Exercise 12.4

Answer 1E.

$\vec{a} = \langle 6, 0, -2 \rangle \quad \vec{b} = \langle 0, 8, 0 \rangle$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 6 & 0 & -2 \\ 0 & 8 & 0 \end{vmatrix}$$

$$= \begin{vmatrix} 0 & -2 \\ 8 & 0 \end{vmatrix} \hat{i} - \begin{vmatrix} 6 & -2 \\ 0 & 0 \end{vmatrix} \hat{j} + \begin{vmatrix} 6 & 0 \\ 0 & 8 \end{vmatrix} \hat{k}$$

$$= (0+16)\hat{i} - (0-0)\hat{j} + (48-0)\hat{k}$$

$$= 16\hat{i} + 48\hat{k}$$

Now

$$(\vec{a} \times \vec{b}).\vec{a} = (16\hat{i} + 48\hat{k}).(6\hat{i} - 2\hat{k})$$

$$= 16(6) - 2(48)$$

$$= 96 - 96 = 0$$

And

$$(\vec{a} \times \vec{b}).\vec{b} = (16\hat{i} + 48\hat{k}).(8\hat{j})$$

$$= 16(0) + (0)(8) + (48)(0) = 0$$

Therefore, $\vec{a} \times \vec{b}$ is orthogonal to both $\vec{a}$ and $\vec{b}$.

Answer 2E.

To find the cross product of **a**=<1,1,-1> and **b**=<2,4,6>

axb= $\begin{vmatrix} i & j & k \\ 1 & 1 & -1 \\ 2 & 4 & 6 \end{vmatrix} = \left(6+4\right)i - \left(6+2\right)j + \left(4-2\right)k = 10i - 8j + 2k$

To prove that **axb** is orthogonal to both **a** and **b** recall

If the dot product of two vectors equals zero then the vectors are orthogonal, so since

$$(a \times b) \cdot a = 10 - 8 - 2 = 0$$

$$(a \times b) \cdot b = 20 - 32 + 12 = 0$$

axb is orthogonal to both **a** and **b**

Q3E.

Given that

a = **i** + 3**j** - 2**k** = <1, 3, -2>

b = -**i** + **5**k = <-1, 0, 5>

From the definition of the cross product,

a x **b** = < (a2b3) - (a3b2), (a3b1) - (a1b3), (a1b2) - (a2b1) >

a x **b** = < (3*5) - (-2*0), (-2*-1) - (1*5), (1*0) - (3*-1)>

a x **b** = < (15 + 0), (2 - 5), (0 + 3) >

a x **b** = <15, -3, 3>

The cross product is a vector, so you can show that the cross product **a** x **b** is orthogonal to both **a** and **b** with the following definition:

Two vectors **a** and **b** are orthogonal only if **a** •**b**=0

In this case, we need to show that vectors **a** x **b** •**a** = 0 and **a** x **b** •**b** = 0.

(a x **b)** •**a** = <15, -3, 3> • <1, 3, -2>

= (15 * 1) + (-3 * 3) + (3 * -2)

= 15 + -9 + -6

= 0

(a x **b)** •**b** = <15, -3, 3> • <-1, 0, 5>

= (15 * -1) + (-3 * 0) + (3 * 5)

= -15 + 0 + 15

= 0

Answer 4E.

Given that

$\mathbf{a} = \mathbf{j} + 7\mathbf{k}$

$\mathbf{b} = 2\mathbf{i} - \mathbf{j} + 4\mathbf{k}$

we need to find the cross product:

$\mathbf{a} \times \mathbf{b} = [(a_2b_3) - (a_3b_2)]i + [(a_3b_1) - (a_1b_3)]j + [(a_1b_2) - (a_2b_1)]k$

$\mathbf{a} \times \mathbf{b} = [(1*4) - (7*-1)]i + [(7*2) - (0*4)]j + [(0*-1) - (1*2)]k$

$\mathbf{a} \times \mathbf{b} = 11i + 14j - 2k$

we need to verify that vectors are orthogonal

so we use $\mathbf{a} \times \mathbf{b} \cdot \mathbf{a} = 0$ and $\mathbf{a} \times \mathbf{b} \cdot \mathbf{b} = 0$.

$(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{a} = \langle 11, 14, -2 \rangle \cdot \langle 0, 1, 7 \rangle$

$= (11 * 0) + (14 * 1) + (-2 * 7)$

$= 0 + 14 - 14 = 0$

$(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{b} = \langle 11, 14, -2 \rangle \cdot \langle 2, -1, 4 \rangle$

$= (11 * 2) + (14 * -1) + (-2 * 4)$

$= 22 - 14 - 8$

$= 0$

so the vectors are orthogonal

Answer 5E.

Consider the two vectors,

$$\mathbf{a} = \mathbf{i} - \mathbf{j} - \mathbf{k} \text{ and } \mathbf{b} = \frac{1}{2}\mathbf{i} + \mathbf{j} + \frac{1}{2}\mathbf{k}.$$

The cross product of $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$ is the vector

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

So, the cross product of $\mathbf{a} = \mathbf{i} - \mathbf{j} - \mathbf{k}$ and $\mathbf{b} = \frac{1}{2}\mathbf{i} + \mathbf{j} + \frac{1}{2}\mathbf{k}$ is the vector

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & -1 \\ \frac{1}{2} & 1 & \frac{1}{2} \end{vmatrix}$$

Evaluate the determinant using expansion by minors

$$\begin{aligned} \mathbf{a} \times \mathbf{b} &= \mathbf{i}\begin{vmatrix} 1 & -1 \\ 1 & \frac{1}{2} \end{vmatrix} - \mathbf{j}\begin{vmatrix} 1 & -1 \\ \frac{1}{2} & \frac{1}{2} \end{vmatrix} + \mathbf{k}\begin{vmatrix} 1 & -1 \\ \frac{1}{2} & -1 \end{vmatrix} \\ &= \mathbf{i}\left(-1\left(\frac{1}{2}\right) - (-1)(1)\right) - \mathbf{j}\left(1\left(\frac{1}{2}\right) - (-1)\left(\frac{1}{2}\right)\right) + \mathbf{k}\left(1(1) - (-1)\left(\frac{1}{2}\right)\right) \\ &= \boxed{\frac{1}{2}\mathbf{i} - \mathbf{j} + \frac{3}{2}\mathbf{k}} \end{aligned}$$

Two vectors $\mathbf{a}$ and $\mathbf{b}$ are orthogonal to each other if and only if their dot product $\mathbf{a} \cdot \mathbf{b}$ is equal to zero.

Compute the dot product $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{a}$, to show that $\mathbf{a} \times \mathbf{b}$ is orthogonal to $\mathbf{a}$.

$$\begin{aligned} (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{a} &= \left\langle \frac{1}{2}, -1, \frac{3}{2} \right\rangle \cdot \langle 1, -1, -1 \rangle \\ &= \left(\frac{1}{2}\right)(1) - 1(-1) - 1\left(\frac{3}{2}\right) \\ &= 0 \end{aligned}$$

Therefore, the vector $\mathbf{a} \times \mathbf{b}$ is orthogonal to $\mathbf{a}$.

Compute the dot product $(\mathbf{a}\times\mathbf{b})\cdot\mathbf{a}$, to show that $\mathbf{a}\times\mathbf{b}$ is orthogonal to $\mathbf{b}$.

$$\begin{aligned}(\mathbf{a}\times\mathbf{b})\cdot\mathbf{b} &= \left\langle \frac{1}{2},-1,\frac{3}{2}\right\rangle\cdot\left\langle \frac{1}{2},1,\frac{1}{2}\right\rangle \\ &= \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)-1(1)+\frac{3}{2}\left(\frac{1}{2}\right) \\ &= 0\end{aligned}$$

Therefore, the vector $\mathbf{a}\times\mathbf{b}$ is orthogonal to $\mathbf{b}$.

Answer 6E.

Consider the following two vectors:

$\mathbf{a} = t\mathbf{i}+\cos t\mathbf{j}+\sin t\mathbf{k}$, and

$\mathbf{b} = \mathbf{i}-\sin t\mathbf{j}+\cos t\mathbf{k}$.

The objective is to find the value of $\mathbf{a}\times\mathbf{b}$. Also verify that it is orthogonal to both $\mathbf{a}$ and $\mathbf{b}$.

The cross product of $\mathbf{a}=a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k}$, and $\mathbf{b}=b_1\mathbf{i}+b_2\mathbf{j}+b_3\mathbf{k}$ is the vector $\mathbf{a}\times\mathbf{b}$, defined by as follows:

$$\mathbf{a}\times\mathbf{b}=(a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k})\times(b_1\mathbf{i}+b_2\mathbf{j}+b_3\mathbf{k})$$

$$=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{vmatrix} \quad \ldots\ldots(1)$$

Then, the cross product of $\mathbf{a} = t\mathbf{i} + \cos t\mathbf{j} + \sin t\mathbf{k}$, and $\mathbf{b} = \mathbf{i} - \sin t\mathbf{j} + \cos t\mathbf{k}$ will be,

$$\mathbf{a} \times \mathbf{b} = (t\mathbf{i} + \cos t\mathbf{j} + \sin t\mathbf{k}) \times (\mathbf{i} - \sin t\mathbf{j} + \cos t\mathbf{k})$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ t & \cos t & \sin t \\ 1 & -\sin t & \cos t \end{vmatrix} \text{ Use (1)}$$

$$= \begin{vmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{vmatrix} \mathbf{i} - \begin{vmatrix} t & \sin t \\ 1 & \cos t \end{vmatrix} \mathbf{j} + \begin{vmatrix} t & \cos t \\ 1 & -\sin t \end{vmatrix} \mathbf{k}$$

$$= \left[(\cos t)(\cos t) - (\sin t)(-\sin t)\right]\mathbf{i} - \left[(t)(\cos t) - (\sin t)(1)\right]\mathbf{j}$$

$$+ \left[(t)(-\sin t) - (\cos t)(1)\right]\mathbf{k}$$

$$= \left[\cos^2 t + \sin^2 t\right]\mathbf{i} - \left[t\cos t - \sin t\right]\mathbf{j} + \left[-t\sin t - \cos t\right]\mathbf{k}$$

$$= 1\mathbf{i} - \left[t\cos t - \sin t\right]\mathbf{j} + \left[-t\sin t - \cos t\right]\mathbf{k}$$

Use the identity; $\cos^2 t + \sin^2 t = 1$

$$= \mathbf{i} + (\sin t - t\cos t)\mathbf{j} - (t\sin t + \cos t)\mathbf{k}.$$

Therefore, the cross product of $\mathbf{a} = t\mathbf{i} + \cos t\mathbf{j} + \sin t\mathbf{k}$, and $\mathbf{b} = \mathbf{i} - \sin t\mathbf{j} + \cos t\mathbf{k}$ is,

$$\mathbf{a} \times \mathbf{b} = \boxed{\mathbf{i} + (\sin t - t\cos t)\mathbf{j} - (t\sin t + \cos t)\mathbf{k}}.$$

To verify that the vector $\mathbf{a}\times\mathbf{b}$ is orthogonal to both $\mathbf{a}$ and $\mathbf{b}$, use the following fact:

"Two vectors **u** and **v** are orthogonal if and only if their dot product, $\mathbf{u}\cdot\mathbf{v}$, is 0, that is, $\mathbf{u}\cdot\mathbf{v}=0$."

To verify that the vector $\mathbf{a}\times\mathbf{b}$ is orthogonal to $\mathbf{a}$, first find their dot product $(\mathbf{a}\times\mathbf{b})\cdot\mathbf{a}$.

The dot product of $\mathbf{a}\times\mathbf{b}=\mathbf{i}+(\sin t-t\cos t)\mathbf{j}-(t\sin t+\cos t)\mathbf{k}$ and $\mathbf{a}=t\mathbf{i}+\cos t\mathbf{j}+\sin t\mathbf{k}$ will be,

$$(\mathbf{a}\times\mathbf{b})\cdot\mathbf{a}=\left(\mathbf{i}+(\sin t-t\cos t)\mathbf{j}-(t\sin t+\cos t)\mathbf{k}\right)\cdot(t\mathbf{i}+\cos t\mathbf{j}+\sin t\mathbf{k})$$

$$=1(t)+(\sin t-t\cos t)(\cos t)-(t\sin t+\cos t)(\sin t)$$

Use $(a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k})\cdot(b_1\mathbf{i}+b_2\mathbf{j}+b_3\mathbf{k})=a_1b_1+a_2b_2+a_3b_3$

$$=t+\sin t\cos t-t\cos^2 t-t\sin^2 t-\sin t\cos t$$

$$=t-t\cos^2 t-t\sin^2 t$$

$$=t-t\left(\cos^2 t+\sin^2 t\right)$$

$$=t-t(1)$$

Use the identity; $\cos^2 t+\sin^2 t=1$

$$=0.$$

Hence, the dot product is $(\mathbf{a}\times\mathbf{b})\cdot\mathbf{a}=0$.

And so, by above fact, the vector $\mathbf{a}\times\mathbf{b}$ is orthogonal to the vector $\mathbf{a}$.

To verify that the vector $\mathbf{a}\times\mathbf{b}$ is orthogonal to **b**, first find their dot product $(\mathbf{a}\times\mathbf{b})\cdot\mathbf{b}$.

The dot product of $\mathbf{a}\times\mathbf{b}=\mathbf{i}+(\sin t-t\cos t)\mathbf{j}-(t\sin t+\cos t)\mathbf{k}$ and $\mathbf{b}=\mathbf{i}-\sin t\mathbf{j}+\cos t\mathbf{k}$ will be,

$$(\mathbf{a}\times\mathbf{b})\cdot\mathbf{b}=(\mathbf{i}+(\sin t-t\cos t)\mathbf{j}-(t\sin t+\cos t)\mathbf{k})\cdot(\mathbf{i}-\sin t\mathbf{j}+\cos t\mathbf{k})$$

$$=1(1)+(\sin t-t\cos t)(-\sin t)-(t\sin t+\cos t)(\cos t)$$

Use $(a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k})\cdot(b_1\mathbf{i}+b_2\mathbf{j}+b_3\mathbf{k})=a_1b_1+a_2b_2+a_3b_3$

$$=1-\sin^2 t+t\sin t\cos t-t\sin t\cos t-\cos^2 t$$

$$=1-\sin^2 t-\cos^2 t$$

$$=1-(\sin^2 t+\cos^2 t)$$

$$=1-1$$

Use the identity; $\cos^2 t+\sin^2 t=1$

$$=0.$$

Hence, the dot product is $(\mathbf{a}\times\mathbf{b})\cdot\mathbf{b}=0$.

And so, by above fact, the vector $\mathbf{a}\times\mathbf{b}$ is orthogonal to the vector **b**.

Therefore, it is verified that, the vector $\mathbf{a}\times\mathbf{b}$ is orthogonal to both **a** and **b**.

Answer 7E.

Consider the following two vectors:

$\mathbf{a} = \left\langle t, 1, \frac{1}{t} \right\rangle$, and

$\mathbf{b} = \left\langle t^2, t^2, 1 \right\rangle$.

The objective is to find the value of $\mathbf{a} \times \mathbf{b}$. Also verify that it is orthogonal to both $\mathbf{a}$ and $\mathbf{b}$.

The cross product of $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$, and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$ is the vector $\mathbf{a} \times \mathbf{b}$, defined by as follows:

$$\mathbf{a} \times \mathbf{b} = \langle a_1, a_2, a_3 \rangle \times \langle b_1, b_2, b_3 \rangle$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \text{. ......(1)}$$

Then, the cross product of $\mathbf{a} = \left\langle t, 1, \frac{1}{t} \right\rangle$, and $\mathbf{b} = \left\langle t^2, t^2, 1 \right\rangle$ will be,

$$\mathbf{a} \times \mathbf{b} = \left\langle t, 1, \frac{1}{t} \right\rangle \times \left\langle t^2, t^2, 1 \right\rangle$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ t & 1 & 1/t \\ t^2 & t^2 & 1 \end{vmatrix} \text{ Use (1)}$$

$$= \begin{vmatrix} 1 & 1/t \\ t^2 & 1 \end{vmatrix} \mathbf{i} - \begin{vmatrix} t & 1/t \\ t^2 & 1 \end{vmatrix} \mathbf{j} + \begin{vmatrix} t & 1 \\ t^2 & t^2 \end{vmatrix} \mathbf{k}$$

$$= \left[(1)(1) - \left(\frac{1}{t}\right)(t^2) \right] \mathbf{i} - \left[(t)(1) - \left(\frac{1}{t}\right)(t^2) \right] \mathbf{j} + \left[(t)(t^2) - (1)(t^2) \right] \mathbf{k}$$

$$= (1-t)\mathbf{i} - (t-t)\mathbf{j} + (t^3 - t^2)\mathbf{k}$$

$$= (1-t)\mathbf{i} - 0\mathbf{j} + (t^3 - t^2)\mathbf{k}$$

$$= (1-t)\mathbf{i} + (t^3 - t^2)\mathbf{k}$$

$$= \langle 1-t, 0, t^3 - t^2 \rangle.$$

Therefore, the cross product of $\mathbf{a} = \left\langle t, 1, \frac{1}{t} \right\rangle$, and $\mathbf{b} = \left\langle t^2, t^2, 1 \right\rangle$ is $\mathbf{a} \times \mathbf{b} = \boxed{(1-t)\mathbf{i} + (t^3 - t^2)\mathbf{k}}$.

