(2) $(0.5, -0.5, 1) = (0.5)(1, -1, 2)$,

Here $k = 0.5 > 0$

$\therefore$ The vectors have the same direction.

(3) If possible, let $(0, 1, -1) = k(1, -1, 0)$, where $k \in R - \{0\}$.

$\therefore$ $0 = k,\ 1 = -k,\ -1 = 0$

$\therefore$ $k = 0,\ k = -1$ and $-1 = 0$ which is not possible.

Thus there is no such k. Hence the vectors have different directions.

(4) $(3, 6, -9) = -3(-1, -2, 3)$. Here $k = -3 < 0$

$\therefore$ The vectors have opposite directions.

(5) As we did in (3) above, for no $k \in R$,

$(1, 0, 0) = k(0, 1, 0)$

$\therefore$ The vectors have different directions.

(6) If possible, suppose, for some $k \in R - \{0\}$.

$(2, 5, 7) = k(-2, 5, -7)$ then

$2 = -2k,\ 5 = 5k,\ 7 = -7k$

$\therefore$ $k = -1,\ k = 1,\ k = -1$

This is not possible as, the first equation is satisfied for $k = -1$, but second one is not satisfied. Thus, the vectors have different directions.

Note : (1) Suppose $\bar{x}$ and $\bar{y}$ are non-zero vectors and $x_i \neq 0$, $y_i \neq 0$ $(i = 1, 2, 3)$

If $\frac{y_1}{x_1} = \frac{y_2}{x_2} = \frac{y_3}{x_3} = k$ then according to $k > 0$ or $k < 0$, $\bar{x}$ and $\bar{y}$ have the same direction or opposite directions. If $\frac{y_1}{x_1} \neq \frac{y_2}{x_2}$ or $\frac{y_2}{x_2} \neq \frac{y_3}{x_3}$ or $\frac{y_3}{x_3} \neq \frac{y_1}{x_1}$, then their directions are different.

(2) If $x_1 = 0 = y_1$ and $\frac{y_2}{x_2} = \frac{y_3}{x_3} = k > 0$, then $\bar{x}$ and $\bar{y}$ have the same direction and if $k < 0$ then $\bar{x}$ and $\bar{y}$ have opposite directions.

$\frac{y_2}{x_2} \neq \frac{y_3}{x_3}$, then their direction are different. Similar results are true, if $x_2 = 0 = y_2$ or $x_3 = 0 = y_3$.

(3) Finally, if $x_1 = x_2 = y_1 = y_2 = 0$, then for $\frac{y_3}{x_3} > 0$, the directions are same and for $\frac{y_3}{x_3} < 0$ the directions are opposite.

We note again that $\bar{0} = (0, 0, 0)$ has no direction.

Example 6 : Find unit vector along the vector $\bar{u} = (6, -7, 6)$.

Solution : Here $|\bar{u}| = \sqrt{6^2 + (-7)^2 + 6^2} = \sqrt{121} = 11$

$\therefore$ The unit vector in the direction of $\bar{u}$ is, $\frac{\bar{u}}{|\bar{u}|} = \left(\frac{6}{11}, \frac{-7}{11}, \frac{6}{11}\right)$.

Example 7 : Find the unit vector in the direction opposite to the direction of $\bar{x} - 2\bar{y}$, given that,

$\bar{x} = (4, 7, -2)$, $\bar{y} = (1, 2, 2)$.

Solution : $\bar{x} - 2\bar{y} = (4, 7, -2) - 2(1, 2, 2) = (2, 3, -6) = \bar{z}$ (say)

Now $|\bar{z}| = \sqrt{2^2 + 3^2 + (-6)^2} = \sqrt{49} = 7$

$\therefore$ The unit vector in the direction opposite to direction of $\bar{z}$ is,

$$-\frac{\bar{z}}{|\bar{z}|} = \left(-\frac{2}{7}, -\frac{3}{7}, \frac{6}{7}\right).$$

Example 8 : For the pairs of points A, B given below find vector $\overrightarrow{AB}$.

(1) A(1, −1), B(1, 2) (2) A(1, −1, 1), B(1, 1, −1)

(3) A(1, 2, 3), B(4, 5, 6) (4) A(1, −2, 1), B(−1, 1, 1)

Solution : $\overrightarrow{AB}$ **= Position vector of B − Position vector of A**

(1) $\overrightarrow{AB} = (1, 2) - (1, -1) = (0, 3)$

(2) $\overrightarrow{AB} = (1, 1, -1) - (1, -1, 1) = (0, 2, -2)$

(3) $\overrightarrow{AB} = (4, 5, 6) - (1, 2, 3) = (3, 3, 3)$

(4) $\overrightarrow{AB} = (-1, 1, 1) - (1, -2, 1) = (-2, 3, 0)$

Exercise 9.3

1. For the following pairs of vectors, determine whether the two vectors have the same or opposite directions or different directions :

(1) (2, −5, 3), (0.4, −1, 0.6) (2) (1, 2, 4), (3, 4, 6)

(3) (2, 4, −6), (−1, −2, 3) (4) (1, 0, 1), (0, 1, 1)

2. Find the unit vector in the direction of the following vectors :

(1) $\bar{x} = (3, -4)$ (2) $\bar{y} = (-3, -4)$ (3) $\bar{x} = (1, 3, 5)$

(4) $\bar{y} = \left(1, \frac{1}{2}, \frac{1}{3}\right)$ (5) $\bar{y} = (1, 0, 0)$ (6) $\bar{y} = (-5, 12)$

3. If $\bar{x} = (x_1, x_2)$ and $\bar{x} = \alpha(1, 2) + \beta(2, 1)$, find α, β.

*

9.10 Distance Formula

Let $\bar{r}_1$ and $\bar{r}_2$ be the position vectors of points A and B respectively and let $\bar{r}_1 = (x_1, y_1, z_1)$ and $\bar{r}_2 = (x_2, y_2, z_2)$. We know that,

$$\overrightarrow{AB} = \text{Position vector of B} - \text{Position vector of A}$$

$$= (x_2, y_2, z_2) - (x_1, y_1, z_1)$$

$$= (x_2 - x_1, y_2 - y_1, z_2 - z_1)$$

$$\therefore \text{AB} = |\overrightarrow{AB}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

This is called distance formula, it gives distance between two points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ in R^3.

Note : In XY-plane z-coordinates of a point is zero. Hence setting $z_1 = z_2 = 0$ in the distance formula we get the distance formula in plane which was studied in std. 10.

Example 9 : Find the distance between points (1, −1, 2) and (−2, 1, 8).

Solution : Taking P(1, −1, 2) and Q(−2, 1, 8) we have

$$PQ = \sqrt{(1-(-2))^2+(-1-1)^2+(2-8)^2} = \sqrt{3^2+(-2)^2+(-6)^2} = \sqrt{49} = 7$$

Thus the distance between two given points is 7.

Example 10 : Using distance formula, show that the points P(4, −3, −1), Q(5, −7, 6) and R(3, 1, −8) are collinear.

Solution : We have,

$$PQ = \sqrt{(4-5)^2+(-3+7)^2+(-1-6)^2} = \sqrt{1+16+49} = \sqrt{66}$$

$$QR = \sqrt{(5-3)^2+(-7-1)^2+(6+8)^2} = \sqrt{4+64+196} = 2\sqrt{66}$$

$$PR = \sqrt{(4-3)^2+(-3-1)^2+(-1+8)^2} = \sqrt{1+16+49} = \sqrt{66}$$

Thus, PQ + PR = QR and hence Q–P–R.

$\therefore$ The given points are collinear.

Example 11 : If A(1, 2, 4), B(1, 2, 0) and C(1, 5, 0), show that ΔABC is a right angled triangle.

Solution : $AB^2 = (1-1)^2 + (2-2)^2 + (4-0)^2 = 16$. So AB = 4

$BC^2 = (1-1)^2 + (2-5)^2 + (0-0)^2 = 9$. So BC = 3

$AC^2 = (1-1)^2 + (5-2)^2 + (0-4)^2 = 25$. So AC = 5

$\therefore$ A, B, C are non-collinear and form a triangle.

Also $AC^2 = AB^2 + BC^2$ and hence ΔABC is a right angled triangle with right angle at B.

Example 12 : Find coordinates of points on X-axis at distance $3\sqrt{3}$ from the point A(2, −1, 1).

Solution : A point on X-axis is $P(x, 0, 0)$. Now $AP = 3\sqrt{3}$

$$\sqrt{(x-2)^2+(0+1)^2+(0-1)^2} = 3\sqrt{3}$$

$\therefore$ $x^2 - 4x + 4 + 1 + 1 = 27$

$\therefore$ $x^2 - 4x + 4 = 25$

$\therefore$ $(x-2)^2 = 5$

$\therefore$ $x - 2 = \pm 5$

$\therefore$ $x = 7$ or $x = -3$

Thus, there are two such points namely P(7, 0, 0) and P(−3, 0, 0).

Example 13 : Find the equation of the set of points which are equidistant from the points (2, −1, 1) and (1, 3, 1).

Solution : Let (x, y, z) be the coordinates of the points equidistant from the given points (2, −1, 1) and (1, 3, 1).

$$(x - 2)^2 + (y + 1)^2 + (z - 1)^2 = (x - 1)^2 + (y - 3)^2 + (z - 1)^2$$

$\therefore \quad x^2 - 4x + 4 + y^2 + 2y + 1 + z^2 - 2z + 1$

$\qquad = x^2 - 2x + 1 + y^2 - 6y + 9 + z^2 - 2z + 1$

$\therefore \quad -4x + 2y + 5 = -2x - 6y + 10$

$\therefore \quad 2x - 8y + 5 = 0$

This is the equation of the required set.

Note : In the plane this type of set is called the **perpendicular bisector line** of the given segment. In space this is called the **perpendicular bisector plane** of the given segment. It is a plane perpendicular to the segment and passes through the mid-point of the segment.

Exercise 9.4

1. Find the distance between the following pairs of points :

(1) (1, −1, 3), (1, −1, 3) (2) (1, 2, 3), (3, 4, 5)

(3) (2, −3, 18), (0, 1, 14) (4) $(1, \sqrt{2}, -1), (3, 3\sqrt{2}, 1)$

(5) (1, −2, 5014), (4, 2, 5014) (6) (1, 1, 0), (0, 1, 0)

2. Using distance formula, determine whether the following points are collinear or not :

(1) P(1, 3, 2), Q(1, 2, 1), R(2, 3, 1) (2) A(0, 1, 0), B(0, −1, 0), C(0, 2, 0)

(3) L(1, 2, 3), M(−3, −1, 1), A(−3, 2, 7) (4) V(1, 2, 3), A(2, 3, 1), H(3, 1, 2)

3. Given that A(0, 7, 10), B(−1, 6, 6), C(−4, 9, 6), determine the type of ΔABC.

4. Find the points on Z-axis which are at a distance $\sqrt{14}$ from the point (−2, 1, 3).

5. Find the equation of the set of points P such that $PA^2 + PB^2 = 2k^2$, where A and B are the points (3, 4, 5) and (−1, 2, 7) respectively, $k \in R$.

6. Show that O(0, 0, 0), A(2, −3, 6), B(0, −7, 0) are vertices of an isosceles triangle.

*

9.11 Section Formula

We have studied section formula for a line segment joining two points in R^2. Now using vectors we will derive section formula for a line segment joining two points in R^3.

Let $\bar{r}_1 = (x_1, y_1, z_1)$ and $\bar{r}_2 = (x_2, y_2, z_2)$ be the position vectors of two points A and B in the space respectively. Suppose $P \in \overleftrightarrow{AB}$ ($P \neq A$, $P \neq B$). As the points A, B and P are on the same line, the directions of $\overrightarrow{AP}$ and $\overrightarrow{PB}$ are same or opposite. Thus, we have

$$\overrightarrow{AP} = k\overrightarrow{PB}, \text{ where } k \neq 0 \qquad \text{(i)}$$

$\therefore \quad |\overrightarrow{AP}| = |k|\,|\overrightarrow{PB}|$ or $AP = |k|\,PB$

$\therefore \quad \frac{AP}{PB} = |k|$

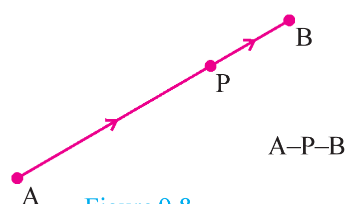

Figure 9.8

Let the position vector of P be $\bar{r} = (x, y, z)$.

Now let P divide $\overline{AB}$ from the side of A in the ratio λ.

(i) If $\lambda > 0$ and A–P–B and $\frac{AP}{PB} = \lambda$, we say that P divides $\overline{AB}$ internally from the side of A in the ratio λ. (figure 9.8)

$\therefore \quad \frac{AP}{PB} = |k| = \lambda$

Further, as $\overrightarrow{AP}$ and $\overrightarrow{PB}$ have the same direction, $k > 0$

So, $|k| = k$.

Since, $|k| = \lambda$, $k = \lambda$

$\therefore \quad \overrightarrow{AP} = \lambda\, \overrightarrow{PB}$. **(using (i))**

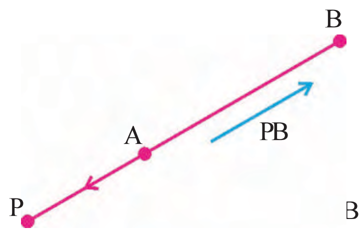

Figure 9.9

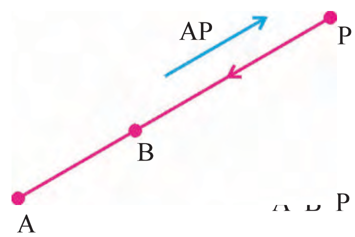

Figure 9.10

(ii) If $\lambda < 0$ and P–A–B or A–B–P and $\frac{AP}{PB} = -\lambda$, we say P divides $\overline{AB}$ externally from the side of A in the ratio λ. As shown in the figures 9.9 and 9.10, it is clear that $\overrightarrow{AP}$ and $\overrightarrow{PB}$ have opposite directions, so $k < 0$.

$\therefore \quad |k| = -k$

$\therefore \quad \frac{AP}{PB} = |k| = -k$ and $\frac{AP}{PB} = -\lambda$

Hence $k = \lambda$

$\overrightarrow{AP} = \lambda\, \overrightarrow{PB}$ **($\overrightarrow{AP} = k\, \overrightarrow{PB}$)**

Thus, in each case $\overrightarrow{AP} = \lambda\, \overrightarrow{PB}$

$\therefore \quad \bar{r} - \bar{r}_1 = \lambda(\bar{r}_2 - \bar{r})$

$\therefore \quad \bar{r} - \bar{r}_1 = \lambda\bar{r}_2 - \lambda\bar{r}$

$\therefore \quad (1 + \lambda)\,\bar{r} = \lambda\bar{r}_2 + \bar{r}_1$

Note that by the definition of division, $\lambda \neq -1$.

$\therefore \quad \bar{r} = \frac{1}{\lambda + 1}(\lambda\bar{r}_2 + \bar{r}_1)$

$$(x, y, z) = \frac{1}{\lambda+1}\left(\lambda(x_2, y_2, z_2) + (x_1, y_1, z_1)\right)$$

$$= \frac{1}{(\lambda+1)}\left(\lambda x_2 + x_1, \lambda y_2 + y_1, \lambda z_2 + z_1\right)$$

$$\therefore \quad (x, y, z) = \left(\frac{\lambda x_2 + x_1}{\lambda+1}, \frac{\lambda y_2 + y_1}{\lambda+1}, \frac{\lambda z_2 + z_1}{\lambda+1}\right)$$

This is called **section formula.** It gives coordinates of the point which divides line segment $\overline{AB}$ in the ratio λ from the side of point $A(x_1, y_1, z_1)$.

If the ratio λ is $m : n$, then above formula gives,

$$\bar{r} = \frac{1}{\frac{m}{n}+1}\left(\frac{m}{n}\bar{r}_2 + \bar{r}_1\right) = \frac{1}{m+n}\left(m\bar{r}_2 + n\bar{r}_1\right); \quad m + n \neq 0$$

$$\therefore \quad (x, y, z) = \left(\frac{mx_2 + nx_1}{m+n}, \frac{my_2 + ny_1}{m+n}, \frac{mz_2 + nz_1}{m+n}\right)$$

9.12 Some Applications of Section Formula

(i) Coordinates of mid-points : If P is the mid-point of $\overline{AB}$, then AP = PB and A–P–B.

$\therefore \quad \frac{AP}{PB} = \lambda = 1$

$\therefore$ Let position vector of P be $\bar{r}$.

If $\bar{r}_1 = (x_1, y_1, z_1)$ and $\bar{r}_2 = (x_2, y_2, z_2)$ and $\bar{r} = (x, y, z)$, then section formula gives

$$(x, y, z) = \frac{1}{2}((x_1, y_1, z_1) + (x_2, y_2, z_2)) \qquad (\lambda = 1)$$

$$= \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right)$$

$\therefore$ The position vector of the mid-point of $\overline{AB}$ is given by $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right)$.

(ii) Centroid of a Triangle : Let ABC be a triangle in R^3. Suppose position vectors of A, B and C are $\bar{r}_1 = (x_1, y_1, z_1)$, $\bar{r}_2 = (x_2, y_2, z_2)$ and $\bar{r}_3 = (x_3, y_3, z_3)$ respectively.

As shown in the figure 9.11, D is mid-point of $\overline{BC}$. Hence its position vector is $\frac{\bar{r}_2 + \bar{r}_3}{2}$.

Let G be the point dividing $\overline{AD}$ in the ratio 2 : 1 from the side of A. The position vector of G is

$$\frac{1}{2+1}\left(2 \cdot \frac{1}{2}(\bar{r}_2 + \bar{r}_3) + \bar{r}_1\right) = \frac{1}{3}(\bar{r}_1 + \bar{r}_2 + \bar{r}_3)$$

Symmetry of this result shows that G is on all the three medians. Thus, the medians of a triangle are concurrent in G.

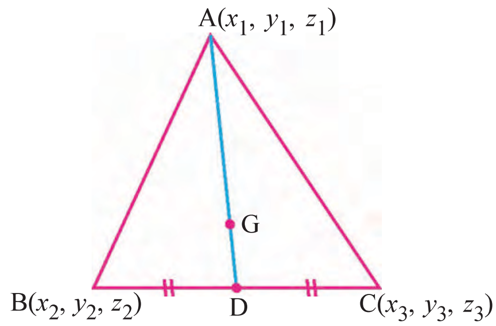

Figure 9.11

Thus, G is the centroid of ΔABC and its position vector is $\frac{1}{3}(\bar{r}_1 + \bar{r}_2 + \bar{r}_3)$. So the coordinates of G are $\left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}, \frac{z_1 + z_2 + z_3}{3}\right)$.

Example 14 : Find the coordinates of the point which divides the segment joining the points A(2, 3, –1) and B(1, –3, 5) from A in the ratio (i) 3 : 5 internally, (ii) 3 : 5 externally.

Solution : (i) Let $P(x, y, z)$ divides $\overline{AB}$ from A in the ratio 3 : 5 internally. Thus $m = 3$, $n = 5$. Now by section formula,

$$x = \frac{3(1) + 5(2)}{3+5} = \frac{3+10}{8} = \frac{13}{8}$$

$$y = \frac{3(-3) + 5(3)}{3+5} = \frac{-9+15}{8} = \frac{6}{8} = \frac{3}{4}$$

$$z = \frac{3(5) + 5(-1)}{3+5} = \frac{15-5}{8} = \frac{10}{8} = \frac{5}{4}$$

Thus, the point $\left(\frac{13}{8}, \frac{3}{4}, \frac{5}{4}\right)$ divides $\overline{AB}$ in the ratio 3 : 5 internally from A.

(ii) Here the division is external. Thus $m = 3$, $n = -5$. Hence the coordinates of required point are

$$x = \frac{3(1) - 5(2)}{3 - 5} = \frac{3 - 10}{-2} = \frac{-7}{-2} = \frac{7}{2}$$

$$y = \frac{3(-3) - 5(3)}{3 - 5} = \frac{-9 - 15}{-2} = 12$$

$$z = \frac{3(5) - 5(-1)}{3 - 5} = \frac{15 + 5}{-2} = -10$$

Thus, the coordinates of the point which divides $\overline{AB}$ from A in the ratio 3 : 5 externally are $\left(\frac{7}{2}, 12, -10\right)$.

Example 15 : Use section formula to examine collinearity of the points (1, −3, 3), (3, 7, 1), (1, 1, 1).

Solution : If A(1, −3, 3), B(3, 7, 1) and C(1, 1, 1) are collinear, then one of them divides the line segment joining the other two in some ratio say $k : 1$. Suppose B divides $\overline{AC}$ in some ratio k.

$$3 = \frac{k(1) + 1}{k + 1} = \frac{k + 1}{k + 1} = 1$$

This is not true. Hence the points are not collinear.

Example 16 : Show that the triangle with vertices (−1, 6, 6), (−4, 9, 6) and (0, 7, 10) is a right angled triangle. Further verify that the mid-point of its hypotenuse is equidistant from all vertices.

Solution : Let A(−1, 6, 6), B(−4, 9, 6) and C(0, 7, 10).

Now, $AB^2 = (-4 + 1)^2 + (9 - 6)^2 + (6 - 6)^2 = 9 + 9 = 18$

$BC^2 = (0 + 4)^2 + (7 - 9)^2 + (10 - 6)^2 = 16 + 4 + 16 = 36$

$AC^2 = (0 + 1)^2 + (7 - 6)^2 + (10 - 6)^2 = 1 + 1 + 16 = 18$

$\therefore \quad AB^2 + AC^2 = BC^2$

Thus, ΔABC is a right angled triangle and $\overline{BC}$ is its hypotenuse.

Let M(x, y, z) be the mid-point of $\overline{BC}$. Then

$$(x, y, z) = \left(\frac{0 - 4}{2}, \frac{7 + 9}{2}, \frac{10 + 6}{2}\right) = (-2, 8, 8).$$

Now as, M is the mid-point of $\overline{BC}$ and $BC = \sqrt{36} = 6$

$BM = CM = 3$

Further $AM = \sqrt{(-2 + 1)^2 + (8 - 6)^2 + (8 - 6)^2} = \sqrt{1 + 4 + 4} = 3$

Thus, AM = BM = CM, i.e. M is equidistant from all the vertices of ΔABC.

Miscellaneous Problems :

In a plane if four points are given, then they form a **quadrilateral** provided any three of them are non-collinear. Using distance formula and section formula, the type of the quadrilateral can be determined. In the case of four points in the space, they may form a quadrilateral if all these points are coplanar. Thus, before determining the type of a quadrilateral, we must make sure that the points are coplanar. Following examples are based on this.

Example 17 : Determine whether the points A(0, 0, 0), B(1, 0, 0), C(0, 1, 0), D(0, 0, 1) are vertices of a quadrilateral or not. If they form a quadrilateral, then determine its type.

Solution : $\overrightarrow{AC} = (0, 1, 0)$, $\overrightarrow{BD} = (-1, 0, 1)$.

$\overrightarrow{AC}$ and $\overrightarrow{BD}$ have different directions. Thus, $\overleftrightarrow{AC} \nparallel \overleftrightarrow{BD}$.

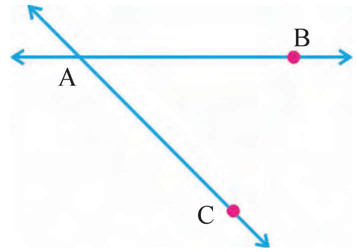

Figure 9.12(i)

Now let us examine if they intersect in a point.

If they intersect in a point, it may happen the point of intersection is A or B or C or D.

$\overrightarrow{AC} = (0, 1, 0)$, $\overrightarrow{AD} = (0, 0, 1)$ **(i)**

$\overrightarrow{AC}$ and $\overrightarrow{AD}$ have different directions.

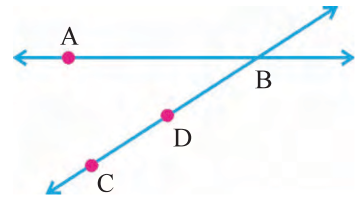

Figure 9.12(ii)

$\therefore$ A, C, D cannot be collinear.

$\overrightarrow{BC} = (-1, 1, 0)$, $\overrightarrow{BD} = (-1, 0, 1)$ **(ii)**

$\therefore$ B, C and D cannot be collinear.

Similarly from (i) and (ii) A, B, C or A, B, D are not collinear.

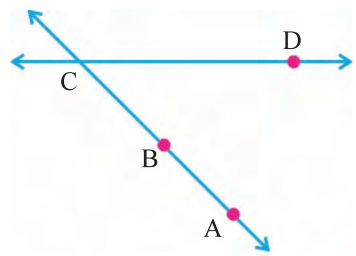

Figure 9.12(iii)

Now suppose, if possible $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ intersect in a point P other than A or B or C or D.

Four distinct points A, B, C, D lie in a plane if either $\overleftrightarrow{AC}$ and $\overleftrightarrow{BD}$ intersect in a point or $\overleftrightarrow{AC} \parallel \overleftrightarrow{BD}$. If possible, suppose they intersect in a point $P(x, y, z)$. Thus $P \in \overleftrightarrow{AC}$ and $P \in \overleftrightarrow{BD}$. Let P divide $\overline{AC}$ from the side of A in the ratio λ and it divide $\overline{BD}$ from the side of B in the ratio $\mu(\lambda \in R - \{0, -1\}, \mu \in R - \{0, -1\})$. By section formula,

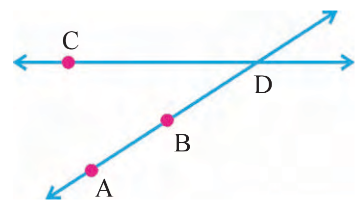

Figure 9.12(iv)

$$\left.\begin{aligned} P \in \overleftrightarrow{AC} \Rightarrow x &= \frac{\lambda(0)+0}{\lambda+1} = 0 \\ y &= \frac{\lambda(1)+0}{\lambda+1} = \frac{\lambda}{\lambda+1} \\ z &= \frac{\lambda(0)+0}{\lambda+1} = 0 \end{aligned}\right\} \quad \textbf{(iii)}$$

$$\left.\begin{aligned} \text{and } P \in \overleftrightarrow{BD} \Rightarrow x &= \frac{\mu(0)+1}{\mu+1} = \frac{1}{\mu+1} \\ y &= \frac{\mu(0)+0}{\mu+1} = 0 \\ z &= \frac{\mu(1)+0}{\mu+1} = \frac{\mu}{\mu+1} \end{aligned}\right\} \quad \textbf{(iv)}$$

Thus, from (iii) and (iv) $x = 0 = \frac{1}{\mu+1}$ which is not possible. Thus, $\overleftrightarrow{AC}$ and $\overleftrightarrow{BD}$ neither intersect nor are parallel. Thus the points A, B, C and D are not coplanar. Hence given points are not vertices of a quadrilateral.

Note : Four non-coplanar points in the space form a geometrical figure called a tetrahedron (figure 9.12(v)). A tetrahedron has four triangular faces and six edges.

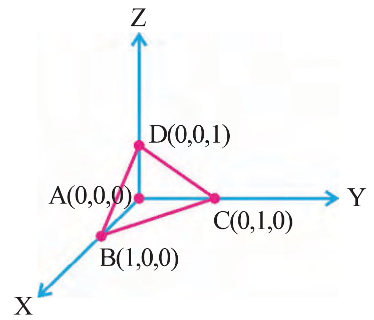

Figure 9.12(v)

Example 18 : Examine coplanarity of the point P(1, 1, 1), Q(−2, 4, 1), R(−1, 5, 5) and S(2, 2, 5). Also determine the type of quadrilateral formed by them, if any.

Solution : The mid-point of $\overline{PR}$ = M(0, 3, 3)

The mid-point of $\overline{QS}$ = M(0, 3, 3)

$\therefore$ $\overleftrightarrow{PR}$ and $\overleftrightarrow{QS}$ intersect in M.

