\sin^n x.1-x\left(n\sin^{n-1}x\cos x\right)}{\sin^{2n} x}$$

$$=\frac{\sin^{n-1}x\left(\sin x-nx\cos x\right)}{\sin^{2n} x}$$

$$=\frac{\sin x-nx\cos x}{\sin^{n+1} x}$$