To verify that the vector $\mathbf{a}\times\mathbf{b}$ is orthogonal to both $\mathbf{a}$ and $\mathbf{b}$, use the following fact:

“Two vectors **u** and **v** are orthogonal if and only if their dot product, $\mathbf{u}\cdot\mathbf{v}$, is 0, that is, $\mathbf{u}\cdot\mathbf{v}=0$.”

To verify that the vector $\mathbf{a}\times\mathbf{b}$ is orthogonal to $\mathbf{a}$, first find their dot product $(\mathbf{a}\times\mathbf{b})\cdot\mathbf{a}$.

The dot product of $\mathbf{a}\times\mathbf{b}=\langle 1-t,0,t^3-t^2\rangle$ and $\mathbf{a}=\left\langle t,1,\frac{1}{t}\right\rangle$ will be,

$$(\mathbf{a}\times\mathbf{b})\cdot\mathbf{a}=\langle 1-t,0,t^3-t^2\rangle\cdot\left\langle t,1,\frac{1}{t}\right\rangle$$

$$=(1-t)(t)+0(1)+(t^3-t^2)\left(\frac{1}{t}\right)$$

Use $\langle a_1,a_2,a_3\rangle\cdot\langle b_1,b_2,b_3\rangle=a_1b_1+a_2b_2+a_3b_3$

$$=t-t^2+0+t^2-t$$
$$=0.$$

Hence, the dot product is $(\mathbf{a}\times\mathbf{b})\cdot\mathbf{a}=0$.

And so, by above fact, the vector $\mathbf{a}\times\mathbf{b}$ is orthogonal to the vector $\mathbf{a}$.

To verify that the vector $\mathbf{a}\times\mathbf{b}$ is orthogonal to **b**, first find their dot product $(\mathbf{a}\times\mathbf{b})\cdot\mathbf{b}$.

The dot product of $\mathbf{a}\times\mathbf{b}=\langle 1-t,0,t^3-t^2\rangle$ and $\mathbf{b}=\langle t^2,t^2,1\rangle$ will be,

The dot product of $\mathbf{a}\times\mathbf{b}=\langle 1-t,0,t^3-t^2\rangle$ and $\mathbf{b}=\langle t^2,t^2,1\rangle$ will be,

$$(\mathbf{a}\times\mathbf{b})\cdot\mathbf{b}=\langle 1-t,0,t^3-t^2\rangle\cdot\langle t^2,t^2,1\rangle$$

$$=(1-t)(t^2)+0(t^2)+(t^3-t^2)(1)$$

Use $\langle a_1,a_2,a_3\rangle\cdot\langle b_1,b_2,b_3\rangle=a_1b_1+a_2b_2+a_3b_3$

$$=t^2-t^3+0+t^3-t^2$$
$$=0.$$

Hence, the dot product is $(\mathbf{a}\times\mathbf{b})\cdot\mathbf{b}=0$.

And so, by above fact, the vector $\mathbf{a}\times\mathbf{b}$ is orthogonal to the vector **b**.

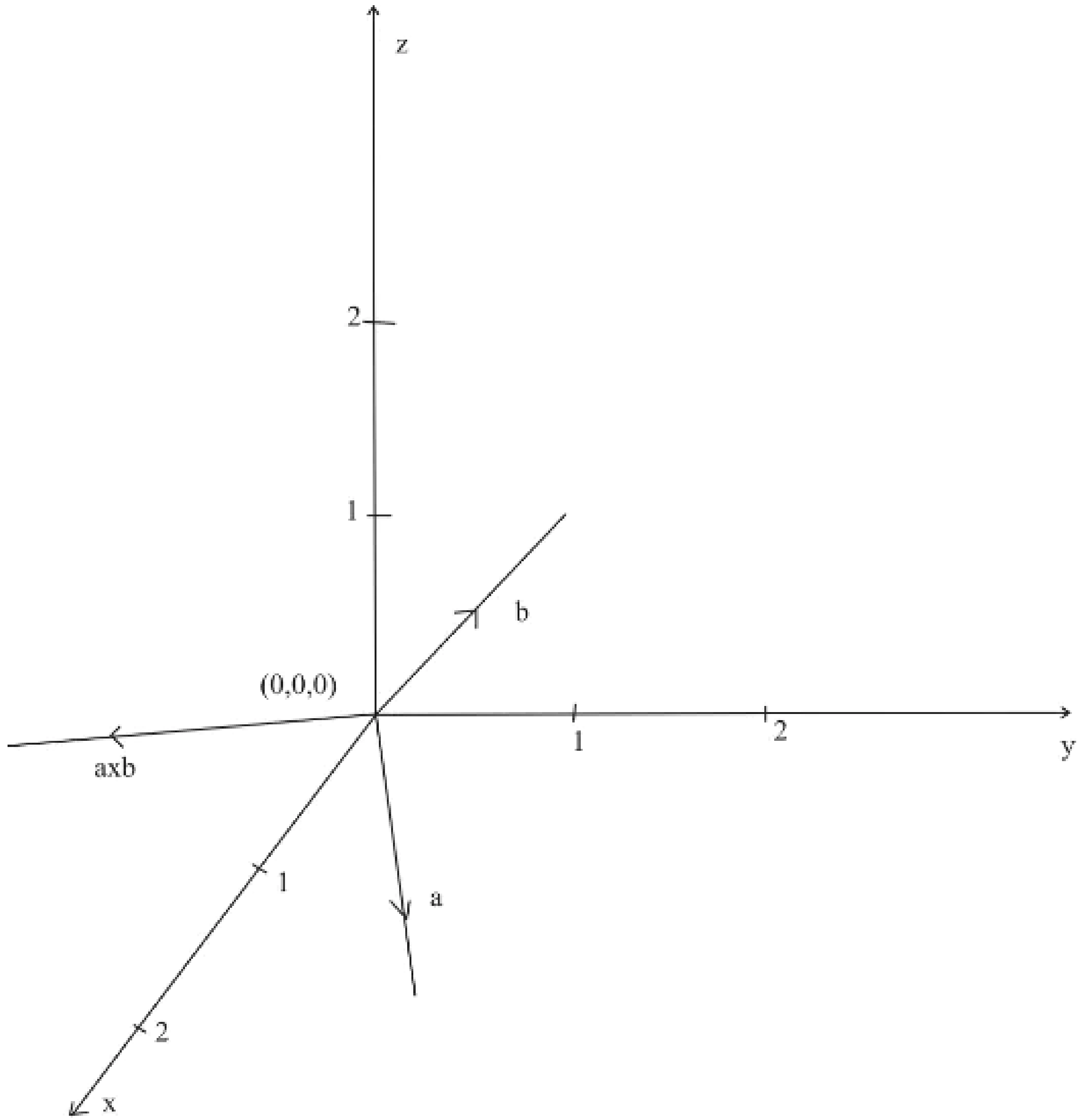
Therefore, it is verified that, the vector $\mathbf{a}\times\mathbf{b}$ is orthogonal to both $\mathbf{a}$ and $\mathbf{b}$.

Answer 8E.

$$\vec{a} = \hat{i} - 2\hat{k}$$
$$\vec{b} = \hat{j} + \hat{k}$$

Then $\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & -2 \\ 0 & 1 & 1 \end{vmatrix}$

$$= \hat{i}(0+2) - \hat{j}(1-0) + \hat{k}(1-0)$$
$$= 2\hat{i} - \hat{j} + \hat{k}$$



Consider the following cross product,

$$(\mathbf{i}\times\mathbf{j})\times\mathbf{k}$$

The object is to find the vector.

Use the following properties of cross product:

$$\mathbf{i}\times\mathbf{j}=\mathbf{k}$$

$$\mathbf{k}\times\mathbf{k}=\mathbf{0}$$

Substitute $\mathbf{i}\times\mathbf{j}=\mathbf{k}$ and $\mathbf{k}\times\mathbf{k}=\mathbf{0}$ in the cross product of $(\mathbf{i}\times\mathbf{j})\times\mathbf{k}$, you get the vector.

$$(\mathbf{i}\times\mathbf{j})\times\mathbf{k}=\mathbf{k}\times\mathbf{k}$$

$$=\mathbf{0}$$

Therefore, the vector is $(\mathbf{i}\times\mathbf{j})\times\mathbf{k}=\boxed{\mathbf{0}}$.

Answer 10E.

Consider the following cross product:

$$\mathbf{k}\times(\mathbf{i}-2\mathbf{j}).$$

Find the vector using properties of cross products.

$$\mathbf{k}\times(\mathbf{i}-2\mathbf{j})=\mathbf{k}\times(\mathbf{i}-\mathbf{j}-\mathbf{j})$$

$$=(\mathbf{k}\times\mathbf{i})-(\mathbf{k}\times\mathbf{j})-(\mathbf{k}\times\mathbf{j})$$

Since $\mathbf{k}\times(a\mathbf{i}+b\mathbf{j}+c\mathbf{k})=a(\mathbf{k}\times\mathbf{i})+b(\mathbf{k}\times\mathbf{j})+c(\mathbf{k}\times\mathbf{k})$

$$=\mathbf{j}-(-\mathbf{i})-(-\mathbf{i})$$ Since $\mathbf{k}\times\mathbf{i}=\mathbf{j},\mathbf{k}\times\mathbf{j}=-\mathbf{i}$

$$=\mathbf{j}+\mathbf{i}+\mathbf{i}$$

$$=\mathbf{j}+2\mathbf{i}$$

Therefore, the required vector is $\boxed{\mathbf{j}+2\mathbf{i}}$.

Answer 11E.

Consider the vector $(\mathbf{j}-\mathbf{k})\times(\mathbf{k}-\mathbf{i})$

The objective is to find the vector by using the properties of cross product.

Now

$$(\mathbf{j}-\mathbf{k})\times(\mathbf{k}-\mathbf{i})=(\mathbf{j}-\mathbf{k})\times\mathbf{k}+(\mathbf{j}-\mathbf{k})\times(-\mathbf{i}) \quad (\text{Since } \mathbf{a}\times(\mathbf{b}+\mathbf{c})=(\mathbf{a}\times\mathbf{b})+(\mathbf{a}\times\mathbf{c}))$$

$$=\mathbf{j}\times\mathbf{k}+(-\mathbf{k})\times\mathbf{k}+\mathbf{j}\times(-\mathbf{i})+(-\mathbf{k})\times(-\mathbf{i})$$

$$(\text{Since } (\mathbf{a}+\mathbf{b})\times\mathbf{c}=(\mathbf{a}\times\mathbf{c})+(\mathbf{b}\times\mathbf{c}))$$

$$=\mathbf{j}\times\mathbf{k}+(-1)(\mathbf{k}\times\mathbf{k})+(-1)(\mathbf{j}\times\mathbf{i})+(\mathbf{k}\times\mathbf{i})$$

$$(\text{Since } (c\mathbf{a})\times\mathbf{b}=c(\mathbf{a}\times\mathbf{b})=\mathbf{a}\times(c\mathbf{b}))$$

$$=\mathbf{i}+\mathbf{j}+\mathbf{k} \qquad \left(\begin{array}{l}\text{Since } \mathbf{j}\times\mathbf{k}=\mathbf{i},\ \ \mathbf{k}\times\mathbf{k}=\mathbf{0},\\ \mathbf{j}\times\mathbf{i}=\mathbf{k} \text{ and } \mathbf{k}\times\mathbf{i}=\mathbf{j}\end{array}\right)$$

Hence, $\boxed{(\mathbf{j}-\mathbf{k})\times(\mathbf{k}-\mathbf{i})=\mathbf{i}+\mathbf{j}+\mathbf{k}}$.

Answer 12E.

Consider the cross product,

$(\mathbf{i}+\mathbf{j})\times(\mathbf{i}-\mathbf{j})$.

Need to find the vector by using the properties of cross product

The cross product of the two vectors is calculated as,

$$(\mathbf{i}+\mathbf{j})\times(\mathbf{i}-\mathbf{j})=(\mathbf{i}+\mathbf{j})\times(\mathbf{i})+(\mathbf{i}+\mathbf{j})\times(-\mathbf{j}) \quad \text{Use } \mathbf{a}\times(\mathbf{b}+\mathbf{c})=(\mathbf{a}\times\mathbf{b})+(\mathbf{a}\times\mathbf{c})$$

$$=\mathbf{i}\times(\mathbf{i})+\mathbf{j}\times(\mathbf{i})+\mathbf{i}\times(-\mathbf{j})+\mathbf{j}\times(-\mathbf{j}) \quad \text{Use } (\mathbf{a}+\mathbf{b})\times\mathbf{c}=(\mathbf{a}\times\mathbf{c})+(\mathbf{b}\times\mathbf{c})$$

$$=(\mathbf{i}\times\mathbf{i})-(\mathbf{i}\times\mathbf{j})-(\mathbf{i}\times\mathbf{j})+(\mathbf{j}\times\mathbf{j}) \quad \text{Use } (c\mathbf{a})\times\mathbf{b}=c(\mathbf{a}\times\mathbf{b})=\mathbf{a}\times(c\mathbf{b})$$

$$=-\mathbf{i}\times\mathbf{j}-\mathbf{i}\times\mathbf{j} \qquad \text{Use } \mathbf{i}\times\mathbf{i}=\mathbf{0} \text{ and } \mathbf{j}\times\mathbf{j}=\mathbf{0}$$

$$=-\mathbf{k}-\mathbf{k}$$

$$=-2\mathbf{k}$$

Hence, $(\mathbf{i}+\mathbf{j})\times(\mathbf{i}-\mathbf{j})=\boxed{-2\mathbf{k}}$.

Answer 13E.

a) Meaningful- The result creates a scalar.

b) Not Meaningful- $b\cdot c$ creates a scalar, and this cannot be crossed with a vector.

c) Meaningful- The result creates a vector.

d) Not Meaningful- $a\cdot b$ creates a scalar, and this cannot be crossed with a vector.

e) Not Meaningful- The scalars produced by $a.b$ and $c\cdot d$ cannot be crossed together.

Answer 14E.

Consider the following figure showing that the two vectors and the angle between them:

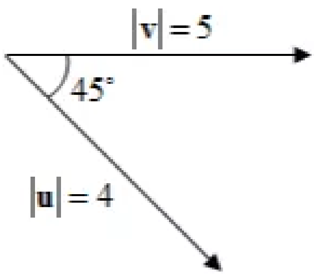


Figure-1

The objective is to find $|\mathbf{u}\times\mathbf{v}|$, and determine whether $\mathbf{u}\times\mathbf{v}$ is directed into the page or out of the page.

From this figure, it is obvious that,

- $|\mathbf{u}|=4$, $|\mathbf{v}|=5$, and
- the angle between **u** and **v** is $\theta=45^\circ$.

Use the following theorem to find the required value:

"If the angle between the vectors $\mathbf{a}$ and $\mathbf{b}$ is θ (where $0\le\theta\le\pi$), then the value of $|\mathbf{a}\times\mathbf{b}|$ is defined by as follows:

$|\mathbf{a}\times\mathbf{b}|=|\mathbf{a}||\mathbf{b}|\sin\theta$." ......(1)

If $|\mathbf{u}|=4$, $|\mathbf{v}|=5$, and $\theta=45^\circ$, then the value of $|\mathbf{u}\times\mathbf{v}|$ will be,

$|\mathbf{u}\times\mathbf{v}|=|\mathbf{u}||\mathbf{v}|\sin\theta$ Use (1)

$=4\cdot5\cdot\sin45^\circ$ Substitute values

$=4\cdot5\cdot\dfrac{\sqrt{2}}{2}$

$=10\sqrt{2}$.

Therefore, the value of $|\mathbf{u}\times\mathbf{v}|$ is $\boxed{10\sqrt{2}}$.

According to the *right-hand rule*, if the fingers of your right hand curl in the direction of rotation from $\mathbf{a}$ to **b**, then your thumb points in the direction of $\mathbf{a}\times\mathbf{b}$.

Hence, by the right hand rule, the vector $\mathbf{u}\times\mathbf{v}$ is directed out of the page.

Answer 15E.

