7 Integrals

Fastrack Revision

▶ **Indefinite Integrals**

➤ Let $f(x)$ be a function. Then, the collection of all its primitives is called the indefinite integral of $f(x)$ and is denoted by $\int f(x)dx$.

➤ $\frac{d}{dx}[F(x)] = f(x) \Leftrightarrow \int f(x)dx = F(x) + k$

where k is integration constant.

The process of finding functions whose derivative is given, is called anti-differentiation or integration.

➤ $\frac{d}{dx}[\int f(x)dx] = f(x)$

➤ $\int [kf(x)]dx = k\int f(x)dx$, where k is a constant.

➤ $\int [f_1(x) \pm f_2(x)]\,dx = \int f_1(x)dx \pm \int f_2(x)\,dx$

Some Standard Formulae

S.No.	Integrand	Integral
1.	x^n	$\frac{x^{(n+1)}}{n+1} + C, n \neq -1$
2.	$\frac{1}{x}$	$\log\lvert x\rvert + C$
3.	e^x	$e^x + C$
4.	a^x	$\frac{a^x}{\log\lvert a\rvert} + C$
5.	$\sin x$	$-\cos x + C$
6.	$\cos x$	$\sin x + C$
7.	$\sec^2 x$	$\tan x + C$
8.	$\text{cosec}^2 x$	$-\cot x + C$
9.	$\sec x \cdot \tan x$	$\sec x + C$
10.	$\text{cosec}\, x \cdot \cot x$	$-\text{cosec}\, x + C$
11.	$\frac{\frac{d}{dx}(f(x))}{f(x)}$	$\log\lvert f(x)\rvert + C$
12.	$\tan x$	$\log\lvert\sec x\rvert + C$ or $-\log\lvert\cos x\rvert + C$
13.	$\cot x$	$\log\lvert\sin x\rvert + C$ or $-\log\lvert\text{cosec}\, x\rvert + C$
14.	$\sec x$	$\log\lvert\sec x + \tan x\rvert + C$ or $\log\left\lvert\tan\left[\frac{\pi}{4} + \frac{x}{2}\right] + C\right\rvert$
15.	$\text{cosec}\, x$	$\log\lvert\text{cosec}\, x - \cot x\rvert + C$ or $\log\left\lvert\tan\frac{x}{2}\right\rvert + C$
16.	$\frac{1}{\sqrt{1-x^2}}$	$\sin^{-1} x + C$ or $-\cos^{-1} x + C$
17.	$\frac{1}{1+x^2}$	$\tan^{-1} x + C$ or $-\cot^{-1} x + C$
18.	$\frac{1}{x\sqrt{x^2-1}}$	$\sec^{-1} x + C$ or $-\text{cosec}^{-1} x + C$
19.	$\frac{1}{x^2-a^2}$	$\frac{1}{2a}\log\left\lvert\frac{x-a}{x+a}\right\rvert + C$
20.	$\frac{1}{a^2-x^2}$	$\frac{1}{2a}\log\left\lvert\frac{a+x}{a-x}\right\rvert + C$
21.	$\frac{1}{x^2+a^2}$	$\frac{1}{a}\tan^{-1}\frac{x}{a} + C$
22.	$\frac{1}{\sqrt{a^2-x^2}}$	$\sin^{-1}\frac{x}{a} + C$
23.	$\frac{1}{\sqrt{x^2+a^2}}$	$\log\left\lvert x + \sqrt{x^2+a^2}\right\rvert + C$
24.	$\frac{1}{\sqrt{x^2-a^2}}$	$\log\left\lvert x + \sqrt{x^2-a^2}\right\rvert + C$
25.	$\sqrt{a^2-x^2}$	$\frac{1}{2}\left[x\sqrt{a^2-x^2} + a^2\sin^{-1}\frac{x}{a}\right] + C$
26.	$\sqrt{x^2 \pm a^2}$	$\frac{1}{2}\left[x\sqrt{x^2 \pm a^2} \pm a^2\log\left\lvert x + \sqrt{x^2 \pm a^2}\right\rvert\right] + C$

▶ **Integration by Substitution**

Sometimes the given integral functions are not in standard form. But sometimes by proper substitution, the given integral becomes standard form. This method is said to be integration by substitution.

▶ **Integration by Parts**

➤ $\int (u \times v)dx = u\int v\,dx - \int\left\{\frac{d}{dx}u \cdot \int v\,dx\right\}dx$, where u is first function and v is second function.

We use the following preferential order for taking the first function: Inverse → Logarithm → Algebraic → Trigonometric → Exponential. In short, we write it **ILATE**.