$\therefore$ P, Q, R, S are coplanar.

Now, $\overrightarrow{PQ} = (-2, 4, 1) - (1, 1, 1) = (-3, 3, 0)$

$\overrightarrow{QR} = (-1, 5, 5) - (-2, 4, 1) = (1, 1, 4)$

$\overrightarrow{SR} = (-1, 5, 5) - (2, 2, 5) = (-3, 3, 0)$

$\overrightarrow{PS} = (2, 2, 5) - (1, 1, 1) = (1, 1, 4)$

Thus, $\overrightarrow{PQ}$ and $\overrightarrow{SR}$ have same directions. $\overrightarrow{QR}$ and $\overrightarrow{PS}$ have same directions.

Further,

$$PQ = \sqrt{(-3)^2 + (3)^2 + 0} = \sqrt{18} = RS$$

$$QR = \sqrt{1^2 + 1^2 + 4^2} = \sqrt{18} = PS$$

Also as seen above diagonals $\overline{PR}$ and $\overline{QS}$ bisect each other and also

$$PR = \sqrt{(1+1)^2 + (1-5)^2 + (1-5)^2} = \sqrt{4+16+16} = 6$$

$$QS = \sqrt{(-2-2)^2 + (4-2)^2 + (1-5)^2} = \sqrt{16+4+16} = 6$$

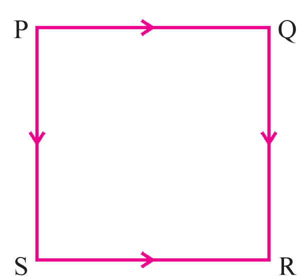

Figure 9.13

Thus for the parallelogram PQRS, all four sides are of equal length and diagonals have equal length. Thus, $\square$ PQRS is a square.

So for collinearity for three points was checked using distance formula and also using section formula. Suppose three distinct points A, B and C are given. Then they are collinear only if one of the following is true.

A–B–C

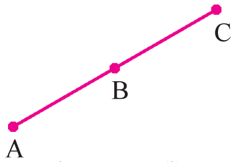

Figure 9.14(i)

A–C–B

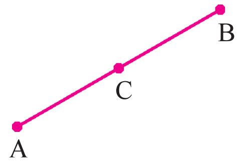

Figure 9.14(ii)

C–A–B

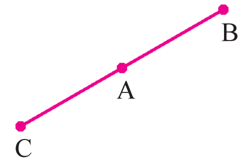

Figure 9.14(iii)

In all three cases $\overrightarrow{AB}$ and $\overrightarrow{BC}$ have the same or opposite directions. Hence three points A, B and C are collinear only if $\overrightarrow{AB}$ and $\overrightarrow{BC}$ have the same or opposite directions. The following examples are based on this fact.

Example 19 : Using directions examine if following points are collinear :

(1) A(0, 2), B(2, 4), C(−2, 0) (2) P(1, −1, 0), Q(−3, 1, 2), R(−1, 0, 1)

(3) A(1, 2, 3), P(5, 2, 2), S(2, 3, 1) (4) L(0, 0), M(1, 0), N(0, 1)

Solution : (1) $\overrightarrow{AB} = (2, 4) - (0, 2) = (2, 2)$

$\overrightarrow{BC} = (-2, 0) - (2, 4) = (-4, -4)$

Obviously $\overrightarrow{BC} = (-2)\overrightarrow{AB}$

Hence $\overrightarrow{AB}$ and $\overrightarrow{BC}$ have opposite directions. Thus A, B and C are collinear. $(\overleftrightarrow{AB} ⋕ \overleftrightarrow{BC})$

(2) $\overrightarrow{PQ} = (-3, 1, 2) - (1, -1, 0) = (-4, 2, 2)$

$\overrightarrow{QR} = (-1, 0, 1) - (-3, 1, 2) = (2, -1, -1)$

Here $\overrightarrow{PQ} = (-2)\overrightarrow{QR}$. So, $\overrightarrow{PQ}$ and $\overrightarrow{QR}$ have opposite directions. Thus, P, Q, R are collinear. $(\overleftrightarrow{PQ} ⋕ \overleftrightarrow{QR})$

(3) $\overrightarrow{AP} = (5, 2, 2) - (1, 2, 3) = (4, 0, -1)$

$\overrightarrow{PS} = (2, 3, 1) - (5, 2, 2) = (-3, 1, -1)$

If possible suppose for a non-zero, $k \in R$

$\overrightarrow{AP} = k(\overrightarrow{PS})$

$\therefore$ $(4, 0, -1) = k(-3, 1, -1)$

$\therefore$ $4 = -3k,\ 0 = k,\ -1 = -k$

For any $k \in R$ all there are not satisfied. So, $\overrightarrow{AP}$ and $\overrightarrow{PS}$ have different directions. Hence A, P and S are not collinear.

(4) $\overrightarrow{LM} = (1, 0) - (0, 0) = (1, 0)$

$\overrightarrow{MN} = (0, 1) - (1, 0) = (-1, 1)$

If possible suppose for some $k \in R - \{0\}$,

$\overrightarrow{LM} = k(\overrightarrow{MN})$

$\therefore$ $(1, 0) = k(-1, 1)$

$\therefore$ $1 = -k,\ k = 0$ which is not possible.

So, $\overrightarrow{LM}$ and $\overrightarrow{MN}$ have different directions. Hence given points are non-collinear.

Example 20 : Prove that A(1, 2, 3), B(−1, −2, −1), C(2, 3, 2) and D(4, 7, 6) forms a parallelogram.

Solution : Mid-point of $\overline{AC} = \left(\frac{3}{2}, \frac{5}{2}, \frac{5}{2}\right)$. Mid-point of $\overline{BD} = \left(\frac{3}{2}, \frac{5}{2}, \frac{5}{2}\right)$.

∴ $\overline{AC}$ and $\overline{BD}$ bisect each other and they intersect at the mid-point. Hence $\overleftrightarrow{AC}$ and $\overleftrightarrow{BD}$ are coplanar.

∴ A, B, C, D form a quadrilateral in a plane and its diagonals bisect each other.

∴ □ ABCD is a parallelogram.

Alternate Method :

$\overrightarrow{AB} = (-2, -4, -4)$, $\overrightarrow{BC} = (3, 5, 3)$, $\overrightarrow{DC} = (-2, -4, -4)$

∴ $\overrightarrow{AB}$ and $\overrightarrow{DC}$ are in the same direction.

∴ $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$ or A, B, C, D are collinear. But $\overrightarrow{AB}$ and $\overrightarrow{BC}$ are in different direction,

∴ $C \notin \overleftrightarrow{AB}$

∴ $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$

Similarly, $\overleftrightarrow{AD} \parallel \overleftrightarrow{BC}$ **($\overrightarrow{AD} = (3, 5, 3)$)**

∴ A, B, C, D are coplanar and □ ABCD is a parallelogram.

Note : Solution given below is not proper :

$AB = \sqrt{4+16+16} = 6$, $CD = \sqrt{4+16+16} = 6$, $AD = \sqrt{9+25+9} = \sqrt{43} = BC$

∴ Opposite sides of □ ABCD congruent. Hence □ ABCD is a parallelogram.

If A, B, C, D are coplanar, then this decision is correct. So it is necessary to prove A, B, C, D are coplanar. See the example given below :

Example 21 : Prove that for O(0, 0, 0), A(1, 1, 0), B(1, 0, 1), C(0, 1, 1), OA = AB = BC = AC = OB = OC, but O, A, B, C do not form a parallelogram.

Solution : $\overrightarrow{OA} = (1, 1, 0)$, $\overrightarrow{OB} = (1, 0, 1)$, $\overrightarrow{OC} = (0, 1, 1)$

$\overrightarrow{AB} = (0, -1, 1)$, $\overrightarrow{BC} = (-1, 1, 0)$, $\overrightarrow{AC} = (-1,$

∴ $OA = OB = OC = AB = BC = AC = \sqrt{2}$

But any two of above vectors are not in the same or in the opposite directions.

∴ O, A, B, C do not form a parallolegram.

That these points are non-coplanar can be proved. Points O, A, B, C form a tetrahedron.

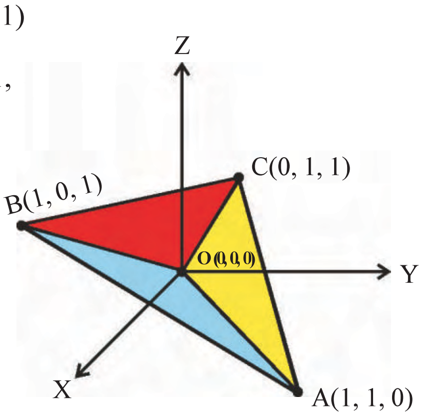

Figure 9.15

Exercise 9.5

1. Find the points of trisection of the segment $\overline{AB}$, where A(1, 3, −2), B(2, 4, −1).
2. Using section formula, check the collinearity of points :

(1) P(1, −1, 1), Q(1, 0, 3), R(2, 0, 0)

(2) A(5, 6, −1), B(1, −1, 3), C(1, 1, 1)

(3) L(2, −3, 4), M(−1, 2, 1), $N\left(-\frac{1}{4}, \frac{3}{4}, \frac{5}{4}\right)$

(4) O(0, 0, 0), A(1, 1, 1), B(2, 2, 2)

(5) L(1, 2, 3), M(−1, −2, −3), N(1, −2, 3)

Exercise 9

1. Show that the points A(−2, −3, −1), B(2, 1, 1), C(−3, −2, −2) and D(−7, −6, −4) form a parallelogram. Is it a rectangle ?

2. Determine the type of ΔABC, given that A(0, 1, 2), B(2, −1, 3), C(1, −3, 1).

3. Find the equation of the set of points at the same distance from the points (1, 2, 3) and (3, 2, −1).

4. Find the lengths of medians and coordinates of the centroid in each of the following triangles :

(1) A(1, 0, 1), B(1, 2, 0), C(1, 1, 2)

(2) P(1, 2, 3), Q(−1, 1, 0), R(0, 0, 3)

(3) L(−1, −2, −3), M(1, 2, 3), N(1, 2, 1)

5. Let P(1, 2, −3), Q(3, 0, 1) and R(−1, 1, 4) be the mid-points of the sides of ΔABC. Find the centroid of ΔABC.

6. Using vectors examine the collinearity of the points given below. If they are collinear, then in which ratio and from which side one point divides segment joining other two ?

(1) A(5, 4, 6), B(1, −1, 3), C(4, 3, 2)

(2) A(2, 3, 4), B(−4, 1, −10), C(−1, 2, −3)

(3) A(1, 2, 3), B(0, 4, 1), C(−1, −1, −1)

(4) L(3, 2, −4), M(5, 4, −6), N(9, 8, −10)

(5) P(2, 3, 4), Q(3, 4, 5), R(1, 2, 3)

7. Select a proper option (a), (b), (c) or (d) from given options and write in the box given on the right so that the statement becomes correct :

(1) The magnitude of sum of vectors $(1, -\sqrt{2})$, $(2, \sqrt{2})$ is ...

(a) −3 (b) 3 (c) 9 (d) −9

(2) Given that the points A(1, 0, 1), B(2, −1, 3) and C(3, −2, 5) are collinear, then the ratio in which C divides $\overline{AB}$ from side of A is ...

(a) 2 : 1 (b) −1 : 2 (c) 1 : 2 (d) −2 : 1

(3) The centroid of the triangle whose vertices are P(1, −2, 1), Q(2, 3, −1), R(1, −1, −1) is ...

(a) (1, 2, 1) (b) $\left(\frac{4}{3}, 0, -\frac{1}{3}\right)$ (c) $\left(\frac{3}{2}, \frac{1}{2}, 0\right)$ (d) $\left(-\frac{4}{3}, -\frac{4}{3}, -\frac{1}{3}\right)$

(4) If the position vectors of A and B are respectively (1, 1, 0) and (0, 1, 1) then $\overrightarrow{AB}$ = ...

(a) (0, 0, 0) (b) (1, 0, −1) (c) (−1, 0, 1) (d) (1, 2, 1)

(5) The direction of (1, 1, 2) and (2, 1, 0) is ......

(a) same (b) opposite (c) different (d) not defined

(6) $<2, 2, 2>$ = ...

(a) $-<-4, -4, -4>$ (b) $<1, 1, -1>$ (c) $<-1, 1, -1>$ (d) $<0, 0, 0>$

(7) $\left\langle \frac{1}{3}, \frac{1}{3}, \frac{1}{3} \right\rangle$ = ...

(a) $<1, 1, -1>$ (b) $<\cos\theta \cos\alpha, \cos\theta \sin\alpha, \sin\theta>$

(c) $<5, 5, 5>$ (d) $<3, 3, -3>$

(8) Unit vector in the direction of (2, 2, −1) is ......

(a) $\left(\frac{2}{3}, \frac{2}{3}, \frac{-1}{3}\right)$ (b) $\left(\frac{-2}{3}, \frac{-2}{3}, \frac{1}{3}\right)$ (c) (2, 2, 1) (d) $\left(\frac{2}{3}, \frac{2}{3}, \frac{1}{3}\right)$

(9) Unit vector in the direction of (1, 0, 0) is ......

(a) (0, 1, 0) (b) (0, 0, 1) (c) (−1, 0, 0) (d) (1, 0, 0)

(10) If the centroid of ΔABC is (0, 0, 0), where A(1, 1, 1), B(2, 1, 2), C(x, y, z) then (x, y, z) = ...

(a) (3, 2, 3) (b) (0, 0, 0) (c) (−3, −2, −3) (d) (1, −1, 1)

(11) If A(1, 1, 2), B(2, 1, 2), C(2, 2, 1) then A, B, C are ......

(a) vertices of a triangle (b) collinear
(c) on axes (d) non-coplanar

(12) If A(1, 2, 1), B(2, 3, 2), C(2, 1, 3), D(3, 2, 4), then directions of $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are ......

(a) same (b) perpendicular to each other
(c) different (d) not defined

(13) If A(1, 2, 1), B(2, 3, 2), C(2, 1, 3), D(3, 2, 4) then ......

(a) $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$ (b) $\overleftrightarrow{AB} = \overleftrightarrow{CD}$
(c) $\overleftrightarrow{AB} \cap \overleftrightarrow{CD}$ is singleton (d) C $\in \overleftrightarrow{AB}$

(14) Vector (0, 0, 0) ....

(a) has no direction (b) has no magnitude
(c) is in the direction of (1, 1, 1) (d) is in opposite direction of (−1, −1, −1)

(15) P(2, 3, 1) and Q(7, 15, 1) then $|\overrightarrow{PQ}|$ = ...

(a) 5 (b) 12 (c) 13 (d) 17

(16) A vector which is in the directions of (3, 6, 2) and has magnitude 4 is ....

(a) $\left(\frac{3}{7}, \frac{6}{7}, \frac{2}{7}\right)$ (b) (12, 24, 8) (c) $\left(\frac{12}{7}, \frac{24}{7}, \frac{8}{7}\right)$ (d) (−12, −24, −8)

(17) A unit vector which is in the opposite direction of (2, −2, 1) is ....

(a) $\left(\frac{-2}{3}, \frac{2}{3}, \frac{-1}{3}\right)$ (b) (−2, 2, −1) (c) $\left(\frac{2}{3}, \frac{-2}{3}, \frac{1}{3}\right)$ (d) $\left(\frac{2}{3}, \frac{2}{3}, \frac{1}{3}\right)$

(18) ($\cos\alpha$, $\sin\alpha$) and ($\cos(\pi + \alpha)$, $\sin(\pi + \alpha)$) ($\alpha \in$ R) have directions ....

(a) same (b) opposite (c) different (d) same as (1, 0)

(19) If $\overline{x}$ is a non-zero vector and $k > 0$, $k \neq 1$, then $\frac{-k\overline{x}}{|\overline{x}|}$ is ....

(a) unit vector in the direction of $\overline{x}$
(b) in the direction of $\overline{x}$ having magnitude k
(c) in the opposite direction of $\overline{x}$ having magnitude k
(d) unit vector in the opposite direction of $\overline{x}$

(20) If $\overline{x}$ is a non-zero vector and $k < 0$, $k \neq -1$, then $\frac{k\overline{x}}{|\overline{x}|}$ is ....

(a) unit vector in the direction of $\overline{x}$
(b) unit vector in the opposite direction of $\overline{x}$
(c) in the opposite direction of $\overline{x}$ having magnitude $|k|$
(d) in the direction of $\overline{x}$ having magnitude $|k|$

*

Summary

We studied following points in this chapter :

1. Set of ordered pairs and ordered triplets of real numbers R^2 and R^3 respectively form a vector space over R.

2. Magnitude of a vector $\bar{x} = (x_1, x_2, x_3)$ is $|\bar{x}| = \sqrt{x_1^2 + x_2^2 + x_3^2}$ and if $\bar{x} = (x_1, x_2)$, then $|\bar{x}| = \sqrt{x_1^2 + x_2^2}$.

3. $|\bar{x}| = 0 \Leftrightarrow \bar{x} = \bar{0}$ and $|k\bar{x}| = |k||\bar{x}|$

4. For two non-zero vectors $\bar{x}$ and $\bar{y}$ if $\bar{x} = k\bar{y}$, then $\bar{x}$, $\bar{y}$ have the same direction, if $k > 0$. They have opposite directions, if $k < 0$.

5. For two points A and B **(in R^3 or R^3)**

$\overrightarrow{AB}$ **= Position vector of B − Position vector of A**

6. Distance between two points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ is given by

$$AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

7. If $\bar{r}_1$ and $\bar{r}_2$ are position vectors of points A and B respectively and point P divides $\overline{AB}$ in the ratio λ from the side of A, then position vector of P is $\frac{1}{\lambda + 1}(\lambda\bar{r}_2 + \bar{r}_1)$.

8. If $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$ and $C(x_3, y_3, z_3)$ then position vector of the centroid is $\left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}, \frac{z_1 + z_2 + z_3}{3}\right)$.

Bhaskara II

- Solutions of Diophantine equations of the second order, such as $61x^2 + 1 = y^2$. This very equation was posed as a problem in 1657 by the French mathematician Pierre de Fermat, but its solution was unknown in Europe until the time of Euler in the 18th century.
- Solved quadratic equations with more than one unknown and found negative and irrational solutions.
- Preliminary concept of infinitesimal calculus, along with notable contributions towards integral calculus.
- Conceived differential calculus, after discovering the derivative and differential coefficient.
- Stated Rolle's theorem, a special case of one of the most important theorems in analysis, the mean value theorem. Traces of the general mean value theorem are also found in his works.
- Calculated the derivatives of trigonometric functions and formulae.
- In *Siddhanta Shiromani*, Bhaskara developed spherical trigonometry along with a number of other trigonometric results.

Bhaskara II gave the formula : $\sqrt{a \pm \sqrt{b}} = \sqrt{\frac{a + \sqrt{a^2 - b}}{2}} \pm \sqrt{\frac{a - \sqrt{a^2 - b}}{2}}$

Bhaskaracharya studied Pell's equation $px^2 + 1 = y^2$ for $p = 8, 11, 32, 61$ and 67. When $p = 61$, he found the solutions $x = 226153980$, $y = 1776319049$. When $p = 67$ he found the solutions $x = 5967$, $y = 48842$. He studied many Diophantine problems.

The topics covered in Lilavati, thirteen chapters of the book are : definitions; arithmetical terms; interest; arithmetical and geometrical progressions; plane geometry; solid geometry; the shadow of the gnomon; the kuttaka; combinations.

Chapter 10

LIMITS

If people do not believe that mathematics is simple, it is only because they do not realise how complicated life is.

– John Louis Von Neumann

10.1 Introduction and History

Now we start with the study of calculus. Whatever we have studied so far is known as pre-calculus. Calculus is a Latin word meaning a small stone used for counting. Calculus is the study of change in the way that geometry is the study of shape and algebra is the study of operations and their applications to solving equations. Calculus has widespread applications in science, economics and engineering.

The ancient period saw some of the ideas that led to integral calculus. Calculations of volumes and areas by integral calculus can be found in the Egyptian ***Moscow Papyrus*** (1820 B.C.). But the formulae are mere instructions and some of them are wrong. From the age of Greek mathematics **Eudoxus** (408-335 B.C.) used the method of exhaustion which prefigures the concept of the limit to calculate areas and volumes. **Archimedes** (287-212 B.C.) developed the idea further. The method of exhaustion was reinvented by **Lie Hui** in China in the third century A.D. to find the area of a circle.

Brahmagupta's Yuktibhasha is considered to be the first book on calculus. **Bhaskar's** work on calculus precedes much before the time of **Leibnitz** and **Newton**. **Bhaskara-2** used principles of differential calculus in problems on Astrotomy. There is a strong evidence that **Bhaskar** was a pioneer on some principles of differential calculus. He stated **Rolle's** Mean value theorem. In his book ***Siddhanta Shiromani***, we find elementary concept of mathematical analysis and infinitesimal calculus.

These ideas were systematized into calculus by **Gottfried Wilhelm Leibnitz**. He independently invented calculus along the same time as **Newton**. **Leibnitz** and **Newton** are both credited with the invention of calculus. **Newton** derived his results first but **Leibnitz** published them first. Both arrived at the results independently. But **Leibnitz** started with integral calculus and **Newton** started with differentiation. The name calculus was given by Leibnitz. In the 19th century calculus was put on a much rigorous footing by **Cauchy, Riemann** and **Weierstrass**. The modern ε-δ definition of limit is due to **Weierstrass**.

The modern notion of the limit of a function dates back to **Bolzano**. He introduced ε-δ technique in 1817 for continuous functions. **Cauchy** discussed limits in his ***cours de' analyse*** in 1821. But he gave only a verbal definition. **Weierstrass** introduced modern ε-δ definition which is studied today. He also gave notations *lim* and lim $x \to x_0$. The modern notation $\lim\limits_{x \to x_0}$ is due to **Hardy** given in his book ***'A Course of Pure Mathematics'*** in 1908.

10.2 Intuitive Idea of Limit

Now we turn to the main idea of calculus namely limits. Before giving definition, we will get intuitive idea of limits. We understand that the discussion that follows only gives some intuitive idea of limits and the examples solved only suggest ideas leading to the concept of limits.

Limit of a function is the **'ultimate'** value of the function, if it exists, when variable changes continuouly in the domain and goes nearer and nearer to a specific value. Let us be more specific. Limit of $f(x) = 3x + 2$ when x approaches 1 is written as $\lim\limits_{x \to 1} (3x + 2)$ and let us see how we 'find' it. Let us tabulate some values of x and $f(x)$ as follows :

x	0.9	0.99	0.999	0.9999	1.1	1.01	1.001	1.0001
$f(x)$	4.7	4.97	4.997	4.9997	5.3	5.03	5.003	5.0003

We observe that as $x \to 1$ through values less than 1, $f(x)$ approaches 5. This we express by saying that limit of $f(x)$ is 5 as x approaches 1 from left and we write $\lim\limits_{x \to 1-} f(x) = 5$ in notation. Similarly the limit of $f(x)$ as x approaches 1 from right is 5 or $\lim\limits_{x \to 1+} f(x) = 5$. Incidentally $f(1) = 3 + 2 = 5$. But this is not necessary.

If $\lim\limits_{x \to a-} f(x)$ and $\lim\limits_{x \to a+} f(x)$ exist and are equal, we say $\lim\limits_{x \to a} f(x)$ exists and is equal to either of $\lim\limits_{x \to a-} f(x)$ or $\lim\limits_{x \to a+} f(x)$.

Let us understand by a graph.

See that as $x \to 1-$, y-coordinate approaches 5 and so is the case with $x \to 1+$. Note that in discussing this limit, $f(1) = 5$ has no bearing on the limit.

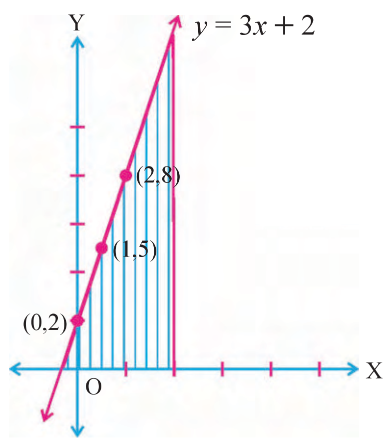

Figure 10.1

Example 1 to 13 are for understanding of concept of limit only. They are not meant to be asked in the examination.

Example 1 : Verify $\lim_{x \to \frac{1}{2}} \frac{4x^2 - 1}{2x - 1} = 2$ by tabulation. $\left(x \neq \frac{1}{2}\right)$

x	0.49	0.499	0.4999	0.51	0.501	0.5001
$f(x)$	1.98	1.998	1.9998	2.02	2.002	2.0002

See that

$$f(x) = \frac{4x^2 - 1}{2x - 1}$$

$$= 2x + 1 \text{ as } x \neq \frac{1}{2}.$$

Hence we can see that

$$\lim_{x \to \frac{1}{2}} f(x) = 2.$$

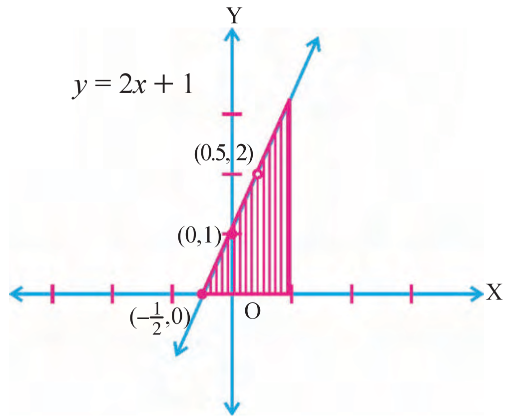

Figure 10.2

Explanation : As x approaches $\frac{1}{2}$ from left or from right $f(x)$ approaches 2. Here the graph does not contain the point corresponding to $x = \frac{1}{2}$ namely $\left(\frac{1}{2}, 2\right)$. All the while 'ultimate' value of $f(x)$, as approaches 1, is 2.

Example 2 : Find $\lim_{x \to 0} |x|$.

Solution : We know $|x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$

Hence,

x	−0.1	−0.01	−0.001	0.1	0.01	0.001
$f(x)$	0.1	0.01	0.001	0.1	0.01	0.001

We can guess that $\lim_{x \to 0} f(x) = 0$.

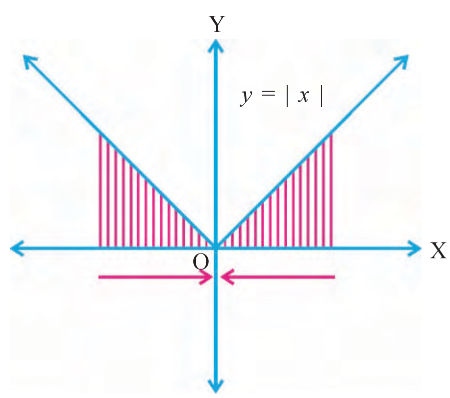

Figure 10.3

See that $f(0) = 0$

Example 3 : Prove that $\lim_{x \to 2} [x]$ does not exist.

Solution : $f(x) = [x] = \begin{cases} 1 & \text{if} \quad 1 \leq x < 2 \\ 2 & \text{if} \quad 2 \leq x < 3 \end{cases}$

x	1.9	1.99	1.999	1.9999	2.1	2.01	2.001	2.0001
$f(x)$	1	1	1	1	2	2	2	2

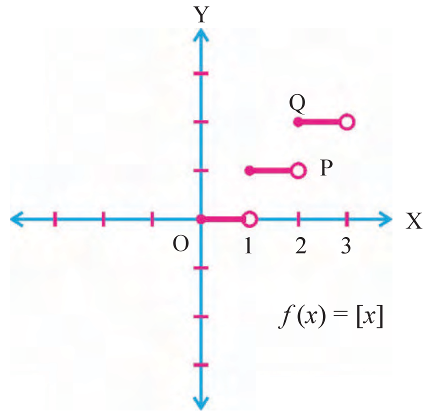

Figure 10.4

So, $\lim_{x \to 2-} f(x) = 1$ and $\lim_{x \to 2+} f(x) = 2$

$\therefore \lim_{x \to 2} f(x)$ does not exist.