Consider the following figure showing that the two vectors and the angle between them:

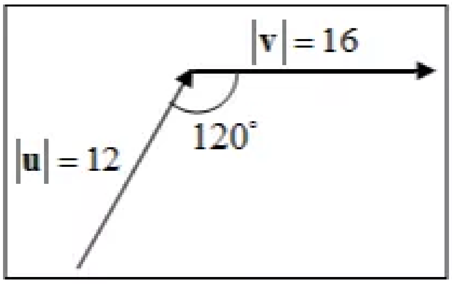


Figure-1

The objective is to find $|\mathbf{u}\times\mathbf{v}|$, and determine whether $\mathbf{u}\times\mathbf{v}$ is directed into the page or out of the page.

First draw the vectors **u** and **v** starting from the same initial point, notice that, the angle between them is $\theta = 60^\circ$.

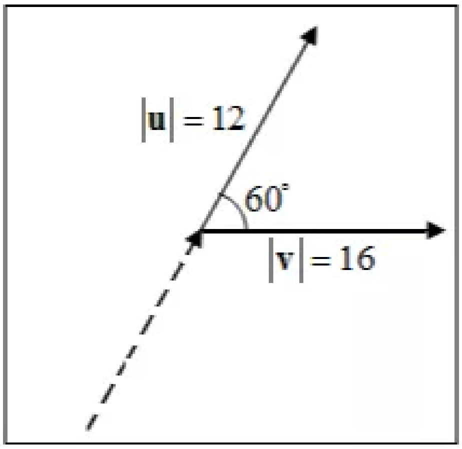


Figure-2

From this figure, it is obvious that,

- $|\mathbf{u}| = 12$, $|\mathbf{v}| = 16$, and
- the angle between **u** and **v** is $\theta = 60^\circ$.

Use the following theorem to find the required value:

"If the angle between the vectors $\mathbf{a}$ and $\mathbf{b}$ is θ (where $0 \le \theta \le \pi$), then the value of $|\mathbf{a} \times \mathbf{b}|$ is defined by as follows:

$|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}|\sin\theta$." ......(1)

If $|\mathbf{u}| = 12$, $|\mathbf{v}| = 16$, and $\theta = 60^\circ$, then the value of $|\mathbf{u} \times \mathbf{v}|$ will be,

$|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}||\mathbf{v}|\sin\theta$ Use (1)

$= 12 \cdot 16 \cdot \sin 60^\circ$ Substitute values

$= 12 \cdot 16 \cdot \frac{\sqrt{3}}{2}$

$= 96\sqrt{3}$.

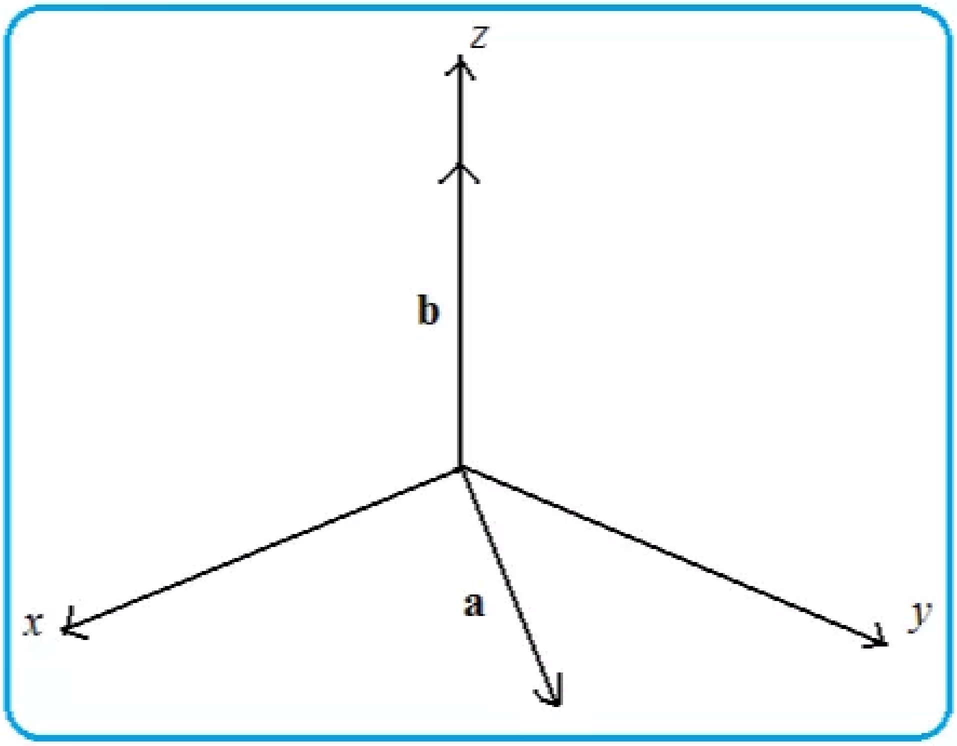
Therefore, the value of $|\mathbf{u} \times \mathbf{v}|$ is $\boxed{96\sqrt{3}}$.

According to the *right-hand rule*, if the fingers of your right hand curl in the direction of rotation from $\mathbf{a}$ to $\mathbf{b}$, then your thumb points in the direction of $\mathbf{a} \times \mathbf{b}$.

Hence, by the right hand rule, the vector $\mathbf{u} \times \mathbf{v}$ is directed into the page.

Answer 16E.

Consider the following figure:



Consider the lengths of the vectors:

$|\mathbf{a}| = 3$ and $|\mathbf{b}| = 2$

(a)

Determine the cross product $|\mathbf{a} \times \mathbf{b}|$.

State the formula for cross product of two vectors as follows:

$$|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}|\sin\theta$$

Use this formula to find $|\mathbf{a} \times \mathbf{b}|$ for the given vectors.

From the figure, notice that the angle between the two vectors is $90°$ as the vector **a** is in *xy*-plane and **b** is in z-direction.

Compute the magnitude of cross product as follows:

$$\begin{aligned}|\mathbf{a} \times \mathbf{b}| &= |\mathbf{a}||\mathbf{b}|\sin\theta \\ &= (3)(2)\sin 90° \\ &= 6(1) \qquad (\text{Since } \sin 90° = 1) \\ &= 6\end{aligned}$$

Therefore, the magnitude of the cross product of vectors **a** and **b** is: $\boxed{|\mathbf{a} \times \mathbf{b}| = 6}$

b)

Use right hand rule to find the components of $\mathbf{a}\times\mathbf{b}$ are positive, negative or zero.

Here, $\mathbf{a}$ is the in the $xy-$ plane, place right hand in the graph so that, the thumb is in the direction of $\mathbf{a}$. Then, it is clear that, the *z* component is zero.

Visualizing the thumb, the thumb is in close enough to the positively oriented *x*-axis. So, the *x*-component is positive. The thumb is away from the positively oriented *y*-axis, so, the *y*-component is negative.

Therefore, using the right hand rule, it is found that the x component will be positive, the z component will be zero and the y component will be negative.

Answer 17E.

Consider the following two vectors:

$\mathbf{a}=\langle 2,-1,3\rangle$, and

$\mathbf{b}=\langle 4,2,1\rangle$.

The objective is to find the value of $\mathbf{a}\times\mathbf{b}$ and $\mathbf{b}\times\mathbf{a}$.

The cross product of $\mathbf{a}=\langle a_1,a_2,a_3\rangle$, and $\mathbf{b}=\langle b_1,b_2,b_3\rangle$ is the vector $\mathbf{a}\times\mathbf{b}$, defined by as follows:

$$\mathbf{a}\times\mathbf{b}=\langle a_1,a_2,a_3\rangle\times\langle b_1,b_2,b_3\rangle$$

$$=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{vmatrix} \text{ ......(1)}$$

Then, the cross product of $\mathbf{a}=\langle 2,-1,3\rangle$, and $\mathbf{b}=\langle 4,2,1\rangle$ will be,

$$\mathbf{a}\times\mathbf{b}=\langle 2,-1,3\rangle\times\langle 4,2,1\rangle$$

$$=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 2 & -1 & 3\\ 4 & 2 & 1\end{vmatrix} \text{ Use (1)}$$

$$=\begin{vmatrix}-1 & 3\\ 2 & 1\end{vmatrix}\mathbf{i}-\begin{vmatrix}2 & 3\\ 4 & 1\end{vmatrix}\mathbf{j}+\begin{vmatrix}2 & -1\\ 4 & 2\end{vmatrix}\mathbf{k}$$

$$=\left[(-1)(1)-(3)(2)\right]\mathbf{i}-\left[(2)(1)-(3)(4)\right]\mathbf{j}+\left[(2)(2)-(-1)(4)\right]\mathbf{k}$$

$$=[-1-6]\mathbf{i}-[2-12]\mathbf{j}+[4-(-4)]\mathbf{k}$$

$$=(-7)\mathbf{i}-(-10)\mathbf{j}+(4+4)\mathbf{k}$$

$$=-7\mathbf{i}+10\mathbf{j}+8\mathbf{k}.$$

Therefore, the cross product of $\mathbf{a}=\langle 2,-1,3\rangle$, and $\mathbf{b}=\langle 4,2,1\rangle$ is $\mathbf{a}\times\mathbf{b}=\boxed{-7\mathbf{i}+10\mathbf{j}+8\mathbf{k}}$.

And, the cross product of $\mathbf{a}=\langle 2,-1,3\rangle$, and $\mathbf{b}=\langle 4,2,1\rangle$ will be,

$$\mathbf{b}\times\mathbf{a}=\langle 4,2,1\rangle\times\langle 2,-1,3\rangle$$

$$=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 4 & 2 & 1\\ 2 & -1 & 3\end{vmatrix} \text{ Use (1)}$$

$$=\begin{vmatrix}2 & 1\\ -1 & 3\end{vmatrix}\mathbf{i}-\begin{vmatrix}4 & 1\\ 2 & 3\end{vmatrix}\mathbf{j}+\begin{vmatrix}4 & 2\\ 2 & -1\end{vmatrix}\mathbf{k}$$

$$=\left[(2)(3)-(1)(-1)\right]\mathbf{i}-\left[(4)(3)-(1)(2)\right]\mathbf{j}+\left[(4)(-1)-(2)(2)\right]\mathbf{k}$$

$$=\left[6-(-1)\right]\mathbf{i}-\left[12-2\right]\mathbf{j}+\left[-4-4\right]\mathbf{k}$$

$$=7\mathbf{i}-10\mathbf{j}-8\mathbf{k}.$$

Therefore, the cross product of $\mathbf{a}=\langle 2,-1,3\rangle$, and $\mathbf{b}=\langle 4,2,1\rangle$ is $\mathbf{b}\times\mathbf{a}=\boxed{7\mathbf{i}-10\mathbf{j}-8\mathbf{k}}$.

Answer 18E.

We know that the vector $\mathbf{a}\times\mathbf{b}$ is orthogonal to $\mathbf{a}$ and $\mathbf{b}$.

If $\mathbf{a}$ equals $a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k}$, and $\mathbf{b}$ equals $b_1\mathbf{i}+b_2\mathbf{j}+b_3\mathbf{k}$, then the cross product of the two vectors $\mathbf{a}$ and $\mathbf{b}$ is given by $\mathbf{a}\times\mathbf{b} = \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{vmatrix}$.

Find $\mathbf{a}\times\mathbf{b}$.

$$\mathbf{a}\times\mathbf{b} = \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 1 & 0 & 1\\ 2 & 1 & -1\end{vmatrix}$$

$$= \begin{vmatrix}0 & 1\\ 1 & -1\end{vmatrix}\mathbf{i} - \begin{vmatrix}1 & 1\\ 2 & -1\end{vmatrix}\mathbf{j} + \begin{vmatrix}1 & 0\\ 2 & 1\end{vmatrix}\mathbf{k}$$

$$= (0-1)\mathbf{i}-(-1-2)\mathbf{j}+(1+0)\mathbf{k}$$

$$= -\mathbf{i}+3\mathbf{j}+\mathbf{k}$$

Now, evaluate $(\mathbf{a}\times\mathbf{b})\times\mathbf{c}$.

$$(\mathbf{a}\times\mathbf{b})\times\mathbf{c} = \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ -1 & 3 & 1\\ 0 & 1 & 3\end{vmatrix}$$

$$= \begin{vmatrix}3 & 1\\ 1 & 3\end{vmatrix}\mathbf{i} - \begin{vmatrix}-1 & 1\\ 0 & 3\end{vmatrix}\mathbf{j} + \begin{vmatrix}-1 & 3\\ 0 & 1\end{vmatrix}\mathbf{k}$$

$$= (9-1)\mathbf{i}-(-3-0)\mathbf{j}+(-1+0)\mathbf{k}$$

$$= 8\mathbf{i}+3\mathbf{j}-\mathbf{k}$$

We thus get $(\mathbf{a}\times\mathbf{b})\times\mathbf{c}$ as $8\mathbf{i}+3\mathbf{j}-\mathbf{k}$.

Now, let us find $\mathbf{b} \times \mathbf{c}$.

$$\mathbf{b} \times \mathbf{c} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & -1 \\ 0 & 1 & 3 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & -1 \\ 1 & 3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 2 & -1 \\ 0 & 3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix}\mathbf{k}$$

$$= (3+1)\mathbf{i} - (6-0)\mathbf{j} + (2-0)\mathbf{k}$$

$$= 4\mathbf{i} - 6\mathbf{j} + 2\mathbf{k}$$

Determine $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$.

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 1 \\ 4 & -6 & 2 \end{vmatrix}$$

$$= \begin{vmatrix} 0 & 1 \\ -6 & 2 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 1 & 1 \\ 4 & 2 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 1 & 0 \\ 4 & -6 \end{vmatrix}\mathbf{k}$$

$$= (0+6)\mathbf{i} - (2-4)\mathbf{j} + (-6-0)\mathbf{k}$$

$$= 6\mathbf{i} + 2\mathbf{j} - 6\mathbf{k}$$

We get $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$ as $6\mathbf{i} + 2\mathbf{j} - 6\mathbf{k}$.

Therefore, we can say that $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) \neq (\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$.

Answer 19E.

Consider the following two vectors:

$\mathbf{a} = \langle 3,2,1 \rangle$, and

$\mathbf{b} = \langle -1,1,0 \rangle$.

The objective is to find two unit vectors orthogonal to both $\mathbf{a} = \langle 3,2,1 \rangle$ and $\mathbf{b} = \langle -1,1,0 \rangle$.

First find a vector that is orthogonal to both $\mathbf{a} = \langle 3,2,1 \rangle$ and $\mathbf{b} = \langle -1,1,0 \rangle$, by using the following theorem:

"The cross product of two vectors is orthogonal to both vectors."

Hence, a vector that is orthogonal to both $\mathbf{a} = \langle 3,2,1 \rangle$ and $\mathbf{b} = \langle -1,1,0 \rangle$, is the cross product of $\mathbf{a}$ and $\mathbf{b}$.

Now, find the value of $\mathbf{a}\times\mathbf{b}$.