➤ To find the integral of the form $\int \frac{dx}{ax^2 + bx + c}$:

Write

$$\int \frac{dx}{ax^2+bx+c} = \frac{1}{a}\int \frac{dx}{x^2 + \frac{bx}{a} + \frac{c}{a}} = \frac{1}{a}\int \frac{dx}{\left(x + \frac{b}{2a}\right)^2 + \left(\frac{c}{a} - \frac{b^2}{4a^2}\right)}$$

which is anyone of the form $\int \frac{dx}{x^2 - A^2}$ or $\int \frac{dx}{x^2 + A^2}$

or $\int \frac{dx}{A^2 - x^2}$ depending upon the sign of $\left(\frac{c}{a} - \frac{b^2}{4a^2}\right)$

and hence can be evaluated.

➤ To find the integral of the form $\int \frac{dx}{\sqrt{ax^2+bx+c}}$:

Proceed using the above steps to reduce the integral of the form $\int \frac{dx}{\sqrt{x^2-A^2}}, \int \frac{dx}{\sqrt{x^2+A^2}}$ or $\int \frac{dx}{\sqrt{A^2-x^2}}$ and hence can be evaluated.

➤ To find the integral of the form $\int \frac{px+q}{ax^2+bx+c}dx$ and $\int \frac{px+q}{\sqrt{ax^2+bx+c}}dx$:

Put $px+q = A\left\{\frac{d}{dx}(ax^2+bx+c)\right\} + B = A(2ax+b)+B$

Find A and B by comparing the coefficients of x and the constant terms. Then, given integral is reduced to one of the known standard forms.

➤ To find the integral of the form $\int \frac{(px^2+qx+r)}{\sqrt{ax^2+bx+c}}dx$:

Let $px^2+qx+r = A(ax^2+bx+c) + B\frac{d}{dx}(ax^2+bx+c) + C$

Find A, B and C by comparing the coefficients and like powers of x and the constant terms. Then, given integral is reduced to one of the known standard forms.

➤ To find the integral of the form $\int \frac{a\sin x + b\cos x}{c\sin x + d\cos x}dx$:

Let $a\sin x + b\cos x = \lambda \frac{d}{dx}$(denominator) $+ \mu$ (denominator)

Find λ and μ by comparing the coefficients of $\sin x$ and $\cos x$. Then, given integral is reduced to one of the known standard form.

➤ To find the integral of the form $\int \frac{a\sin x + b\cos x + c}{p\sin x + q\cos x + r}dx$:

Let $a\sin x + b\cos x + c = \lambda \frac{d}{dx}$ (denominator) $+ \mu$ (denominator) $+ r$

Find λ, μ and r by comparing the coefficients of $\sin x$ and $\cos x$ and the constant terms. Then, given integral is reduced to one of the known standard form.

➤ To find the integral of the form $\int \frac{dx}{a+b\sin^2 x}$, $\int \frac{dx}{a+b\cos^2 x}$, $\int \frac{dx}{a\sin^2 x + b\cos^2 x}$, $\int \frac{dx}{a\sin^2 x + b\cos^2 x + c}$, $\int \frac{dx}{(a\sin x + b\cos x)^2}$:

Divide numerator and denominator by $\cos^2 x$, replace $\sec^2 x$ if any in denominator by $1+\tan^2 x$. Put $\tan x = t$ and integrate.

➤ To find the integral of the form $\int \frac{dx}{a+b\cos x}$ or $\int \frac{dx}{a+b\sin x}$ or $\int \frac{dx}{a\sin x + b\cos x}$ or $\int \frac{dx}{a\sin x + b\cos x + c}$:

Use $\cos x = \frac{1-\tan^2 x/2}{1+\tan^2 x/2}$ or $\sin x = \frac{2\tan x/2}{1+2\tan^2 x/2}$

Then put $\tan\frac{x}{2} = t$.

Rules of Partial Fractions

S.No.	Factor in the denominator	Corresponding partial fraction
1.	$(x-a)$	$\frac{A}{(x-a)}$
2.	$(x-a)(x-b)$	$\frac{A}{(x-a)} + \frac{B}{(x-b)}$
3.	$(x-b)^2$	$\frac{A}{(x-b)} + \frac{B}{(x-b)^2}$
4.	$(x-c)^3$	$\frac{A}{(x-c)} + \frac{B}{(x-c)^2} + \frac{C}{(x-c)^3}$
5.	$(x-b)^2(x-c)$	$\frac{A}{(x-b)} + \frac{B}{(x-b)^2} + \frac{C}{(x-c)}$
6.	(x^2+bx+c)	$\frac{Ax+B}{x^2+bx+c}$
7.	$(x-a)(x^2+bx+c)$	$\frac{A}{x-a} + \frac{Bx+C}{x^2+bx+c}$

➤ A special type of integral of the form

$$\int \frac{x^2+1}{x^4+kx^2+1}dx \quad \text{or} \quad \int \frac{x^2-1}{x^4+kx^2+1}dx$$

Divide the numerator and denominator by x^2 and put $\left(x+\frac{1}{x}\right) = t$ or $\left(x-\frac{1}{x}\right) = t$.