Explanation : Observe that there is a 'gap' between P and Q. Left limit and right limit do not coincide.

Example 4 : What can you say about $\lim_{x \to 0} \frac{|x|}{x}$? $(x \neq 0)$

Solution : Here $f(x) = \frac{|x|}{x} = \begin{cases} 1 & \text{if} \quad x > 0 \\ -1 & \text{if} \quad x < 0 \end{cases}$

f is not defined for $x = 0$.

x	–0.1	–0.01	–0.001	0.1	0.01	0.001
$f(x)$	–1	–1	–1	1	1	1

Obviously, $\lim_{x \to 0+} f(x) = 1$, $\lim_{x \to 0-} f(x) = -1$

$\therefore \lim_{x \to 0} f(x)$ does not exist.

Note : In the example 1, $f\left(\frac{1}{2}\right)$ is not defined but $\lim_{x \to \frac{1}{2}} f(x)$ exists.

In the example 2, $f(0)$ is defined and $\lim_{x \to 0} f(x) = f(0)$.

In the example 3, $\lim_{x \to 2-} f(x) \neq \lim_{x \to 2+}, f(x)$ but $f(2)$ exists. But limit does not exist.

In the example 4, $\lim_{x \to 0+} f(x) \neq \lim_{x \to 0-} f(x)$ and $f(0)$ does not exist. Limit does not exist.

So we have enough ground to conclude that existence or value of $\lim_{x \to a} f(x)$ is not affected by its value at a, namely $f(a)$.

Example 5 : Find $\lim_{x \to 0} f(x)$ where $f(x) = \begin{cases} x + 3 & x < 0 \\ 3 - x & x \geq 0 \end{cases}$

Solution : Here for $x < 0$, $f(x) = x + 3$ and for $x > 0$, $f(x) = 3 - x$.

$\therefore$ The table of values will be as follows :

x	–0.1	–0.01	–0.001	0.1	0.01	0.001
$f(x)$	2.9	2.99	2.999	2.9	2.99	2.999

$$\therefore \quad \lim_{x \to 3-} f(x) = \lim_{x \to 3+} f(x) = 3$$

and also $f(0) = 3 - 0 = 3$.

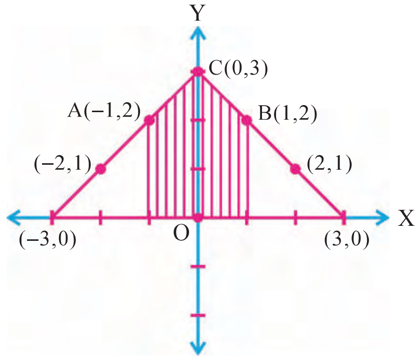

Figure 10.5

Explanation : (0, 3) is on the graph. As $x \to 0-$, point A move towards C and as $x \to 0+$, point B moves towards C. Hence $\lim_{x \to 0+} f(x)$ and $\lim_{x \to 0-} f(x)$ coincide.

Also $f(0) = 3$. All the three coincide.

Example 6 : Find $\lim_{x \to 1} f(x)$ where $f(x) = \begin{cases} x + 3 & x > 1 \\ 10 & x = 1 \\ x + 5 & x < 1 \end{cases}$

Solution :

x	0.9	0.99	0.999	1.1	1.01	1.001
$f(x)$	5.9	5.99	5.999	4.1	4.01	4.001

$x < 1$ $\qquad$ $x > 1$

Thus, $\lim_{x \to 1-} f(x)$ seems to be 6 and $\lim_{x \to 1+} f(x)$ appears to be 4. Thus $\lim_{x \to 1} f(x)$ does not exist.

Also $f(1) = 10$. All the three are distinct.

Explanation :

Hence, $\lim_{x \to 1-} f(x) \neq \lim_{x \to 1+} f(x)$

and the two are different.

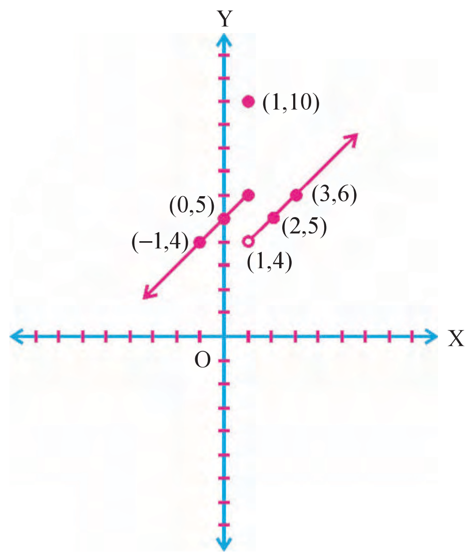

Figure 10.6

Example 7 : Find $\lim_{x \to 1} (x^2 - x)$.

Solution :

x	0.9	0.99	0.999	1.1	1.01	1.001
$f(x)$	−0.09	−0.0099	−0.000999	0.11	0.0101	0.000101

Thus, $\lim_{x \to 1-} f(x) = \lim_{x \to 1+} f(x) = 0$, $f(1) = 1^2 - 1 = 0$

$$\therefore \quad \lim_{x \to 1} f(x) = 0 = f(1)$$

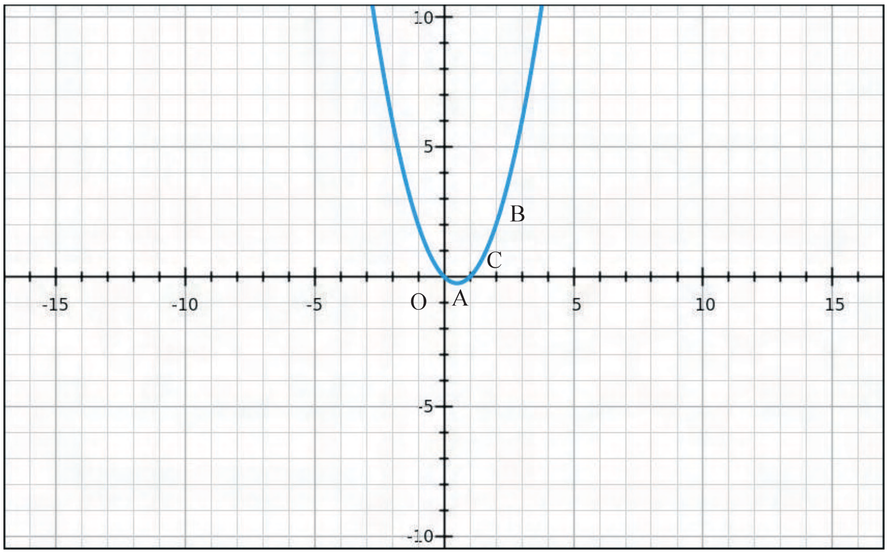

Figure 10.7

Explanation : As $x \to 1-$, A approaches C and $x \to 1+$, B approaches C.

$\therefore \lim_{x \to 1} f(x) = 0$

Example 8 : $f : R \to R$. $f(x) = 5$, Find $\lim_{x \to 10} f(x)$.

Solution :

x	9.9	9.99	9.999	10.1	10.01	10.001
$f(x)$	5	5	5	5	5	5

$\therefore \lim_{x \to 10} f(x) = 5$

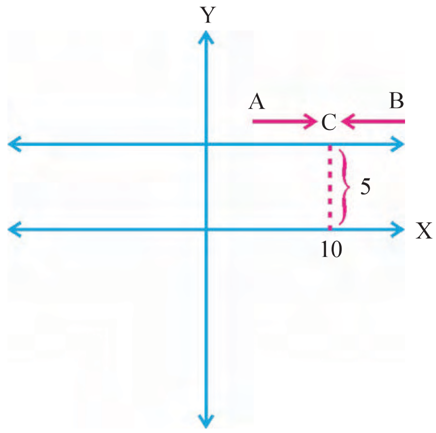

Figure 10.8

Explanation : As $x \to 10-$, A approaches C and as $x \to 10+$, B approaches C.

C is (10, 5).

$\lim_{x \to 10} f(x) = 5$

Example 9 : Find $\lim_{x \to \frac{\pi}{2}} \cos x$.

Solution :

x	$\frac{\pi}{2} - 0.1$	$\frac{\pi}{2} - 0.01$	$\frac{\pi}{2} - 0.001$	$\frac{\pi}{2} + 0.1$	$\frac{\pi}{2} + 0.01$	$\frac{\pi}{2} + 0.001$
$f(x)$	0.099833	0.009999833	0.0009999998	−0.099833	−0.009999833	−0.0009999998

Obviously $\lim_{x \to \frac{\pi}{2}} \cos x = 0$

Explanation : Look at the graph of $\cos x$.

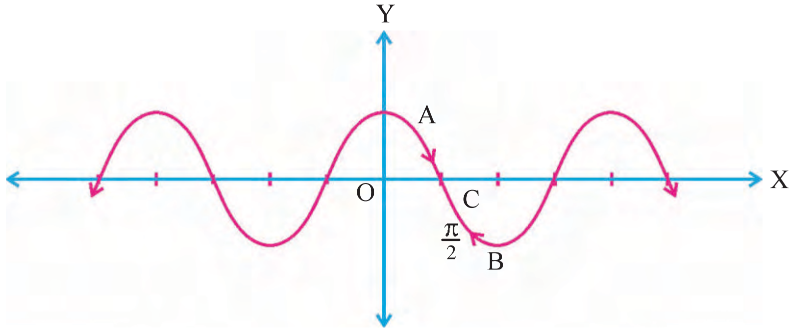

Figure 10.9

As before A approaches C and B tends to C as $x \to \frac{\pi}{2}-$ and $x \to \frac{\pi}{2}+$ respectively.

$\therefore \lim_{x \to \frac{\pi}{2}} \cos x = 0$

Example 10 : Verify $\lim_{x \to 0} \frac{\sin x}{x} = 1$. $(x \neq 0)$

Solution :

x	−0.7	−0.2	−0.05	1.4	0.3	0.03	0.01
$f(x)$	0.92031	0.993347	0.999583	0.97275	0.98506	0.99985	0.999983

Explanation : Note that $\frac{\sin x}{x}$ is an even function i.e. $\frac{\sin(-x)}{-x} = \frac{-\sin x}{-x} = \frac{\sin x}{x}$

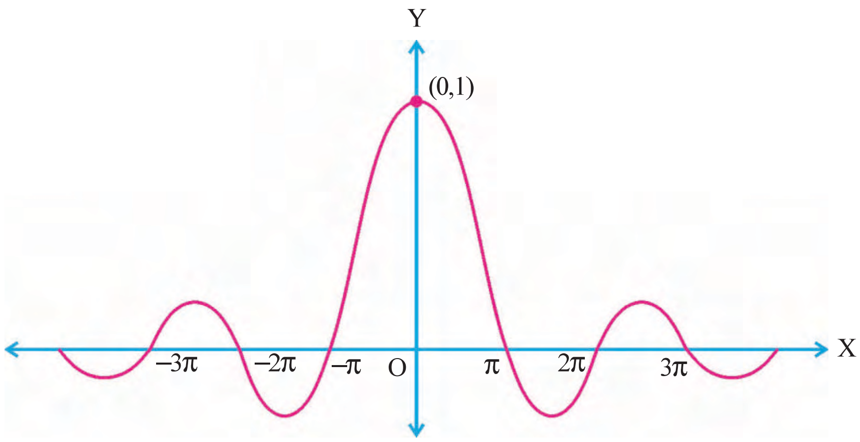

Figure 10.10

So we need consider $x > 0$ only. It is apparent that $\lim_{x \to 0} \frac{\sin x}{x} = 1$. This is reflected in the graph given also. In fact we will prove this in this chapter later on.

Example 11 : Find $\lim_{x \to 0} (x + \cos x)$.

Solution :

x	−0.1	−0.01	−0.001	0.1	0.01	0.001
$f(x)$	0.895004165	0.98995	0.9989995	1.095004165	1.009995	1.0009995

Explanation : From the graph as well the table we infer that $\lim_{x \to 0} (x + \cos x) = 1$.

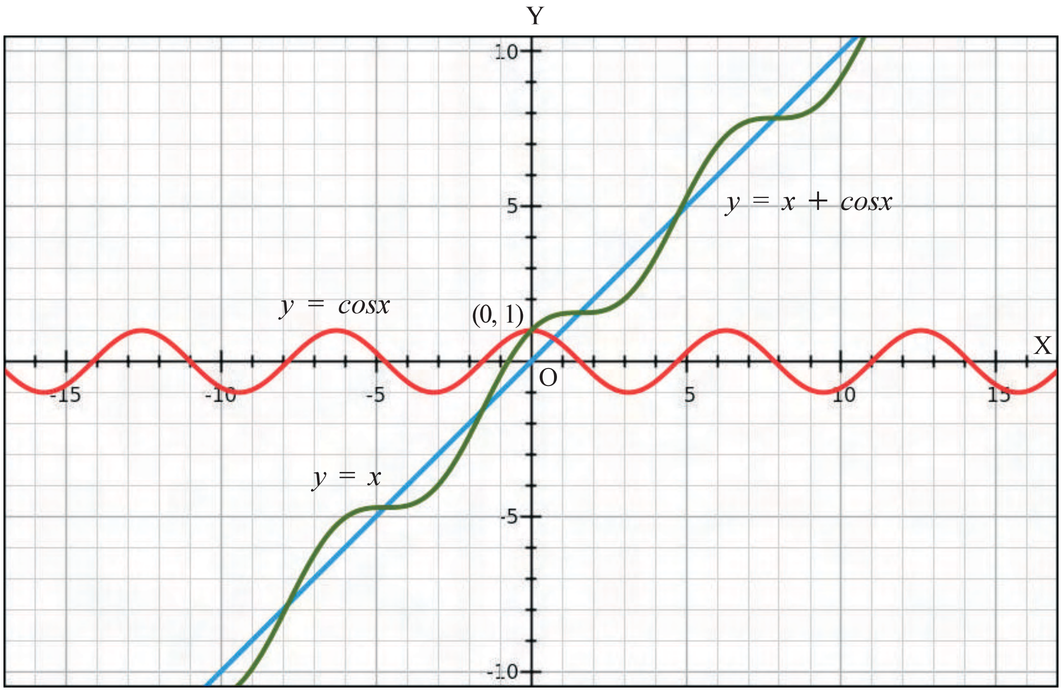

Figure 10.11

See $\lim_{x \to 0} x = 0$, $\lim_{x \to 0} \cos x = 1$.

$\therefore \quad \lim_{x \to 0} (x + \cos x) = \lim_{x \to 0} x + \lim_{x \to 0} \cos x$

Example 12 : Discuss existence of $\lim_{x \to 0} \frac{1}{x}$.

Solution :

x	−0.1	−0.01	−0.001	0.1	0.01	0.001
$f(x)$	−10	−100	−1000	10	100	1000

Explanation : Here we observe that as $x \to 0+$, $\frac{1}{x}$ increases 'unboundedly' and as $x \to 0-$, we say $\frac{1}{x}$ decreases 'unboundedly'. So $\lim_{x \to 0+} \frac{1}{x}$ or $\lim_{x \to 0-} \frac{1}{x}$ do not exist. We say as $x \to 0+$, $\frac{1}{x} \to \infty$ and as $x \to 0-$, $\frac{1}{x} \to -\infty$. $\lim_{x \to 0} \frac{1}{x}$ does not exist.

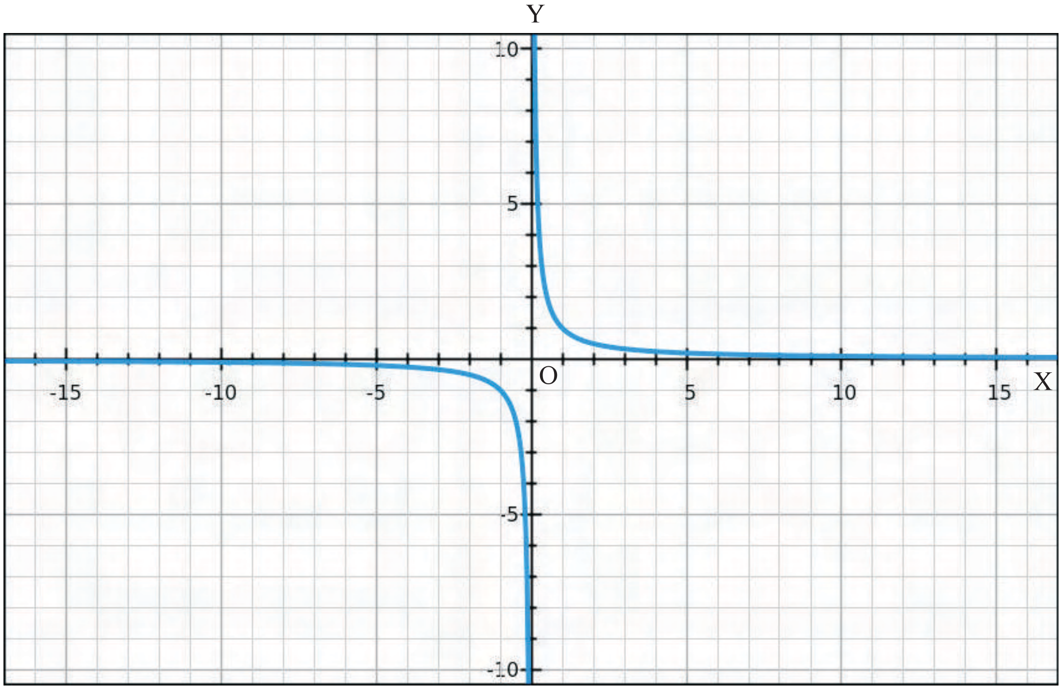

Figure 10.12

It is incorrect to say $\lim_{x \to 0+} \frac{1}{x} = \infty$ or $\lim_{x \to 0-} \frac{1}{x} = -\infty$. Note that ∞ and $-\infty$ are merely symbols or members of the extended real number system. We are dealing with limits in real number system only.

Example 13 : Discuss $\lim_{x \to 0} \frac{1}{x^2}$.

Solution :

x	-0.1	-0.01	-0.001	0.1	0.01	0.001
$f(x)$	10^2	10^4	10^6	10^2	10^4	10^6

Explanation : In this case, whether $x \to 0+$ or $x \to 0-$, $\frac{1}{x^2}$ increases unboundedly or $\frac{1}{x^2} \to \infty$.

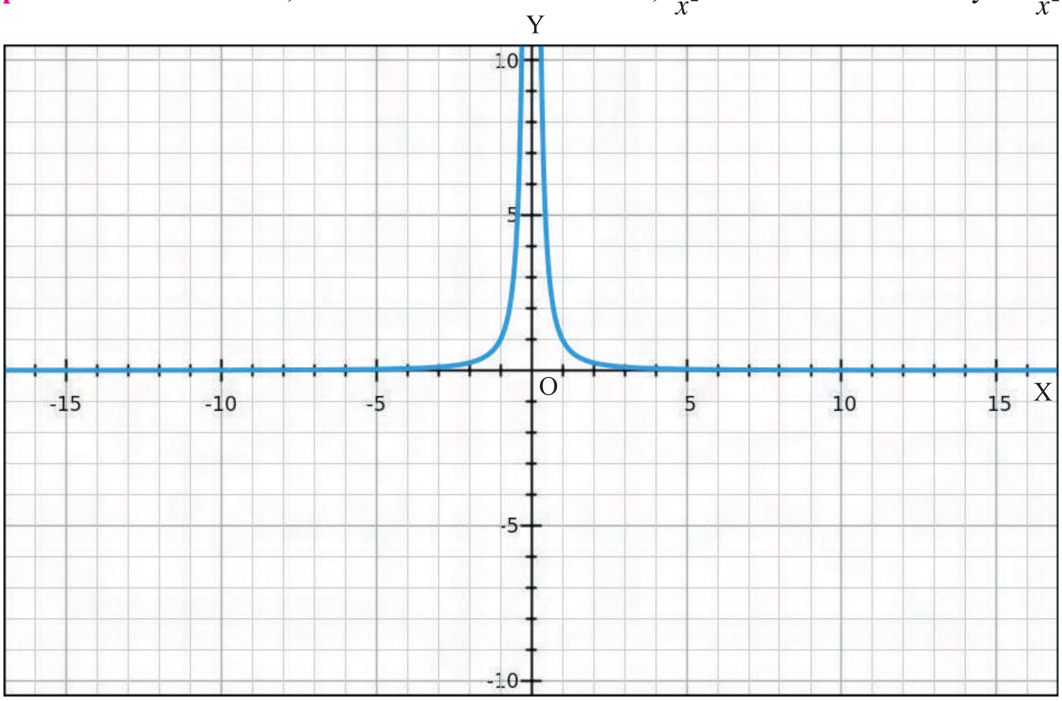

Figure 10.13

Again we do not write $\lim_{x \to 0} \frac{1}{x^2} = \infty$. $\lim_{\infty \to 0} \frac{1}{x^2}$ does not exist.

10.3 Formal Definition of Limit

Now we are ready to give formal definition of limit. So far we had inferred certain limits by observing some tabulated values and graphs. But in practice it is not possible even in simple examples and this tabulation may even mislead. Look at the graph of $sin\frac{1}{x}$ (figure 10.14).

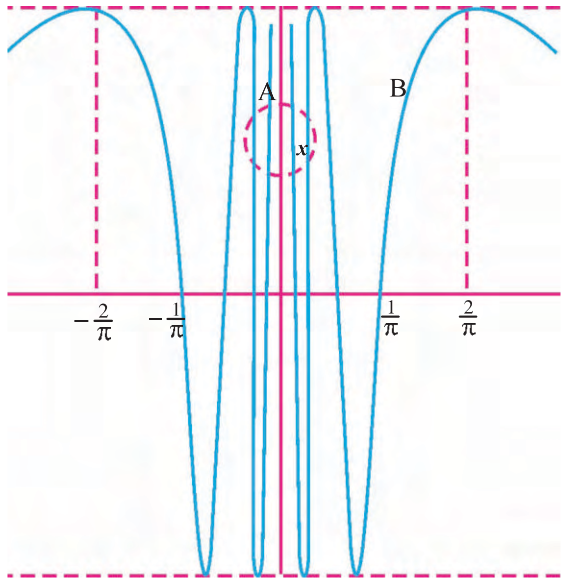

Figure 10.14

Can we infer anything about $\lim_{x \to 0} sin\frac{1}{x}$? When x takes a sequence of values $\frac{1}{k\pi}$, $k \in Z - \{0\}$, $sin\frac{1}{x} = 0$, for $x = \frac{2}{(4m+1)\pi}$, $sin\frac{1}{x} = 1$ and for $x = \frac{2}{(4m+3)\pi}$, $sin\frac{1}{x} = -1$. Other values of x also exist which we have not considered. So it is difficult to guess anything about $\lim_{x \to 0} sin\frac{1}{x}$.

Definition : Limit of a function : Let $f(x)$ be a function defined on a domain containing some interval containing a but a may not be in the domain of f. If for every $\varepsilon > 0$, there exists some $\delta > 0$ such that whenever $a - \delta < x < a + \delta$, $x \neq a \Rightarrow l - \varepsilon < f(x) < l + \varepsilon$, we say $\lim_{x \to a} f(x) = l$ or limit of $f(x)$ as x tends to a is l.

See that $\delta > 0$ is any positive number. Hence $f(x)$ can be brought as near to l as we please. $-\varepsilon < f(x) - l < \varepsilon$ or $|f(x) - l| < \varepsilon$ just by proper selection of δ such that $a - \delta < x < a + \delta$, $x \neq a$ or $-\delta < x - a < \delta$, $x \neq a$ i.e. $|x - a| < \delta$, $x \neq a$.

Thus $f(x)$ can be brought as near to l as we please if we can choose $\delta > 0$ such that x should be brought near to a.

Left limit of a function : If $f(x)$ is a function defined in some interval $(a - h, a)$, $(h > 0)$ and if for every $\varepsilon > 0$, there exists $\delta > 0$ such that $l - \varepsilon < f(x) < l + \varepsilon$ whenever $x \in (a - \delta, a)$ and $\delta < h$, we say left limit of $f(x)$ is l as $x \to a-$ or $\lim_{x \to a-} f(x) = l$.

Right limit of a function : If $f(x)$ is a function defined in an interval $(a, a + k)$, $(k > 0)$ and for every $\varepsilon > 0$, there exists $\delta > 0$ such that $l - \varepsilon < f(x) < l + \varepsilon$ whenever $x \in (a, a + \delta)$, $\delta < k$, then we say right limit of $f(x)$ is l as $x \to a+$ or $\lim_{x \to a+} f(x) = l$.

Notes : (1) Nowhere in the definition, it is required that a be in domain of f. $f(x)$ must be defined 'around' a. f must be defined in an interval containing a except for possibly at $x = a$. f may or may not be defined at $x = a$.

(2) $\varepsilon > 0$ is any given number and $\delta > 0$ is to be found out depending upon f.

Let us understand the definition more closely by some examples.

Example 14 : Prove : $\lim\limits_{x \to 2} (3x + 2) = 8$

Solution : Let $\varepsilon > 0$ be any positive number.

We require $8 - \varepsilon < 3x + 2 < 8 + \varepsilon$ **($l = 8$)**

$$8 - \varepsilon < 3x + 2 < 8 + \varepsilon \Leftrightarrow 6 - \varepsilon < 3x < 6 + \varepsilon$$

$$\Leftrightarrow 2 - \frac{\varepsilon}{3} < x < 2 + \frac{\varepsilon}{3}.$$

Comparing with $2 - \delta < x < 2 + \delta$, we are motivated to let $\delta = \frac{\varepsilon}{3}$. **($a = 2$)**

Now let $\delta = \frac{\varepsilon}{3}$

$$\therefore \quad 2 - \delta < x < 2 + \delta, x \neq 2 \Rightarrow 2 - \frac{\varepsilon}{3} < x < 2 + \frac{\varepsilon}{3}$$

$$\Rightarrow 6 - \varepsilon < 3x < 6 + \varepsilon$$

$$\Rightarrow 8 - \varepsilon < 3x + 2 < 8 + \varepsilon$$

This is what we wanted and $\delta = \frac{\varepsilon}{3}$ exists for every $\varepsilon > 0$ such that

$$2 - \delta < x < 2 + \delta, x \neq 2 \Rightarrow 8 - \varepsilon < 3x + 2 < 8 + \varepsilon$$

$\therefore \quad \lim\limits_{x \to 2} (3x + 2) = 8.$

Example 15 : Prove : $\lim\limits_{x \to a} x = a$

Solution : Let $\varepsilon = \delta$, $\varepsilon > 0$. Then, $a - \delta < x < a + \delta, x \neq a \Rightarrow a - \varepsilon < x < a + \varepsilon$

$\therefore \quad \lim\limits_{x \to a} x = a$

Note : It is not obvious that $x \to a$, as $x \to a$, we have proved it using definition.

Example 16 : Prove : $\lim\limits_{x \to a} (mx + c) = ma + c \qquad (m \neq 0)$

Solution : Let $\delta = \frac{\varepsilon}{|m|}$, $\varepsilon > 0$.

$$a - \delta < x < a + \delta, x \neq a \Rightarrow a - \frac{\varepsilon}{|m|} < x < a + \frac{\varepsilon}{|m|}$$

$$\Rightarrow ma - \frac{\varepsilon}{|m|}m < mx < ma + \frac{\varepsilon}{|m|}m$$ **($m > 0$)**

$$\Rightarrow ma - \varepsilon < mx < ma + \varepsilon$$

$$\Rightarrow ma - \varepsilon + c < mx + c < ma + \varepsilon + c$$

Let $l = ma + c$

$\therefore \quad a - \delta < x < a + \delta, x \neq a \Rightarrow l - \varepsilon < mx + c < l + \varepsilon$

$\therefore \quad$ If $m > 0$, $\lim\limits_{x \to a} (mx + c) = ma + c$

Similarly if $m < 0$ we can prove.

$$a - \delta < x < a + \delta, x \neq a \Rightarrow ma + c + \varepsilon > mx + c > ma + c - \varepsilon \qquad (|m| = -m)$$

$$\Rightarrow ma + c - \varepsilon < mx + c < ma + c + \varepsilon$$

$\therefore$ If $m < 0$, $\lim_{x \to a} (mx + c) = ma + c$.