The cross product of $\mathbf{a}=\langle a_1,a_2,a_3\rangle$, and $\mathbf{b}=\langle b_1,b_2,b_3\rangle$ is the vector $\mathbf{a}\times\mathbf{b}$, defined by as follows:

$$\mathbf{a}\times\mathbf{b}=\langle a_1,a_2,a_3\rangle\times\langle b_1,b_2,b_3\rangle$$

$$=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{vmatrix}.$$

Then, the cross product of $\mathbf{a}=\langle 3,2,1\rangle$ and $\mathbf{b}=\langle -1,1,0\rangle$ will be,

$$\mathbf{a}\times\mathbf{b}=\langle 3,2,1\rangle\times\langle -1,1,0\rangle$$

$$=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 3 & 2 & 1\\ -1 & 1 & 0\end{vmatrix}$$

$$=\begin{vmatrix}2 & 1\\ 1 & 0\end{vmatrix}\mathbf{i}-\begin{vmatrix}3 & 1\\ -1 & 0\end{vmatrix}\mathbf{j}+\begin{vmatrix}3 & 2\\ -1 & 1\end{vmatrix}\mathbf{k}$$

$$=\left[(2)(0)-(1)(1)\right]\mathbf{i}-\left[(3)(0)-(1)(-1)\right]\mathbf{j}+\left[(3)(1)-(2)(-1)\right]\mathbf{k}$$

$$=\left[0-1\right]\mathbf{i}-\left[0-(-1)\right]\mathbf{j}+\left[3-(-2)\right]\mathbf{k}$$

$$=-1\mathbf{i}-1\mathbf{j}+5\mathbf{k}$$

$$=\langle -1,-1,5\rangle.$$

Hence, a vector that is orthogonal to both $\mathbf{a}=\langle 3,2,1\rangle$ and $\mathbf{b}=\langle -1,1,0\rangle$, is

$$\mathbf{a}\times\mathbf{b}\ =\langle -1,-1,5\rangle.$$

A unit vector that has the same direction as the vector $\mathbf{a}$ is defined by;

$$\mathbf{u}=\frac{1}{|\mathbf{a}|}\mathbf{a}.$$

Then, a unit vector that has the same direction as the vector $\mathbf{a}\times\mathbf{b}$ is defined by;

$$\mathbf{u}=\frac{1}{|\mathbf{a}\times\mathbf{b}|}(\mathbf{a}\times\mathbf{b}). \quad \text{......(1)}$$

Because, the vector $\mathbf{a}\times\mathbf{b}$ is orthogonal to both $\mathbf{a}$ and $\mathbf{b}$, so by (1), the two unit vectors orthogonal to both $\mathbf{a}$ and $\mathbf{b}$, are defined by as follows:

$$\mathbf{u}=\pm\frac{1}{|\mathbf{a}\times\mathbf{b}|}(\mathbf{a}\times\mathbf{b}). \quad \text{......(2)}$$

Next, use (2), and find the two unit vectors orthogonal to both $\mathbf{a}$ and $\mathbf{b}$ as follows:

$$\mathbf{u} = \pm\frac{1}{|\mathbf{a}\times\mathbf{b}|}(\mathbf{a}\times\mathbf{b})$$

$$= \pm\frac{1}{|\langle -1,-1,5\rangle|}\langle -1,-1,5\rangle$$

$$= \pm\frac{1}{\sqrt{(-1)^2+(-1)^2+5^2}}\langle -1,-1,5\rangle \quad \text{Use } |\langle a_1,a_2,a_3\rangle| = \sqrt{a_1^2+a_2^2+a_3^2}$$

$$= \pm\frac{1}{\sqrt{1+1+25}}\langle -1,-1,5\rangle$$

$$= \pm\frac{1}{\sqrt{27}}\langle -1,-1,5\rangle$$

$$= \pm\frac{1}{3\sqrt{3}}\langle -1,-1,5\rangle$$

$$= \pm\left\langle -\frac{1}{3\sqrt{3}},-\frac{1}{3\sqrt{3}},\frac{5}{3\sqrt{3}}\right\rangle, \text{ that is,}$$

$$= \left\langle -\frac{1}{3\sqrt{3}},-\frac{1}{3\sqrt{3}},\frac{5}{3\sqrt{3}}\right\rangle \text{ and } \left\langle \frac{1}{3\sqrt{3}},\frac{1}{3\sqrt{3}},-\frac{5}{3\sqrt{3}}\right\rangle.$$

Therefore, the two unit vectors orthogonal to both $\mathbf{a}=\langle 3,2,1\rangle$ and $\mathbf{b}=\langle -1,1,0\rangle$, are,

$$\left\langle -\frac{1}{3\sqrt{3}},-\frac{1}{3\sqrt{3}},\frac{5}{3\sqrt{3}}\right\rangle \text{ and } \left\langle \frac{1}{3\sqrt{3}},\frac{1}{3\sqrt{3}},-\frac{5}{3\sqrt{3}}\right\rangle.$$

Answer 20E.

We know that the vector $\mathbf{a}\times\mathbf{b}$ is orthogonal to $\mathbf{a}$ and $\mathbf{b}$.

If $\mathbf{a}$ equals $a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k}$, and $\mathbf{b}$ equals $b_1\mathbf{i}+b_2\mathbf{j}+b_3\mathbf{k}$, then the cross product of the two vectors $\mathbf{a}$ and $\mathbf{b}$ is given by $\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$.

Find $\mathbf{a}\times\mathbf{b}$.

$$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & -1 \\ 1 & 1 & 0 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & -1 \\ 1 & 0 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 0 & -1 \\ 1 & 0 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix}\mathbf{k}$$

$$= (0+1)\mathbf{i} - (0+1)\mathbf{j} + (0-1)\mathbf{k}$$

$$= \mathbf{i} - \mathbf{j} - \mathbf{k}$$

Now, let us find $|\mathbf{a}\times\mathbf{b}|$.

$$|\mathbf{a}\times\mathbf{b}| = \sqrt{(1)^2+(-1)^2+(-1)^2}$$
$$= \sqrt{3}$$

Thus, a unit vector orthogonal to **a** on **b** is $\boxed{\frac{\mathbf{i}}{\sqrt{3}}-\frac{\mathbf{j}}{\sqrt{3}}-\frac{\mathbf{k}}{\sqrt{3}}}$.

The unit vector opposite in direction to $\frac{\mathbf{i}}{\sqrt{3}}-\frac{\mathbf{j}}{\sqrt{3}}-\frac{\mathbf{k}}{\sqrt{3}}$ will also be orthogonal to the given vectors.

Therefore, the second vector orthogonal to **a** and **b** is $\boxed{-\frac{\mathbf{i}}{\sqrt{3}}+\frac{\mathbf{j}}{\sqrt{3}}+\frac{\mathbf{k}}{\sqrt{3}}}$.

Answer 21E.

Consider the following vectors in V_3

$\mathbf{0}=\langle 0,0,0\rangle$, and $\mathbf{a}=\langle a_1,a_2,a_3\rangle$

Calculate $\mathbf{0}\times\mathbf{a}$

If $\mathbf{a}=\langle a_1,a_2,a_3\rangle$, and $\mathbf{b}=\langle b_1,b_2,b_3\rangle$, then

$$\mathbf{a}\times\mathbf{b}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{vmatrix}$$
$$=(a_2b_3-a_3b_2)\mathbf{i}-(a_1b_3-a_3b_1)\mathbf{j}+(a_1b_2-a_2b_1)\mathbf{k}$$

Use above definition to calculate $\mathbf{0}\times\mathbf{a}$

$$\mathbf{0}\times\mathbf{a}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 0 & 0 & 0\\ a_1 & a_2 & a_3\end{vmatrix}$$
$$=[(0)(a_3)-(0)(a_2)]\mathbf{i}-[(a_1)(0)-(0)(a_3)]\mathbf{j}+[(0)(a_2)-(0)(a_1)]\mathbf{k}$$
$$=(0-0)\mathbf{i}-(0-0)\mathbf{j}+(0-0)\mathbf{k}$$
$$=0\mathbf{i}-0\mathbf{j}+0\mathbf{k}$$
$$=\mathbf{0}$$

Calculate $\mathbf{a}\times\mathbf{0}$

Use above definition to calculate $\mathbf{a}\times\mathbf{0}$

$$\mathbf{a}\times\mathbf{0}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ a_1 & a_2 & a_3\\ 0 & 0 & 0\end{vmatrix}$$
$$=\left[(a_2)(0)-(a_3)(0)\right]\mathbf{i}-\left[(a_1)(0)-(a_3)(0)\right]\mathbf{j}+\left[(a_1)(0)-(a_2)(0)\right]\mathbf{k}$$
$$=(0-0)\mathbf{i}-(0-0)\mathbf{j}+(0-0)\mathbf{k}$$
$$=0\mathbf{i}-0\mathbf{j}+0\mathbf{k}$$
$$=\mathbf{0}$$

Therefore $\mathbf{0}\times\mathbf{a}=\mathbf{a}\times\mathbf{0}=0$ for all vectors $\mathbf{a}$ in V_3

Answer 22E.

Consider the following vectors in V_3.

$\mathbf{a}=\langle a_1,a_2,a_3\rangle$, and $\mathbf{b}=\langle b_1,b_2,b_3\rangle$

Calculate $\mathbf{a}\times\mathbf{b}$.

If $\mathbf{a}=\langle a_1,a_2,a_3\rangle$, and $\mathbf{b}=\langle b_1,b_2,b_3\rangle$, then

$$\mathbf{a}\times\mathbf{b}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{vmatrix}$$
$$=(a_2b_3-a_3b_2)\mathbf{i}-(a_1b_3-a_3b_1)\mathbf{j}+(a_1b_2-a_2b_1)\mathbf{k}$$

Calculate $(\mathbf{a}\times\mathbf{b})\cdot\mathbf{b}$

$$(\mathbf{a}\times\mathbf{b})\cdot\mathbf{b}=\left[(a_2b_3-a_3b_2)\mathbf{i}-(a_1b_3-a_3b_1)\mathbf{j}+(a_1b_2-a_2b_1)\mathbf{k}\right]\cdot(b_1\mathbf{i}+b_2\mathbf{j}+b_3\mathbf{k})$$
$$=(a_2b_3-a_3b_2)b_1-(a_1b_3-a_3b_1)b_2+(a_1b_2-a_2b_1)b_3$$
$$=a_2b_1b_3-a_3b_1b_2-a_1b_2b_3+a_3b_1b_2+a_1b_2b_3-a_2b_1b_3$$
$$=0$$

Therefore $(\mathbf{a}\times\mathbf{b})\cdot\mathbf{b}=0$ for any vectors $\mathbf{a}$, $\mathbf{b}$ in V_3

Answer 23E.

If $\mathbf{a}$ and $\mathbf{b}$ are two vectors, then prove that $\mathbf{a}\times\mathbf{b}=-(\mathbf{b}\times\mathbf{a})$.

Consider the vectors: $\mathbf{a}=\langle a_1,a_2,a_3\rangle$ and $\mathbf{b}=\langle b_1,b_2,b_3\rangle$.

By the definition of the cross product of two vectors, $\mathbf{a}=\langle a_1,a_2,a_3\rangle$ and $\mathbf{b}=\langle b_1,b_2,b_3\rangle$

$$\begin{aligned}\mathbf{a}\times\mathbf{b}&=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\a_1&a_2&a_3\\b_1&b_2&b_3\end{vmatrix}\\&=\mathbf{i}\begin{vmatrix}a_2&a_3\\b_2&b_3\end{vmatrix}-\mathbf{j}\begin{vmatrix}a_1&a_3\\b_1&b_3\end{vmatrix}+\mathbf{k}\begin{vmatrix}a_1&a_2\\b_1&b_2\end{vmatrix}\\&=(a_2b_3-b_2a_3)\mathbf{i}-(a_1b_3-a_3b_1)\mathbf{j}+(a_1b_2-a_2b_1)\mathbf{k}\end{aligned}$$

Therefore the cross product of the vectors $\mathbf{a}=\langle a_1,a_2,a_3\rangle$ and $\mathbf{b}=\langle b_1,b_2,b_3\rangle$ is

$$\mathbf{a}\times\mathbf{b}=(a_2b_3-b_2a_3)\mathbf{i}-(a_1b_3-a_3b_1)\mathbf{j}+(a_1b_2-a_2b_1)\mathbf{k}.$$

Now, find $\mathbf{b}\times\mathbf{a}$.

$$\begin{aligned}\mathbf{b}\times\mathbf{a}&=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\b_1&b_2&b_3\\a_1&a_2&a_3\end{vmatrix}\\&=\mathbf{i}\begin{vmatrix}b_2&b_3\\a_2&a_3\end{vmatrix}-\mathbf{j}\begin{vmatrix}b_1&b_3\\a_1&a_3\end{vmatrix}+\mathbf{k}\begin{vmatrix}b_1&b_2\\a_1&a_2\end{vmatrix}\\&=(b_2a_3-a_2b_3)\mathbf{i}-(b_1a_3-b_3a_1)\mathbf{j}+(b_1a_2-b_2a_1)\mathbf{k}\end{aligned}$$

Therefore the cross product of the vectors $\mathbf{b}=\langle b_1,b_2,b_3\rangle$ and $\mathbf{a}=\langle a_1,a_2,a_3\rangle$ is

$$\mathbf{b}\times\mathbf{a}=(b_2a_3-a_2b_3)\mathbf{i}-(b_1a_3-b_3a_1)\mathbf{j}+(b_1a_2-b_2a_1)\mathbf{k}.$$

Previously, we find the cross product,

$$\begin{aligned}\mathbf{a}\times\mathbf{b}&=(a_2b_3-b_2a_3)\mathbf{i}-(a_1b_3-a_3b_1)\mathbf{j}+(a_1b_2-a_2b_1)\mathbf{k}\\&=-\left[(b_2a_3-a_2b_3)\mathbf{i}-(a_3b_1-a_1b_3)\mathbf{j}+(a_2b_1-a_1b_2)\mathbf{k}\right]\\&=-(\mathbf{b}\times\mathbf{a})\end{aligned}$$

Hence the result is, $\boxed{\mathbf{a}\times\mathbf{b}=-(\mathbf{b}\times\mathbf{a})}$.

Answer 24E.

To prove $\left(c\vec{a}\right)\times\vec{b}=c\left(\vec{a}\times\vec{b}\right)=\vec{a}\times\left(c\vec{b}\right)$

Where $\vec{a}$ and $\vec{b}$ are vectors and c a scalar

Let $\vec{a}=\langle a_1,a_2,a_3\rangle$

$\vec{b}=\langle b_1,b_2,b_3\rangle$

Consider $\left(c\vec{a}\right)\times\vec{b}=\langle ca_1,ca_2,ca_3\rangle\times\langle b_1,b_2,b_3\rangle$

$=\langle (ca_2)b_3-(ca_3)b_2,(ca_3)b_1-(ca_1)b_3,(ca_1)b_2-(ca_2)b_1\rangle$

...... (*)

$=\langle c(a_2b_3-a_3b_2),c(a_3b_1-a_1b_3),c(a_1b_2-a_2b_1)\rangle$

$=c\langle a_2b_3-a_3b_2,a_3b_1-a_1b_3,a_1b_2-a_2b_1\rangle$

$=c\left(\vec{a}\times\vec{b}\right)$

Also $\left(c\vec{a}\right)\times\vec{b}=\langle (ca_2)b_3-(ca_3)b_2,(ca_3)b_1-(ca_1)b_3,(ca_1)b_2-(ca_2)b_1\rangle$

(from *)

$=\langle a_2(cb_3)-a_3(cb_2),a_3(cb_1)-a_1(cb_3),a_1(cb_2)-a_2(cb_1)\rangle$

$=\langle a_1,a_2,a_3\rangle\times\langle cb_1,cb_2,cb_3\rangle$

$=\vec{a}\times\left(c\vec{b}\right)$

Hence $\left(c\vec{a}\right)\times\vec{b}=c\left(\vec{a}\times\vec{b}\right)=\vec{a}\times\left(c\vec{b}\right)$

If $\vec{a}=\langle a_1,a_2,a_3\rangle,\vec{b}=\langle b_1,b_2,b_3\rangle$ *and*

$\vec{c}=\langle c_1,c_2,c_3\rangle$ are there vectors in V_3

Then show that

$\vec{a}\times\left(\vec{b}+\vec{c}\right)=\vec{a}\times\vec{b}+\vec{a}\times\vec{c}$

$$\vec{b}+\vec{c}=\langle b_1+c_1, b_2+c_2, b_3+c_3\rangle$$

L.H.S

$$\vec{a}\times\left(\vec{b}+\vec{c}\right)=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ a_1 & a_2 & a_3\\ b_1+c_1 & b_2+c_2 & b_3+c_3\end{vmatrix}$$

$$=\left[a_2\left(b_3+c_3\right)-a_3\left(b_2+c_2\right)\right]\hat{i}-\left[a_1\left(b_3+c_3\right)-a_3\left(b_1+c_1\right)\right]\hat{j}$$

$$+\left[a_1\left(b_2+c_2\right)-a_2\left(b_1+c_1\right)\right]\hat{k}$$

$$=\left(a_2b_3-a_3b_2+a_1c_3-a_3c_2\right)\hat{i}-\left(a_1b_3-a_3b_1+a_1c_3-c_3c_1\right)\hat{j}$$

$$+\left(a_1b_2-a_2b_1+a_1c_2-a_2c_1\right)\hat{k}$$

$$=\left(a_2b_3-a_3b_2\right)\hat{i}+\left(a_2c_3-a_3c_2\right)\hat{i}$$

$$-\left(a_1b_3-a_3b_1\right)\hat{j}-\left(a_1c_3-a_3c_1\right)\hat{j}$$

$$+\left(a_1b_2-a_2b_1\right)\hat{k}+\left(a_1c_2-a_2c_1\right)\hat{k}$$

$$=\vec{a}\times\vec{b}+\vec{a}\times\vec{c}$$

Hence $\vec{a}\times\left(\vec{b}+\vec{c}\right)=\vec{a}\times\vec{b}+\vec{a}\times\vec{c}$

Answer 26E.

Prove the property of $(\mathbf{a}+\mathbf{b})\times\mathbf{c}=\mathbf{a}\times\mathbf{c}+\mathbf{b}\times\mathbf{c}.$

Prove the property as follows:

The vectors $\mathbf{a}=a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k}, \mathbf{b}=b_1\mathbf{i}+b_2\mathbf{j}+b_3\mathbf{k},$ and $\mathbf{c}=c_1\mathbf{i}+c_2\mathbf{j}+c_3\mathbf{k}$ are three vectors in V_3.