➤ Integrals of the form $\int \frac{dx}{a\sin x + b\cos x}$:

Let $a = r\cos\theta$, $b = r\sin\theta$, then calculate $r = \sqrt{a^2+b^2}$ and $\theta = \tan^{-1}\left(\frac{b}{a}\right)$ and convert given integrals in the form of $\operatorname{cosec}(\theta \pm x)$ or $\sec(\theta \mp x)$.

▶ Integration of Irrational Algebraic Function

➤ $\int \frac{dx}{(Ax+B)\sqrt{Cx+D}}$, substitute $Cx+D = t^2$, then the substitution will reduce the given integral into the form of $2\int \frac{dt}{At^2 - AD + BC}$.

➤ $\int \frac{dx}{(Ax^2+B)\sqrt{Cx+D}}$, substitute $Cx+D = t^2$, then the substitution will reduce the given integral into the form of $\int \frac{2Cdt}{At^4 - 2DAt^2 + (AD^2 + BC^2)}$.

➤ $\int \frac{dx}{(x-k)^r\sqrt{ax^2+bx+c}}$, substitute $x-k = \frac{1}{t}$ then the substitution will reduce the given integral into the form of $\int \frac{t^{r-1}}{At^2+Bt+C}dt$.

➤ $\int \frac{dx}{(Ax^2+B)\sqrt{Cx^2+D}}$, substitute $x = \frac{1}{t}$ then the substitution will reduce the given integral into the form of $\int \frac{tdt}{(A+Bt^2)\sqrt{C+Dt^2}}dt$. Again substitute $C+Dt^2 = u^2$ reduces into the form of $\int \frac{du}{u^2 \pm a^2}$.

- $\int \frac{ax^2+bx+c}{(dx+e)\sqrt{fx^2+gx+h}}dx$

  Here, we write $ax^2+bx+c=A(dx+e)\frac{d}{dx}(fx^2+gx+h)$
  $+B(dx+e)+C$

  Find A, B and C by comparing the coefficients of like powers of x and constant terms. Then, given integral is reduced to one of the known standard forms.

- **Properties of Definite Integrals**
  - $\int_a^b f(x)dx=F(b)-F(a)$
  - $\int_a^b f(x)dx=\int_a^b f(y)dy$
  - $\int_a^b f(x)dx=-\int_b^a f(x)dx$
  - $\int_a^b f(x)dx=\int_a^c f(x)dx+\int_c^b f(x)dx, a<c<b$
  - $\int_a^a f(x)dx=0$
  - $\int_0^a f(x)dx=\int_0^a f(a-x)dx$
  - $\int_a^b f(x)dx=\int_a^b f(a+b-x)dx$
  - $\int_0^{2a} f(x)dx=\int_0^a f(x)dx+\int_0^a f(2a-x)dx$
  - $\int_a^b \{f(x)\pm g(x)\}dx=\int_a^b f(x)dx\pm\int_a^b g(x)dx$
  - $\int_{-a}^a f(x)dx=$
    $\begin{cases} 0, & \text{if } f(x) \text{ is an odd function } i.e., f(-x)=-f(x) \\ 2\int_0^a f(x)dx, & \text{if } f(x) \text{ is an even function } i.e., f(-x)=f(x) \end{cases}$
  - $\int_0^{2a} f(x)dx=\begin{cases} 2\int_0^a f(x)dx, & \text{if } f(2a-x)=f(x) \\ 0, & \text{if } f(2a-x)=-f(x) \end{cases}$

Practice Exercise

Multiple Choice Questions

Q 1. $\int e^{5\log x}dx$ **is equal to:** *(CBSE 2023)*

a. $\frac{x^5}{5}+C$ b. $\frac{x^6}{6}+C$
c. $5x^4+C$ d. $6x^5+C$

Q 2. $\int 4^x 3^x\, dx$ **equals:** *(CBSE 2020)*

a. $\frac{12^x}{\log 12}+C$ b. $\frac{4^x}{\log 4}+C$
c. $\left(\frac{4^x\cdot 3^x}{\log 4\cdot\log 3}\right)+C$ d. $\frac{3^x}{\log 3}+C$

Q 3. **If** $f'(x)=x+\frac{1}{x}$**, then** $f(x)$ **is:** *(CBSE SQP 2022-23)*

a. $x^2+\log|x|+C$ b. $\frac{x^2}{2}+\log|x|+C$
c. $\frac{x}{2}+\log|x|+C$ d. $\frac{x}{2}-\log|x|+C$