10.4 Algebra of Limits

It is tedious and difficult to find limits using definition. So some working rules are derived. They can be proved but we will not prove them.

Let $\lim_{x \to a} f(x)$ exist and be equal to l and let $\lim_{x \to a} g(x)$ exist and be equal to m.

Then (1) $\lim_{x \to a} (f(x) + g(x))$ exists and

$$\lim_{x \to a} (f(x) + g(x)) = \lim_{x \to a} f(x) + \lim_{x \to a} g(x) = l + m$$

(2) $\lim_{x \to a} (f(x)\, g(x))$ exists and

$$\lim_{x \to a} (f(x)\, g(x)) = \lim_{x \to a} f(x) \lim_{x \to a} g(x) = lm$$

(3) If $m \neq 0$, $\lim_{x \to a} \frac{f(x)}{g(x)}$ exists and $\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)} = \frac{l}{m}$

Example 17 : Prove if $f(x)$ is a constant function and if $f(x) = c$, then $\lim_{x \to a} f(x) = c$ or in other words $\lim_{x \to a} c = c$.

Deduce $\lim_{x \to a} cf(x) = c \lim_{x \to a} f(x)$, if $\lim_{x \to a} f(x)$ exists.

Solution : Let $f(x) = c$ and $x \in (a - \delta, a + \delta) - \{a\}$. Let $l = c$.

$$a - \delta < x < a + \delta, x \neq a \Rightarrow |f(x) - l| = |c - c| = 0 < \varepsilon \text{ as } 0 < \varepsilon.$$

$\therefore$ $\lim_{x \to a} f(x) = c$ i.e. $\lim_{x \to a} c = c$

If $\lim_{x \to a} f(x)$ exists, then $\lim_{x \to a} cf(x) = \lim_{x \to a} c \lim_{x \to a} f(x)$

$$= c \lim_{x \to a} f(x)$$

Note : Using $\lim_{x \to a} (f(x) + g(x)) = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$ and

$\lim_{x \to a} cf(x) = c \lim_{x \to a} f(x)$, we can prove

$$\lim_{x \to a} (f(x) - g(x)) = \lim_{x \to a} f(x) - \lim_{x \to a} g(x)$$

If $c = -1$,

$$\lim_{x \to a} (f(x) - g(x)) = \lim_{x \to a} (f(x) + (-1)g(x))$$

$$= \lim_{x \to a} f(x) + \lim_{x \to a} (-1)g(x)$$

$$= \lim_{x \to a} f(x) - \lim_{x \to a} g(x)$$

Theorem 1 : Prove $\lim_{x \to a} x^n = a^n \qquad n \in N$

Let P(n) : $\lim_{x \to a} x^n = a^n \qquad n \in N$

We have proved $\lim_{x \to a} x = a$

$\therefore$ P(1) is true.

Let P(k) be true.

$\therefore \lim_{x \to a} x^k = a^k$

Let $n = k + 1$

$\therefore \lim_{x \to a} x^{k+1} = \lim_{x \to a} x^k \cdot x$

$= \lim_{x \to a} x^k \lim_{x \to a} x$ **(Product rule for limits)**

$= a^k \cdot a = a^{k+1}$ **(P(k) and P(1))**

$\therefore$ P($k + 1$) is true.

$\therefore$ P(n) is true, $\forall n \in N$ by P.M.I.

Theorem 2 : $\lim_{x \to a} (f_1(x) + f_2(x) + ... + f_n(x)) = \lim_{x \to a} f_1(x) + \lim_{x \to a} f_2(x) + ... + \lim_{x \to a} f_n(x)$,

Let P(n) : $\lim_{x \to a} (f_1(x) + f_2(x) + ... + f_n(x)) = \lim_{x \to a} f_1(x) + \lim_{x \to a} f_2(x) + ... + \lim_{x \to a} f_n(x)$

if individual limits $\lim_{x \to a} f_i(x)$ exist **($i = 1, 2, 3, ..., n$)**

For $n = 1$ the result is obvious.

Let P(k) be true.

$\therefore \lim_{x \to a} (f_1(x) + f_2(x) + ... + f_k(x)) = \lim_{x \to a} f_1(x) + \lim_{x \to a} f_2(x) + ... + \lim_{x \to a} f_k(x)$

Let $n = k + 1$

$\therefore \lim_{x \to a} (f_1(x) + ... + f_k(x) + f_{k+1}(x))$

$= \lim_{x \to a} (f_1(x) + ... + f_k(x)) + \lim_{x \to a} f_{k+1}(x)$ **$\left(\lim_{x \to a} (f(x) + g(x)) = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)\right)$**

$= \lim_{x \to a} f_1(x) + \lim_{x \to a} f_2(x) + ... + \lim_{x \to a} f_k(x) + \lim_{x \to a} f_{k+1}(x)$ **(P(k))**

$\therefore$ P($k + 1$) is true.

$\therefore$ P(n) is true, $\forall n \in N$ by P.M.I.

Limit of a Polynomial :

We know $f(x) = c_n x^n + c_{n-1} x^{n-1} + ... + c_0$, $x \in R$ **($c_n \neq 0$, $c_0, c_1, ... c_n \in R$)**

is called a polynomial of degree n.

$$\lim_{x \to a} f(x) = \lim_{x \to a} (c_n x^n + c_{n-1} x^{n-1} + ... + c_0)$$

$$= \lim_{x \to a} c_n x^n + \lim_{x \to a} c_{n-1} x^{n-1} + ... + \lim_{x \to a} c_0 \quad \textbf{(Lemma 2)}$$

$$\left(\lim_{x \to a} (f(x) + g(x)) = \lim_{x \to a} f(x) + \lim_{x \to a} g(x) \right)$$

$$= \lim_{x \to a} c_n \lim_{x \to a} x^n + \lim_{x \to a} c_{n-1} \lim_{x \to a} x^{n-1} + ... + \lim_{x \to a} c_0$$

$$\left(\lim_{x \to a} (f(x)\, g(x)) = \lim_{x \to a} f(x) \lim_{x \to a} g(x) \right)$$

$$= c_n a^n + c_{n-1} a^{n-1} + ... + c_0 \quad \left(\lim_{x \to a} x^n = a^n, \lim_{x \to a} c_k = c_k \right)$$

$$= f(a)$$

Thus, limit of a polynomial as $x \longrightarrow a$ is obtained by just substituting $x = a$ in the polynomial.

(This is called **'continuity'** of polynomials.)

Example 18 : Find $\lim_{x \to 2} (2x^3 + 3x^2 - 5x + 1)$.

Solution : $\lim_{x \to 2} (2x^3 + 3x^2 - 5x + 1) = 2 \cdot 2^3 + 3 \cdot 2^2 - 5 \cdot 2 + 1$

$= 16 + 12 - 10 + 1$

$= 19$

Limit of Rational Functions :

If $p(x)$ and $q(x)$ are polynomials defined over a domain in which $q(x) \neq 0$, then $h(x) = \frac{p(x)}{q(x)}$ is called a rational function.

If $p(x)$ and $q(x)$ are polynomials defined in a domain containing a and $q(a) \neq 0$ then

$$\lim_{x \to a} h(x) = \lim_{x \to a} \frac{p(x)}{q(x)} = \frac{\lim_{x \to a} p(x)}{\lim_{x \to a} q(x)} = \frac{p(a)}{q(a)} = h(a).$$

In other words rational function $h(x)$ is also a **'continuous'** function.

Example 19 : Find $\lim_{x \to 1} \frac{x^2 + 1}{x^2 + 3x + 4}$.

Solution : Here $x^2 + 3x + 4 \neq 0$ for $x = 1$.

$$\therefore \quad \lim_{x \to 1} \frac{x^2 + 1}{x^2 + 3x + 4} = \frac{2}{8} = \frac{1}{4}$$

Hence in case of a rational function $h(x) = \frac{p(x)}{q(x)}$ if $q(a) \neq 0$, then $\lim_{x \to a} h(x) = h(a)$ is obtained by just substituting $x = a$ in $h(x)$. But what happens if $q(a) = 0$?

By remainder theorem, we know that $x - a$ is a factor of $q(x)$. Now we consider some cases.

Case (1) : $p(x) = (x - a)^k f(x)$

$q(x) = (x - a)^k g(x)$, $f(a) \neq 0$, $g(a) \neq 0$, $k \in N$

Now $\lim_{x \to a} h(x) = \lim_{x \to a} \frac{p(x)}{q(x)}$

$$= \lim_{x \to a} \frac{(x-a)^k f(x)}{(x-a)^k g(x)}$$

$$= \lim_{x \to a} \frac{f(x)}{g(x)} \qquad \text{(while discussing limit } x \neq a\text{)}$$

$$= \frac{f(a)}{g(a)}$$

Thus, if $(x - a)$ occurs to the same index in both numerator and denominator, we can cancel it and have the limit by substituting $x = a$ after cancellation of the factor $(x - a)^k$.

Example 20 : Find $\lim_{x \to 0} \frac{x^3 - 3x^2 + x}{4x^3 - 5x^2 + 3x}$.

Solution : Here $\lim_{x \to 0} \frac{x^3 - 3x^2 + x}{4x^3 - 5x^2 + 3x} = \lim_{x \to 0} \frac{x(x^2 - 3x + 1)}{x(4x^2 - 5x + 3)}$

$$= \lim_{x \to 0} \frac{x^2 - 3x + 1}{4x^2 - 5x + 3}$$

$$= \frac{1}{3}$$

Example 21 : Find $\lim_{x \to 1} \frac{x^4 - 7x^3 + 8x^2 - 3x + 1}{3x^4 - 5x^3 + 6x^2 - 10x + 6}$.

Solution : $\lim_{x \to 1} \frac{x^4 - 7x^3 + 8x^2 - 3x + 1}{3x^4 - 5x^3 + 6x^2 - 10x + 6} = \lim_{x \to 1} \frac{(x-1)(x^3 - 6x^2 + 2x - 1)}{(x-1)(3x^3 - 2x^2 + 4x - 6)}$

$$= \lim_{x \to 1} \frac{x^3 - 6x^2 + 2x - 1}{3x^3 - 2x^2 + 4x - 6}$$

$$= \frac{-4}{-1} = 4$$

Note : Here $p(1) = q(1) = 0$. Hence $(x - 1)$ is a factor of $p(x)$ and $q(x)$. After factorisation of $p(x)$ and $q(x)$, we remove the factor $(x - 1)$ and substitute $x = 1$.

Example 22 : Find $\lim_{x \to 2} \frac{x^3 - 5x^2 + 8x - 4}{2x^3 - 9x^2 + 12x - 4}$.

Solution : $p(2) = 8 - 20 + 16 - 4 = 0$, $q(2) = 16 - 36 + 24 - 4 = 0$

$\therefore$ $(x - 2)$ is a factor of $p(x)$ and $q(x)$.

$$\lim_{x \to 2} \frac{x^3 - 5x^2 + 8x - 4}{2x^3 - 9x^2 + 12x - 4} = \lim_{x \to 2} \frac{(x-2)^2(x-1)}{(x-2)^2(2x-1)}$$

$$= \lim_{x \to 2} \frac{x-1}{2x-1} = \frac{1}{3}$$

Here $(x - 2)^2$ is a factor of both $p(x)$ and $q(x)$.

Case (2) : Let us see what happens if $(x - a)^k$ and $(x - a)^m$ are factors of $p(x)$ and $q(x)$ respectively where $k \neq m$ and $\frac{p(x)}{(x-a)^k}$ and $\frac{q(x)}{(x-a)^m}$ do not have $x - a$ as a factor.

Now $h(x) = \dfrac{p(x)}{q(x)} = \dfrac{(x-a)^k f(x)}{(x-a)^m g(x)} = \dfrac{(x-a)^{k-m} f(x)}{g(x)}$ if $k > m$.

Here $k - m \in N$

$\therefore$ Also $f(a) \neq 0$, $g(a) \neq 0$.

$\therefore \lim\limits_{x \to a} h(a) = \dfrac{0 \cdot f(a)}{g(a)} = 0$

Thus, if $(x - a)$ occurs to higher index in $p(x)$, then $\lim\limits_{x \to a} h(x) = 0$.

Case (3) : If $p(x) = (x - a)^k f(x)$, $q(x) = (x - a)^m g(x)$ with $k < m$ and $\dfrac{p(x)}{(x-a)^k} = f(x)$ and $\dfrac{q(x)}{(x-a)^m} = g(x)$ are non-zero for $x - a$, we proceed as follows :

$$\lim_{x \to a} h(x) = \lim_{x \to a} \frac{(x-a)^k f(x)}{(x-a)^m g(x)} = \lim_{x \to a} \frac{f(x)}{(x-a)^{m-k} g(x)}$$

Now $f(a)$ is a real number. $(a - a)^{m-k} g(a) = 0$

$\therefore$ Denominator of $h(x)$ becomes unbounded as $x \to a$ and we say $\lim\limits_{x \to a} h(x)$ does not exist.

Example 23 : Find $\lim\limits_{x \to 1} \dfrac{x^3 - 3x^2 + 3x - 1}{x^2 - 1}$.

Solution : $\lim\limits_{x \to 1} \dfrac{x^3 - 3x^2 + 3x - 1}{x^2 - 1} = \lim\limits_{x \to 1} \dfrac{(x-1)^3}{(x-1)(x+1)}$

$$= \lim_{x \to 1} \frac{(x-1)^2}{x+1} = \frac{0}{2} = 0$$

Example 24 : Find $\lim\limits_{x \to 0} \dfrac{x^4 - x^3 + x^2}{x^6 - x^5 + x}$.

Solution : $\lim\limits_{x \to 0} \dfrac{x^4 - x^3 + x^2}{x^6 - x^5 + x} = \lim\limits_{x \to 0} \dfrac{x^2(x^2 - x + 1)}{x(x^5 - x^4 + 1)}$

$$= \lim_{x \to 0} \frac{x(x^2 - x + 1)}{x^5 - x^4 + 1} = \frac{0 \cdot 1}{1} = 0$$

An Important Limit :

$$\lim_{x \to a} \frac{x^n - a^n}{x - a} = na^{n-1}, \quad n \in N \ (x \neq a),\ x,\ a \in R$$

We can see that this is a rational function.

$$\lim_{x \to a} \frac{x^n - a^n}{x - a} = \lim_{x \to a} \frac{(x-a)(x^{n-1} + x^{n-2}a + x^{n-3}a^2 + \ldots + a^{n-1})}{x - a}$$

$$= \lim_{x \to a} (x^{n-1} + x^{n-2}a + x^{n-3}a^2 + \ldots + a^{n-1})$$

$$= a^{n-1} + a^{n-2}a + a^{n-3}a^2 + \ldots + a^{n-1} = na^{n-1}$$

Note : This result is true even for $n \in$ R. But then $x \in$ R$^+$, $a \in$ R$^+$, $x \neq a$.
We will use this extended result in future.

Example 25 : Find $\lim\limits_{x \to 1} \frac{x^{18} - 1}{x^{16} - 1}$.

Solution : $\lim\limits_{x \to 1} \frac{x^{18} - 1}{x^{16} - 1} = \lim\limits_{x \to 1} \frac{x^{18} - 1}{x - 1} \times \frac{x - 1}{x^{16} - 1}$

$$= \frac{\lim\limits_{x \to 1} \frac{x^{18} - 1}{x - 1}}{\lim\limits_{x \to 1} \frac{x^{16} - 1}{x - 1}}$$

$$= \frac{18(1)^{17}}{16(1)^{15}} = \frac{18}{16} = \frac{9}{8}$$

Example 26 : Find $\lim\limits_{x \to -2} \frac{x^5 + 32}{x^3 + 8}$.

Solution : $\lim\limits_{x \to -2} \frac{x^5 + 32}{x^3 + 8} = \lim\limits_{x \to -2} \frac{x^5 - (-2)^5}{x^3 - (-2)^3}$

$$= \frac{\lim\limits_{x \to -2} \frac{x^5 - (-2)^5}{x - (-2)}}{\lim\limits_{x \to -2} \frac{x^3 - (-2)^3}{x - (-2)}}$$

$$= \frac{5(-2)^4}{3(-2)^2} = \frac{5 \cdot 16}{3 \cdot 4} = \frac{20}{3}$$

Example 27 : $\lim\limits_{x \to 2} \frac{x^4 - 16}{x^3 - 3x^2 + 3x - 2}$.

Solution : $\lim\limits_{x \to 2} \frac{x^4 - 16}{x^3 - 3x^2 + 3x - 2} = \frac{\lim\limits_{x \to 2} \frac{x^4 - 2^4}{x - 2}}{\lim\limits_{x \to 2} \frac{(x-2)(x^2 - x + 1)}{x - 2}} = \frac{4 \cdot 2^3}{4 - 2 + 1} = \frac{32}{3}$

Rule of Substitution or Rule of Limit of a Composite Function :

Suppose $\lim\limits_{x \to a} f(x)$ exists and $\lim\limits_{x \to a} f(x) = b$ and $\lim\limits_{y \to b} g(y)$ exists and $\lim\limits_{y \to b} g(y) = l$. Then $\lim\limits_{x \to a} g(f(x)) = l$.

Here $\lim\limits_{x \to a} f(x)$ exists means f is defined in $(a - \delta, a + \delta) - \{a\}$ for some $\delta > 0$ and $y = f(x)$. g is defined in $(b - \delta', b + \delta') - \{b\}$ for some $\delta' > 0$.

Example 28 : Find $\lim\limits_{x \to 0} \frac{(x + 2)^5 - 32}{x}$.

Solution : Let $y = f(x) = x + 2$. Then $\lim\limits_{x \to 0} f(x) = 2 = b$.

$$\lim_{y \to 2} g(y) = \lim_{y \to 2} \frac{y^5 - 2^5}{y - 2}$$

$$= 5 \cdot 2^4 = 80$$

$$\therefore \quad \lim_{x \to 0} g(f(x)) = \lim_{x \to 0} \frac{(x+2)^5 - 32}{x} = 80$$

In practice, we just take the substitution $y = x + 2$ and write $y \to 2$ as $x \to 0$ in the example. This is valid for so called **'continuous'** functions.

Another Method :

$$\lim_{x \to 0} \frac{(x+2)^5 - 32}{x} = \lim_{x \to 0} \frac{x^5 + \binom{5}{1}x^4 \cdot 2 + \binom{5}{2}x^3 \cdot 2^2 + \binom{5}{3}x^2 \cdot 2^3 + \binom{5}{4}x \cdot 2^4 + \binom{5}{5}2^5 - 32}{x}$$

$$= \lim_{x \to 0} \left(x^4 + \binom{5}{1}2x^3 + \binom{5}{2}4x^2 + \binom{5}{3}8x + \binom{5}{4}2^4\right)$$

$$= 5 \cdot 16 = 80$$

Example 29 : Find $\lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h}$.

Solution : Let $y = x + h$. Then $y \to x$ as $h \to 0$.

$$\therefore \quad \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} = \lim_{y \to x} \frac{y^{\frac{1}{2}} - x^{\frac{1}{2}}}{y - x} = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$$

Example 30 : Find $\lim_{x \to 2} \frac{x^3 - 8}{\sqrt{x^2 + x + 2} - \sqrt{3x + 2}}$.

Solution : $\lim_{x \to 2} \frac{x^3 - 8}{\sqrt{x^2 + x + 2} - \sqrt{3x + 2}}$

$$= \lim_{x \to 2} \frac{(x-2)(x^2+2x+4)\left(\sqrt{x^2+x+2} + \sqrt{3x+2}\right)}{\left(\sqrt{x^2+x+2} - \sqrt{3x+2}\right)\left(\sqrt{x^2+x+2} + \sqrt{3x+2}\right)}$$

$$= \lim_{x \to 2} \frac{(x-2)(x^2+2x+4)\left(\sqrt{x^2+x+2} + \sqrt{3x+2}\right)}{(x^2+x+2) - (3x+2)}$$

$$= \lim_{x \to 2} \frac{(x-2)(x^2+2x+4)\left(\sqrt{x^2+x+2} + \sqrt{3x+2}\right)}{x^2 - 2x}$$

$$= \lim_{x \to 2} \frac{(x-2)(x^2+2x+4)\left(\sqrt{x^2+x+2} + \sqrt{3x+2}\right)}{x(x-2)}$$

$$= \lim_{x \to 2} \frac{(x^2+2x+4)\left(\sqrt{x^2+x+2} + \sqrt{3x+2}\right)}{x}$$

$$= \frac{(12)(\sqrt{8} + \sqrt{8})}{2} = 6(4\sqrt{2}) = 24\sqrt{2}$$

$\left(\lim_{x \to 2} \sqrt{x^2 + x + 2} = \sqrt{\lim_{x \to 2} (x^2 + x + 2)} = \sqrt{8}\right.$ by rule of limit of composite function.$\left.\right)$

Two Important Rules :

(1) If $f(x) < g(x)$, $\forall x$ in the same domain and both $\lim\limits_{x \to a} f(x)$ and $\lim\limits_{x \to a} g(x)$ exist, then

$$\lim_{x \to a} f(x) \le \lim_{x \to a} g(x).$$

(2) If $g(x) < f(x) < h(x)$, $\forall x$ in the same domain and if $\lim\limits_{x \to a} g(x)$ and $\lim\limits_{x \to a} h(x)$ exist and are both equal to l, then $\lim\limits_{x \to a} f(x)$ exists and is equal to l.

This is known as **Sandwich Theorem or Squeeze Theorem.**

(We do not prove it.)

Example 31 : Prove $\lim\limits_{x \to 0} x sin\frac{1}{x} = 0$. $(x \neq 0)$

Solution : $-1 \le sin\frac{1}{x} \le 1$

$\therefore \quad -x \le x sin\frac{1}{x} \le x \qquad\qquad (x > 0)$

$$\lim_{x \to 0+} x = 0, \quad \lim_{x \to 0+} -x = -\lim_{x \to 0+} x = 0$$

$\therefore$ **By sandwich theorem** $\lim\limits_{x \to 0+} x sin\frac{1}{x} = 0$

Similarly $\lim\limits_{x \to 0-} x sin\frac{1}{x} = 0$

$\therefore \quad \lim\limits_{x \to 0} x sin\frac{1}{x} = 0$

Note : It is incorrect to argue as follows :

$$\lim_{x \to 0} x sin\frac{1}{x} = \lim_{x \to 0} x \lim_{x \to 0} sin\frac{1}{x}$$

$= 0$ (a number between -1 and 1)

$= 0$

Product rule for limit applies only if both the factors have limits. Here $\lim\limits_{x \to 0} sin\frac{1}{x}$ does not exist. (Look at the graph 10.14)

Example 32 : Prove $\lim\limits_{x \to 0} x^2 sin\frac{1}{x} = 0$. $(x \neq 0)$

Solution : $-1 \le sin\frac{1}{x} \le 1$

$\therefore \quad -x^2 \le x^2 sin\frac{1}{x} \le x^2 \qquad\qquad$ **$(x^2 > 0)$**

$$\lim_{x \to 0} x^2 = 0, \quad \lim_{x \to 0} -x^2 = -\lim_{x \to 0} x^2 = 0$$

$\therefore$ **By sandwich theorem** $\lim\limits_{x \to 0} x^2 sin\frac{1}{x} = 0$

Note : Tabulating values will not offer rigorous results. Thus we can proceed by definition.

Let $\delta = \sqrt{\varepsilon}$

Since $\varepsilon > 0$, δ exists.

$0 < |x - 0| < \delta \Rightarrow 0 < |x| < \sqrt{\varepsilon}$

$\Rightarrow 0 < |x|^2 < \varepsilon$

Now, $\left| x^2 sin\frac{1}{x} - 0 \right| = \left| x^2 sin\frac{1}{x} \right| \leq |x|^2 < \varepsilon$ as $\left| sin\frac{1}{x} \right| \leq 1$ and $0 < |x - 0| < \delta$

$\therefore \lim_{x \to 0} x^2 sin\frac{1}{x} = 0$

10.5 Trigonometric Limits

We proceed to prove some lemmas.

Lemma 1 : $cosx < \frac{sinx}{x} < 1; \quad 0 < |x| < \frac{\pi}{2}$.

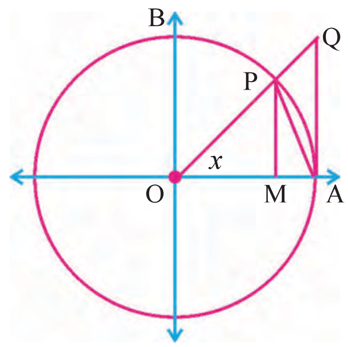

Figure 10.15

Proof : Let x be the radian measure of $\angle AOP$ such that $0 < x < \frac{\pi}{2}$. Then $P(x) \in \overset{\frown}{AB}$. $\odot(O, OA)$ is unit circle.

Let $\overrightarrow{OP}$ intersect tangent at A at Q.

Let $\overline{PM} \perp$ X-axis and $M \in \overline{OA}$.

Obviously area of ΔOAP < area of sector OAP < area of ΔOAQ **(i)**

Now $PM = sinx$, $AQ = \frac{AQ}{OA} \cdot OA = OA\ tanx = tanx$

$\therefore \frac{1}{2}OA \cdot PM < \frac{1}{2}(OA)^2 x < \frac{1}{2} OA \cdot AQ$ **(from (i) and area of a sector = $\frac{1}{2}r^2\theta$)**

$sinx < x < tanx$ **(OA = 1)**

$\therefore 1 < \frac{x}{sinx} < \frac{1}{cosx}$ **($sinx > 0$)**

$\therefore cosx < \frac{sinx}{x} < 1 \qquad 0 < x < \frac{\pi}{2}$

If $x < 0$, let $x = -y$, $y > 0$

$\therefore cosy < \frac{siny}{y} < 1 \qquad 0 < y < \frac{\pi}{2}$

$\therefore cos(-x) < \frac{sin(-x)}{-x} < 1 \qquad 0 < -x < \frac{\pi}{2}$

$\therefore cosx < \frac{sinx}{x} < 1 \qquad 0 < |x| < \frac{\pi}{2}$ **($|x| = -x$)**

Lemma 2 : $|sinx| \le |x| \qquad \forall x \in R$

Proof : If $|x| \ge 1$, then $|sinx| \le 1 \le |x|$ is true.

For $x = 0 \quad |sinx| = 0 \le 0 = |0|$

Thus, we have to prove the result for $0 < |x| < 1$.

We have, $\frac{sinx}{x} < 1 \qquad 0 < |x| < \frac{\pi}{2}$

Let $0 < x < 1$

$\therefore \quad 0 < x < 1 < \frac{\pi}{2}$

$\therefore \quad \frac{sinx}{x} < 1$

$\therefore \quad sinx < x$ **($x > 0$)**

$\therefore \quad |sinx| \le |x|$ as $sinx > 0$, $x > 0$ for $0 < x < \frac{\pi}{2}$.