First, compute the value of $(\mathbf{a}+\mathbf{b})$ as follows:

$$\begin{aligned}\mathbf{a}+\mathbf{b}&=\left(a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k}\right)+\left(b_1\mathbf{i}+b_2\mathbf{j}+b_3\right)\\&=\left(a_1+b_1\right)\mathbf{i}+\left(a_2+b_2\right)\mathbf{j}+\left(a_3+b_3\right)\mathbf{k}\\&=\langle a_1+b_1, a_2+b_2, a_3+b_3\rangle\end{aligned}$$

Next, find the value of $(\mathbf{a}+\mathbf{b})\times\mathbf{c}$ as follows:

$$\begin{aligned}(\mathbf{a}+\mathbf{b})\times\mathbf{c}&=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ a_1+b_1 & a_2+b_2 & a_3+b_3\\ c_1 & c_2 & c_3\end{vmatrix}\\&=\left[c_3\left(a_2+b_2\right)-c_2\left(a_3+b_3\right)\right]\mathbf{i}\\&\quad-\left[c_3\left(a_1+b_1\right)-c_1\left(a_3+b_3\right)\right]\mathbf{j}\\&\quad+\left[c_2\left(a_1+b_1\right)-c_1\left(a_2+b_2\right)\right]\mathbf{k}\end{aligned}$$

Rewrite the above result as follows:

$$\begin{aligned}(\mathbf{a}+\mathbf{b})\times\mathbf{c} &= [a_2c_3+b_2c_3-a_3c_2-b_3c_2]\mathbf{i} \\ &\quad -[a_1c_3+b_1c_3-a_3c_1-b_3c_3]\mathbf{j} \\ &\quad +[a_1c_2+b_1c_2-a_2c_1-a_2b_2]\mathbf{k} \\ &= [a_2c_3+b_2c_3-a_3c_2-b_3c_2]\mathbf{i} \\ &\quad +[-a_1c_3-b_1c_3+a_3c_1+b_3c_3]\mathbf{j} \\ &\quad +[a_1c_2+b_1c_2-a_2c_1-a_2b_2]\mathbf{k}\end{aligned}$$

Continue the above step,

$$\begin{aligned}(\mathbf{a}+\mathbf{b})\times\mathbf{c} &= (a_2c_3-a_3c_2)\mathbf{i}+(a_3c_1-a_1c_3)\mathbf{j}+(a_1c_2-a_2c_1)\mathbf{k} \\ &\quad +(b_2c_3-b_3c_2)\mathbf{i}+(b_3c_1-b_1c)\mathbf{j}+(b_1c_2-b_2c_1)\mathbf{k}\end{aligned}$$

Definition:

If $\mathbf{a}=a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k}$ and $\mathbf{b}=b_1\mathbf{i}+b_2\mathbf{j}+b_3\mathbf{k}$, then the cross product of $\mathbf{a}$ and $\mathbf{b}$ is the vector $\mathbf{a}\times\mathbf{b}=\langle a_2b_3-a_3b_2, a_3b_1-a_1b_3, a_1b_2-a_2b_1\rangle$.

Use the above definition,

$$\begin{aligned}(\mathbf{a}+\mathbf{b})\times\mathbf{c} &= (a_2c_3-a_3c_2)\mathbf{i}+(a_3c_1-a_1c_3)\mathbf{j}+(a_1c_2-a_2c_1)\mathbf{k} \\ &\quad +(b_2c_3-b_3c_2)\mathbf{i}+(b_3c_1-b_1c)\mathbf{j}+(b_1c_2-b_2c_1)\mathbf{k} \\ &= (\mathbf{a}\times\mathbf{c})+(\mathbf{b}\times\mathbf{c})\end{aligned}$$

Therefore,

$$\boxed{(\mathbf{a}+\mathbf{b})\times\mathbf{c}=\mathbf{a}\times\mathbf{c}+\mathbf{b}\times\mathbf{c}}$$

Answer 27E.

$$A(-2,1),\qquad B(0,4),\qquad C(4,2),\qquad D(2,-1)$$

$$\overrightarrow{AB}=(0-(-2),4-1)=(2,3)$$

$$\overrightarrow{BC}=(4-0,2-4)=(4,-2)$$

The area of parallelogram with adjacent sides $\overrightarrow{AB}$ and $\overrightarrow{BC}$ is the length of their cross product $\left|\overrightarrow{AB}\times\overrightarrow{BC}\right|$

Now

$$\left|\overrightarrow{AB}\times\overrightarrow{BC}\right| = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & 0 \\ 4 & -2 & 0 \end{vmatrix}$$

$$= (0-0)\hat{i} - (0-0)\hat{j} + (-4-12)\hat{k}$$

$$= -16\hat{k}$$

$$\left|\overrightarrow{AB}\times\overrightarrow{BC}\right| = \sqrt{(-16)^2}$$

$$= 16$$

Answer 28E.

$K(1,2,3), L(1,3,6), M(3,8,6)\, and\ N(3,7,3)$

$\overrightarrow{KL} = (1-1, 3-2, 6-3) = (0,1,3)$ $\overrightarrow{LM} = (3-1, 8-3, 6-6) = (2,5,0)$

Now

$$\overrightarrow{KL}\times\overrightarrow{LM} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 1 & 3 \\ 2 & 5 & 0 \end{vmatrix}$$

$$= (0-15)\hat{i} - (0-6)\hat{j} + (0-2)\hat{k}$$

$$= -15\hat{i} + 6\hat{j} - 2\hat{k}$$

The area of parallelogram is

$$\left|\overrightarrow{KL}\times\overrightarrow{LM}\right| = \sqrt{(-15)^2 + 6^2 + (-2)^2}$$

$$= \sqrt{225+36+4}$$

$$= \sqrt{265}$$

Answer 29E.

(a) The volume of the parallelepiped determined by the vectors $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$, $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$, and $\mathbf{c} = \langle c_1, c_2, c_3 \rangle$ is given by the scalar triple product $V = |\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|$.

The scalar triple product $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$ is given by $\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$.

Let us consider the sides $\overrightarrow{PQ}$ and $\overrightarrow{PR}$.

$$\begin{aligned} \overrightarrow{PQ} &= (-2-1)\mathbf{i} + (1-0)\mathbf{j} + (3-1)\mathbf{k} \\ &= -3\mathbf{i} + \mathbf{j} + 2\mathbf{k} \\ \overrightarrow{PR} &= (4-1)\mathbf{i} + (2-0)\mathbf{j} + (5-1)\mathbf{k} \\ &= 3\mathbf{i} + 2\mathbf{j} + 4\mathbf{k} \end{aligned}$$

We know that $\overrightarrow{PQ} \times \overrightarrow{PR}$ is orthogonal to both $\overrightarrow{PQ}$ and $\overrightarrow{PR}$.

Find $\overrightarrow{PQ} \times \overrightarrow{PR}$.

$$\begin{aligned} \overrightarrow{PQ} \times \overrightarrow{PR} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -3 & 1 & 2 \\ 3 & 2 & 4 \end{vmatrix} \\ &= \begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix}\mathbf{i} - \begin{vmatrix} -3 & 2 \\ 3 & 4 \end{vmatrix}\mathbf{j} + \begin{vmatrix} -3 & 1 \\ 3 & 2 \end{vmatrix}\mathbf{k} \\ &= (4-4)\mathbf{i} - (-12-6)\mathbf{j} + (-6-3)\mathbf{k} \\ &= 0\mathbf{i} + 18\mathbf{j} - 9\mathbf{k} \\ &= 18\mathbf{j} - 9\mathbf{k} \end{aligned}$$

Thus, $\langle 0, 18, -9 \rangle$ is the vector orthogonal to the plane through the points P, Q, and R.

(b) Find the area of the triangle given by $A = \frac{1}{2}\left|\overrightarrow{PQ} \times \overrightarrow{PR}\right|$.

$$A = \frac{1}{2}\sqrt{0^2 + 18^2 + (-9)^2}$$
$$= \frac{1}{2}\sqrt{0 + 324 + 81}$$
$$= \frac{1}{2}\sqrt{405}$$
$$= \frac{9}{2}\sqrt{5}$$

Thus, the area of the triangle is obtained as $\boxed{\frac{9}{2}\sqrt{5}}$.

Answer 30E.

(a)

Consider the points,

$P(0,0,-3), Q(4,2,0), R(3,3,1)$.

Need to find a non-zero vector orthogonal to the plane through the points $P, Q,$ and R.

Recall that, the vector which is orthogonal to the points $P, Q,$ and R is

$\mathbf{PQ} \times \mathbf{PR}$.

Let $\mathbf{OP} = (0,0,-3), \mathbf{OQ} = (4,2,0), \mathbf{OR} = (3,3,1)$ be the position vectors of the points $P, Q,$ and R.

The sides $\mathbf{PQ}$ and $\mathbf{PR}$ are calculated as,

$$\begin{aligned}\mathbf{PQ} &= \mathbf{OQ} - \mathbf{OP} \\ &= (4,2,0) - (0,0,-3) \\ &= (4,2,3) \\ &= 4\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}\end{aligned}$$

$$\begin{aligned}\mathbf{PR} &= \mathbf{OR} - \mathbf{OP} \\ &= (3,3,1) - (0,0,-3) \\ &= (3,3,4) \\ &= 3\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}\end{aligned}$$

So the orthogonal vector to the plane through the points $P, Q,$ and R is,

$$\begin{aligned}\mathbf{PQ} \times \mathbf{PR} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & 2 & 3 \\ 3 & 3 & 4 \end{vmatrix} \\ &= \mathbf{i}(8-9) - \mathbf{j}(16-9) + \mathbf{k}(12-6) \\ &= -\mathbf{i} - 7\mathbf{j} + 6\mathbf{k}\end{aligned}$$

Hence, the vector $-\mathbf{i} - 7\mathbf{j} + 6\mathbf{k}$ is orthogonal to the plane through the points $P, Q,$ and R.

(b)

Consider the points,

$P(0,0,-3), Q(4,2,0), R(3,3,1)$.

Need to find the area of the triangle through the points $P, Q,$ and R.

Recall that, the area of the triangle through the points $P, Q,$ and R is given by

$$A = \frac{1}{2}|\mathbf{PQ} \times \mathbf{PR}|.$$

From part (a), $\mathbf{PQ} \times \mathbf{PR} = -\mathbf{i} - 7\mathbf{j} + 6\mathbf{k}$.

The modulus of this vector $\mathbf{PQ} \times \mathbf{PR}$ is

$$\begin{aligned} |\mathbf{PQ} \times \mathbf{PR}| &= \sqrt{(-1)^2 + (-7)^2 + 6^2} \\ &= \sqrt{1 + 49 + 36} \\ &= \sqrt{86} \end{aligned}$$

Therefore, the area of the triangle is

$$\begin{aligned} A &= \frac{1}{2}|\mathbf{PQ} \times \mathbf{PR}| \\ &= \boxed{\frac{1}{2}\sqrt{86} \text{ Square units}}. \end{aligned}$$

Answer 31E.

$$P(0,-2,0), \quad Q(4,1,-2), \quad R(5,3,1)$$

$$\begin{aligned} \overrightarrow{PQ} &= (4-0)\hat{i} + (1+2)\hat{j} + (-2-0)\hat{k} \\ &= 4\hat{i} + 3\hat{j} - 2\hat{k} \\ \overrightarrow{PR} &= (5-0)\hat{i} + (3+2)\hat{j} + (1-0)\hat{k} \\ &= 5\hat{i} + 5\hat{j} + \hat{k} \end{aligned}$$

$$\begin{aligned} \overrightarrow{PQ} \times \overrightarrow{PR} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 4 & 3 & -2 \\ 5 & 5 & 1 \end{vmatrix} \\ &= (3+10)\hat{i} - (4+10)\hat{j} + (20-15)\hat{k} \\ &= 13\hat{i} - 14\hat{j} + 5\hat{k} \end{aligned}$$

Since the vector $\overrightarrow{PQ}\times\overrightarrow{PR}$ is perpendicular to both $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ and is therefore perpendicular to the plane through P, Q and R.

So the vector <13,-14, 5> is perpendicular to the given plane. Now area of parallelogram is

$$\left|\overrightarrow{PQ}\times\overrightarrow{PR}\right| = \sqrt{13^2+(-14)^2+5^2}$$
$$= \sqrt{169+196+25}$$
$$= \sqrt{390}$$

The area of triangle PQR is half the area of this parallelogram, That is

$$= \frac{1}{2}\sqrt{390}$$

Answer 32E.

We are given that three points

$P(-1, 3, 1)$; $Q(0, 5, 2)$ and $R(4, 3, -1)$

(a)

We know that the vector $\overrightarrow{PQ}\times\overrightarrow{PR}$ is perpendicular to both $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ and is therefore orthogonal to the plane through P, Q and R

We know that

$$\overrightarrow{PQ} = (0+1)i + (5-3)j + (2-1)k$$
$$= i+2j+k$$
$$\Rightarrow \overrightarrow{PQ} = i+2j+k$$

And

$$\overrightarrow{PR} = (4+1)i + (3-3)j + (-1-1)k$$
$$= 5i+0j-2k$$
$$\Rightarrow \overrightarrow{PR} = 5i+0j-2k$$

We compute the cross vector of these vectors

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} i & j & k \\ 1 & 2 & 1 \\ 5 & 0 & -2 \end{vmatrix}$$

$$= i(-4) + (5+2)j + (0-10)k$$

$$= -4i + 7j - 10k$$

$$\Rightarrow \overrightarrow{PQ} \times \overrightarrow{PR} = -4i + 7j - 10k$$

So, the vector $\langle -4, 7, -10 \rangle$ is orthogonal to the plane through the points

P, Q and R

Answer 33E.

The volume V of a parallelepiped with vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ as adjacent edges is given by $V = |\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|$.

For $\mathbf{a}$ equals $a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$, $\mathbf{b}$ equals $b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$, and $\mathbf{c}$ equals $c_1\mathbf{i} + c_2\mathbf{j} + c_3\mathbf{k}$, the value of the triple scalar product is given by

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}.$$

Substitute the known values in the equation to determine $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$.

$$\begin{aligned} \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) &= \begin{vmatrix} 1 & 2 & 3 \\ -1 & 1 & 2 \\ 2 & 1 & 4 \end{vmatrix} \\ &= 1(4-2) - 2(-4-4) + 3(-1-2) \\ &= 2 + 16 - 9 \\ &= 9 \end{aligned}$$

Therefore, the volume of the parallelepiped is $\boxed{9 \text{ cubic units}}$.

The volume V of a parallelepiped with vectors **a**, **b** and **c** as adjacent edges is given by $V = |\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|$.

For **a** equals $a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$, **b** equals $b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$, and **c** equals $c_1\mathbf{i} + c_2\mathbf{j} + c_3\mathbf{k}$, the value of the triple scalar product is given by

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}.$$

Substitute the known values in the equation to determine $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$.

$$\begin{aligned} \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) &= \begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{vmatrix} \\ &= 1(1-1) - 1(0-1) + 0(0-1) \\ &= 0 + 1 + 0 \\ &= 1 \end{aligned}$$

Therefore, the volume of the parallelepiped is $\boxed{1 \text{ cubic unit}}$.

Answer 35E.

The volume of the parallelepiped determined by the vectors $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$, $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$, and $\mathbf{c} = \langle c_1, c_2, c_3 \rangle$ is given by the scalar triple product $V = |\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|$.

The scalar triple product $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$ is given by $\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$.

Consider any three adjacent sides of the parallelepiped.

Let us consider the sides, $\overrightarrow{PQ}$, $\overrightarrow{PR}$, and $\overrightarrow{PS}$.

$$\overrightarrow{PQ} = \langle 4, 2, 2 \rangle$$
$$\overrightarrow{PR} = \langle 3, 3, -1 \rangle$$
$$\overrightarrow{PS} = \langle 5, 5, 1 \rangle$$

Find $\overrightarrow{PQ} \cdot \left(\overrightarrow{PR} \times \overrightarrow{PS}\right)$.

$$\overrightarrow{PQ} \cdot \left(\overrightarrow{PR} \times \overrightarrow{PS}\right) = \begin{vmatrix} 4 & 2 & 2 \\ 3 & 3 & -1 \\ 5 & 5 & 1 \end{vmatrix}$$

$$= 4\begin{vmatrix} 3 & -1 \\ 5 & 1 \end{vmatrix} - 2\begin{vmatrix} 3 & -1 \\ 5 & 1 \end{vmatrix} + 2\begin{vmatrix} 3 & 3 \\ 5 & 5 \end{vmatrix}$$

$$= 4(3+5) - 2(3+5) + 2(15-15)$$

$$= 32 - 16$$

$$= 16$$

Therefore, the volume of the parallelepiped is $\boxed{16}$.

Answer 36E.

Consider the following points:

$$P(3,0,1), Q(-1,2,5), R(5,1,-1), S(0,4,2)$$

Consider any three adjacent sides of the parallelepiped.

Let us consider the sides, $\overline{PQ}$, $\overline{PR}$, and $\overline{PS}$

Find the volume of the parallelepiped with adjacent edges $PQ, PR,$ and PS as shown below:

The volume of the parallelepiped is determined by the vectors, $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$,

$\mathbf{b} = \langle b_1, b_2, b_3 \rangle$, and $\mathbf{c} = \langle c_1, c_2, c_3 \rangle$, and is given by the scalar triple product

$V = \left|\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})\right|$.