Q 4. $\int x^2e^{x^3}dx$ **equals:** *(NCERT EXERCISE; CBSE 2020)*

a. $\frac{1}{3}e^{x^3}+C$ b. $\frac{1}{3}e^{x^4}+C$
c. $\frac{1}{2}e^{x^3}+C$ d. $\frac{1}{2}e^{x^2}+C$

Q 5. $\int \frac{\sin^2 x-\cos^2 x}{\sin x\cos x}dx$ **equals:** *(CBSE 2017)*

a. $-\log|\cos 2x|$ b. $-\log|\sin 2x|$
c. $-\log|\sec 2x|$ d. None of these

Q 6. $\int \frac{\sec x}{\sec x-\tan x}dx$ **equals:** *(CBSE 2023)*

a. $\sec x-\tan x+C$ b. $\sec x+\tan x+C$
c. $\tan x-\sec x+C$ d. $-(\sec x+\tan x)+C$

Q 7. **Integrate the function** $\frac{1-\tan x}{1+\tan x}$ **with respect to** x**.**

a. $\log|\cos x+\sin x|+C$ b. $\log|\sin x-\cos x|+C$
c. $-\log|\cos x+\sin x|+C$ d. $\frac{1}{2}\log|\sin x-\cos x|+C$

Q 8. $\int \sin 2x\cos 3x\, dx$ **equals:** *(NCERT EXERCISE)*

a. $\frac{1}{2}\left[\frac{-\cos 5x}{5}+\cos x\right]+C$
b. $\frac{1}{2}\left[-\frac{\sin 5x}{5}+\sin x\right]+C$
c. $\frac{1}{10}\left[\frac{\cos 5x}{5}-\cos x\right]+C$
d. None of the above

Q 9. $\frac{\cos 2x}{\sqrt{1+\sin 2x}}dx$ **equals:** *(NCERT EXEMPLAR)*

a. $\sin x-\cos x+C$ b. $\cot x+\cos x+C$
c. $\tan x+\sec x+C$ d. $\sin x+\cos x+C$

Q 10. $\int \sin^3 x\cos^2 x\, dx$ **equals:** *(NCERT EXERCISE)*

a. $\frac{1}{5}\cos^5 x-\frac{1}{3}\cos^3 x+C$ b. $\frac{1}{3}\cos^5 x-\frac{1}{5}\cos^3 x+C$
c. $\frac{1}{5}\sin^5 x-\frac{1}{3}\sin^3 x+C$ d. $\frac{1}{5}\sin^5 x-\frac{1}{3}\cos^3 x+C$

Q 11. **The value of** $\int \frac{1}{\sqrt{1-e^{-2x}}}dx$ **is:**

a. $\log|e^x+\sqrt{e^x-1}|+C$ b. $\log|e^x-\sqrt{e^x-1}|+C$
c. $\log|e^{2x}+\sqrt{e^x-1}|+C$ d. $\log|e^x+\sqrt{e^{2x}-1}|+C$

Q 12. $\int x\log x\, dx$ **equals:** *(NCERT EXERCISE)*

a. $\frac{x^2}{2}(\log x-2)+C$ b. $\frac{x^2}{4}(2\log x-1)+C$
c. $\frac{1}{4x}(\log x-2)+C$ d. None of these

Q 13. $\int \frac{xe^x}{(x+1)^2} dx$ **equals:** *(NCERT EXERCISE)*

a. $-\frac{e^x}{(x+1)}+C$ b. $\frac{e^{-x}}{(x+1)}+C$

c. $\frac{e^x}{(x+1)^2}+C$ d. $\frac{e^x}{(x+1)}+C$

Q 14. $\int \frac{1-\cos x}{\cos x\,(1+\cos x)} dx$ **equals:**

a. $\log\tan\left(\frac{x}{2}+\frac{\pi}{4}\right)-2\operatorname{cosec} x+2\cot x+C$

b. $\log\tan\left(\frac{x}{2}+\frac{\pi}{4}\right)-2\sin x+2+C$

c. $\log\tan\frac{x}{2}-2\sin x+2\tan x+C$

d. None of the above

Q 15. $\int \frac{dx}{1+\cos x+\sin x}$ **equals:**

a. $\log\left|1+\cot\frac{x}{2}\right|+C$ b. $\log\left|1+\tan\frac{x}{2}\right|+C$