Let $-1 < x < 0$. Let $x = -y$

Then $-1 < -y < 0$ or $0 < y < 1$

$\therefore \quad |siny| < |y|$

$\therefore \quad |sin(-x)| < |-x|$

$\therefore \quad |-sinx| < |-x|$ Hence $|sinx| < |x|$

$\therefore \quad |sinx| \le |x| \qquad \forall x \in R$

Lemma 3 : $\lim\limits_{x \to 0} |x| = 0$

Proof : Let $\varepsilon = \delta$. Then, $-\delta < x < \delta \Rightarrow |x| < \delta$

$\Rightarrow |x| < \varepsilon$ **($\delta = \varepsilon$)**

$\Rightarrow \big||x|\big| < \varepsilon$ **($(\big||x|\big|) = |x|$)**

$\Rightarrow \big||x| - 0\big| < \varepsilon$

$\therefore \quad \lim\limits_{x \to 0} |x| = 0$

Lemma 4 : If $\lim\limits_{x \to 0} |f(x)| = 0$, $\lim\limits_{x \to 0} f(x) = 0$

Proof : $-|f(x)| \le f(x) \le |f(x)|$

$\lim\limits_{x \to 0} -|f(x)| = -\lim\limits_{x \to 0} |f(x)| = 0$, $\lim\limits_{x \to 0} |f(x)| = 0$

$\therefore$ **By sandwich theorem** $\lim\limits_{x \to 0} f(x) = 0$

Lemma 5 : $\lim_{x \to 0} \sin x = 0$

Proof : $0 \leq |\sin x| \leq |x| \quad \forall x \in R$

$\lim_{x \to 0} 0 = 0, \quad \lim_{x \to 0} |x| = 0$ **(Sandwich theorem)**

$\lim_{x \to 0} |\sin x| = 0$

$\therefore \lim_{x \to 0} \sin x = 0$ **(Lemma 4)**

Lemma 6 : $1 - \frac{x^2}{2} \leq \cos x \leq 1 \quad \forall x \in R$

Proof : We know $1 - \cos x = 2\sin^2 \frac{x}{2}$

$|\sin x| \leq |x|$

$\therefore \left|\sin \frac{x}{2}\right| \leq \left|\frac{x}{2}\right|$

$\therefore \sin^2 \frac{x}{2} \leq \frac{x^2}{4}$

$\therefore 1 - \cos x = 2\sin^2 \frac{x}{2} \leq 2 \times \frac{x^2}{4} = \frac{x^2}{2}$

$\therefore 1 - \frac{x^2}{2} \leq \cos x \leq 1$

Theorem 3 : $\lim_{x \to 0} \cos x = 1$

Proof : $1 - \frac{x^2}{2} \leq \cos x \leq 1$

$\lim_{x \to 0} 1 - \frac{x^2}{2} = 1 - 0 = 1$ **(limit of a polynomial)**

$\therefore \lim_{x \to 0} 1 = 1$

$\therefore$ **By sandwich theorem** $\lim_{x \to 0} \cos x = 1$

Theorem 4 : $\lim_{x \to 0} \frac{\sin x}{x} = 1$

Proof : $\cos x < \frac{\sin x}{x} < 1 \quad 0 < |x| < \frac{\pi}{2}$

$\therefore \lim_{x \to 0} \cos x = 1, \quad \lim_{x \to 0} 1 = 1$

$\therefore$ **By sandwich theorem** $\lim_{x \to 0} \frac{\sin x}{x} = 1$

Theorem 5 : $\lim_{x \to a} \sin x = \sin a$

Proof : Let $x - a = h$. Then $x = a + h$

$\therefore$ As $x \to a, \quad h \to 0$

$\therefore \quad \lim_{h \to 0} \sin h = 0$

$$\lim_{h \to 0} \sin(a+h) = \lim_{h \to 0} (\sin a \cos h + \cos a \sin h)$$

$$= \sin a \lim_{h \to 0} \cos h + \cos a \lim_{h \to 0} \sin h \qquad \textbf{(algebra of limits)}$$

$$= \sin a \cdot 1 + \cos a \cdot 0 \qquad \left(\lim_{h \to 0} \cos h = 1, \lim_{h \to 0} \sin h = 0\right)$$

$$= \sin a$$

$\therefore \quad \lim_{x \to a} \sin x = \sin a$

Theorem 6 : $\lim_{x \to a} \cos x = \cos a$

Proof : Again let $x = a + h$

$\therefore$ As $x \to a, \quad h \to 0$

$\therefore \quad \lim_{h \to 0} \cos h = 1$ and $\lim_{h \to 0} \sin h = 0$

$$\lim_{x \to a} \cos x = \lim_{h \to 0} \cos(a+h) = \lim_{h \to 0} (\cos a \cos h - \sin a \sin h)$$

$$= \cos a \lim_{h \to 0} \cos h - \sin a \lim_{h \to 0} \sin h \qquad \textbf{(algebra of limits)}$$

$$= \cos a \cdot 1 + \sin a \cdot 0$$

$$= \cos a$$

$\therefore \quad \lim_{x \to a} \cos x = \cos a$

Theorem 7 : $\lim_{x \to 0} \frac{\tan x}{x} = 1$

Proof : $\lim_{x \to 0} \frac{\tan x}{x} = \lim_{x \to 0} \frac{\sin x}{x \cos x}$

$$= \frac{\lim_{x \to 0} \frac{\sin x}{x}}{\lim_{x \to 0} \cos x} = \frac{1}{1} = 1 \qquad \left(\lim_{x \to 0} \frac{\sin x}{x} = 1, \lim_{x \to 0} \cos x = 1\right)$$

$\therefore \quad \lim_{x \to 0} \frac{\tan x}{x} = 1$

Now we will apply these results to examples.

Example 33 : Find $\lim_{x \to 0} \frac{\sin ax}{\sin bx}$, $\quad a, b \neq 0$

Solution : $\lim_{x \to 0} \frac{\sin ax}{\sin bx} = \frac{\lim_{x \to 0} \frac{\sin ax}{ax} \cdot a}{\lim_{x \to a} \frac{\sin bx}{bx} \cdot b}$

$$= \frac{1 \cdot a}{1 \cdot b} = \frac{a}{b} \qquad \left(\lim_{x \to 0} \frac{\sin x}{x} = 1\right)$$

Example 34 : Find $\lim\limits_{x\to 0}\dfrac{1-\cos 2x}{1-\cos x}$.

Solution : $\lim\limits_{x\to 0}\dfrac{1-\cos 2x}{1-\cos x} = \lim\limits_{x\to 0}\dfrac{2\sin^2 x}{2\sin^2\frac{x}{2}}$

$$= \lim_{x\to 0}\frac{\sin x}{x}\cdot\frac{\sin x}{x}\cdot\frac{\frac{x}{2}\cdot 2}{\sin\frac{x}{2}}\cdot\frac{\frac{x}{2}\cdot 2}{\sin\frac{x}{2}}$$

$$= 1\cdot 1\cdot 2\cdot 2 = 4$$

Another Method : $\lim\limits_{x\to 0}\dfrac{1-\cos 2x}{1-\cos x} = \lim\limits_{x\to 0}\dfrac{(1-\cos 2x)(1+\cos 2x)(1+\cos x)}{(1+\cos 2x)(1-\cos x)(1+\cos x)}$

$$= \lim_{x\to 0}\frac{\sin^2 2x(1+\cos x)}{\sin^2 x(1+\cos 2x)}$$

$$= \lim_{x\to 0}\frac{\sin^2 2x}{4x^2}\,\frac{4x^2}{\sin^2 x}\,\frac{(1+\cos x)}{(1+\cos 2x)}$$

$$= 1\cdot 4\cdot\frac{(2)}{(2)} = 4$$

Example 35 : Find $\lim\limits_{x\to 0}\dfrac{\sin ax+bx}{ax+\sin bx}$.

Solution : $\lim\limits_{x\to 0}\dfrac{\sin ax+bx}{ax+\sin bx} = \lim\limits_{x\to 0}\dfrac{\frac{\sin ax}{x}+b}{a+\frac{\sin bx}{x}}$

$$= \lim_{x\to 0}\frac{a\frac{\sin ax}{ax}+b}{a+b\frac{\sin bx}{bx}}$$

$$= \frac{a+b}{a+b} = 1$$

Example 36 : Find $\lim\limits_{x\to\frac{\pi}{2}}\dfrac{\tan 2x}{\frac{\pi}{2}-x}$.

Solution : Let $\frac{\pi}{2} - x = \alpha$. Then as $x \to \frac{\pi}{2}$, $\alpha \to 0$

$\therefore \quad \lim\limits_{x\to\frac{\pi}{2}}\dfrac{\tan 2x}{\frac{\pi}{2}-x} = \lim\limits_{\alpha\to 0}\dfrac{\tan 2\left(\frac{\pi}{2}-\alpha\right)}{\alpha}$

$$= \lim_{\alpha\to 0}\frac{\tan(\pi-2\alpha)}{\alpha} = \lim_{\alpha\to 0}\frac{-\tan 2\alpha}{2\alpha}\cdot 2 = -2$$

Example 37 : Find $\lim\limits_{x\to 0}\dfrac{\tan x-\sin x}{x^3}$.

Solution : $\lim\limits_{x\to 0}\dfrac{\tan x-\sin x}{x^3} = \lim\limits_{x\to 0}\dfrac{\frac{\sin x}{\cos x}-\sin x}{x^3}$

$$= \lim_{x\to 0}\frac{\sin x(1-\cos x)}{\cos x\cdot x^3}$$

$$= \lim_{x\to 0}\frac{\tan x\cdot 2\sin\frac{x}{2}\cdot\sin\frac{x}{2}}{x\cdot 2\cdot\frac{x}{2}\cdot\frac{x}{2}\cdot 2}$$

$$= \frac{2}{4} = \frac{1}{2}$$

Example 38 : Find $\lim_{x \to 0} \frac{cosecx - cotx}{x}$.

Solution : $\lim_{x \to 0} \frac{cosecx - cotx}{x} = \lim_{x \to 0} \frac{1 - cosx}{sinx \cdot x}$

$$= \lim_{x \to 0} \frac{2sin\frac{x}{2} \cdot sin\frac{x}{2}}{2 \cdot \frac{x}{2} \cdot \frac{x}{2} \cdot 2} \frac{x}{sinx}$$

$$= \frac{2}{4} = \frac{1}{2}$$

or $\lim_{x \to 0} \frac{1 - cosx}{sinx \cdot x} = \lim_{x \to 0} \frac{(1 - cosx)(1 + cosx)}{(1 + cosx)\ xsinx}$

$$= \lim_{x \to 0} \frac{sin^2 x}{xsinx\ (1 + cosx)}$$

$$= \lim_{x \to 0} \frac{sinx}{x(1 + cosx)}$$

$$= \frac{1}{1+1} = \frac{1}{2}$$

Example 39 : Find $\lim_{x \to \frac{\pi}{4}} \frac{sinx - cosx}{\frac{\pi}{4} - x}$.

Solution : $\lim_{x \to \frac{\pi}{4}} \frac{sinx - cosx}{\frac{\pi}{4} - x} = \lim_{x \to \frac{\pi}{4}} \frac{\sqrt{2}\left(\frac{1}{\sqrt{2}} sinx - \frac{1}{\sqrt{2}} cosx\right)}{\frac{\pi}{4} - x}$

$$= \lim_{x \to \frac{\pi}{4}} \frac{\sqrt{2}\left(sinx\ cos\frac{\pi}{4} - sin\frac{\pi}{4} cosx\right)}{\frac{\pi}{4} - x}$$

$$= \lim_{x \to \frac{\pi}{4}} \frac{\sqrt{2}\, sin\left(x - \frac{\pi}{4}\right)}{-\left(x - \frac{\pi}{4}\right)}$$

$$= \lim_{\alpha \to 0} \frac{\sqrt{2}\, sin\,\alpha}{-\alpha} \qquad \left(\textbf{take } \alpha = x - \frac{\pi}{4},\ \alpha \to 0\right)$$

$$= -\sqrt{2}$$

Miscellaneous Problems :

Example 40 : Find $\lim_{x \to 1} \left(\frac{1}{x-1} - \frac{2}{x^2 - 1}\right)$.

Solution : $\lim_{x \to 1} \left(\frac{1}{x-1} - \frac{2}{x^2 - 1}\right) = \lim_{x \to 1} \frac{x + 1 - 2}{x^2 - 1}$

$$= \lim_{x \to 1} \frac{(x-1)}{(x-1)(x+1)}$$

$$= \lim_{x \to 1} \frac{1}{x+1}$$

$$= \frac{1}{2}$$

Example 41 : Find $\lim\limits_{x \to -2} \dfrac{x^3 + x^2 + 4x + 12}{x^3 - 3x + 2}$.

Solution : $\lim\limits_{x \to -2} \dfrac{x^3 + x^2 + 4x + 12}{x^3 - 3x + 2} = \lim\limits_{x \to -2} \dfrac{(x+2)(x^2 - x + 6)}{(x+2)(x^2 - 2x + 1)}$

$= \lim\limits_{x \to -2} \dfrac{x^2 - x + 6}{x^2 - 2x + 1}$

$= \dfrac{4 + 2 + 6}{4 + 4 + 1}$

$= \frac{12}{9} = \frac{4}{3}$

Example 42 : Find $\lim\limits_{x \to 0} \dfrac{\sqrt{x^2 + x + 1} - \sqrt{x + 1}}{x^2}$.

Solution : $\lim\limits_{x \to 0} \dfrac{\left(\sqrt{x^2 + x + 1} - \sqrt{x + 1}\right)\left(\sqrt{x^2 + x + 1} + \sqrt{x + 1}\right)}{x^2\left(\sqrt{x^2 + x + 1} + \sqrt{x + 1}\right)}$

$= \lim\limits_{x \to 0} \dfrac{x^2 + x + 1 - x - 1}{x^2\left(\sqrt{x^2 + x + 1} + \sqrt{x + 1}\right)}$

$= \lim\limits_{x \to 0} \dfrac{1}{\sqrt{x^2 + x + 1} + \sqrt{x + 1}}$

$= \frac{1}{2}$

Example 43 : Find $\lim\limits_{x \to \frac{\pi}{2}} \left(x\tan x - \frac{\pi}{2}\sec x\right)$.

Solution : $\lim\limits_{x \to \frac{\pi}{2}} \left(x\tan x - \frac{\pi}{2}\sec x\right) = \lim\limits_{x \to \frac{\pi}{2}} \dfrac{x\sin x - \frac{\pi}{2}}{\cos x}$

$= \lim\limits_{\alpha \to 0} \dfrac{\left(\frac{\pi}{2} - \alpha\right)\cos\alpha - \frac{\pi}{2}}{\sin\alpha}$ $\qquad \left(\frac{\pi}{2} - x = \alpha, \alpha \to 0\right)$

$= \lim\limits_{\alpha \to 0} \dfrac{\frac{\pi}{2}(\cos\alpha - 1)}{\sin\alpha} - \dfrac{\alpha\cos\alpha}{\sin\alpha}$

$= \lim\limits_{\alpha \to 0} \left(\dfrac{-\frac{\pi}{2}\left(2\sin^2\frac{\alpha}{2}\right)}{2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}} - \dfrac{\alpha}{\tan\alpha}\right)$

$= \lim\limits_{\alpha \to 0} \left(-\frac{\pi}{2}\tan\frac{\alpha}{2} - \dfrac{\alpha}{\tan\alpha}\right)$

$= -1$

Example 44 : Find $\lim\limits_{x \to 1} (1 - x)\tan\frac{\pi x}{2}$.

Solution : Let $1 - x = \alpha$, $\alpha \to 0$ as $x \to 1$.

$$\therefore \quad \lim_{x \to 1} (1 - x) \tan \frac{\pi x}{2} = \lim_{\alpha \to 0} \alpha \tan \frac{\pi}{2}(1 - \alpha)$$

$$= \lim_{\alpha \to 0} \alpha \tan\left(\frac{\pi}{2} - \frac{\pi\alpha}{2}\right)$$

$$= \lim_{\alpha \to 0} \alpha \cot \frac{\pi\alpha}{2}$$

$$= \lim_{\alpha \to 0} \frac{\frac{\pi}{2}\alpha}{\frac{\pi}{2} \tan \frac{\pi\alpha}{2}}$$

$$= \frac{1}{\frac{\pi}{2}} = \frac{2}{\pi}$$

Example 45 : Find $\lim_{x \to 1} \left(\frac{m}{1-x^m} - \frac{n}{1-x^n}\right)$; $(m, n \in \text{N})$.

Solution : $\lim_{x \to 1} \left(\frac{m}{1-x^m} - \frac{n}{1-x^n}\right) = \lim_{x \to 1} \frac{m(1-x^n) - n(1-x^m)}{(1-x^n)(1-x^m)}$

Let $x = 1 + h$ so that $h \to 0$ as $x \to 1$

$$\therefore \quad \lim_{x \to 1} \left(\frac{m}{1-x^m} - \frac{n}{1-x^n}\right)$$

$$= \lim_{h \to 0} \frac{m[1-(1+h)^n] - n[1-(1+h)^m]}{[(1+h)^m - 1][(1+h)^n - 1]}$$

$$= \lim_{h \to 0} \frac{m\left(1-1-nh-\binom{n}{2}h^2-\binom{n}{3}h^3-\ldots-h^n\right)-n\left(1-1-\binom{m}{1}h-\binom{m}{2}h^2-\ldots-h^m\right)}{\left(\binom{m}{1}h+\binom{m}{2}h^2+\ldots+h^m\right)\left(\binom{n}{1}h+\binom{n}{2}h^2+\ldots+h^n\right)}$$

$$= \lim_{h \to 0} \frac{h\left(-mn-m\binom{n}{2}h-m\binom{n}{3}h^2-\ldots-mh^{n-1}+nm+n\binom{m}{2}h+n\binom{m}{3}h^2+\ldots+nh^{m-1}\right)}{h\left(\binom{m}{1}+\binom{m}{2}h+\ldots+h^{m-1}\right)\cdot h\left(\binom{n}{1}+\binom{n}{2}h+\ldots+h^{n-1}\right)}$$

$$= \lim_{h \to 0} \frac{h\left(-m\binom{n}{2}-m\binom{n}{3}h-\ldots-mh^{n-2}+n\binom{m}{2}+n\binom{m}{3}h+\ldots+nh^{m-2}\right)}{h\left(\binom{m}{1}+\binom{m}{2}h+\ldots+h^{m-1}\right)\left(\binom{n}{1}+\binom{n}{2}h+\ldots+h^{n-1}\right)}$$

$$= \frac{-m\binom{n}{2}+n\binom{m}{2}}{\binom{m}{1}\binom{n}{1}}$$

$$= \frac{\frac{-mn(n-1)}{2}+\frac{nm(m-1)}{2}}{mn}$$

$$= \frac{m-1-n+1}{2}$$

$$= \frac{m-n}{2}$$

Exercise 10

Using algebra of limits and the definition of limit prove the following : **(1 to 3)**

1. $\lim_{x \to 2} x^2 = 4$ **2.** $\lim_{x \to 1} |x|^2 = 1$ **3.** $\lim_{x \to 3} x^3 = 27$

Prove following limits do not exist : **(4 to 6)**

4. $\lim_{x \to 0} \frac{|x|}{x}$ **5.** $\lim_{x \to 3} \frac{|x-3|}{x-3}$ **6.** $\lim_{x \to 2} [x]$

7. For $f(x) = \frac{x^2 - 9}{x - 3}$, $x \neq 3$, $f(3) = 6$, prove $\lim_{x \to 3} f(x) = f(3)$.

8. For $f(x) = \frac{x^2 - 1}{x + 1}$, $x \neq -1$, $f(-1) = 5$, prove $\lim_{x \to -1} f(x) \neq f(-1)$.

9. If $\lim_{x \to a} f(x) = \lim_{x \to a} g(x)$ where $x \in (a - \delta, a + \delta) - \{a\}$ for some $\delta > 0$, can we say $f(x) = g(x)$ for all $x \in (a - \delta, a + \delta) - \{a\}$.

10. If $x^2 + 1 \leq f(x) \leq 2x^4 + x^2 + 1$, prove $\lim_{x \to 0} f(x) = 1$.

Find following limits : **(11 to 32)**

11. $\lim_{x \to 64} \frac{x^{\frac{1}{6}} - 2}{\sqrt{x} - 8}$ **12.** $\lim_{x \to 0} \frac{\tan mx}{\tan nx}$

13. $\lim_{x \to \frac{\pi}{3}} \frac{\sqrt{3}\cos x - \sin x}{x - \frac{\pi}{3}}$

14. $\lim_{x \to \alpha} \frac{9\sin x - 40\cos x}{x - \alpha}$ where $\tan\alpha = \frac{40}{9}$, $0 < \alpha < \frac{\pi}{2}$

15. $\lim_{h \to 0} \frac{(x+h)^{\frac{1}{4}} - x^{\frac{1}{4}}}{h}$ **16.** $\lim_{h \to 0} \frac{(x+h)^{\frac{1}{3}} - x^{\frac{1}{3}}}{h}$

17. $\lim_{x \to 1} \frac{x^4 - 3x^3 + 2}{x^3 - 5x^2 + 3x + 1}$ **18.** $\lim_{x \to \frac{\pi}{2}} (\sec x - \tan x)$

19. $\lim_{x \to 1+} \frac{\sqrt{x-1}}{\sqrt{x^2 - 1} + \sqrt{x^3 - 1}}$ **(why $x \to 1+$?)**

20. $\lim_{x \to 1} \frac{x^{n+1} - (n+1)x + n}{(x-1)^2}$, $n \in N$ **21.** $\lim_{x \to 0} \frac{2\sin x - \sin 2x}{x^3}$

22. $\lim_{x \to \frac{\pi}{4}} \frac{\sin 3x + \cos 3x}{x - \frac{\pi}{4}}$ **23.** $\lim_{x \to 0} \frac{(1+mx)^n - (1+nx)^m}{x^2}$ $m, n \in N$

24. $\lim_{x \to \pi} \dfrac{\sqrt{10 + cosx} - 3}{(\pi - x)^2}$

25. $\lim_{x \to 0} \dfrac{cos5x - cos7x}{x^2}$

26. $\lim_{x \to 0} \dfrac{\sqrt{2} - \sqrt{1 + cosx}}{x^2}$

27. $\lim_{x \to \frac{\pi}{2}} \dfrac{secx - tanx}{\frac{\pi}{2} - x}$

28. $\lim_{h \to 0} \dfrac{sin(a + 3h) - 3sin(a + 2h) + 3sin(a + h) - sina}{h^3}$

29. $\lim_{x \to 0} \dfrac{sin(3 + x) - sin(3 - x)}{x}$

30. $\lim_{x \to a} \dfrac{\sqrt{2a + 3x} - \sqrt{x + 4a}}{\sqrt{3a + 2x} - \sqrt{4x + a}}$

31. $\lim_{h \to 0} \dfrac{(a + h)^2 \; sin\,(a + h) - a^2 sina}{h}$

32. $\lim_{h \to 0} \dfrac{(x + h) \; sec\,(x + h) - xsecx}{h}$

33. Select a proper option (a), (b), (c) or (d) from given options and write in the box given on the right so that the statement becomes correct :

(1) $\lim_{x \to \frac{\pi}{2}} \dfrac{1 - sinx}{cosx} = ...$ ☐

(a) 1 (b) 0 (c) −1 (d) 2

(2) $\lim_{x \to 0} \dfrac{|x|}{x}$ .... ☐

(a) is 1 (b) is −1 (c) is zero (d) does not exist

(3) $\lim_{x \to \pi} \dfrac{tanx}{\pi - x}$ .... ☐

(a) is 1 (b) is −1 (c) does not exist (d) is 0

(4) If $\lim_{x \to 2} \dfrac{x^n - 2^n}{x - 2} = 80$, then n = ... ☐

(a) −3 (b) 2 (c) 5 (d) 6

(5) $\lim_{x \to 0} \dfrac{1 - cosmx}{1 - cosnx} = ...$ ☐

(a) $\dfrac{m}{n}$ (b) $\dfrac{m^2}{n^2}$ (c) $\dfrac{m^3}{n^3}$ (d) 0

(6) $\lim_{x \to 0+} \dfrac{|sinx|}{x}$ ...... ☐

(a) is 1 (b) is −1 (c) does not exist (d) is 0

(7) $\lim_{x \to 0-} \frac{\sin[x]}{[x]}$ .... $(-1 < x < 0,\ x \in R)$

(a) is 1 (b) is zero

(c) is -1 (d) is $\sin 1$

(8) $\lim_{x \to 0} \frac{\sin x}{\sqrt{x+1} - \sqrt{1-x}} = ...$

(a) 1 (b) 2 (c) 0 (d) -1

(9) $\lim_{x \to 1} \frac{(\sqrt{x} - 1)(2x - 3)}{2x^2 + x - 3}$ ....

(a) does not exist (b) is 1 (c) is $\frac{1}{10}$ (d) is $-\frac{1}{10}$

(10) $\lim_{x \to 0} \frac{\sin x - 2\sin 3x + \sin 5x}{x} = ...$

(a) 5 (b) 6 (c) 0 (d) 10

(11) If $1 \le f(x) \le x^2 + 2x + 2 \quad \forall x \in R$, $\lim_{x \to -1} f(x) = ...$

(a) 2 (b) 0 (c) -1 (d) 1

(12) $\lim_{x \to 0-} \left(\frac{1}{x} + \frac{1}{|x|}\right) = ...$

(a) 2 (b) 1 (c) 0 (d) -1

(13) $\lim_{x \to 2} f(x) = ...$ where $f(x) = \begin{cases} 2x + 3 & x < 2 \\ 5 & x = 2 \\ 3x + 2 & x > 2 \end{cases}$

(a) 5 (b) 3 (c) 2 (d) does not exist

(14) $\lim_{x \to 0+} f(x) = ...$ where $f(x) = \begin{cases} 3x^2 - 1 & x < 0 \\ 3x^2 + 1 & x \ge 0 \end{cases}$

(a) 1 (b) -1 (c) 0 (d) $\frac{1}{3}$

(15) $\lim_{x \to 5+} [x] = ...$

(a) 6 (b) 5 (c) -5 (d) 4

(16) $\lim_{x \to -4-} [x] = ...$

(a) 5 (b) -5 (c) -4 (d) 4

(17) $\lim_{x \to a} \frac{\sin x - \sin a}{x - a} = ...$

(a) $\cos a$ (b) $\sin a$ (c) a (d) 0

(18) $\lim_{x \to a} \frac{\sin x - \sin a}{\sqrt{x} - \sqrt{a}}$ $(a > 0)$ = ...

(a) $\cos a$ (b) $\frac{\cos a}{2\sqrt{a}}$ (c) $2\sqrt{a}\cos a$ (d) $2\sqrt{a}$

(19) $\lim_{x \to 0} \frac{\tan x - 5x}{7x - \sin x}$ = ...

(a) $\frac{2}{3}$ (b) $\frac{-2}{3}$ (c) $\frac{5}{7}$ (d) $\frac{-5}{7}$

(20) $\lim_{x \to a} \frac{x^{\frac{1}{3}} - a^{\frac{1}{3}}}{x^{\frac{1}{5}} - a^{\frac{1}{5}}}$ $(a > 0)$ = ...

(a) $\frac{1}{3}a^{\frac{3}{5}}$ (b) $\frac{1}{5}a^{\frac{1}{15}}$ (c) $\frac{5}{3}a^{\frac{5}{3}}$ (d) $\frac{5}{3}a^{\frac{2}{15}}$

*

Summary

We studied following points in this chapter :

1. History of limits

2. Graphical and tabulation for inference of limit

3. Formal definition of limit and applications

4. Algebra of limits, if $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ exist, then

$$\lim_{x \to a} (f(x) \pm g(x)) = \lim_{x \to a} f(x) \pm \lim_{x \to a} g(x),$$

$$\lim_{x \to a} (f(x)\ g(x)) = \lim_{x \to a} f(x) \lim_{x \to a} g(x),$$

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)} \quad (\text{where } \lim_{x \to a} g(x) \neq 0)$$

5. $\lim_{x \to a} \frac{x^n - a^n}{x - a} = na^{n-1}$ and rule of substitution

6. Sandwich theorem and trigonometric limits

$$\lim_{x \to 0} \frac{\sin x}{x} = 1, \quad \lim_{x \to 0} \frac{\tan x}{x} = 1, \quad \lim_{x \to 0} \sin x = 0, \quad \lim_{x \to 0} \cos x = 1$$

$$\lim_{x \to a} \sin x = \sin a, \quad \lim_{x \to a} \cos x = \cos a$$

Bhaskara I

Bhaskara stated theorems about the solutions of today's so called Pell equations. For instance, he posed the problem : "Tell me, O mathematician, what is that square which multiplied by 8 becomes - together with unity - a square?" In modern notation, he asked for the solutions of the Pell equation $8x^2 + 1 = y^2$. It has the simple solution $x = 1$, $y = 3$, or shortly $(x,y) = (1,3)$, from which further solutions can be constructed, e.g., $(x,y) = (6,17)$.