The scalar triple product $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$ is given by $\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$.

Let us consider the sides, $\overline{PQ}$, $\overline{PR}$, and $\overline{PS}$.

$$P(3,0,1), Q(-1,2,5), R(5,1,-1), S(0,4,2)$$

$$\begin{aligned}\overline{PQ} &= \langle x_2-x_1,\ y_2-y_1,\ z_2-z_1\rangle \\ &= \langle -1-3,\ 2-0,\ 5-1\rangle \\ &= \langle -4,\ 2,\ 4\rangle\end{aligned}$$

$$\begin{aligned}\overline{PR} &= \langle x_2-x_1,\ y_2-y_1,\ z_2-z_1\rangle \\ &= \langle 5-3,\ 1-0,\ -1-1\rangle \\ &= \langle 2,\ 1,\ -2\rangle\end{aligned}$$

$$\begin{aligned}\overline{PS} &= \langle x_2-x_1,\ y_2-y_1,\ z_2-z_1\rangle \\ &= \langle 0-3,\ 4-0,\ 2-1\rangle \\ &= \langle -3,\ 4,\ 1\rangle\end{aligned}$$

Compute $\overline{PQ}\cdot\left(\overline{PR}\times\overline{PS}\right)$ as shown below:

$$\begin{aligned}\overline{PQ}\cdot\left(\overline{PR}\times\overline{PS}\right) &= \begin{vmatrix}-4 & 2 & 4\\ 2 & 1 & -2\\ -3 & 4 & 1\end{vmatrix} \\ &= -4\begin{vmatrix}1 & -2\\ 4 & 1\end{vmatrix} - 2\begin{vmatrix}2 & -2\\ -3 & 1\end{vmatrix} + 4\begin{vmatrix}2 & 1\\ -3 & 4\end{vmatrix} \\ &= -4(1+8) - 2(2-6) + 4(8+3) \\ &= -36+8+44 \\ &= 16\end{aligned}$$

Therefore, the volume of the parallelepiped with adjacent edges is $\boxed{16}$.

Answer 37E.

Consider the vectors,

$\mathbf{u} = \mathbf{i} + 5\mathbf{j} - 2\mathbf{k}$, $\mathbf{v} = 3\mathbf{i} - \mathbf{j}$ and $\mathbf{w} = 5\mathbf{i} + 9\mathbf{j} - 4\mathbf{k}$

Write the scalar triple product equation.

$$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = \begin{vmatrix}a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\\ c_1 & c_2 & c_3\end{vmatrix}$$

Magnitude of the scalar triple product $|\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})|$ is the volume of the parallelepiped formed by the vectors, $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$.

Evaluate scalar triple product for the given vectors.

$$\mathbf{u}\times(\mathbf{v}\times\mathbf{w})=\begin{vmatrix}1 & 5 & -2\\ 3 & -1 & 0\\ 5 & 9 & -4\end{vmatrix}$$

$$=1(4-0)-5(-12-0)-2(27+5)$$

$$=1(4)-5(-12)-2(32)$$

$$=4+60-64$$

$$=0$$

Volume of the parallelepiped is,

$$|\mathbf{u}\times(\mathbf{v}\times\mathbf{w})|=|0|$$

$$=0$$

Volume of the parallelepiped formed by the vectors $\mathbf{u},\mathbf{v}$ and $\mathbf{w}$ is zero.

That means they lie on the same plane.

Thus the vectors are co-planar.

Answer 38E.

Consider the following four points:

$A(1,3,1), B(3,-1,6), C(5,2,0)$ and $D(3,6,-4)$

The objective is to test whether the points lie in the same plane.

Consider the following four points:

$A(1,3,1), B(3,-1,6), C(5,2,0)$ and $D(3,6,-4)$

The objective is to test whether the points lie in the same plane.

The scalar triple product is,

$$\overline{AB}\cdot(\overline{AC}\times\overline{AD})=\begin{vmatrix}2 & -4 & 4\\ 4 & -1 & -2\\ 2 & 3 & -6\end{vmatrix}$$

$$=2(6+6)+4(-24+4)+4(12+2)$$

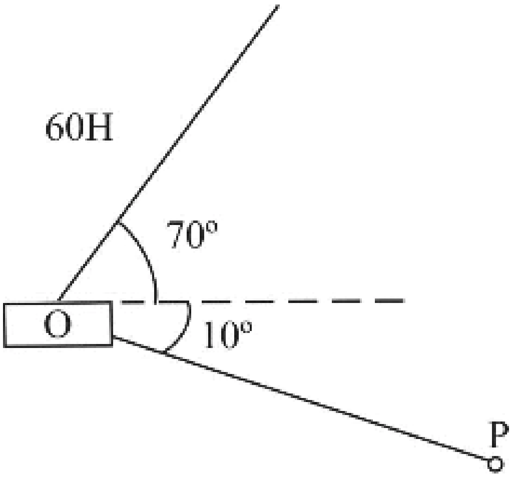
$$=24-80+56$$

$$=\boxed{0}$$

Hence, the given points are **Coplanar.**

In other words the points lie on the same plane.

Answer 39E.



Force $\left|\vec{F}\right| = 60\text{ N}$

Shaft of pedal $\left|\vec{r}\right| = 18\text{cm}$

$= 0.18\text{m}$

Angle between position of shaft and force vector, is $\theta = 80^o$

Then the torque $\left|\vec{\tau}\right| = \left|\vec{r}\times\vec{F}\right|$

$= \left|\vec{r}\right|\left|\vec{F}\right|\sin\theta$

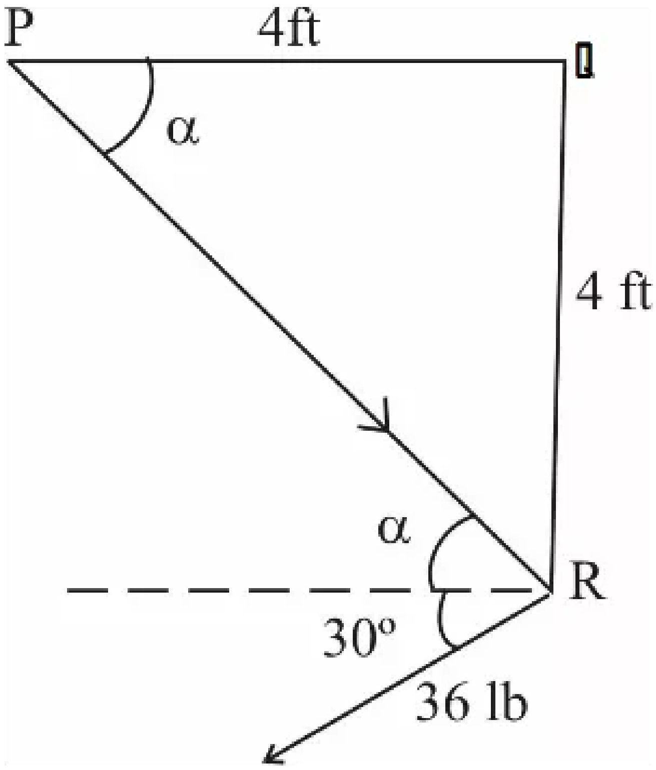
$= \left((0.18)(60)\sin 80^0\right)\text{Nm}$

$= 10.8\sin 80^o J$

$= 10.6J$

Hence magnitude of torque about P is $\boxed{10.6\text{J}}$

Answer 40E.



Let the force is applied at point R

Now $PQ = QR = 4\text{ft}$

Then by Pythagoras theorem,

$$PR = \sqrt{4^2 + 4^2}\ ft$$
$$= \sqrt{32}\ ft$$
$$= 4\sqrt{2}\ ft$$

Then $\tan\alpha = \frac{4}{4} = 1$

$$\Rightarrow \alpha = 45^o$$

Therefore angle between force vector and the position vector

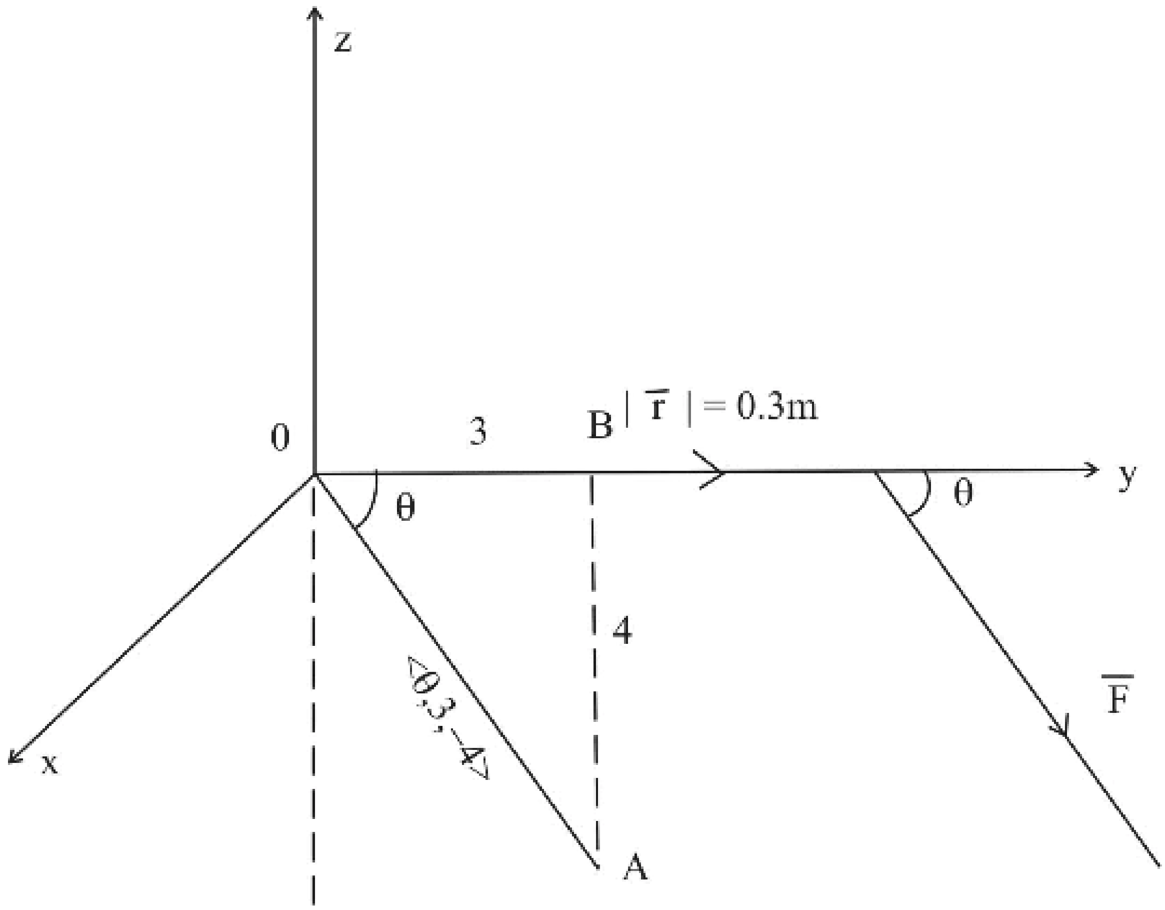
$\overrightarrow{PR}$ is $180^o - (30^o + 45^o) = 105^o$

Hence the magnitude of torque is

$$|\vec{\tau}| = |\overrightarrow{PR} \times \vec{F}|$$
$$= |\overrightarrow{PR}||\vec{F}|\sin 105^o$$
$$= 4\sqrt{2}(36)(0.965)\ ft - lb$$
$$= 196.7\ ft - lb$$

Hence $\boxed{|\vec{\tau}| = 196.7\ ft - lb}$.

Answer 41E.



Let $\vec{r}$ is the position vector of the end of wrench where the force is applied. Then $|\vec{r}| = 0.3m$.

Now the force $\vec{F}$ is parallel to $\langle 0,3,-4 \rangle$

Let $\overrightarrow{OA} = \langle 0, 3, -4 \rangle$, consider right angled triangle OBA

where OB = 3 and AB = 4

Then $\tan\theta = \frac{4}{3} = 1.33$

Then $\theta = \tan^{-1}(1.33)$

i.e. $\theta = 53.13^o$

Therefore the angle between $\vec{r}$ *and* $\vec{F} = 53.13^o$.

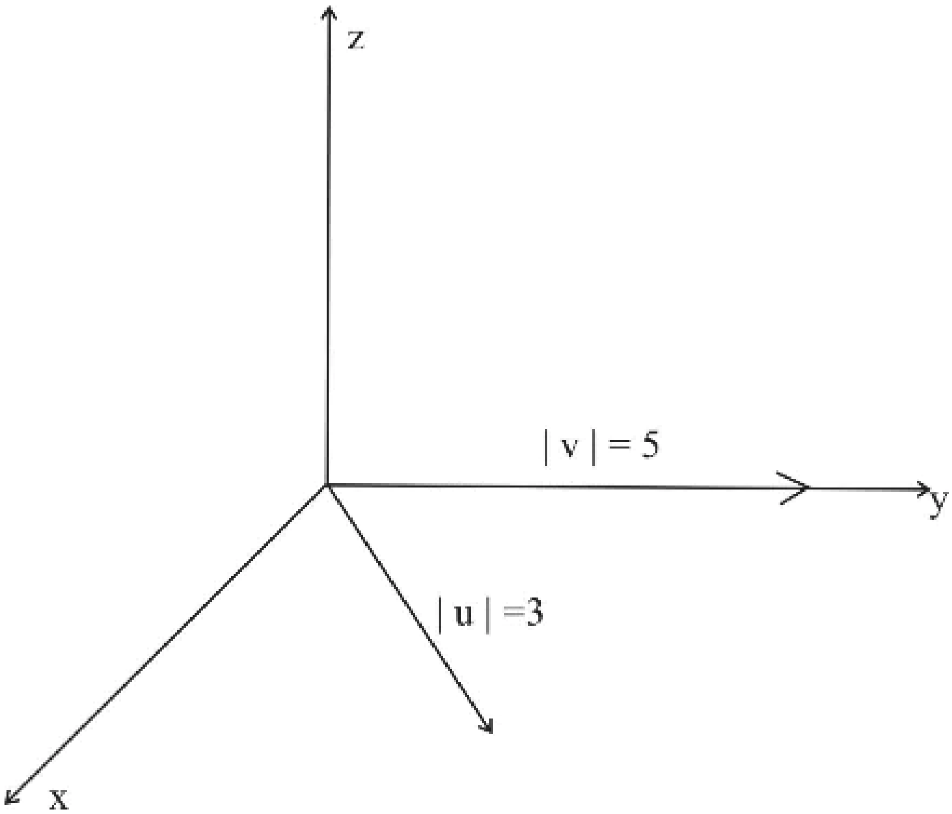
Now $|\vec{\tau}| = |\vec{r}||\vec{F}|\sin\theta$

i.e. $100J = (0.3m)|\vec{F}|\sin(53.13^o)$

Then $|\vec{F}| = \frac{100}{(0.3)(0.8)} N$

i.e. $\boxed{|\vec{F}| \approx 417N}$

Answer 42E.



Now $\vec{v} = 5\hat{j}$ *Then* $|\vec{v}| = 5$

and $|\vec{u}| = 3$

Then length of $\vec{u} \times \vec{v}$ is

$$|\vec{u} \times \vec{v}| = |\vec{u}||\vec{v}|\sin\theta$$

where θ is the angle between $\vec{u}$ and $\vec{v}$.

$$\Rightarrow |\vec{u} \times \vec{v}| = 15\sin\theta$$

Now $\left|\vec{u}\times\vec{v}\right|$ will be maximum and minimum depending upon sin θ. Since $0\le\theta\le 2\pi$. Then $[\sin\theta]\max = 1$ and $(\sin\theta)\min = 0$.

Hence $\left|\vec{u}\times\vec{v}\right|\max = 15$

And $\left|\vec{u}\times\vec{v}\right|\min = 0$

Also $\vec{u}\times\vec{v}$ point in the direction of positive z-axis.

Answer 44E.

(a)

Let $\mathbf{v}$ be a vector defined by $\langle v_1, v_2, v_3\rangle$. Then $\langle 1, 2, 1\rangle\times\langle v_1, v_2, v_3\rangle = \langle 3, 1, -5\rangle$.

The objective is to find all the vectors, $\mathbf{v}$.

Simplify the equation and find the expressions for v_1, v_2, v_3.

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 1 \\ v_1 & v_2 & v_3 \end{vmatrix} = \langle 3, 1, -5\rangle$$

$$(2v_3 - v_2)\mathbf{i} - (v_3 - v_1)\mathbf{j} + (v_2 - 2v_1)\mathbf{k} = \langle 3, 1, -5\rangle$$

Equate the corresponding coordinates.

$$\begin{aligned} 2v_3 - v_2 &= 3 \\ v_1 - v_3 &= 1 \\ v_2 - 2v_1 &= -5 \end{aligned}$$

Solve the equations.