c. $\log\left|1-\cot\frac{x}{2}\right|+C$ d. None of these

Q 16. $\int \frac{\sin(x-a)}{\sin(x+a)} dx$**, where a is constant, equals:**

a. $x\cos 2a-\sin 2a\cdot\log|\sin(x+a)|+C$

b. $x\sin 2a-\cos 2a\cdot\log|\cos(x+a)|+C$

c. $x\cos 2a-\cos 2a\cdot\log|\cos(x+a)|+C$

d. $x\sin 2a-\sin 2a\cdot\log|\sin(x+a)|+C$

Q 17. $\int \frac{\cos x}{\sqrt{3+\cos^2 x}} dx$ **equals:** *(NCERT EXERCISE)*

a. $\sin^{-1}\left(\frac{1}{2}\cos x\right)+C$ b. $\cos^{-1}\left(\frac{1}{2}\sin x\right)+C$

c. $\sin^{-1}\left(\frac{1}{2}\sin x\right)+C$ d. $\cos^{-1}\left(\frac{1}{2}\cos x\right)+C$

Q 18. $\int \frac{dx}{2x^{1/2}+x^{3/2}}$ **equals:**

a. $\sqrt{2}\tan^{-1}\left(\sqrt{\frac{x}{2}}\right)+C$ b. $\frac{1}{\sqrt{2}}\tan^{-1}\left(\sqrt{\frac{x}{2}}\right)+C$

c. $\frac{1}{\sqrt{2}}\cot^{-1}\left(\frac{\sqrt{x}}{2}\right)+C$ d. None of these

Q 19. $\int \frac{dx}{\sqrt{3x^2+13x-10}}$ **equals:** *(NCERT EXERCISE)*

a. $\frac{1}{\sqrt{3}}\log\left|\left(x+\frac{13}{6}\right)+\sqrt{x^2+\frac{13}{3}x-\frac{10}{3}}\right|+C$

b. $\sqrt{3}\log\left|\left(x-\frac{13}{6}\right)+\sqrt{x^2-\frac{13}{3}x+\frac{10}{3}}\right|+C$

c. $\frac{1}{\sqrt{3}}\log\left|\left(x-\frac{13}{6}\right)-\sqrt{x^2-\frac{13}{3}x+\frac{10}{3}}\right|+C$

d. None of the above

Q 20. $\int \frac{x^2+1}{x^2-1} dx$ **equals:**

a. $x-\log\left|\frac{x-1}{x+1}\right|+C$ b. $x+\log\left|\frac{x-1}{x+1}\right|+C$

c. $x-2\log\left|\frac{x-1}{x+1}\right|+C$ d. None of these

Q 21. $\int \frac{x^2}{(x^2+1)(x^2+4)} dx$ **equals:** *(NCERT EXERCISE)*

a. $\frac{2}{3}\tan^{-1}\frac{x}{2}-\frac{1}{3}\tan^{-1}x+C$

b. $\frac{2}{3}\cot^{-1}\frac{x}{2}-\frac{1}{3}\cot^{-1}x+C$

c. $\frac{3}{2}\tan^{-1}\frac{x}{2}+\frac{1}{3}\cot^{-1}x+C$

d. None of the above

Q 22. $\int \frac{\sec^2 x}{(1+\tan x)(2+\tan x)} dx$ **equals:**

a. $\log\left|\frac{1+\tan x}{2+\cot x}\right|+C$ b. $\log\left|\frac{1+\cos x}{2+\tan x}\right|+C$

c. $\log\left|\frac{1+\tan x}{2+\tan x}\right|+C$ d. None of these

Q 23. $\int \frac{1}{x(2+x^5)} dx$ **equals:** *(NCERT EXERCISE)*

a. $\frac{1}{5}\log\left|\frac{x^5}{2+x^5}\right|+C$ b. $\frac{1}{10}\log\left|\frac{x^5}{1+x^5}\right|+C$

c. $\frac{1}{5}\log\left|\frac{x^5}{1+x^5}\right|+C$ d. $\frac{1}{10}\log\left|\frac{x^5}{2+x^5}\right|+C$

Q 24. $\int \cot^{-1} x\, dx$ **equals:**

a. $x\cot^{-1}x+\frac{1}{2}\log|1+x^2|+C$

b. $x\tan^{-1}x+\log|1+x^2|+C$

c. $x\tan^{-1}x+\frac{1}{2}\log|1+x^2|+C$

d. None of the above

Q 25. $\int x\tan^2 x\, dx$ **equals:**

a. $x\cot x-\log|\operatorname{cosec} x|-\frac{x^2}{4}+C$

b. $x\tan x-\log|\sec x|-\frac{x^2}{2}+C$

c. $x\tan x-\log|\operatorname{cosec} x|-\frac{x^2}{4}+C$

d. None of the above

Q 26. $\int \frac{x+\sin x}{1+\cos x} dx$ **equals:** *(NCERT EXEMPLAR)*

a. $x\sec\frac{x}{2}+C$ b. $x\cot\frac{x}{2}+C$

c. $x\tan\frac{x}{2}+C$ d. $x\operatorname{cosec}\frac{x}{2}+C$

Q 27. $\int \sqrt{3-2x-2x^2}\, dx$ **equals:**

a. $\frac{1}{\sqrt{2}}\left(x+\frac{1}{2}\right)\sqrt{3-2x-2x^2}$

$+\frac{7}{4\sqrt{2}}\sin^{-1}\left\{\frac{2}{\sqrt{7}}\left(x+\frac{1}{2}\right)\right\}+C$