Chapter 11

DERIVATIVE

Mathematics is as much an aspect of culture as it is a collection of algorithms.
*– **Carl Boyer** (in a Calculus Textbook)*

11.1 Introduction

In calculus the derivative is a measure of how a function changes as its input changes. Loosely speaking we can think of a derivative as how much one quantity changes in response to changes in some other quantity. The derivative of the position of a moving object with respect to time is its instantaneous velocity.

The derivative of a function at a chosen input value describes the best linear approximation to the function near the input value. For a real function of a real variable, the derivative at a point is equal to the slope of the tangent line to the graph of the function at that point.

For a 'small' h the line passing through $(a, f(a))$ and $(a + h, f(a + h))$ is called a ***secant*** line. Its slope for a value of h near to zero, gives a good approximation to the slope of the tangent line to the curve $y = f(x)$ at $(a, f(a))$ and smaller the value of h, we get a better approximation.

Slope m of the secant line at $(a, f(a))$ is given by

$$m = \frac{f(a+h) - f(a)}{h}$$

This is called Newton's difference quotient.

$\lim\limits_{h \to 0} \dfrac{f(a+h) - f(a)}{h}$ is the derivative of f at a and is denoted by $f'(a)$, if this limit exists. This represents slope of the tangent to $y = f(x)$ at $(a, f(a))$.

We can also say

$$\lim_{h \to 0} \frac{f(a+h) - f(a) - hf'(a)}{h} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} - f'(a)$$

$$= f'(a) - f'(a)$$

$$= 0$$

This gives best linear approximation $f(a + h) \cong f(a) + hf'(a)$ for f near 'small' h.

If we write $Q(h) = \frac{f(a+h) - f(a)}{h}$, $Q(h)$ is the slope of the secant line joining points $(a, f(a))$ and $(a + h, f(a + h))$ on the graph of $y = f(x)$. If the graph of f is a unbroken curve with no gaps, then $\lim_{h \to 0} Q(h)$, if it exists, is called the derivative of f at a and we say f is differentiable at $x = a$.

Rocket scientists need to compute the accurate velocity with which the satellite needs to be shot out from the rocket knowing the height of the rocket. Derivative is a word regularly used in stock market. Financial institutes predict the change in the value of a stock knowing its present value. All these require the knowledge of change in one quantity called dependent variable depending upon the change in another quantity called independent variable.

11.2 Formal Definition and Examples

Definition : Let f be real valued function defined on an interval (a, b). Let $c \in (a, b)$. Let h be sufficiently small so that $c + h \in (a, b)$.

If $\lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$ exists, it is called the derivative of f at c and is denoted by $f'(c)$.

Example 1 : Find $f'(1)$ for $f : R \to R$, $f(x) = 3x + 5$, if it exists.

Solution : $\lim_{h \to 0} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0} \frac{3(1+h) + 5 - 8}{h}$ **($f(1) = 8$)**

$= \lim_{h \to 0} \frac{3h}{h} = 3$

$\therefore$ $f'(1)$ exists and $f'(1) = 3$

Example 2 : For $f : R \to R$, $f(x) = 2x^2 + 3x - 1$, find $f'(0)$, if it exists.

Solution : $\lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{2h^2 + 3h - 1 - (-1)}{h}$ **($f(0) = -1$)**

$= \lim_{h \to 0} \frac{2h^2 + 3h}{h}$

$= \lim_{h \to 0} (2h + 3) = 3$

$\therefore$ $f'(0)$ exists and $f'(0) = 3$

Example 3 : For $f : R \to R$, $f(x) = sinx$, find $f'(0)$, if it exists.

Solution : $\lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{sinh - 0}{h}$ **($sin0 = 0$)**

$= \lim_{h \to 0} \frac{sinh}{h} = 1$

$\therefore$ $f'(0)$ exists and $f'(0) = 1$

Example 4 : For $f : R \to R$, $f(x) = \begin{cases} x\sin\frac{1}{x}, & x \neq 0 \\ 0 & , x = 0 \end{cases}$, find $f'(0)$, if it exists.

Solution : $\lim_{h \to 0} \frac{f(h) - f(0)}{h} = \lim_{h \to 0} \frac{h\sin\frac{1}{h}}{h}$ **($f(0) = 0$)**

$= \lim_{h \to 0} \sin\frac{1}{h}$ does not exist. (Refer chapter 10)

$\therefore$ $f(x) = \begin{cases} x\sin\frac{1}{x}, & x \neq 0 \\ 0 & , x = 0 \end{cases}$ has no derivative at $x = 0$.

Example 5 : For $f : R \to R$, $f(x) = \begin{cases} x^2\sin\frac{1}{x}, & x \neq 0 \\ 0 & , x = 0 \end{cases}$, find $f'(0)$, if it exists.

Solution : $\lim_{h \to 0} \frac{f(h) - f(0)}{h} = \lim_{h \to 0} \frac{h^2\sin\frac{1}{h}}{h}$ **($f(0) = 0$)**

$= \lim_{h \to 0} h\sin\frac{1}{h}$

Now $0 \leq \left|\sin\frac{1}{h}\right| \leq 1 \Rightarrow 0 \leq \left|h\sin\frac{1}{h}\right| \leq |h|$ and $\lim_{h \to 0} 0 = 0$, $\lim_{h \to 0} |h| = 0$

$\therefore$ $\lim_{h \to 0} \left|h\sin\frac{1}{h}\right| = 0$

$\therefore$ $\lim_{h \to 0} h\sin\frac{1}{h} = 0$

$\therefore$ $f'(0)$ exists and $f'(0) = 0$

Definition : Let f be defined on (a, b). Let $x \in (a, b)$ and h be sufficiently small so that $x + h \in (a, b)$. If $\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$ exists, we say f is differentiable at x and call this limit the derivative of f at x. This gives us a function $\frac{d}{dx} f(x)$ defined at all points of $x \in (a, b)$ where f is differentiable and we write $f'(x) = \frac{d}{dx} f(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$, at all points of (a, b) where f is differentiable. (Assuming that f is differentiable at at least one point of (a, b).)

If we write $y = f(x)$, $\frac{d}{dx} f(x)$ may be written as $\frac{dy}{dx}$. Its value at $x = c$ can be written as $\left[\frac{d}{dx} f(x)\right]_{x = c}$ or $\left(\frac{dy}{dx}\right)_{x = c}$ or sometimes $[D f(x)]_{x = c}$ or $f'(c)$.

Example 6 : For $f : R \to R$, $f(x) = ax^2 + bx + c$, find $f'(x)$ and $f'(0)$.

Solution : $\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{[a(x+h)^2 + b(x+h) + c] - (ax^2 + bx + c)}{h}$

$= \lim_{h \to 0} \frac{[a(x^2 + 2hx + h^2) + bx + bh + c] - (ax^2 + bx + c)}{h}$

$= \lim_{h \to 0} \frac{2ahx + ah^2 + bh}{h}$

$$= \lim_{h \to 0} (2ax + ah + b)$$

$$= 2ax + b$$

$\therefore$ $f'(x)$ exists for $\forall x \in$ R and $f'(x) = 2ax + b$

$\therefore$ $f'(0) = b$ **(taking $x = 0$ in $f'(x)$)**

Note : If we obtain $f'(0)$ as $\lim_{h \to 0} \frac{f(h) - f(0)}{h}$, then also we will get b as the answer.

Example 7 : For f : R $\to$ R, $f(x) = ax + b$, find $f'(x)$.

Solution : $\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{a(x+h) + b - (ax+b)}{h}$

$$= \lim_{h \to 0} \frac{ah}{h} = a$$

$\therefore$ $f'(x)$ exists and $f'(x) = a$, $\forall x \in$ R.

Example 8 : For f : R $\to$ R, $f(x) = \frac{ax+b}{cx+d}$, find $f'(x)$ and $f'(0)$. $\left(x \neq -\frac{d}{c}\right)$

Solution : $\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

$$= \lim_{h \to 0} \frac{\frac{a(x+h)+b}{c(x+h)+d} - \frac{ax+b}{cx+d}}{h}$$

$$= \lim_{h \to 0} \frac{(ax + ah + b)(cx + d) - (ax + b)(cx + ch + d)}{(cx + ch + d)(cx + d)h}$$

$$= \lim_{h \to 0} \frac{h(acx + ad - acx - bc)}{(cx + ch + d)(cx + d)h}$$

$$= \lim_{h \to 0} \frac{(ad - bc)}{(cx + ch + d)(cx + d)}$$

$$= \frac{(ad - bc)}{(cx + d)^2}$$

$\therefore$ $f'(x)$ exists and $f'(x) = \frac{ad - bc}{(cx + d)^2}$ **(i)**

$\therefore$ $f'(0) = \frac{ad - bc}{d^2}$

Note : $\frac{d}{dx} \frac{1}{x} = \frac{-1}{x^2}$ (taking $a = 0$, $b = 1$, $c = 1$, $d = 0$ in (i))

11.3 Algebra of Derivatives

Let f and g be differentiable in (a, b).

Then **(1)** $f + g$ is also differentiable in (a, b) and

$$\frac{d}{dx}(f(x) + g(x)) = \frac{d}{dx} f(x) + \frac{d}{dx} g(x)$$

(2) $f - g$ is also differentiable in (a, b) and

$$\frac{d}{dx}(f(x) - g(x)) = \frac{d}{dx} f(x) - \frac{d}{dx} g(x)$$

(3) $f \times g$ is also differentiable in (a, b) and

$$\frac{d}{dx}(f(x)g(x)) = g(x)\frac{d}{dx}f(x) + f(x)\frac{d}{dx}g(x)$$

(4) $\frac{f}{g}$ is also differentiable in (a, b), if $g(x) \neq 0$, and

$$\frac{d}{dx}\frac{f(x)}{g(x)} = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$$

Some Important Results :

(1) $f'(x) = \lim\limits_{t \to x} \frac{f(t) - f(x)}{t - x}$

We have $f'(x) = \lim\limits_{h \to 0} \frac{f(x+h) - f(x)}{h}$

Let $x + h = t$. Then $t \to x$ as $h \to 0$.

$\therefore \quad f'(x) = \lim\limits_{t \to x} \frac{f(t) - f(x)}{t - x}$

(2) The derivative of a constant function is zero.

Let $f(x) = c$, $\forall x \in \mathbb{R}$.

$\therefore \quad \lim\limits_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim\limits_{h \to 0} \frac{c - c}{h} = \lim\limits_{h \to 0} 0 = 0$

$\therefore \quad \frac{d}{dx} c = 0$

(3) $\frac{d}{dx} kf(x) = k\frac{d}{dx}f(x)$ **($k \in \mathbb{R}$ is a constant.)**

$$\frac{d}{dx} kf(x) = k\frac{d}{dx}f(x) + f(x)\frac{d}{dx}k$$

$$= k\frac{d}{dx}f(x) + f(x) \cdot 0 \qquad \textbf{(by (2))}$$

$$= k\frac{d}{dx}f(x)$$

Example 9 : Prove : $\frac{d}{dx}(f(x) - g(x)) = \frac{d}{dx}f(x) - \frac{d}{dx}g(x)$ using

$\frac{d}{dx}(f(x) + g(x)) = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)$ and $\frac{d}{dx} kf(x) = k\frac{d}{dx}f(x)$

Solution : $\frac{d}{dx}(f(x) - g(x)) = \frac{d}{dx}(f(x) + (-1)g(x))$

$$= \frac{d}{dx}f(x) + \frac{d}{dx}(-1)g(x)$$

$$= \frac{d}{dx}f(x) + (-1)\frac{d}{dx}g(x)$$

$$= \frac{d}{dx}f(x) - \frac{d}{dx}g(x)$$

Example 10 : Prove : $\frac{d}{dx} kf(x) = k\frac{d}{dx} f(x),\ k \in \mathrm{R}$

Solution : $\lim\limits_{h \to 0} \frac{kf(x+h) - kf(x)}{h} = \lim\limits_{h \to 0} k \lim\limits_{h \to 0} \frac{f(x+h) - f(x)}{h}$ **(Rules of limit)**

$= k\frac{d}{dx} f(x)$

$\therefore\ \frac{d}{dx} kf(x) = k\frac{d}{dx} f(x)$

Some Standard Forms :

(1) $\frac{d}{dx} x^n = nx^{n-1},\ n \in \mathrm{N},\ x \in \mathrm{R}$

Proof : $\frac{d}{dx} x^n = \lim\limits_{h \to 0} \frac{(x+h)^n - x^n}{h}$

$= \lim\limits_{h \to 0} \frac{\left(x^n + \binom{n}{1}x^{n-1}h + \binom{n}{2}x^{n-2}h^2 + \ldots + h^n\right) - x^n}{h}$

$= \lim\limits_{h \to 0} \frac{nx^{n-1}h + \binom{n}{2}x^{n-2}h^2 + \ldots + h^n}{h}$

$= \lim\limits_{h \to 0} \left(nx^{n-1} + \frac{n(n-1)}{2} x^{n-2} h + \binom{n}{3} x^{n-3} h^2 + \ldots + h^{n-1}\right) = nx^{n-1}$

Second Proof : Let P(n) : $\frac{d}{dx} x^n = nx^{n-1}$

We have $\frac{d}{dx} x^1 = \lim\limits_{h \to 0} \frac{x+h-x}{h} = \lim\limits_{h \to 0} \frac{h}{h} = 1$. Also $1 \cdot x^{1-1} = 1 \cdot 1 = 1$ **($x \neq 0$)**

$\therefore$ P(1) is true.

Let P(k) be true.

$\therefore\ \frac{d}{dx} x^k = kx^{k-1}$

Let $n = k + 1$

$\frac{d}{dx} x^{k+1} = \frac{d}{dx} x^k \cdot x$

$= x^k \frac{d}{dx} x + x \frac{d}{dx} x^k$

$= x^k \cdot 1 + x \cdot kx^{k-1}$

$= x^k + kx^k$

$= (k + 1)x^k$

$\therefore$ P(k + 1) is true.

$\therefore$ P(n) is true, $\forall n \in \mathrm{N}$ by P.M.I.

$\therefore$ By the principle of mathematical induction P(n) is true, $\forall n \in \mathrm{N}$.

Third Proof : $\frac{d}{dx}x^n = \lim_{t \to x} \frac{t^n - x^n}{t - x}$ **(formula (2))**

$$= \lim_{t \to x} \frac{(t - x)(t^{n-1} + t^{n-2}x + t^{n-3}x^2 + ... + x^{n-1})}{t - x}$$

$$= \lim_{t \to x} (t^{n-1} + t^{n-2}x + t^{n-3}x^2 + ... + x^{n-1})$$

$$= x^{n-1} + x^{n-2} \cdot x + x^{n-3} \cdot x^2 + ... + x^{n-1}$$

$$= nx^{n-1}$$

Note : We have given the proof for $n \in$ N, $x \in$ R, but the result is valid for $n \in$ R, $x \in$ R$^+$. We will not prove it.

(2) Derivative of a Polynomial :

Let $P(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + ... + a_0$, $x \in$ R, $n \in$ N, $a_n \neq 0$, $a_i \in$ R **(i = 0, 1, 2, ..., n)**

be a polynomial of degree n.

$$\therefore \quad \frac{d}{dx}P(x) = \frac{d}{dx}(a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + ... + a_0)$$

$$= \frac{d}{dx}a_n x^n + \frac{d}{dx}a_{n-1}x^{n-1} + \frac{d}{dx}a_{n-2}x^{n-2} + ... + \frac{d}{dx}a_0 \quad \textbf{(Derivative of sum)}$$

$$= a_n \frac{d}{dx}x^n + a_{n-1}\frac{d}{dx}x^{n-1} + a_{n-2}\frac{d}{dx}x^{n-2} + ... + \frac{d}{dx}a_0$$

$$= na_n x^{n-1} + (n-1)a_{n-1}x^{n-2} + (n-2)a_{n-2}x^{n-3} + ... + 0)$$

$$\left(\frac{d}{dx}x^n = nx^{n-1}\right)$$

(3) Derivative of a Rational Function :

Let $h(x) = \frac{p(x)}{q(x)}$ be a rational function, where $p(x)$ and $q(x)$ are polynomial functions. $q(x) \neq 0$.

$\therefore \quad h'(x) = \frac{q(x)\,p'(x) - p(x)\,q'(x)}{[q(x)]^2}$ and $p'(x)$ and $q'(x)$ can be obtained by (2).

(4) $\frac{d}{dx}sinx = cosx$, $x \in$ R

$$\lim_{h \to 0} \frac{sin(x+h) - sinx}{h} = \lim_{h \to 0} \frac{2cos\frac{2x+h}{2}\,sin\frac{h}{2}}{h}$$

$$= \lim_{h \to 0} \frac{cos\left(x + \frac{h}{2}\right) sin\frac{h}{2}}{\frac{h}{2}} = cosx$$

$$\therefore \quad \frac{d}{dx}sinx = cosx$$

(5) $\frac{d}{dx}\cos x = -\sin x,\ x \in \mathbb{R}$

$$\lim_{h \to 0} \frac{\cos(x+h) - \cos x}{h} = \lim_{h \to 0} \frac{-2\sin\left(x+\frac{h}{2}\right)\sin\frac{h}{2}}{h}$$

$$= -\lim_{h \to 0} \frac{\sin\left(x+\frac{h}{2}\right)\sin\frac{h}{2}}{\frac{h}{2}}$$

$$= -\sin x$$

$\therefore\ \frac{d}{dx}\cos x = -\sin x$

(6) $\frac{d}{dx}\tan x = \sec^2 x,\quad x \in \mathbb{R} - \left\{(2k-1)\frac{\pi}{2} \,\middle|\, k \in \mathbb{Z}\right\}$

$$\frac{d}{dx}\tan x = \frac{d}{dx}\frac{\sin x}{\cos x}$$

$$= \frac{\cos x \frac{d}{dx}\sin x - \sin x \frac{d}{dx}\cos x}{\cos^2 x} \qquad \left(\textbf{by rule for } \frac{d}{dx}\frac{f}{g}\right)$$

$$= \frac{\cos x(\cos x) - \sin x(-\sin x)}{\cos^2 x}$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x}$$

$$= \frac{1}{\cos^2 x}$$

$$= \sec^2 x$$

(7) $\frac{d}{dx}\cot x = -\mathrm{cosec}^2 x,\quad x \in \mathbb{R} - \{k\pi \mid k \in \mathbb{Z}\}$

$$\frac{d}{dx}\cot x = \frac{d}{dx}\frac{\cos x}{\sin x}$$

$$= \frac{\sin x \frac{d}{dx}\cos x - \cos x \frac{d}{dx}\sin x}{\sin^2 x} \qquad \left(\textbf{by rule for } \frac{d}{dx}\frac{f}{g}\right)$$

$$= \frac{\sin x(-\sin x) - \cos x \cdot \cos x}{\sin^2 x}$$

$$= \frac{-(\sin^2 x + \cos^2 x)}{\sin^2 x}$$

$$= \frac{-1}{\sin^2 x}$$

$$= -\mathrm{cosec}^2 x$$

(8) $\frac{d}{dx}\sec x = \sec x \tan x,\quad x \in \mathbb{R} - \left\{(2k-1)\frac{\pi}{2} \,\middle|\, k \in \mathbb{Z}\right\}$

$$\frac{d}{dx}\sec x = \frac{d}{dx}\frac{1}{\cos x}$$

$$= \frac{\cos x \frac{d}{dx}1 - 1\frac{d}{dx}\cos x}{\cos^2 x}$$

$$= \frac{cosx \cdot 0 - 1(-sinx)}{cos^2 x}$$

$$= \frac{sinx}{cos^2 x}$$

$$= secx \ tanx$$

(9) $\frac{d}{dx} cosecx = -cosecx \ cotx, \quad x \in R - \{k\pi \mid k \in Z\}$

$$\frac{d}{dx} cosecx = \frac{d}{dx} \frac{1}{sinx}$$

$$= \frac{sinx \frac{d}{dx} 1 - 1 \frac{d}{dx} sinx}{sin^2 x}$$

$$= \frac{sinx \cdot 0 - 1(cosx)}{sin^2 x}$$

$$= \frac{-cosx}{sin^2 x}$$

$$= -cosecx \ cotx$$

Note : $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$ is called the derivative of $f(x)$ obtained using definition or from first principle. Above standard forms can also be obtained from first principle.

Also we can extended the rule $\frac{d}{dx}(f_1(x) + f_2(x)) = \frac{d}{dx} f_1(x) + \frac{d}{dx} f_2(x)$ as $\frac{d}{dx}(f_1(x) + f_2(x) + ... + f_n(x)) = \frac{d}{dx} f_1(x) + \frac{d}{dx} f_2(x) + ... + \frac{d}{dx} f_n(x)$, using principle of mathematical induction and we have used it in obtaining derivative of a polynomial.

Also we note that this result is true only for a finite sum of n terms and for infinite sum $\frac{d}{dx}(f_1(x) + f_2(x) + ...) = \frac{d}{dx} f_1(x) + \frac{d}{dx} f_2(x) + ...$ may not be valid. This would require advance discussion on convergence and **uniform convergence** of a series which we are not able to do here at this stage.

Some Miscellaneous Problems :

Example 11 : Find the derivative of $f(x) = cos^2 x$.

Solution : $\frac{d}{dx} cos^2 x = \frac{d}{dx} cosx \ cosx$

$$= cosx \frac{d}{dx} cosx + cosx \frac{d}{dx} cosx$$

$$= 2cosx \ (-sinx)$$

$$= -2sinx \ cosx$$

$$= -sin2x$$

Example 12 : Find the derivative of $xsin\,x$ from first principle.

Solution : $\frac{d}{dx} xsin\,x = \lim_{h \to 0} \frac{(x+h)\, sin(x+h) - xsin\,x}{h}$

$$= \lim_{h \to 0} \frac{(x+h)\sin(x+h) - (x+h)\sin x + (x+h)\sin x - x\sin x}{h}$$

$$= \lim_{h \to 0} (x + h)\left(\frac{\sin(x+h) - \sin x}{h}\right) + \lim_{h \to 0} \frac{(x+h-x)\sin x}{h}$$

$$= \lim_{h \to 0} \frac{(x+h)\, 2\cos\left(x+\frac{h}{2}\right)\sin\frac{h}{2}}{h} + \lim_{h \to 0} \sin x$$

$$= \lim_{h \to 0} \frac{(x+h)\cos\left(x+\frac{h}{2}\right)\sin\frac{h}{2}}{\frac{h}{2}} + \sin x \qquad \left(\lim_{h \to 0} c = c\right)$$

$$= x\cos x + \sin x$$

Example 13 : Find $\frac{d}{dx}\tan x$ from first principle. $x \in R - \left\{(2k-1)\frac{\pi}{2} \,\middle|\, k \in Z\right\}$

Solution : $\frac{d}{dx}\tan x = \lim_{h \to 0} \frac{\tan(x+h) - \tan x}{h}$

$$= \lim_{h \to 0} \frac{\tan(x+h-x)}{h}(1 + \tan x \tan(x + h)) \qquad (\tan(A + B) \text{ formula})$$

$$= \lim_{h \to 0} \frac{\tan h}{h} \lim_{h \to 0} (1 + \tan x \tan(x + h))$$

$$= 1 \cdot (1 + \tan^2 x)$$

$$= \sec^2 x$$

Example 14 : Find $\frac{d}{dx}\sec x$ from first principle. $x \in R - \left\{(2k-1)\frac{\pi}{2} \,\middle|\, k \in Z\right\}$

Solution : $\frac{d}{dx}\sec x = \lim_{h \to 0} \frac{\sec x(x+h) - \sec x}{h}$

$$= \lim_{h \to 0} \frac{\frac{1}{\cos(x+h)} - \frac{1}{\cos x}}{h}$$

$$= \lim_{h \to 0} \frac{\cos x - \cos(x+h)}{h\cos x\cos(x+h)}$$

$$= \lim_{h \to 0} \frac{-2\sin\left(\frac{-h}{2}\right)\sin\left(x+\frac{h}{2}\right)}{h\cos x\cos(x+h)}$$

$$= \lim_{h \to 0} \frac{\sin\frac{h}{2}}{\frac{h}{2}} \frac{\sin\left(x+\frac{h}{2}\right)}{\cos x\cos(x+h)}$$

$$= \frac{1 \cdot \sin x}{\cos x \cos x}$$

$$= \sec x \tan x$$

Example 15 : Find $\frac{d}{dx} sin2x$.

Solution : $\frac{d}{dx} sin2x = \frac{d}{dx} 2sinx\ cosx$

$= 2\left[sinx \frac{d}{dx} cosx + cosx \frac{d}{dx} sinx\right]$

$= 2[sinx(-sinx) + cosx \cdot cosx]$

$= 2(cos^2x - sin^2x)$

$= 2cos\,2x$

Example 16 : Find the derivative of $\frac{x^n}{n} + \frac{x^{n-1}}{n-1} + \frac{x^{n-2}}{n-2} + ... + x + 1$

Solution : $\frac{d}{dx}\left(\frac{x^n}{n} + \frac{x^{n-1}}{n-1} + ... + x + 1\right) = \frac{nx^{n-1}}{n} + \frac{(n-1)\,x^{n-2}}{n-1} + \frac{(n-2)\,x^{n-3}}{n-2} + 1 + 0$

$= x^{n-1} + x^{n-2} + x^{n-3} + ... + 1$

$= \frac{x^n - 1}{x - 1}$ as a sum of G.P.

Example 17 : Find $\frac{d}{dx}(ax + b)^n$. Deduce value of $\frac{d}{dx}(ax + b)^m (cx + d)^n$.

Solution :

$$\frac{d}{dx}(ax + b)^n = \frac{d}{dx}\left((ax)^n + \binom{n}{1}(ax)^{n-1} b + \binom{n}{2}(ax)^{n-2} b^2 + ... + \binom{n}{n-1} ax \cdot b^{n-1} + b^n\right)$$

$$= a^n n \cdot x^{n-1} + n(n-1)\,x^{n-2} a^{n-1} b + (n-2)\binom{n}{2}x^{n-3} a^{n-2} b^2$$

$$+ ... + \binom{n}{n-1} a \cdot 1 \cdot b^{n-1} + 0$$

$$= na\left[(ax)^{n-1} + (n-1)(ax)^{n-2} b + \frac{(n-2)(n-1)}{2}(ax)^{n-3} b^2 + ... + b^{n-1}\right]$$

$$= na\left((ax)^{n-1} + \binom{n-1}{1}(ax)^{n-2} b + \binom{n-1}{2}(ax)^{n-3} b^2 + ... + b^{n-1}\right)$$

$$= na(ax + b)^{n-1}$$

Now, $\frac{d}{dx}(ax + b)^m (cx + d)^n = (cx + d)^n \frac{d}{dx}(ax + b)^m + (ax + b)^m \frac{d}{dx}(cx + d)^n$

$= (cx + d)^n\, ma(ax + b)^{m-1} + (ax + b)^m\, nc(cx + d)^{n-1}$

$= (ax + b)^{m-1} (cx + d)^{n-1} [ma(cx + d) + nc(ax + b)]$

Example 18 : Find $\frac{d}{dx}\left(\frac{a + bsinx}{c + dsinx}\right)$. **$(c + dsinx \neq 0)$**

Solution : $\frac{d}{dx}\left(\frac{a + bsinx}{c + dsinx}\right) = \frac{(c + dsinx)\frac{d}{dx}(a + bsinx) - (a + bsinx)\frac{d}{dx}(c + dsinx)}{(c + dsinx)^2}$

$= \frac{(c + dsinx)\, bcosx - (a + bsinx)\, dcosx}{(c + dsinx)^2}$

$$= \frac{bc\cos x + bd\sin x\cos x - ad\cos x - bd\sin x\cos x}{(c + d\sin x)^2}$$

$$= \frac{-(ad - bc)\cos x}{(c + d\sin x)^2}$$

Example 19 : Find $\frac{d}{dx}\frac{x}{\sin^n x}$. $(\sin x \neq 0)$, $n \in$ N

Solution : First of all we prove $\frac{d}{dx}\sin^n x = n\sin^{n-1}x\cos x$ by P.M.I.