Multiply $v_1 - v_3 = 1$ by 2 and add it to $v_2 - 2v_1 = -5$

$$\begin{aligned} v_2 - 2v_1 &= -5 \\ \underline{2v_1 - 2v_3} &\underline{= 2} \\ v_2 - 2v_3 &= -3 \end{aligned}$$

It is similar to the equation $2v_3 - v_2 = 3$.

Let $v_3 = k$, where k is an arbitrary constant.

$$\begin{aligned} v_1 &= 1 + v_3 \\ &= 1 + k \end{aligned}$$

And

$$\begin{aligned} v_2 &= 2v_1 - 5 \\ &= 2(1+k) - 5 \\ &= 2k - 3 \end{aligned}$$

(b)

Suppose the vector $\mathbf{v}$ satisfies $\langle 1, 2, 1\rangle \times \langle v_1, v_2, v_3\rangle = \langle 3, 1, 5\rangle$.

The objective is to find all the vectors, $\mathbf{v}$.

Rewrite the equation, $\langle 1, 2, 1\rangle \times \langle v_1, v_2, v_3\rangle = \langle 3, 1, 5\rangle$ as follows,

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 1 \\ v_1 & v_2 & v_3 \end{vmatrix} = \langle 3, 1, 5\rangle$$

$$(2v_3 - v_2)\mathbf{i} - (v_3 - v_1)\mathbf{j} + (v_2 - 2v_1)\mathbf{k} = \langle 3, 1, 5\rangle$$

Equate the corresponding coordinates.

$$2v_3 - v_2 = 3$$
$$v_1 - v_3 = 1$$
$$v_2 - 2v_1 = 5$$

Solve the equations.

Multiply $v_1 - v_3 = 1$ by 2 and add it to $v_2 - 2v_1 = 5$

$$v_2 - 2v_1 = 5$$
$$\underline{2v_1 - 2v_3 = 2}$$
$$v_2 - 2v_3 = 7$$

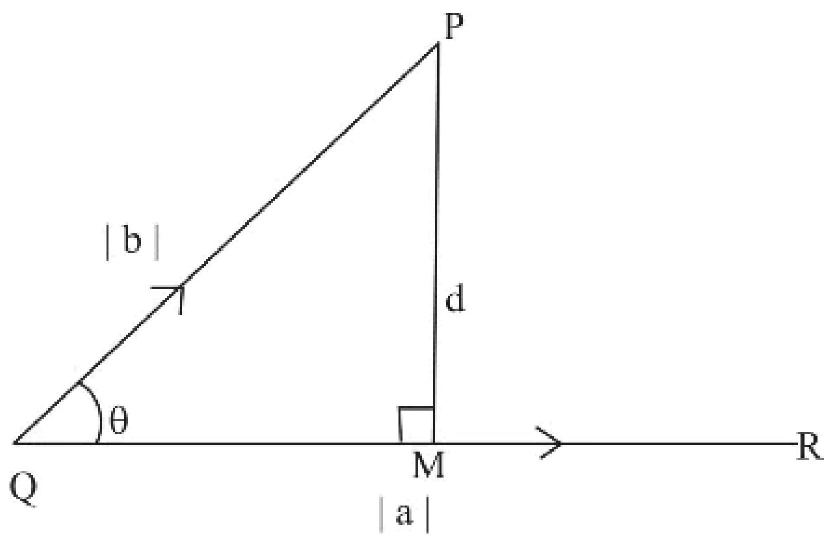
Or $2v_3 - v_2 = -7$

This is not possible, because $2v_3 - v_2 = 3$.

In other words, the system is inconsistent.

Therefore, there is no $\mathbf{v}$ such that $\langle 1, 2, 1\rangle \times \mathbf{v} = \langle 3, 1, 5\rangle$.

Answer 45E.



Let θ be the angle between $\vec{a}$ and $\vec{b}$.
Now in right angled ΔPMQ,

$$\sin\theta = \frac{d}{|\vec{b}|} \qquad ---(1)$$

Also we know $\quad |\vec{a}\times\vec{b}| = |\vec{a}||\vec{b}|\sin\theta$

i.e. $$\sin\theta = \frac{|\vec{a}\times\vec{b}|}{|\vec{a}||\vec{b}|} \qquad ---(2)$$

Form (1) *and* (2)

$$\frac{d}{|\vec{b}|} = \frac{|\vec{a}\times\vec{b}|}{|\vec{a}||\vec{b}|}$$

i.e. $$\boxed{d = \frac{|\vec{a}\times\vec{b}|}{|\vec{a}|}}$$

(B)

$$P \equiv (1,1,1), \qquad Q \equiv (0,6,8), \qquad R \equiv (-1,4,7)$$

Then $\overrightarrow{PQ} = \langle -1,5,7\rangle$

$\overrightarrow{QR} = \langle -1,-2,-1\rangle$

Then $\vec{a} = \overrightarrow{QR} = \langle -1,-2,-1\rangle$

$\vec{b} = \overrightarrow{QP} = \langle 1,-5,-7\rangle$

Then $|\vec{a}| = \sqrt{1+4+1} = \sqrt{6}$

$|\vec{b}| = \sqrt{1+25+49} = \sqrt{75} = 5\sqrt{3}$

Now $\vec{a}\times\vec{b} = \langle 14-5,-7-1,5+2\rangle$

$= \langle 9,-8,7\rangle$

Then $|\vec{a}\times\vec{b}| = \sqrt{81+64+49}$

$= \sqrt{194}$

Therefore distance of point P from the line passing through Q and R is

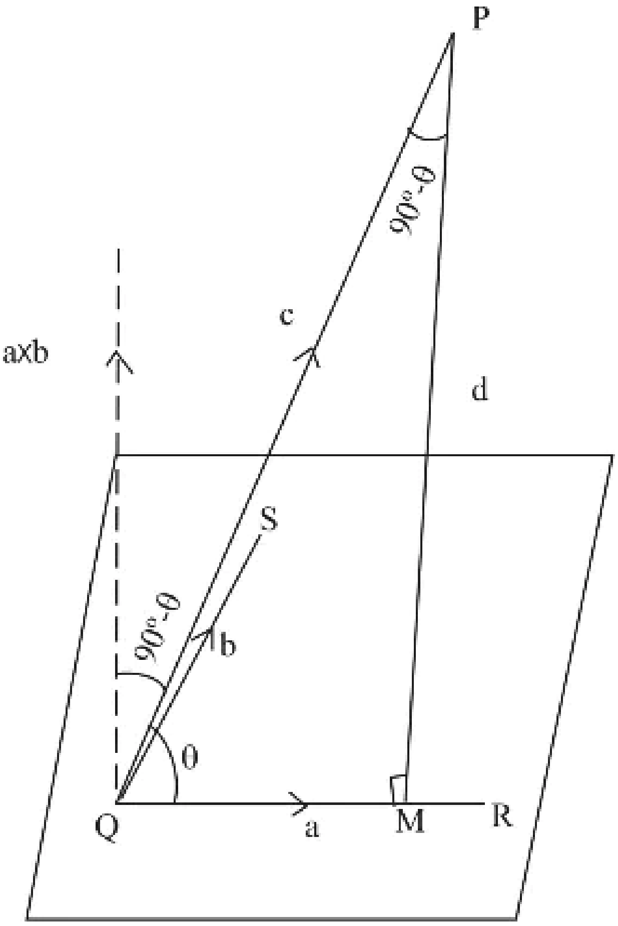
$$= \frac{\sqrt{194}}{\sqrt{6}}$$

$$= \frac{\sqrt{2\times 97}}{\sqrt{2\times 3}}$$

$$= \sqrt{\frac{97}{3}}$$

Hence $\boxed{d = \sqrt{\frac{97}{3}}}$

Answer 46E.



Now $\overrightarrow{QR} = \vec{a}$, $\overrightarrow{QS} = \vec{b}$, $\overrightarrow{QP} = \vec{c}$

Let θ be the angle between $\vec{a}$ and $\vec{c}$

Let the perpendicular from P meets the plane through S,Q and R in M. Then PM =d .

Now in right angled ΔPMQ,

$$\sin\theta = \frac{PM}{QP}$$

$$= \frac{d}{|\vec{c}|} \quad \ldots\ldots\ldots\ldots(1)$$

Now $\vec{a}\times\vec{b}$ is perpendicular to the given plane and is in upwards direction. This gives that $\vec{a}\times\vec{b}$ is parallel to $\overrightarrow{MP}$. Then angle between $\vec{a}\times\vec{b}$ *and* $\vec{c}$ same as the angle between $\overrightarrow{MP}$ *and* c which is $90^o-\theta$.

Then $\left(\vec{a}\times\vec{b}\right).\vec{c}=\left|\vec{a}\times\vec{b}\right|\left|\vec{c}\right|\cos\left(90^o-\theta\right)$

$$=\left|\vec{a}\times\vec{b}\right|\left|\vec{c}\right|\sin\theta$$

$$\Rightarrow \sin\theta=\frac{\left(\vec{a}\times\vec{b}\right).\vec{c}}{\left|\vec{a}\times\vec{b}\right|\left|\vec{c}\right|} \qquad ---(2)$$

From (1) and (2),

$$\left|\frac{d}{\vec{c}}\right|=\frac{\left(\vec{a}\times\vec{b}\right).\vec{c}}{\left|\vec{a}\times\vec{b}\right|\left|\vec{c}\right|}$$

$$\Rightarrow d=\frac{\left(\vec{a}\times\vec{b}\right).\vec{c}}{\left|\vec{a}\times\vec{b}\right|}$$

Since d is the distance and cannot be negative, then

$$\boxed{d=\frac{\left|\left(\vec{a}\times\vec{b}\right).\vec{c}\right|}{\left|\vec{a}\times\vec{b}\right|}}$$

(B)

$$P\equiv(2,1,4),\quad Q\equiv(1,0,0),\quad R\equiv(0,2,0),\quad S\equiv(0,0,3)$$

$$\vec{a}=\overrightarrow{QR}=\langle -1,2,0\rangle$$

$$\vec{b}=\overrightarrow{QS}=\langle -1,0,3\rangle$$

$$\vec{c}=\overrightarrow{QP}=\langle 1,1,4\rangle$$

Then

$$d=\frac{\left|\left(\vec{a}\times\vec{b}\right).\vec{c}\right|}{\left|\vec{a}\times\vec{b}\right|}$$

$$\left(\vec{a}\times\vec{b}\right).\vec{c}=\begin{vmatrix}1 & 1 & 4\\ -1 & 2 & 0\\ -1 & 0 & 3\end{vmatrix}$$

$$=1(6)-1(-3)+4(2)$$

$$=6+3+8=17$$

And $\vec{a}\times\vec{b}=\langle 6,3,2\rangle$

Then $\left|\vec{a}\times\vec{b}\right|=\sqrt{36+9+4}$

$$=\sqrt{49}$$

$$=7$$

Hence $d = \dfrac{\left|\left(\vec{a}\times\vec{b}\right).\vec{c}\right|}{\left|\vec{a}\times\vec{b}\right|}$

$$\boxed{d = \frac{17}{7}}.$$

Answer 47E.

We know that $\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta$ and $|\mathbf{a}\times\mathbf{b}| = |\mathbf{a}||\mathbf{b}|\sin\theta$.

Square both sides of $|\mathbf{a}\times\mathbf{b}| = |\mathbf{a}||\mathbf{b}|\sin\theta$.

$$\left(|\mathbf{a}\times\mathbf{b}|\right)^2 = |\mathbf{a}|^2|\mathbf{b}|^2\sin^2\theta$$

$$|\mathbf{a}\times\mathbf{b}|^2 = |\mathbf{a}|^2|\mathbf{b}|^2\left(1-\cos^2\theta\right)$$

$$|\mathbf{a}\times\mathbf{b}|^2 = |\mathbf{a}|^2|\mathbf{b}|^2 - |\mathbf{a}|^2|\mathbf{b}|^2\cos^2\theta$$

We have $\left(\mathbf{a}\cdot\mathbf{b}\right)^2 = |\mathbf{a}|^2|\mathbf{b}|^2\cos^2\theta$.

$$|\mathbf{a}\times\mathbf{b}|^2 = |\mathbf{a}|^2|\mathbf{b}|^2 - \left(\mathbf{a}\cdot\mathbf{b}\right)^2$$

Thus, we have proved that $|\mathbf{a}\times\mathbf{b}|^2 = |\mathbf{a}|^2|\mathbf{b}|^2 - \left(\mathbf{a}\cdot\mathbf{b}\right)^2$.

Answer 48E.

We have $\mathbf{a}+\mathbf{b}+\mathbf{c}=0$. Then, $\mathbf{b}=-\mathbf{a}-\mathbf{c}$.

Find $\mathbf{a}\times\mathbf{b}$.

$$\begin{aligned}\mathbf{a}\times\mathbf{b} &= \mathbf{a}\times\left(-\mathbf{a}-\mathbf{c}\right)\\ &= \left(\mathbf{a}\times-\mathbf{a}\right)-\left(\mathbf{a}\times\mathbf{c}\right)\\ &= \mathbf{c}\times\mathbf{a}\end{aligned}$$

We know that $\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta$ and $|\mathbf{a}\times\mathbf{b}| = |\mathbf{a}||\mathbf{b}|\sin\theta$.

Also, we have $\mathbf{a}=-\mathbf{b}-\mathbf{c}$.

Simplify $\mathbf{a}\times\mathbf{b}$.

$$\begin{aligned}\mathbf{a}\times\mathbf{b} &= \left(-\mathbf{b}-\mathbf{c}\right)\times\mathbf{b}\\ &= \left(-\mathbf{b}\times\mathbf{b}\right)-\left(\mathbf{c}\times\mathbf{b}\right)\\ &= \mathbf{b}\times\mathbf{c}\end{aligned}$$

Thus, we have proved that if $\mathbf{a}+\mathbf{b}+\mathbf{c}=0$, then $\mathbf{a}\times\mathbf{b} = \mathbf{c}\times\mathbf{a} = \mathbf{b}\times\mathbf{c}$.

Answer 49E.

Consider $\vec{a} = \langle a_1, a_2, a_3\rangle$

$\vec{b} = \langle b_1, b_2, b_3\rangle$ in V_3.

$$\vec{a}-\vec{b} = \langle a_1-b_1, a_2-b_2, a_3-b_3\rangle$$

$$\vec{a}+\vec{b} = \langle a_1+b_1, a_2+b_2, a_3+b_3\rangle$$

Now

$$\left(\vec{a}-\vec{b}\right)\times\left(\vec{a}+\vec{b}\right)=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ a_1-b_1, a_2-b_2 & a_3-b_3 \\ a_1+b_1, a_2+b_2 & a_3+b_3\end{vmatrix}$$

$$=\left[(a_2-b_2)(a_3+b_3)-(a_2+b_2)(a_3-b_3)\right]\hat{i}$$
$$-\left[(a_1-b_1)(a_3+b_3)-(a_1+b_1)(a_3-b_3)\right]\hat{j}$$
$$+\left[(a_1-b_1)(a_2+b_2)-(a_1+b_1)(a_2-b_2)\right]\hat{k}$$
$$=\left(\cancel{a_2a_3}+a_2b_3-b_2a_3-\cancel{b_2b_3}-\cancel{a_2a_3}+a_2b_3-b_2a_3+\cancel{b_2b_3}\right)\hat{i}$$
$$-\left(\cancel{a_1a_3}+a_1b_3-b_1a_3-\cancel{b_1b_3}-\cancel{a_1a_3}+a_1b_3-b_1a_3+\cancel{b_1b_3}\right)\hat{j}$$
$$+\left(\cancel{a_1a_2}+a_1b_2-b_1a_2-\cancel{b_1b_2}-\cancel{a_1a_2}+a_1b_2-b_1a_2+\cancel{b_1b_2}\right)\hat{k}$$
$$=2(a_2b_3-b_2a_3)\hat{i}-2(a_1b_3-b_1a_3)\hat{j}$$
$$+2(a_1b_2-b_1a_2)\hat{k}$$
$$=2\left[(a_2b_3-b_2a_3)\hat{i}-(a_1b_3-b_1a_3)\hat{j}+(a_1b_2-b_1a_2)\hat{k}\right]$$
$$=2\left(\vec{a}\times\vec{b}\right)$$

Answer 50E.

$$\vec{a}\times\left(\vec{b}\times\vec{c}\right)=\left(\vec{a}.\vec{c}\right)\vec{b}-\left(\vec{a}.\vec{b}\right)\vec{c}$$

Consider $\vec{a}=\langle a_1,a_2,a_3\rangle, \vec{b}=\langle b_1,b_2,b_3\rangle$