b. $\frac{1}{2}\left(x+\frac{1}{2}\right)\sqrt{3-2x-2x^2}$
$+\frac{7}{4\sqrt{2}}\log\left\{\left(x+\frac{1}{2}\right)+\sqrt{3-2x-2x^2}\right\}+C$

c. $\frac{1}{2}\left(x+\frac{1}{2}\right)\sqrt{3-2x-2x^2}$
$+\frac{1}{\sqrt{2}}\log\left\{\left(x+\frac{1}{2}\right)-\sqrt{3-2x-2x^2}\right\}+C$

d. None of the above

Q 28. Find the value of $\int_0^1 x(1-x)^n dx$. *(CBSE 2020)*

a. $-\frac{1}{(n+1)(n+2)}$
b. $\frac{n}{(n+1)(n+2)}$
c. $\frac{1}{(n+1)(n+2)}$
d. $\frac{n+1}{n+2}$

Q 29. $\int_0^4 (e^{2x}+x)\,dx$ is equal to: *(CBSE 2023)*

a. $\frac{15+e^8}{2}$
b. $\frac{16-e^8}{2}$
c. $\frac{e^8-15}{2}$
d. $\frac{-e^8-15}{2}$

Q 30. If $\int_0^a 3x^2dx = 8$, then the value of 'a' is: *(CBSE 2023)*

a. 2 b. 4 c. 8 d. 10

Q 31. The value of $\int_0^{\pi/4}(\sin 2x)\,dx$ is: *(CBSE 2023)*

a. 0
b. 1
c. $\frac{1}{2}$
d. $-\frac{1}{2}$

Q 32. Find the value of $\int_0^{\pi/2}\frac{\sin^4 x}{\sin^4 x+\cos^4 x}dx$.

a. $\frac{\pi}{2}$
b. $\frac{\pi^2}{4}$
c. $\frac{\pi}{4}$
d. None of these

Q 33. $\int_{-\pi/4}^{\pi/4}\sin^2 x\,dx$ equals: *(NCERT EXERCISE)*

a. $\frac{\pi}{4}-\frac{1}{2}$
b. $\frac{\pi}{4}+1$
c. $\frac{\pi}{4}+\frac{1}{2}$
d. $\frac{\pi}{4}$

Q 34. The value of $\int_2^3\frac{x}{x^2+1}dx$ is: *(CBSE SQP 2022-23)*

a. $\log 4$
b. $\log\frac{3}{2}$
c. $\frac{1}{2}\log 2$
d. $\log\frac{9}{4}$

Q 35. $\int_{-1}^{1}\frac{|x-2|}{x-2}dx,\ x\neq 2$ is equal to: *(CBSE 2023)*

a. 1
b. -1
c. 2
d. -2

Q 36. For any integer n, the value of $\int_0^{\pi}e^{\sin^2 x}\cos^3(2n+1)x\,dx$ is: *(CBSE SQP 2023-24)*

a. -1
b. 0
c. 1
d. 2

Q 37. $\int_0^{\pi/4}\log(\sin 2\theta)\,d\theta$ equals:

a. $\frac{\pi}{2}\log 2$
b. $-\frac{\pi}{2}\log 2$
c. $-\frac{\pi}{4}\log 2$
d. $\frac{\pi}{4}\log 2$

Q 38. $\int_0^{\pi/2}(\sin 2x\cdot\log\tan x)\,dx$ equals:

a. $\frac{\pi}{2\sqrt{2}}$
b. $\frac{\pi}{\sqrt{2}}$
c. $\frac{\pi^2}{2\sqrt{2}}$
d. 0

Q 39. $\int_{-1}^{1}\sin^5 x\cdot\cos^4 x\,dx$ equals: *(NCERT EXERCISE)*

a. 0 b. 1 c. $\frac{1}{2}$ d. $\frac{\pi}{2}$

Assertion & Reason Type Questions

Directions (Q. Nos. 40-47): *In the following questions, each question contains Assertion (A) and Reason (R). Each question has 4 choices (a), (b), (c) and (d) out of which only one is correct. The choices are:*

a. Both Assertion (A) and Reason (R) are true and Reason (R) is the correct explanation of Assertion (A)
b. Both Assertion (A) and Reason (R) are true but Reason (R) is not the correct explanation of Assertion (A)
c. Assertion (A) is true and Reason (R) is false
d. Assertion (A) is false and Reason (R) is true