For $n = 1$, $\frac{d}{dx}\sin x = \cos x = 1 \cdot \sin^0 x\cos x$

$\therefore$ P(1) is true.

Let P(k) be true for some $k \in$ N. So $\frac{d}{dx}\sin^k x = k\sin^{k-1}x\cos x$

For $n = k + 1$, $\frac{d}{dx}\sin^{k+1}x = \frac{d}{dx}\sin^k x \cdot \sin x$

$$= \sin x\frac{d}{dx}\sin^k x + \sin^k x\frac{d}{dx}\sin x$$

$$= \sin x \cdot k\sin^{k-1}x\cos x + \sin^k x\cos x$$

$$= k \cdot \sin^k x\cos x + \sin^k x\cos x$$

$$= (k + 1)\sin^k x\cos x$$

$\therefore$ P(k + 1) is true.

$\therefore$ P(n) is true, $\forall n \in$ N by P.M.I.

Now, $$\frac{d}{dx}\frac{x}{\sin^n x} = \frac{\sin^n x\frac{d}{dx}x - x\frac{d}{dx}\sin^n x}{\sin^{2n}x}$$

$$= \frac{\sin^n x - x \cdot n\sin^{n-1}x\cos x}{\sin^{2n}x}$$

$$= \frac{\sin^{n-1}x(\sin x - nx\cos x)}{\sin^{2n}x}$$

$$= \frac{\sin x - nx\cos x}{\sin^{n+1}x}$$

Example 20 : Find the derivative of $\sqrt{\sin x}$ from first principle. **($\sin x > 0$)**

Solution : $$\frac{d}{dx}\sqrt{\sin x} = \lim_{t \to x}\frac{\sqrt{\sin t} - \sqrt{\sin x}}{t - x}$$

$$= \lim_{t \to x}\frac{\sin t - \sin x}{(\sqrt{\sin t} + \sqrt{\sin x})(t - x)}$$

$$= \lim_{t \to x}\frac{2\cos\frac{t+x}{2}\sin\frac{t-x}{2}}{(\sqrt{\sin t} + \sqrt{\sin x})\left(\frac{t-x}{2}\right)2}$$

$$= \frac{\cos x \cdot 1}{2\sqrt{\sin x}} = \frac{\cos x}{2\sqrt{\sin x}}$$

Example 21 : Find $\frac{d}{dx} x^2 sinx$ by definition and verify using rules.

Solution :

$$\frac{d}{dx} x^2 sinx = \lim_{h \to 0} \frac{(x+h)^2 \sin(x+h) - x^2 sinx}{h}$$

$$= \lim_{h \to 0} \frac{(x+h)^2 \sin(x+h) - (x+h)^2 sinx + (x+h)^2 \sin x - x^2 sinx}{h}$$

$$= \lim_{h \to 0} (x+h)^2 \left(\frac{\sin(x+h) - sinx}{h}\right) + \lim_{h \to 0} \frac{[(x+h)^2 - x^2] \sin x}{h}$$

$$= \lim_{h \to 0} \frac{(x+h)^2 \left(2cos\left(x+\frac{h}{2}\right) sin\frac{h}{2}\right)}{2 \cdot \frac{h}{2}} + \lim_{h \to 0} \frac{(2hx + h^2) \, sinx}{h}$$

$$= \lim_{h \to 0} \frac{(x+h)^2 \cos\left(x+\frac{h}{2}\right) \sin\frac{h}{2}}{\frac{h}{2}} + \lim_{h \to 0} (2x + h) \, sinx$$

$$= x^2 cosx + 2x \, sinx$$

Now, $\frac{d}{dx} x^2 sinx = x^2 \frac{d}{dx} sinx + sinx \frac{d}{dx} x^2$

$$= x^2 \, cosx + 2x \, sinx$$

Example 22 : Find $\frac{d}{dx} \frac{cosx}{1 + sinx}$. **($sinx \neq -1$)**

Solution : $\frac{d}{dx} \frac{cosx}{1+sinx} = \frac{(1+sinx)\frac{d}{dx} cosx - cosx \frac{d}{dx}(1+sinx)}{(1+sinx)^2}$

$$= \frac{(1+sinx)(-sinx) - cosx \cdot cosx}{(1+sinx)^2}$$

$$= \frac{-sinx - sin^2 x - cos^2 x}{(1+sinx)^2}$$

$$= \frac{-(1+sinx)}{(1+sinx)^2}$$ **($sin^2 x + cos^2 x = 1$)**

$$= \frac{-1}{1+sinx}$$

Example 23 : For $f(x) = x^{100} + x^{99} + x^{98} + ... + 1$, find $f'(1)$.

Solution : $f(x) = x^{100} + x^{99} + x^{98} + ... + 1$

$f'(x) = 100x^{99} + 99x^{98} + ... + 0$

$\therefore \quad f'(1) = 100 + 99 + 98 + ... + 1$

$$= \frac{100(101)}{2} = 5050$$ **$\left(\Sigma n = \frac{n(n+1)}{2}\right)$**

Example 24 : Find $\frac{d}{dx}\left(\frac{sinx + cosx}{sinx - cosx}\right)$. **($sinx \neq cosx$)**

Solution :

$$\frac{d}{dx}\frac{sinx + cosx}{sinx - cosx} = \frac{(sinx - cosx)\frac{d}{dx}(sinx + cosx) - (sinx + cosx)\frac{d}{dx}(sinx - cosx)}{(sinx - cosx)^2}$$

$$= \frac{(sinx - cosx)(cosx - sinx) - (sinx + cosx)(cosx + sinx)}{(sinx - cosx)^2}$$

$$= \frac{-[(sinx - cosx)^2 + (sinx + cosx)^2]}{(sinx - cosx)^2}$$

$$= \frac{-2}{(sinx - cosx)^2} \qquad (sin^2x + cos^2x = 1)$$

Example 25 : For $f(x) = |x|$, find $f'(0)$, if it exists.

Solution : We want $\lim\limits_{h \to 0} \frac{f(h) - f(0)}{h} = \lim\limits_{h \to 0} \frac{|h|}{h}$

$$\lim_{h \to 0+} \frac{|h|}{h} = \lim_{h \to 0+} \frac{h}{h} = 1,$$

$$\lim_{h \to 0-} \frac{|h|}{h} = \lim_{h \to 0-} \frac{-h}{h} = -1$$

$\therefore$ $\lim\limits_{h \to 0} \frac{f(h) - f(0)}{h}$ does not exist.

$\therefore$ $f(x) = |x|$ is not differntiable at $x = 0$.

Example 26 : $f : R \to Z$, $f(x) = [x]$. Find $f'(1)$, if it exists. Find $f'\left(\frac{1}{2}\right)$, if it exists.

Solution : $f(x) = \begin{cases} 0 & \text{if} \quad 0 \leq x < 1 \\ 1 & \text{if} \quad 1 \leq x < 2 \end{cases}$

$$\lim_{h \to 0+} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0} \frac{1-1}{h} = 0$$ **(since $h > 0$, $1 + h > 1$ and $[1 + h] = 1$)**

$$\lim_{h \to 0-} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0} \frac{0-1}{h} \text{ does not exist.}$$ **($h < 0$, $1 + h < 1$)**

$\therefore$ $f'(1)$ does not exist.

For $\frac{1}{2} - h < x < \frac{1}{2} + h$ $\left(h < \frac{1}{2}\right)$, $f(x) = 0$

$\therefore$ $f'(x) = 0$ since f is a constant function in

$\left(\frac{1}{2} - h, \frac{1}{2} + h\right)$.

$\therefore$ $f'(x) = 0$. Look at the graph.

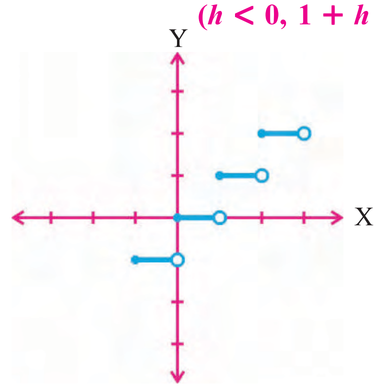

Figure 11.1

Exercise 11

1. Find following derivatives from first principle at given point :

(1) $\sin x$ at $x = 0$ (2) $\frac{1}{x}$ at $x = 1$

(3) $2x + 3$ at $x = 2$ (4) $\frac{3x+2}{2x+3}$ at $x = 1$

(5) $3x^2 - 2x + 1$ at $x = -1$ (6) $\cos x$ at $x = \frac{\pi}{2}$

(7) $\tan x$ at $x = \frac{\pi}{4}$ (8) $\sec x$ at $x = \frac{\pi}{3}$

(9) $\cot x$ at $x = \frac{5\pi}{4}$ (10) $\operatorname{cosec} x$ at $x = \frac{\pi}{6}$

2. Find following derivatives from definition : (on proper domain)

(1) $10x$ (2) $\sec x + \tan x$ (3) $\operatorname{cosec} x - \cot x$

(4) $2\sin^2 x + 3\cos x + 1$ (5) $\cos 2x$ (6) $\sin 2x$

(7) $\tan 2x$ (8) $\frac{1-\cos x}{\sin x}$ (9) $\frac{\cos x}{1-\sin x}$

(10) x^3 (11) x^4 (12) x^6

(13) $\sin^4 x$ (14) $\cos^4 x$ (15) $\sec^2 x$

3. If $f(x) - g(x)$ is a constant function, prove that $f'(x) = g'(x)$.

4. Find $\frac{d}{dx}\cos 2x$ by definition and also verify by using $\cos 2x = \cos^2 x - \sin^2 x$.

5. Find $\frac{d}{dx}\frac{x^n - 1}{x - 1}$, $x \neq 1$

6. $\frac{d}{dx}\frac{x^n - 1}{x - 1} = \frac{d}{dx}(x^{n-1} + x^{n-2} + x^{n-3} + ... + x + 1)$

$= (n-1)x^{n-2} + (n-2)x^{n-3} + (n-3)x^{n-4} + ... + 1 + 0$

$\therefore \frac{d}{dx}\frac{x^n - 1}{x - 1}$ at $x = 1$ is $(n-1) + (n-2) + (n-3) + ... + 1 = \frac{n(n-1)}{2}$.

Comment !

Obtain following derivatives where the function is defined :

7. $\frac{x^2 - 1}{x^2 + 1}$ **8.** $\frac{x^n - a^n}{x - a}$ $(x \neq a)$ **9.** $x^{-5}(7 + 3x)$

10. $x^{-6}(4x^2 - 8x^3)$ **11.** $2\sec x - 3\tan x + 5\sin x \cos x$ **12.** $\frac{\sec x - 1}{\sec x + 1}$

13. $\frac{4x + 7\sin x}{5x + 8\cos x}$ **14.** $\frac{x}{1 + \cot x}$ **15.** $(x^2 - 1)\sin^2 x + (x^2 + 1)\cos^2 x$

16. $(ax^2 + bx + \sin x)(p + q\tan x)$ **17.** $\sin(x + a)$

18. $\frac{\sin(x + a)}{\cos x}$ **19.** $\tan(x + a)$

20. Select a proper option (a), (b), (c) or (d) from given options and write in the box given on the right so that the statement becomes correct :

(1) For $f(x) = sin^2x$, $f'\left(\frac{\pi}{2}\right) = ...$

(a) -1 (b) 0 (c) 1 (d) $\frac{1}{2}$

(2) For $f(x) = \sqrt{x} + \frac{1}{\sqrt{x}}$, $f'(1) = ...$

(a) $-\frac{1}{2}$ (b) $\frac{1}{2}$ (c) 0 (d) 1

(3) If $f(x) = 1 + x + x^2 + x^3 + ... + x^{99} + x^{100}$, then $f'(-1) = ...$

(a) -50 (b) 50 (c) 5050 (d) -5050

(4) $\frac{d}{dx} cos^nx = ...$

(a) $ncos^{n-1}x$ (b) $nsin^{n-1}x$

(c) $ncos^{n-1}x \, sinx$ (d) $-ncos^{n-1}x \, sinx$

(5) $\frac{d}{dx}(sin^2x + cos^2x) = ...$

(a) $sin2x + cos2x$ (b) $sin2x - cos2x$ (c) 0 (d) $sinx + cosx$

(6) If $y = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ... + \frac{x^n}{n!}$, then $\frac{dy}{dx} = ...$

(a) y (b) $y - x$ (c) $y - \frac{x^n}{n!}$ (d) $y - \frac{x^n}{(n-1)!}$

(7) If $y = \sqrt{\frac{1-cos2x}{1+cos2x}}$, $x \in \left(\frac{\pi}{2}, \pi\right)$ then $\frac{dy}{dx} = ...$

(a) sec^2x (b) $-sec^2x$ (c) cos^2x (d) $|tanx|$

(8) If f is differentiable at a, $\lim\limits_{x \to a} \frac{xf(a) - af(x)}{x - a} = ...$

(a) $af'(a)$ (b) $f(a) - af'(a)$ (c) $f'(a)$ (d) $\frac{f'(a)}{a}$

(9) If $f(x) = x^{n-1} + x^{n-2} + ... + 1$, $-1 < x < 1$, then $f'(x) = ...$

(a) $\frac{1}{(x-1)^2}$ (b) $\frac{1}{x-1}$

(c) $\frac{1}{x^n - 1}$ (d) $\frac{(n-1)x^n - nx^{n-1} + 1}{(1-x)^2}$

(10) If $f(4) = 16$, $f'(4) = 2$ and f is differentiable at 4, $\lim\limits_{x \to 4} \frac{\sqrt{f(x)} - 4}{x - 4} = ...$

(a) 2 (b) 1 (c) $\frac{1}{4}$ (d) $\frac{1}{16}$

(11) If $f(x) = \frac{x^2}{|x|}$, $x \in [3, 5]$, then $f'(x) = ...$

(a) 1 (b) -1 (c) does not exist (d) 0

(12) If $\pi < x < \frac{3\pi}{2}$, $\frac{d}{dx}\sqrt{\frac{1+cos2x}{2}} = ...$

(a) $-sinx$ (b) $sinx$ (c) $cosx$ (d) $sin2x$

(13) If $\pi < x < 2\pi$, $\frac{d}{dx}\sqrt{\frac{1-cos2x}{2}} = ...$

(a) $sinx$ (b) $cosx$ (c) $-cosx$ (d) $-sin2x$

(14) $\frac{d}{dx}\sqrt{x^2-2x+1}$ $(x \in [-2, -1]) = ...$

(a) does not exist (b) 0 (c) 1 (d) -1

(15) $\frac{d}{dx}(x + |x|)|x|$ $(x < 0) = ...$

(a) 1 (b) 0 (c) 2 (d) 4

(16) $\frac{d}{dx}(x + |x|)|x|$ $(x > 0) = ...$

(a) $-4x$ (b) $4x$ (c) $2x^2$ (d) x^2

(17) $\frac{d}{dx}|x|^2 = ...$ (at $x = 0$)

(a) 0 (b) does not exist (c) 2 (d) 1

(18) $\frac{d}{dx}x|x|$ $(x > 0) = ...$

(a) x^2 (b) $-2x$ (c) $2x$ (d) 0

(19) $\frac{d}{dx}(cos^2x - sin^2x) = ...$

(a) $sin2x$ (b) $cos2x$ (c) $-cos2x$ (d) $-2sin2x$

(20) $\frac{d}{dx}(3sinx - 4sin^3x) = ...$

(a) $3cos3x$ (b) $cos3x$ (c) $3sin3x$ (d) $-3cos3x$

(21) $\frac{d}{dx}sin18^{o} = ...$

(a) $cos18^{o}$ (b) $-sin18^{o}$ (c) $-cos18^{o}$ (d) 0

(22) $\frac{d}{dx}sinx^{o} = ...$

(a) $cosx^{o}$ (b) $-sinx^{o}$ (c) $\frac{\pi}{180}cosx^{o}$ (d) 0

(23) $\frac{d}{dx}(2x + 3)^n = ...$

(a) $n(2x + 3)^{n-1}$ (b) $2n(2x + 3)^{n-1}$ (c) $3n(2x + 3)^{n-1}$ (d) $2^n n(2x + 3)^{n-1}$

(24) $\frac{d}{dx}\sqrt{sinx}$, $(0 < x < \frac{\pi}{2})$ = ... ☐

(a) $\sqrt{cosx}$ (b) $\sqrt{sinx}$ (c) $\frac{cosx}{2\sqrt{sinx}}$ (d) $\frac{sinx}{2\sqrt{cosx}}$

(25) $\frac{d}{dx}\ tan^2x$ = ... ☐

(a) $2tanx$ (b) sec^2x (c) cot^2x (d) $2tanx\ sec^2x$

*

Summary

We studied following points in this chapter :

1. Formal definition of derivative and examples based on it.
2. Algebra of derivatives and examples based on rules.
   If $f(x)$ and $g(x)$ are differentiable in (a, b),

   (1) $\frac{d}{dx}(f(x) + g(x)) = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)$

   (2) $\frac{d}{dx}(f(x) - g(x)) = \frac{d}{dx}f(x) - \frac{d}{dx}g(x)$

   (3) $\frac{d}{dx}(f(x)\ g(x)) = g(x)\frac{d}{dx}f(x) + f(x)\frac{d}{dx}g(x)$

   (4) $\frac{d}{dx}\frac{f(x)}{g(x)} = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$, $g(x) \neq 0$

   (5) $\frac{d}{dx}\ kf(x) = k\frac{d}{dx}f(x)$, $k \in$ R
3. Some standard forms :

   (1) $\frac{d}{dx}c = 0$

   (2) $\frac{d}{dx}x^n = nx^{n-1}$, $n \in$ N, $x \in$ R

   (3) $\frac{d}{dx}sinx = cosx$

   (4) $\frac{d}{dx}cosx = -sinx$

   (5) $\frac{d}{dx}tanx = sec^2x$

   (6) $\frac{d}{dx}cotx = -cosec^2x$

   (7) $\frac{d}{dx}secx = secx\ tanx$

   (8) $\frac{d}{dx}cosecx = -cosecx\ cotx$
4. Derivative of a polynomial and a rational function.

Bhaskara I

Bhaskara wrote three astronomical contributions. In 629 he created the Aryabhatiya, written in verses, about mathematical astronomy. The comments referred exactly to the 33 verses dealing with mathematics. There he considered variable equations and trigonometric formulae.

His work *Mahabhaskariya* is divided into eight chapters about mathematical astronomy. In chapter 7, he gives a remarkable approximation formula for *sin x*, that is

$$sinx \sim \frac{16x(\pi - x)}{5\pi^2 - 4x(\pi - x)}\ \left(0 \le x \le \frac{\pi}{2}\right)$$

ANSWERS

(Answers to questions involving some calculations only are given.)

Exercise 1

31. (1) d (2) d (3) b (4) a (5) d (6) b (7) d (8) d (9) d (10) d

Exercise 2.1

1. (1) $-2i$ (2) $-1 + 8i$ (3) $2 + i$ (4) $\frac{5}{13} + \frac{14}{13}i$ (5) $-\frac{2}{5}$ (6) $-\frac{1}{2}$ (7) -4

(8) $2i$ (9) $\frac{77}{25} + \frac{36}{25}i$ (10) $-\frac{7}{\sqrt{2}}i$

2. (1) $x = 4, y = 1$ (2) $x = -\frac{16}{23}, y = \frac{29}{23}$ (3) $x = 4, y = -2$

(4) $\left\{(2, 3)\left(-2, \frac{1}{3}\right)\right\}$ (5) $x = \frac{14}{15}, y = \frac{1}{5}$

3. (1) $\frac{3}{13} + \frac{2}{13}i$ (2) $-\frac{1}{4} - \frac{\sqrt{3}}{4}i$ (3) $\frac{11}{25} - \frac{27}{25}i$ (4) $-\frac{5}{169} + \frac{12}{169}i$ (5) i

Exercise 2.2

1. (1) $\sqrt{2}, \frac{3\pi}{4}$ (2) $\frac{1}{2}, \frac{\pi}{2}$ (3) $2, -\frac{\pi}{6}$ (4) $2, \frac{5\pi}{6}$ (5) $6, \frac{3\pi}{4}$

6. z_1 may not be equal to z_2 **8.** 40 **12.** $-2\sqrt{3} + 2i$ **13.** $z_1 = 2 + i, z_2 = 2 - i$ **15.** $\frac{2}{5}$

Exercise 2.3

1. (1) $\pm\sqrt{2}\,i$ (2) $\frac{-1 \pm \sqrt{3}\,i}{2}$ (3) $\frac{-1 \pm \sqrt{19}i}{2\sqrt{5}}$ (4) $\frac{-1 \pm \sqrt{2\sqrt{2} - 1}\,i}{2}$ (5) $\frac{-1 \pm \sqrt{7}i}{2\sqrt{2}}$ (6) $\frac{2 \pm 4i}{3}$

2. (1) $\pm\sqrt{2}(\sqrt{3} + i)$ (2) $\pm(3 - 2i)$ (3) $\pm(1 + 7i)$ (4) $\pm(2\sqrt{2} - \sqrt{5}\,i)$

(5) $\pm\frac{1}{\sqrt{2}}\left(\sqrt{\sqrt{2} - 1} - i\sqrt{\sqrt{2} + 1}\right)$ (6) $\pm\sqrt{2}(1 + i)$ (7) $\pm(2\sqrt{2} - 2\sqrt{2}\;i)$ (8) $\pm 5\,i$ (9) $\pm\sqrt{10}\,i$

Exercise 2

1. (1) $2 - 2i$ (2) $\frac{307 + 599i}{442}$ **2.** 2 **4.** $-\frac{3}{20}, \frac{1}{20}$ **7.** 1 **8.** $b = \frac{-\beta}{(\alpha - 1)^2 + \beta^2}$

10. (1) $1 \pm \frac{\sqrt{2}}{2}i$ (2) $\frac{5 \pm \sqrt{2}i}{27}$ (3) $\frac{2}{3} \pm \frac{\sqrt{14}}{21}i$

11. Maximum value is 5, Minimum value is 1 **12.** -48 **13.** 4 **15.** $\frac{3}{2} - 2i$

21. (1) c (2) b (3) a (4) d (5) c (6) c (7) a (8) c (9) b (10) b
(11) d (12) b (13) a (14) c (15) b

Exercise 3.1

1. (1) $x^{10} + 5x^7 + 10x^4 + 10x + \frac{5}{x^2} + \frac{1}{x^5}$ (2) $1 - 8x + 24x^2 - 32x^3 + 16x^4$

(3) $729x^6 - 2916x^5 + 4860x^4 - 4320x^3 + 2160x^2 - 576x + 64$

(4) $x^5 - \frac{5}{2}x^3 + \frac{5}{2}x - \frac{5}{4x} + \frac{5}{16x^3} - \frac{1}{32x^5}$

2. (1) $x^8 + 4x^7 + 10x^6 + 16x^5 + 19x^4 + 16x^3 + 10x^2 + 4x + 1$

(2) $x^6 - 3x^5 + 6x^4 - 7x^3 + 6x^2 - 3x + 1$

3. (1) 0.92236816 (2) 96059601 (3) 1061520150601 **4.** $(1.01)^{10000}$ is larger.