And $\vec{c}=\langle c_1,c_2,c_3\rangle$ in V_3

L.H.S

$$\vec{a}\times\left(\vec{b}\times\vec{c}\right),$$

$$\vec{b}\times\vec{c}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ b_1 & b_2 & b_3\\ c_1 & c_2 & c_3\end{vmatrix}$$
$$=(b_2c_3-c_2b_3)\hat{i}-(b_1c_3-c_1b_3)\hat{j}+(b_1c_2-c_1b_2)\hat{k}$$

Now

$$\vec{a}\times\left(\vec{b}\times\vec{c}\right)=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ a_1 & a_2 & a_3\\ b_2c_3-c_2b_3 & c_1b_3-b_1c_3 & b_1c_2-c_1b_2\end{vmatrix}$$

$$=\left[a_2(b_1c_2-c_1b_2)-a_3(c_1b_3-b_1c_3)\right]\hat{i}$$
$$-\left[a_1(b_1c_2-c_1b_2)-a_3(b_2c_3-c_2b_3)\right]\hat{j}$$
$$+\left[a_1(c_1b_3-b_1c_3)-a_2(b_2c_3-c_2b_3)\right]\hat{k}$$

R.H.S $\left(\vec{a}.\vec{c}\right)\vec{b}-\left(\vec{a}.\vec{b}\right)\vec{c}$

$$=\left(a_1c_1+a_2c_2+a_3c_3\right)\left(b_1\hat{i}+b_2\hat{j}+b_3\hat{k}\right)$$

$$-\left(a_1b_1+a_2b_2+a_3b_3\right)\left(c_1\hat{i}+c_2\hat{j}+c_3\hat{k}\right)$$

$$=\left(a_2c_2b_1-a_2b_2c_1+a_3c_3b_1-a_3b_3c_1\right)\hat{i}$$

$$+\left(a_1c_1b_2-a_1b_1c_2+a_3c_3b_2-a_3b_3c_2\right)\hat{j}$$

$$+\left(a_1c_1b_3-a_1b_2c_3+a_2c_2b_3-a_2b_2c_3\right)\hat{k}$$

$$=\left[a_2\left(b_1c_2-c_1b_2\right)-a_3\left(c_1b_3-c_3b_1\right)\right]\hat{i}$$

$$-\left[a_1\left(b_1c_2-c_1b_2-\right)-a_3\left(b_2c_3-b_3c_2\right)\right]\hat{j}$$

$$+\left[a_1\left(c_1b_3-b_1c_3\right)-a_2\left(b_2c_3-c_2b_3\right)\right]\hat{k}$$

We can see that the L.H.S =R.H.S

That is $\vec{a}\times\left(\vec{b}\times\vec{c}\right)=\left(\vec{a}.\vec{c}\right)\vec{b}-\left(\vec{a}.\vec{b}\right)\vec{c}$

Answer 51E.

L.H.S

$$\vec{a}\times\left(\vec{b}\times\vec{c}\right)+\vec{b}\times\left(\vec{c}\times\vec{a}\right)+\vec{c}\times\left(\vec{a}\times\vec{b}\right)$$

$$=\left[\left(\vec{a}.\vec{c}\right)\vec{b}-\left(\vec{a}.\vec{b}\right)\vec{c}\right]+\left[\left(\vec{b}.\vec{a}\right)\vec{c}-\left(\vec{b}.\vec{c}\right)\vec{a}\right]$$

$$+\left[\left(\vec{c}.\vec{b}\right)\vec{a}-\left(\vec{c}.\vec{a}\right)\vec{b}\right]$$

Because $\vec{a}.\vec{c}=\vec{c}.\vec{a}$

$\vec{a}.\vec{b}=\vec{b}.\vec{a}$

$\vec{b}.\vec{c}=\vec{c}.\vec{b}$

Then

$$\vec{a}\times\left(\vec{b}\times\vec{c}\right)+\vec{b}\times\left(\vec{c}\times\vec{a}\right)+\vec{c}\times\left(\vec{a}\times\vec{b}\right)=0$$

Answer 52E.

Let $\vec{c}\times\vec{d}=\vec{n}$

Therefore,

$$\left(\vec{a}\times\vec{b}\right)\cdot\left(\vec{c}\times\vec{d}\right)=\left(\vec{a}\times\vec{b}\right)\cdot\vec{n}$$

$$=\vec{a}\cdot\left(\vec{b}\times\vec{n}\right)\qquad\left[\text{since }\vec{a}\cdot\left(\vec{b}\times\vec{c}\right)=\left(\vec{a}\times\vec{b}\right)\cdot\vec{c}\right]$$

$$=\vec{a}\cdot\left\{\vec{b}\times\left(\vec{c}\times\vec{d}\right)\right\}$$

$$=\vec{a}\cdot\left\{\left(\vec{b}\cdot\vec{d}\right)\vec{c}-\left(\vec{b}\cdot\vec{c}\right)\vec{d}\right\}$$

Since $\vec{b}\cdot\vec{c}$ and $\vec{b}\cdot\vec{c}$ are scalars therefore dot product of $\vec{a}$ will be with $\vec{c}$ and $\vec{d}$.

Thus,

$$\left(\vec{a}\times\vec{b}\right)\cdot\left(\vec{c}\times\vec{d}\right)=\vec{a}\cdot\left\{\left(\vec{b}\cdot\vec{d}\right)\vec{c}-\left(\vec{b}\cdot\vec{c}\right)\vec{d}\right\}$$

$$=\left(\vec{a}\cdot\vec{c}\right)\left(\vec{b}\cdot\vec{d}\right)-\left(\vec{a}\cdot\vec{d}\right)\left(\vec{b}\cdot\vec{c}\right)$$

$$=\begin{vmatrix}\vec{a}\cdot\vec{c} & \vec{b}\cdot\vec{c}\\ \vec{a}\cdot\vec{d} & \vec{b}\cdot\vec{d}\end{vmatrix}$$

Hence,

$$\boxed{\left(\vec{a}\times\vec{b}\right)\cdot\left(\vec{c}\times\vec{d}\right)=\begin{vmatrix}\vec{a}\cdot\vec{c} & \vec{b}\cdot\vec{c}\\ \vec{a}\cdot\vec{d} & \vec{b}\cdot\vec{d}\end{vmatrix}}$$

Answer 53E.

Suppose that $a\neq 0$.

(a)

If $\mathbf{a}\cdot\mathbf{b}=\mathbf{a}\cdot\mathbf{c}$, does it follow that $\mathbf{b}=\mathbf{c}$

The given statement is $\boxed{\text{false}}$ because,

Let us consider the counterexample,

$\mathbf{a}=\langle 1,1,1\rangle$

$\mathbf{b}=\langle 1,0,0\rangle$

$\mathbf{c}=\langle 0,0,1\rangle$

Then, $\mathbf{a}\cdot\mathbf{b}=1$

$\mathbf{a}\cdot\mathbf{c}=1$

In this example $\mathbf{a}\cdot\mathbf{b}=\mathbf{a}\cdot\mathbf{c}$ but $\mathbf{b}\neq\mathbf{c}$.

Hence, if $\mathbf{a}\cdot\mathbf{b}=\mathbf{a}\cdot\mathbf{c}$ then it does not follow $\mathbf{b}=\mathbf{c}$.

(b)

Consider the following statement,

If $\mathbf{a}\times\mathbf{b}=\mathbf{a}\times\mathbf{c}$ then it follows $\mathbf{b}=\mathbf{c}$.

The given statement is false because,

$$\mathbf{a}\times\mathbf{b}=\mathbf{a}\times\mathbf{c}$$
$$\mathbf{a}\times\mathbf{b}-\mathbf{a}\times\mathbf{c}=0$$
$$\mathbf{a}\times(\mathbf{b}-\mathbf{c})=0$$

This implies that $\mathbf{a}$ is parallel to $(\mathbf{b}-\mathbf{c})$ and this can't happen if $\mathbf{b}\neq\mathbf{c}$.

Therefore, if $\mathbf{a}\times\mathbf{b}=\mathbf{a}\times\mathbf{c}$ then it does not follow $\mathbf{b}=\mathbf{c}$.

(c)

Consider the following statement,

If $\mathbf{a}\cdot\mathbf{b}=\mathbf{a}\cdot\mathbf{c}$ and $\mathbf{a}\times\mathbf{b}=\mathbf{a}\times\mathbf{c}$ then it follows $\mathbf{b}=\mathbf{c}$.

The given statement is true because,

Since we have $\mathbf{a}\cdot\mathbf{c}=\mathbf{a}\cdot\mathbf{b}$ and that $\mathbf{a}$ is perpendicular to $\mathbf{b}\cdot\mathbf{c}$ and that $\mathbf{a}$ is parallel to $\mathbf{b}-\mathbf{c}$ we have,

$$\mathbf{b}-\mathbf{c}=0$$
$$\mathbf{b}=\mathbf{c}$$

Therefore, if $\mathbf{a}\cdot\mathbf{b}=\mathbf{a}\cdot\mathbf{c}$ and $\mathbf{a}\times\mathbf{b}=\mathbf{a}\times\mathbf{c}$ then it follows $\mathbf{b}=\mathbf{c}$.

Answer 54E.

(A)

Consider $$\vec{k_1}.\vec{v_2}=\frac{\left(\vec{v_2}\times\vec{v_3}\right).\vec{v_2}}{\vec{v_1}\cdot\left(\vec{v_2}\times\vec{v_3}\right)}$$

$$=\frac{\vec{v_2}\cdot\left(\vec{v_2}\times\vec{v_3}\right)}{\vec{v_1}\cdot\left(\vec{v_2}\times\vec{v_3}\right)} \qquad \left\{as\ \vec{a}.\vec{b}=\vec{b}.\vec{a}\right.$$

$$=\frac{\left(\vec{v_2}\times\vec{v_2}\right).\vec{v_3}}{\vec{v_1}\cdot\left(\vec{v_2}\times\vec{v_3}\right)} \qquad \left\{as\ \vec{a}.\left(\vec{b}\times\vec{c}\right)=\left(\vec{a}\times\vec{b}\right).\vec{c}\right.$$

$$=0 \qquad \left\{as\ \vec{a}\times\vec{a}=0\right.$$

Now consider $\vec{k_1}.\vec{v_3} = \dfrac{\left(\vec{v_2}\times\vec{v_3}\right).\vec{v_3}}{\vec{v_1}\cdot\left(\vec{v_2}\times\vec{v_3}\right)}$

$$= \frac{\vec{v_2}\cdot\left(\vec{v_3}\times\vec{v_3}\right)}{\vec{v_1}\cdot\left(\vec{v_2}\times\vec{v_3}\right)}$$

$$= 0$$

Similarly $\vec{k_2}.\vec{v_1} = 0, \vec{k_2}.\vec{v_3} = 0, \vec{k_3}.\vec{v_1} = 0, \vec{k_3}.\vec{v_2} = 0$

(B)

Consider $\vec{k_1}.\vec{v_1} = \dfrac{\left(\vec{v_2}\times\vec{v_3}\right).\vec{v_1}}{\vec{v_1}\cdot\left(\vec{v_2}\times\vec{v_3}\right)}$

$$= \frac{\vec{v_1}\cdot\left(\vec{v_2}\times\vec{v_3}\right)}{\vec{v_1}\cdot\left(\vec{v_2}\times\vec{v_3}\right)} = 1$$

Consider $\vec{k_2}.\vec{v_2} = \dfrac{\left(\vec{v_3}\times\vec{v_1}\right).\vec{v_2}}{\vec{v_1}\cdot\left(\vec{v_2}\times\vec{v_3}\right)} = \dfrac{\vec{v_2}\cdot\left(\vec{v_3}\times\vec{v_1}\right)}{\vec{v_1}\cdot\left(\vec{v_2}\times\vec{v_3}\right)}$

$$= \frac{\left(\vec{v_2}\times\vec{v_3}\right).\vec{v_1}}{\vec{v_1}\cdot\left(\vec{v_2}\times\vec{v_3}\right)} = \frac{\vec{v_1}\cdot\left(\vec{v_2}\times\vec{v_3}\right)}{\vec{v_1}\cdot\left(\vec{v_2}\times\vec{v_3}\right)} = 1$$

Similarly $\vec{k_3}.\vec{v_3} = 1$

(C)

Consider $\vec{k_1}\cdot\left(\vec{k_2}\times\vec{k_3}\right)$

$$= \frac{\vec{v_2}\times\vec{v_3}}{\vec{v_1}\cdot\left(\vec{v_2}\times\vec{v_3}\right)}\cdot\left[\frac{\left(\vec{v_3}\times\vec{v_1}\right)\times\left(\vec{v_1}\times\vec{v_2}\right)}{\left[\vec{v_1}\cdot\left(\vec{v_2}\times\vec{v_3}\right)\right]\left[\vec{v_1}\cdot\left(\vec{v_2}\times\vec{v_3}\right)\right]}\right]$$

$$= \frac{\left(\vec{v_2}\times\vec{v_3}\right)\cdot\left[(\vec{v_3}\times\vec{v_1})\times\left(\vec{v_1}\times\vec{v_2}\right)\right]}{\left[\vec{v_1}\cdot\left(\vec{v_2}\times\vec{v_3}\right)\right]^3}$$

$$= \frac{\left(\vec{v_2}\times\vec{v_3}\right)\cdot\left[\left(\vec{v_3}\times\vec{v_1}\right).\vec{v_2} - \left(\left(\vec{v_3}\times\vec{v_1}\right).\vec{v_1}\right)\vec{v_2}\right]}{\left[\vec{v_1}\cdot\left(\vec{v_2}\times\vec{v_3}\right)\right]^3}$$

$$= \frac{\left(v_2\times v_3\right).v_1\left[\left(\vec{v_3}\times\vec{v_1}\right).\vec{v_2}\right] - \left(\vec{v_2}\times\vec{v_3}\right).\vec{v_2}\left[\left(\vec{v_3}\times\vec{v_1}\right).\vec{v_1}\right]}{\left[\vec{v_1}\cdot\left(\vec{v_2}\times\vec{v_3}\right)\right]^3}$$

i.e. $$\overrightarrow{k_1}\cdot\left(\overrightarrow{k_2}\times\overrightarrow{k_3}\right)=\frac{\overrightarrow{v_1}\cdot\left(\overrightarrow{v_2}\times\overrightarrow{v_3}\right)\left[\overrightarrow{v_2}\cdot\left(\overrightarrow{v_3}\times\overrightarrow{v_1}\right)\right]-\overrightarrow{v_2}\cdot\left(\overrightarrow{v_2}\times\overrightarrow{v_3}\right)\left[\overrightarrow{v_1}\cdot\left(\overrightarrow{v_3}\times\overrightarrow{v_1}\right)\right]}{\left[\overrightarrow{v_1}\cdot\left(\overrightarrow{v_2}\times\overrightarrow{v_3}\right)\right]^3}$$

$$=\frac{\left[\overrightarrow{v_1}\cdot\left(\overrightarrow{v_2}\times\overrightarrow{v_3}\right)\right]\left[\left(\overrightarrow{v_2}\times\overrightarrow{v_3}\right)\cdot\overrightarrow{v_1}\right]-\left[\left(\overrightarrow{v_2}\times\overrightarrow{v_2}\right)\cdot\overrightarrow{v_3}\right]\left[\overrightarrow{v_1}\cdot\left(\overrightarrow{v_3}\times\overrightarrow{v_1}\right)\right]}{\left[\overrightarrow{v_1}\cdot\left(\overrightarrow{v_2}\times\overrightarrow{v_3}\right)\right]^3}$$

$$=\frac{\left[\overrightarrow{v_1}\cdot\left(\overrightarrow{v_2}\times\overrightarrow{v_3}\right)\right]\left[\overrightarrow{v_1}\cdot\left(\overrightarrow{v_2}\times\overrightarrow{v_3}\right)\right]-0\left[\overrightarrow{v_1}\cdot\left(\overrightarrow{v_3}\times\overrightarrow{v_1}\right)\right]}{\left[\overrightarrow{v_1}\cdot\left(\overrightarrow{v_2}\times\overrightarrow{v_3}\right)\right]^3}$$

$$=\frac{\left[\overrightarrow{v_1}\cdot\left(\overrightarrow{v_2}\times\overrightarrow{v_3}\right)\right]^2}{\left[\overrightarrow{v_1}\cdot\left(\overrightarrow{v_2}\times\overrightarrow{v_3}\right)\right]^3}$$

$$=\frac{1}{\overrightarrow{v_1}\cdot\left(\overrightarrow{v_2}\times\overrightarrow{v_3}\right)}$$

Hence $$\overrightarrow{k_1}\cdot\left(\overrightarrow{k_2}\times\overrightarrow{k_3}\right)=\frac{1}{\overrightarrow{v_1}\cdot\left(\overrightarrow{v_2}\times\overrightarrow{v_3}\right)}$$