Q 40. Assertion (A):

$$\int\sin 3x\cos 5x\,dx = \frac{-\cos 8x}{16}+\frac{\cos 2x}{4}+C$$

Reason (R): $2\cos A\sin B = \sin(A+B)-\sin(A-B)$

Q 41. Let $I=\int\frac{\sqrt{\cos 2x}}{\sin x}dx$

Assertion (A):

$$I=\ln\left[\left(\frac{t}{1+\sqrt{1-t^2}}\right)\left(\frac{\sqrt{2}+\sqrt{1-t^2}}{\sqrt{2}-\sqrt{1-t^2}}\right)^{\frac{1}{\sqrt{2}}}\right]+C$$

where, $t=\tan x$

Reason (R): $\int\cot\theta\,d\theta = \log|\sin\theta|+C$

Q 42. Let $F(x)$ be an indefinite integral of $\sin^2 x$.

Assertion (A): The function $F(x)$ satisfies $F(x+\pi)=F(x)$ for all real x.

Reason (R): $\sin^2(x+\pi)=\sin^2 x$ for all real x.

Q 43. Assertion (A): $\int_2^8\frac{\sqrt{10-x}}{\sqrt{x}+\sqrt{10-x}}dx=3$

Reason(R): $\int_a^b f(x)\,dx=\int_a^b f(a+b-x)\,dx$

(CBSE 2023)

Q 44. Assertion (A): $I=\int_0^1 \frac{dx}{\sqrt[3]{1+x^3}}=\int_0^{2^{-1/3}} \frac{dt}{1-t^3}$

Reason (R): The integrand of the integral I becomes rational by the substitution $t=\frac{x}{\sqrt[3]{1+x^3}}$.

Q 45. Assertion (A): $\int_0^{\pi} \cos 6x \cdot \cos 6x \, dx=\frac{\pi}{32}$

Reason (R): $\int_0^{\pi} \cos mx \cos nx \, dx=0$, $m \neq n, m, n \in Z$.

Q 46. Assertion (A):

$$\int_{-\pi/3}^{\pi/3} \frac{(3+4x^3)\,dx}{2-\cos\left(|x|+\frac{\pi}{3}\right)}=4\sqrt{3}\tan^{-1}\left(\frac{1}{2}\right)$$

Reason (R): $\int_{-a}^{a} f(x)\,dx=\int_0^a [f(x)+f(-x)]\,dx$

Q 47. Assertion (A): $\int_0^{2\pi} \sin^3 x \, dx=0$

Reason (R): $\sin^3 x$ is an odd function.

Answers

1. (c)	2. (a)	3. (b)	4. (a)	5. (b)	6. (b)	7. (a)	8. (a)	9. (d)	10. (a)
11. (d)	12. (b)	13. (d)	14. (a)	15. (b)	16. (a)	17. (c)	18. (a)	19. (a)	20. (b)
21. (a)	22. (c)	23. (d)	24. (a)	25. (b)	26. (c)	27. (a)	28. (c)	29. (a)	30. (a)
31. (c)	32. (c)	33. (a)	34. (c)	35. (d)	36. (b)	37. (c)	38. (d)	39. (a)	40. (a)
41. (d)	42. (d)	43. (a)	44. (a)	45. (d)	46. (a)	47. (b)			

Case Study Based Questions

Case Study 1

Following paragraph given to student by the teacher. The given integral $\int f(x)\,dx$ can be transformed into another form by changing the independent variable x to t by substituting $x=g(t)$.

Consider $I=\int f(x)\,dx$

Put $x=g(t)$ so that $\frac{dx}{dt}=g'(t)$

We write $dx=g'(t)\,dt$

Thus, $I=\int f(x)\,dx=\int f[g(t)]g'(t)\,dt$

This change of variable formula is one of the important tools available to us in the name of integration by substitution.