Exercise 3.2

1. (1) 672 (2) 1365 **2.** (1) $\frac{5}{81}$ (2) $\frac{7}{18}$ **3.** $n = 55$

4. (1) $\frac{280}{81}x^{12}$, $\frac{-560}{27}x^9$ (2) $\frac{2835}{8}x^4y^4$ (3) $\binom{20}{10}x^{10}$ (4) $720\,x^2y^3$, $1080\,x^3y^2$

5. $n = 6$ **6.** $n = 14$ or 7

Exercise 3

1. 2 : 1 **2.** $r = 3$ or 15 **3.** $n = 6, x = 2, y = 5$ **4.** $a = 2, b = 3, n = 5$ **6.** $n = 11$ **7.** 135 **8.** $n = 10$

12. (1) c (2) b (3) a (4) c (5) a (6) c (7) b (8) d (9) a (10) b

Exercise 4.1

1. (1) $-\frac{1}{\sqrt{2}}$ (2) $\frac{1}{\sqrt{3}}$ (3) $-\frac{1}{2}$ (4) $\frac{2}{\sqrt{3}}$ (5) $-\sqrt{2}$ (6) $-\frac{1}{\sqrt{3}}$ **17.** (1) 3 (2) 0 (3) $\frac{1}{2}$ (4) 1

18. (1) Negative (2) Positive (3) Negative (4) Negative **19.** $\frac{3}{7}$

Exercise 4.2

1. (1) $\frac{1}{2\sqrt{2}}$ (2) $\frac{-1}{2\sqrt{2}}$ (3) $\frac{\sqrt{6}-\sqrt{2}}{4}$ **4.** (1) Fourth quadrant (2) Fourth quadrant

5. $\frac{2}{11}$, First quadrant **6.** (1) $[-25, 25]$ (2) $[0, 2]$ **8.** $r = 2$, $\alpha = \frac{\pi}{3}$

9. $r = 2$, $\theta = -\frac{\pi}{3}$ **20.** -1, $\frac{1}{7}$

Exercise 4.3

1. (1) $sin10\theta + sin4\theta$ (2) $sin3\theta - sin2\theta$ (3) $sin8\theta - sin2\theta$ (4) $sin6\theta + sin\theta$

(5) $cos14\theta + cos8\theta$ (6) $cos4\theta + cos\theta$ (7) $\frac{1}{2}(cos2\theta - cos20\theta)$ (8) $cos\theta - cos8\theta$ (9) $sin2\theta$

2. (1) $\frac{1}{2}$ (2) $-\frac{1}{2}$ (3) $\frac{2+\sqrt{3}}{2}$ (4) $\frac{\sqrt{3}-2}{2}$ (5) $\sqrt{2}$ (6) 1 **5.** 1

Exercise 4.4

1. (1) $2sin5\theta\ cos2\theta$ (2) $2sin\theta\ cos\frac{\theta}{2}$ (3) $-2cos4\theta\ sin\theta$ (4) $2cos\frac{5\theta}{2}\ sin\theta$

(5) $2cos10\theta\ cos\theta$ (6) $2cos4\theta\ cos\frac{3\theta}{2}$ (7) $2sin8\theta\ sin3\theta$ (8) $2sin\theta\ sin\frac{\theta}{2}$

(9) $-2sin^2\frac{\theta}{2}$ (10) $2sin\left(\frac{\theta}{2}+\frac{\pi}{4}\right)\ cos\left(\frac{\theta}{2}-\frac{\pi}{4}\right)$ (11) $\sqrt{2}\,cos\left(\frac{\pi}{4}-\theta\right)$ (12) $\sqrt{2}\,sin\left(\theta-\frac{\pi}{4}\right)$

Exercise 4

9. $\sqrt{19}$, $-\sqrt{19}$

14. (1) c (2) a (3) d (4) d (5) c (6) b (7) c (8) a (9) a (10) a

(11) b (12) c (13) b (14) c (15) d (16) c (17) d (18) d (19) d

Exercise 5.1

20. $\frac{24}{25}, \frac{7}{25}, \frac{24}{7}, \frac{336}{625}$

Exercise 5.2

1. $\frac{3}{\sqrt{10}}, \frac{-1}{\sqrt{10}}, -3$ **2.** $\frac{1}{65}, \frac{64}{65}$

Exercise 5

23. (1) a (2) b (3) c (4) b (5) c (6) d (7) a (8) a (9) d (10) a
(11) b (12) b (13) a (14) c (15) d (16) a (17) b (18) c (19) a (20) d

Exercise 6.1

1. $\left\{k\pi \pm \frac{3\pi}{8} \,\middle|\, k \in Z\right\}$ **2.** $\left\{(2k + 1)\frac{\pi}{2} \,\middle|\, k \in Z\right\} \cup \left\{2k\pi \pm \frac{5\pi}{6} \,\middle|\, k \in Z\right\}$

3. $\{2k\pi \mid k \in Z\} \cup \left\{2k\pi \pm \frac{\pi}{3} \,\middle|\, k \in Z\right\}$ **4.** $\left\{2k\pi \pm \frac{\pi}{3} \,\middle|\, k \in Z\right\}$

5. $\left\{\frac{k\pi}{3} + (-1)^k \frac{\pi}{12} \,\middle|\, k \in Z\right\}$ **6.** $\{k\pi \mid k \in Z\} \cup \left\{k\pi + (-1)^k \frac{\pi}{6} \,\middle|\, k \in Z\right\}$

7. $\left\{k\pi + (-1)^k \frac{\pi}{6} \,\middle|\, k \in Z\right\} \cup \left\{k\pi + (-1)^k \frac{\pi}{2} \,\middle|\, k \in Z\right\}$

8. $\left\{(2k + 1)\frac{\pi}{2} \,\middle|\, k \in Z\right\} \cup \left\{k\pi - (-1)^k \frac{\pi}{6} \,\middle|\, k \in Z\right\}$

9. $\left\{\frac{k\pi}{3} \,\middle|\, k \in Z\right\} \cup \left\{\frac{k\pi}{2} \pm \frac{\pi}{12} \,\middle|\, k \in Z\right\}$

10. $\left\{(2k + 1)\frac{\pi}{2} \,\middle|\, k \in Z\right\} \cup \{2k\pi \mid k \in Z\}$

11. $\left\{\frac{k\pi}{2} + \frac{\pi}{6} \,\middle|\, k \in Z\right\}$ **12.** $\left\{(2k + 1)\frac{\pi}{2} \,\middle|\, k \in Z\right\} \cup \left\{k\pi + \frac{\pi}{6} \,\middle|\, k \in Z\right\}$

13. $\left\{k\pi + \frac{\pi}{4} \,\middle|\, k \in Z\right\} \cup \left\{k\pi + \frac{\pi}{3} \,\middle|\, k \in Z\right\}$

14. $\{2k\pi \mid k \in Z\} \cup \left\{2k\pi + \frac{\pi}{2} \,\middle|\, k \in Z\right\}$

15. $\left\{2k\pi + \frac{5\pi}{12} \,\middle|\, k \in Z\right\} \cup \left\{2k\pi - \frac{13\pi}{12} \,\middle|\, k \in Z\right\}$

16. $\emptyset$ **17.** $\left\{\frac{k\pi}{5} \pm \frac{\pi}{30} \,\middle|\, k \in Z\right\}$ **18.** $\left\{\frac{k\pi}{2} \pm \frac{\pi}{8} \,\middle|\, k \in Z\right\}$

19. $\left\{(8k \pm 3)\frac{\pi}{16} \,\middle|\, k \in Z\right\}$ **20.** $\left\{(2k + 1)\frac{\pi}{4} \,\middle|\, k \in Z\right\}$

Exercise 6.2

16. $\frac{2\pi}{3}$ **17.** $1 : \sqrt{3} : 2$ **18.** $\frac{5\pi}{12}$ **20.** $\frac{\pi}{3}$

Exercise 6

1. $\left\{2k\pi \pm \frac{\pi}{4} \,\middle|\, k \in Z\right\} \cup \left\{2k\pi \pm \frac{3\pi}{4} \,\middle|\, k \in Z\right\}$ **2.** $\left\{(4k + 1)\frac{\pi}{2} \,\middle|\, k \in Z\right\}$

3. $\{k\pi \mid k \in Z\} \cup \left\{(3k \pm 1)\frac{\pi}{9} \,\middle|\, k \in Z\right\}$ **4.** $\left\{2k\pi \pm \frac{\pi}{3} \,\middle|\, k \in Z\right\}$

5. $\left\{(6k + 1)\frac{\pi}{30} \,\middle|\, k \in Z\right\}$ **6.** $\left\{2k\pi + \frac{\pi}{2} \,\middle|\, k \in Z\right\}$ **7.** $\left\{(4k \pm 1)\frac{\pi}{8} \,\middle|\, k \in Z\right\}$

8. $\left\{(4k+1)\frac{\pi}{12} \;\middle|\; k \in Z\right\}$ **9.** $\left\{(4k+1)\frac{\pi}{8} \;\middle|\; k \in Z\right\}$ **10.** $\left\{(12k \pm 5)\frac{\pi}{6} \;\middle|\; k \in Z\right\}$

21. (1) a (2) c (3) d (4) c (5) c (6) b (7) c (8) a (9) d (10) b
(11) a (12) a (13) b (14) d (15) b

Exercise 7.1

1. (1) 4, 7, 10, 13, 16 (2) 1, $\frac{1}{2}$, 2, $\frac{3}{2}$, 3 (3) 2, 3, 5, 7, 11

2. 2, 3, 5, 8 **3.** (1) −5, −9, −17 (2) $\frac{5}{2}$, $\frac{13}{2}$, $\frac{41}{2}$ **4.** (1) 0, 3, 5, 19 (2) 1, 2, 3, 10

5. (1) $a_n = ar^{n-1}$, $n \in N$ (2) $a_1 = 0$, $a_n = 16(-3)^{n-2}$, $n \geq 2$

Exercise 7.2

1. (1) 43 (2) −49 (3) $\frac{33}{2}$ **2.** 510 **3.** 23,700 **4.** $d = -4$, $t_8 = -24$

5. 27 **6.** $-(m+n)$ **7.** 0 **8.** 1 : 2 **9.** 5 : 11 **10.** 6000 **11.** 1 **12.** −1, 3, 7

13. 2, 6, 10, 14 **14.** ₹ 7800 **15.** $n = 10$, ₹ 1287.50 **16.** 660 *cm*

Exercise 7.3

1. (1) 256 (2) $\frac{7}{1024}$ (3) $-16\sqrt{2}$ **2.** (1) 768 (2) 13 (3) 5 (4) $\frac{405}{4}$

3. 93 **4.** $\frac{3}{2}$, 3, 6, 12, 24, ... **5.** (1) $\frac{7}{9}\left[\frac{10}{9}(10^n - 1) - n\right]$ (2) $3\left[n + \frac{10}{9}(10^{n-1} - 1)\right]$

6. $\dfrac{a^2(a^{2n}-1)}{a^2-1} + \dfrac{ab(a^n b^n - 1)}{ab-1}$ **7.** $\frac{2}{9}$, $\frac{2}{3}$, 2, 6, 18 **8.** $\sqrt{mn}$ **9.** $2\sqrt{2}$

12. $\frac{1}{4}$, 1, 4, 16 **13.** ₹ 39,366

Exercise 7.4

1. $\frac{19}{6}$, $\frac{10}{3}$, $\frac{7}{2}$, $\frac{11}{3}$, $\frac{23}{6}$ **2.** 5, 13, 21 **3.** $\frac{1}{4}$, $\frac{1}{2}$, 1, 2, 4 **4.** $\sqrt{2}$, 1, $\frac{1}{\sqrt{2}}$

5. 45, 5 **6.** $x^2 - 20x + 64 = 0$

Exercise 7.5

1. (1) 800 (2) 465 (3) 1070 (4) −2704

2. (1) $\frac{n}{3}(16n^2 + 12n - 1)$ (2) $\frac{n}{4}(27n^3 - 18n^2 - 9n + 4)$ (3) $\frac{n}{2}(4n^2 + n - 1)$

(4) $\frac{10n}{3}(n^2 + 6n + 11)$ (5) $12n(n+1)(9n^2 + 9n + 8)$ (6) $\frac{n}{36}(4n^2 + 15n + 17)$

(7) $2n^2 + n$ (8) $\dfrac{n(n+1)}{12}(3n^2 + 11n + 10)$ (9) $\dfrac{n^2(n^2-1)}{4}$

3. (1) −6479 (2) −465

Exercise 7

1. −140, 42 **2.** −2, 4, 10, 16, ... **3.** 9 hr **4.** 16 rows, 5 blocks **6.** 1 : 2 : 3

7. $\frac{20n}{3} - \frac{20}{27} + \frac{20}{27} \times 10^{-n}$ **9.** 740 **10.** $\frac{1}{2} - \frac{1}{\sqrt{2}}$ **11.** $\frac{n}{2}(1 - 5n)$

12. 11, 14, 17, 20, ... **13.** $(3 + 2\sqrt{2}) : (3 - 2\sqrt{2})$ **14.** $\frac{25025}{2}$

15. 3, 5, 7, 9, 11, 13 **16.** 48, 12, 3, $\frac{3}{4}$, $\frac{3}{16}$

17. (1) c (2) d (3) a (4) a (5) c (6) c (7) b (8) a (9) b (10) d

(11) a (12) d (13) c (14) a (15) c (16) b (17) a (18) c

Exercise 8.1

1. (1) $x^2 + y^2 + 4x - 6y - 12 = 0$ (2) $x^2 + y^2 + 2x - 2y = 0$

(3) $x^2 + y^2 + 8x\,cos\alpha - 8y\,sin\alpha - 9 = 0$ (4) $x^2 + y^2 + 2\sqrt{2}x + 2\sqrt{5}y + 2 = 0$

(5) $x^2 + y^2 - 2x = 0$

2. $x^2 + y^2 - 6x + 4y - 12 = 0$ **3.** $x^2 + y^2 + 4x + 10y + 25 = 0$

4. $x^2 + y^2 + 6x + 6y + 9 = 0$ **5.** $x^2 + y^2 - 2\sqrt{5}x = 0$

Exercise 8.2

1. (1) Not a circle. (2) Circle, Centre (0, 0), radius 1

(3) Circle, Centre (1, 1), radius 1 (4) Not a circle.

(5) Not a circle. (6) Not a circle.

(7) Circle, Centre $\left(\frac{1}{2}, -\frac{1}{2}\right)$, radius = $\frac{1}{\sqrt{2}}$ (8) Not a circle.

(9) Circle, Centre = $(tan\alpha, -sec\alpha)$, radius = 1

(10) Case-1 : $\alpha = 0$ Centre (0, −1), radius = 1

Case-2 : $\alpha \neq 0$ Not a circle.

2. $x^2 + y^2 - 6x - 8y = 0$ **3.** $x^2 + y^2 - 10y - 15 = 0$

4. $x^2 + y^2 + 6x - 6y + 9 = 0$ and $x^2 + y^2 + 30x - 30y + 225 = 0$

Exercise 8.3

1. (1) Focus $\left(\frac{1}{8}, 0\right)$, directrix $8x + 1 = 0$ (2) Focus (0, −1), directrix $y = 1$

(3) Focus $\left(0, -\frac{1}{16}\right)$, directrix $16y - 1 = 0$ (4) Focus (3, 0), directrix $x + 3 = 0$

2. (1) $x^2 = -8y$ (2) $y^2 = 16x$

3. (1) $x^2 + y^2 + 2xy + 2x - 6y + 9 = 0$ (2) $16x^2 + 9y^2 + 24xy + 180x + 160y + 600 = 0$

4. 4, $y + 3 = 0$ **5.** 18 **6.** $\left(\frac{a}{t_1^2}, \frac{-2a}{t_1}\right)$ **7.** $(3, \pm 6)$

Exercise 8.4

1. (1) $\frac{x^2}{16} + \frac{y^2}{12} = 1$ (2) $\frac{x^2}{25} + \frac{y^2}{9} = 1$ (3) $\frac{x^2}{100} + \frac{y^2}{36} = 1$ (4) $\frac{x^2}{9} + \frac{y^2}{25} = 1$

(5) $\frac{4x^2}{81} + \frac{4y^2}{45} = 1$ (6) $\frac{x^2}{16} + \frac{y^2}{12} = 1$ (7) $\frac{x^2}{64} + \frac{y^2}{100} = 1$

2. $\frac{x^2}{18} + \frac{y^2}{9} = 1$

3.

No.	e	Foci	Directrices	Length of a Latus-rectum
(1)	$\frac{\sqrt{5}}{3}$	$(0, \pm\sqrt{5})$	$y = \pm\frac{9}{\sqrt{5}}$	$\frac{8}{3}$
(2)	$\frac{2}{3}$	$(\pm 4, 0)$	$x = \pm 9$	$\frac{20}{3}$
(3)	$\frac{1}{\sqrt{2}}$	$(\pm 5\sqrt{2}, 0)$	$x = \pm 10\sqrt{2}$	10
(4)	$\frac{3}{4}$	$\left(0, \pm\frac{3\sqrt{43}}{\sqrt{7}}\right)$	$y = \pm\frac{16\sqrt{43}}{3\sqrt{7}}$	$\frac{\sqrt{301}}{2}$
(5)	$\frac{2}{3}$	$\left(\pm\frac{6}{\sqrt{5}}, 0\right)$	$x = \pm\frac{27}{2\sqrt{5}}$	$2\sqrt{5}$

4. $e = \frac{1}{\sqrt{3}}$ 5. $5 : 3,\ x = \pm\frac{50}{3}$ 7. $7x^2 + 15y^2 = 247$ 8. $4x^2 + 3y^2 - 24x - 6y + 27 = 0$

9. Foci : $(2, 1 \pm \sqrt{5})$, Directrices : $y = 1 \pm \frac{9}{\sqrt{5}}$

Exercise 8.5

In answer 1, $\theta \in (-\pi, \pi]$

1. (1) $x = 4cos\theta,\ y = 3sin\theta$ (2) $x = 4cos\theta,\ y = 2\sqrt{3}\,sin\theta$
   (3) $x = 2cos\theta,\ y = \sqrt{3}\,sin\theta$ (4) $x = 4cos\theta,\ y = \sqrt{7}\,sin\theta$ (5) $x = 3\sqrt{2}\,cos\theta,\ y = 3sin\theta$

2. (1) $e = \frac{\sqrt{5}}{3}$, Foci : $(0, \pm\sqrt{5})$ (2) $e = \frac{\sqrt{184}}{25}$, Foci : $\left(\pm\frac{\sqrt{184}}{15}, 0\right)$ (3) $e = \frac{\sqrt{7}}{4}$, Foci : $(\pm\sqrt{7}, 0)$

3. $\frac{x^2}{16} + \frac{y^2}{15} = 1$

Exercise 8.6

1.

No.	Foci	Directrices	Length of a latus-rectum	Length of transverse axis	Length of conjugate axis
(1)	$(\pm 5\sqrt{5}, 0)$	$x = \pm 4\sqrt{5}$	5	20	10
(2)	$(\pm 8\sqrt{2}, 0)$	$x = \pm 4\sqrt{2}$	16	16	16
(3)	$\left(\pm\frac{5}{\sqrt{6}}, 0\right)$	$x = \pm\sqrt{\frac{3}{2}}$	$\frac{2\sqrt{10}}{3}$	$\sqrt{10}$	$2\sqrt{\frac{5}{3}}$
(4)	$(0, \pm 5)$	$y = \pm\frac{16}{5}$	$\frac{9}{2}$	8	6
(5)	$(0, \pm 8)$	$y = \pm\frac{25}{8}$	$\frac{78}{5}$	10	$2\sqrt{39}$

In answer 2 and 4, $\theta \in (-\pi, \pi] - \left\{\frac{-\pi}{2}, \frac{\pi}{2}\right\}$

2. (1) $\frac{y^2}{49} - \frac{9x^2}{343} = 1;\ x = \frac{\sqrt{343}}{3}\,tan\theta,\ y = 7sec\theta$ (2) $\frac{x^2}{9} - \frac{y^2}{4} = 1;\ x = 3sec\theta,\ y = 2tan\theta$
   (3) $\frac{x^2}{25} - \frac{y^2}{20} = 1;\ x = 5sec\theta,\ y = \sqrt{20}\,tan\theta$ (4) $\frac{y^2}{32} - \frac{x^2}{32} = 1;\ x = 4\sqrt{2}\,tan\theta,\ y = 4\sqrt{2}\,sec\theta$
   (5) $\frac{y^2}{16} - \frac{x^2}{9} = 1;\ x = 3tan\theta,\ y = 4sec\theta$

4. $\frac{x^2}{16} - \frac{y^2}{9} = 1$ 5. $x = 4tan\theta,\ y = 3sec\theta$

Exercise 8

1. $x^2 + y^2 - 3x + y - 4 = 0$ **2.** $x^2 + y^2 - 6y - 16 = 0$

3. $x^2 + y^2 - 4x - 6y + 4 = 0$ **4.** Focus : $\left(\frac{1}{4}, 0\right)$, Length of lectus-rectum = 1

5. $\frac{x^2}{144} + \frac{y^2}{128} = 1$ **6.** $\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$

7. $y^2 = -12(x + 1)$ **8.** (a) $y^2 = 10x$ (b) $2\sqrt{110}$ **9.** (6, 0) **10.** 3.2 m

11. Ellipse, $\frac{x^2}{25} + \frac{y^2}{9} = 1$

12. (1) a (2) d (3) a (4) b (5) d (6) b (7) c (8) c (9) b (10) a
(11) b (12) a (13) b (14) b (15) c (16) a (17) a (18) b (19) c (20) d

Exercise 9.1

1. (1) (x_1, x_2) (2) (x, y, z) (3) (5, −2, 2) (4) (4, −4, −4) (5) (−1, −4, −7) (6) (1, −5, −2)

2. (1) $x = 1, y = -1$ (2) $x = 0, y = 0$ (3) $x = \frac{1}{5}, y = \frac{8}{5}$ (4) $x = 0, y = 0$

3. (1) $\sqrt{3}$ (2) $\sqrt{3}$ (3) 5 (4) $\sqrt{14}$ (5) $\sqrt{38}$

4. (1) $|\bar{x} + \bar{y}| < |\bar{x}| + |\bar{y}|$ (2) $|\bar{x} + \bar{y}| = |\bar{x}| + |\bar{y}|$ **5.** $k = 1$ **6.** $\left(\frac{-11}{6}, \frac{47}{15}, 0\right)$

Exercise 9.2

1. (1) OXYZ (2) OXY'Z' (3) OXYZ' (4) OX'Y'Z (5) OX'Y'Z' **2.** (0, 0, 0)

Exercise 9.3

1. (1) Same directions (2) Different directions (3) Opposite directions (4) Different directions

2. (1) $\left(\frac{3}{5}, \frac{-4}{5}\right)$ (2) $\left(\frac{-3}{5}, \frac{-4}{5}\right)$ (3) $\left(\frac{1}{\sqrt{35}}, \frac{3}{\sqrt{35}}, \frac{5}{\sqrt{35}}\right)$ (4) $\left(\frac{6}{7}, \frac{3}{7}, \frac{2}{7}\right)$ (5) (1, 0, 0) (6) $\left(\frac{-5}{13}, \frac{12}{13}\right)$

3. $\alpha = \frac{2x_2 - x_1}{3}$, $\beta = \frac{2x_1 - x_2}{3}$

Exercise 9.4

1. (1) 0 (2) $2\sqrt{3}$ (3) 6 (4) 4 (5) 5 (6) 1

2. (1) Non-collinear (2) Collinear (3) Non-collinear (4) Non-collinear

3. Isosceles right triangle **4.** (0, 0, 0) or (0, 0, 6) **5.** $x^2 + y^2 + z^2 - 2x - 6y - 12z + 52 = k^2$

Exercise 9.5

1. $\left(\frac{4}{3}, \frac{10}{3}, \frac{-5}{3}\right)$ and $\left(\frac{5}{3}, \frac{11}{3}, \frac{-4}{3}\right)$

2. (1) Non-collinear (2) Non-collinear (3) Non-collinear (4) Collinear (5) Non-collinear

Exercise 9

1. Parallelogram, not a rectangle **2.** Isosceles right triangle **3.** $x = 2z$

4. (1) $\frac{3}{2}, \frac{3}{\sqrt{2}}, \frac{3}{2}$; (1, 1, 1) (2) $\frac{3\sqrt{3}}{2}, \frac{3\sqrt{5}}{2}, \frac{3}{\sqrt{2}}$; (0, 1, 2) (3) $3\sqrt{5}, \sqrt{21}, \sqrt{6}$; $\left(\frac{1}{3}, \frac{2}{3}, \frac{1}{3}\right)$

5. $\left(1, 1, \frac{2}{3}\right)$

6. (1) Non-collinear

(2) Collinear, A divides in the ratio −2 : 1 from B and −1 : 2 from C

B divides in the ratio −2 : 1 from A and −1 : 2 from C

C divides in the ratio 1 : 1 from A and 1 : 1 from B

(3) Non-collinear

(4) Collinear, L divides in the ratio −1 : 3 from M and −3 : 1 from N

M divides in the ratio 1 : 2 from L and 2 : 1 from N

N divides in the ratio −3 : 2 from L and −2 : 3 from M

(5) Collinear, P divides in the ratio 1 : 1 from Q and 1 : 1 from R

Q divides in the ratio −1 : 2 from P and −2 : 1 from R

R divides in the ratio −1 : 2 from P and −2 : 1 from Q

7. (1) b (2) d (3) b (4) c (5) c (6) a (7) c (8) a (9) d (10) c (11) a (12) a (13) a (14) a (15) c (16) c (17) a (18) b (19) c (20) c

Exercise 10

11. $\frac{1}{12}$ **12.** $\frac{m}{n}$ **13.** -2 **14.** 41 **15.** $\frac{1}{4} \cdot x^{\frac{-3}{4}}$ **16.** $\frac{1}{3} \cdot x^{\frac{-2}{3}}$ **17.** $\frac{5}{4}$ **18.** 0

19. $\sqrt{3} - \sqrt{2}$ **20.** $\frac{n(n+1)}{2}$ **21.** 1 **22.** $-3\sqrt{2}$ **23.** $\frac{mn(n-m)}{2}$ **24.** $\frac{1}{12}$

25. 12 **26.** $\frac{1}{4\sqrt{2}}$ **27.** $\frac{1}{2}$ **28.** $-cosa$ **29.** $2cos3$ **30.** -1

31. $2asina + a^2cosa$ **32.** $secx(xtanx + 1)$

33. (1) b (2) d (3) b (4) c (5) b (6) a (7) d (8) a (9) d (10) c (11) d (12) c (13) d (14) a (15) b (16) b (17) a (18) c (19) b (20) d

Exercise 11

1. (1) 1 (2) −1 (3) 2 (4) $\frac{1}{5}$ (5) −8 (6) −1 (7) 2 (8) $2\sqrt{3}$ (9) −2 (10) $-2\sqrt{3}$

2. (1) 10 (2) $secx\ tanx + sec^2x$ (3) $cosec^2x - cosecx\ cotx$

(4) $4sinx\ cosx - 3sinx$ (5) $-2sin2x$ (6) $2cos2x$ (7) $2sec^22x$

(8) $\frac{1}{1+cosx}$ (9) $\frac{1}{1-sinx}$ (10) $3x^2$ (11) $4x^3$ (12) $6x^5$ (13) $4sin^3x\ cosx$

(14) $-4cos^3x\ sinx$ (15) $2sec^2x\ tanx$

4. $-2sin2x$ **5.** $\frac{(n-1)x^n - n \cdot x^{n-1} + 1}{(x-1)^2}$ **7.** $\frac{4x}{(x^2+1)^2}$ **8.** $\frac{(n-1)x^n - a \cdot nx^{n-1} + a^n}{(x-a)^2}$

9. $-35x^{-6} - 12x^{-5}$ **10.** $-16x^{-5} + 24x^{-4}$ **11.** $2secx\ tanx - 3sec^2x + 5cos2x$

12. $\frac{2secx\ tanx}{(secx+1)^2}$ **13.** $\frac{56 + 35(xcosx - sinx) + 32(cosx + sinx)}{(5x - 8cosx)^2}$ **14.** $\frac{1 + cotx + xcosec^2x}{(1+cotx)^2}$

15. $2(x - sin2x)$ **16.** $(p + qtanx)(2ax + b + cosx) + (ax^2 + bx + sinx)qsec^2x$

17. $cos(x + a)$ **18.** $cosa \cdot sec^2x$ **19.** $sec^2(x + a)$

20. (1) b (2) c (3) a (4) d (5) c (6) c (7) b (8) b (9) d (10) c (11) a (12) b (13) c (14) d (15) b (16) b (17) a (18) c (19) d (20) a (21) d (22) c (23) b (24) c (25) d

TERMINOLOGY

(In Gujarati)

Addition Formulae	સરવાળાનાં સૂત્રો
Allied Numbers	સંબંધિત સંખ્યાઓ
Argand diagram	આર્ગન્ડ આકૃતિ
Argument	કોણાંક
Arithmetic Progression (A.P.)	સમાંતર શ્રેણી
Binomial Theorem	દ્વિપદી પ્રમેય
Bound Vector	નિયત સદિશ
Branch	શાખા
Calculus	કલનશાસ્ત્ર
Central Conic	કેન્દ્રીય શાંકવ
Centroid	મધ્યકેન્દ્ર
Chord	જીવા
Circumcentre	પરિકેન્દ્ર
Common Difference	સામાન્ય તફાવત
Complex Numbers	સંકર સંખ્યાઓ
Conic / Conic Section	શાંકવ
Conjugate Axis	અનુબદ્ધ અક્ષ
Conjugate Hyperbola	અનુબદ્ધ અતિવલય
Conjugate of a Complex Number	અનુબદ્ધ સંકર સંખ્યા
Coordinate	યામ
Coordinate Axis	યામાક્ષ
Derivative	વિકલિત
Differentiation	વિકલન
Direction	દિશા
Directrices	નિયામિકાઓ
Directrix	નિયામિકા
Divisible	વિભાજ્ય
Eccentricity	ઉત્કેન્દ્રતા
Ellipse	ઉપવલય
Factor Formulae	અવયવ સૂત્રો
Focal Chord	નાભિજીવા
Foci	નાભિઓ
Focus	નાભિ
Free Vector	મુક્ત સદિશ
Geometric Progression (G.P.)	સમગુણોત્તર શ્રેણી
Graph	આલેખ
Hyperbola	અતિવલય
Imaginary Part	કાલ્પનિક ભાગ
Incentre	અંતઃકેન્દ્ર
Instantaneous Velocity	તાત્ક્ષણિક વેગ
Latera-recta	નાભિલંબો
Latus-rectum	નાભિલંબ
Law of Trichotomy	ત્રિવિધ વિકલ્પનો નિયમ
Limit	લક્ષ
Magnitude	માન
Major Axis	પ્રધાન અક્ષ
Mathematical Induction	ગણિતીય અનુમાન
Mean	મધ્યક
Minor Axis	ગૌણ અક્ષ
Modulus of a Complex Number	સંકર સંખ્યાનો માનાંક
Multiple	ગુણિત
Parabola	પરવલય
Parameter	પ્રચલ
Polar Form	ધ્રુવીય સ્વરૂપ
Position Vector	સ્થાન સદિશ
Projection Formula	પ્રક્ષેપ સૂત્ર
Purely Imaginary Number	શુદ્ધ કાલ્પનિક સંખ્યા
Real Part	વાસ્તવિક ભાગ
Rectangular Hyperbola	લંબાતિવલય
Recurrence Relation	આવૃત્ત સંબંધ
Rule of Substitution	આદેશનો નિયમ
Scalar	અદિશ
Secant	છેદિકા
Sequence	શ્રેણી
Series	શ્રેઢી
Slope	ઢાળ
Space	અવકાશ
Square Root	વર્ગમૂળ
Submultiple	ઉપગુણિત
Symmetric	સંમિત
Tangent	સ્પર્શક
Transverse Axis	મુખ્ય અક્ષ
Vector	સદિશ
Vertex	શિરોબિંદુ