Based on the above information, solve the following questions:

Q 1. $\int 2x \sin(x^2+1)\,dx$ **is equal to:**

a. $-\sin(x^2+1)+C$ b. $-\cos(x^2+1)+C$

c. $\sin(x^2+1)+C$ d. None of these

Q 2. $\int \frac{x}{\sqrt{32-x^2}}\,dx$ **is equal to:**

a. $-\sqrt{32-x^2}+C$ b. $\sqrt{32+x^2}+C$

c. $\sqrt{64-x^2}+C$ d. $\sqrt{32-x^2}+C$

Q 3. $\int \frac{\sin(2\tan^{-1}x)}{1+x^2}\,dx$ **is equal to:**

a. $-\frac{\sin 2(\tan^{-1}x)}{2}+C$ b. $-\frac{\cos 2(\tan^{-1}x)}{2}+C$

c. $\frac{\cos 2(\tan^{-1}x)}{2}+C$ d. $\frac{\sin 2(\tan^{-1}x)}{2}+C$

Q 4. $\int \frac{\sqrt{1+x^2}}{x^4}\,dx$ **is equal to:**

a. $-\frac{1}{3}\left(1+\frac{1}{x^2}\right)^{3/2}+C$ b. $\frac{1}{3}\left(1+\frac{1}{x^2}\right)^{3/2}+C$

c. $\frac{1}{3}\left(1+\frac{1}{x^2}\right)^{1/2}+C$ d. None of these

Q 5. $\int \frac{\cos^{-1}x}{\sqrt{1-x^2}}\,dx$ **is equal to:**

a. $\frac{(\sin^{-1}x^2)^2}{2}+C$ b. $\frac{(-\sin^{-1}x)^3}{2}+C$

c. $\frac{-(\cos^{-1}x)^2}{2}+C$ d. $\frac{(\tan^{-1}x)^2}{x}+C$

Solutions

1. Let $I=\int 2x\sin(x^2+1)\,dx$

Put $x^2+1=t \Rightarrow 2x\,dx=dt$

[Differentiating both sides w.r.t. x]

Now, $I=\int \sin t\,dt$

$=-\cos t+C=-\cos(x^2+1)+C$

So, option (b) is correct.

2. Let $I=\int \frac{x}{\sqrt{32-x^2}}\,dx$

Put $32-x^2=t$

$\Rightarrow -2x\,dx=dt$

$\Rightarrow x\,dx=\frac{-1}{2}dt$

Now, $I=-\frac{1}{2}\int \frac{dt}{\sqrt{t}}$

$=\frac{-1}{2}\cdot 2\sqrt{t}+C$

$=-\sqrt{t}+C$

$=-\sqrt{32-x^2}+C$

So, option (a) is correct.

3. Let $I = \int \frac{\sin(2\tan^{-1}x)}{1+x^2} dx$

Put $\tan^{-1}x = t \Rightarrow \frac{1}{1+x^2}dx = dt$

$\therefore$ $I = \int \sin 2t\, dt = -\frac{\cos 2t}{2} + C$

$= -\frac{\cos 2(\tan^{-1}x)}{2} + C$

So, option (b) is correct.

4. Let $I = \int \frac{\sqrt{1+x^2}}{x^4} dx = \int \frac{\sqrt{1+x^2}}{x} \cdot \frac{1}{x^3} dx$

$= \int \sqrt{\frac{1+x^2}{x^2}} \cdot \frac{1}{x^3} dx = \int \sqrt{\frac{1}{x^2}+1} \cdot \frac{1}{x^3} dx$

Now, putting $1 + \frac{1}{x^2} = t^2$

$\Rightarrow$ $\frac{-2}{x^3}dx = 2t\,dt \Rightarrow -\frac{1}{x^3}dx = t\,dt$

$\therefore$ $I = \int -t^2 dt = -\frac{t^3}{3} + C$

$= -\frac{1}{3}\left(1+\frac{1}{x^2}\right)^{3/2} + C$

So, option (a) is correct.

5. Let $I = \int \frac{\cos^{-1}x}{\sqrt{1-x^2}} dx$

Put $\cos^{-1}x = t$

$\Rightarrow$ $\frac{-1}{\sqrt{1-x^2}}dx = dt \Rightarrow \frac{1}{\sqrt{1-x^2}}dx = -dt$

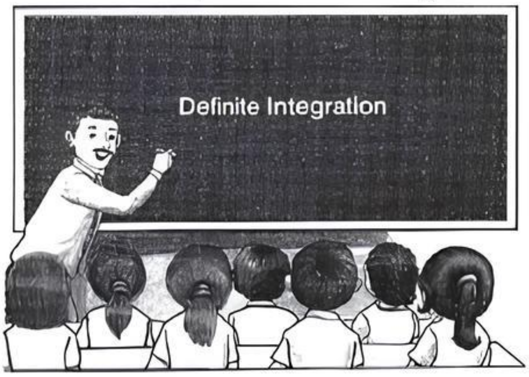
$\therefore$ $I = \int t(-dt) = -\int t\,dt$

$= \frac{-t^2}{2} + C = \frac{-(\cos^{-1}x)^2}{2} + C$

So, option (c) is correct.

Case Study 2

In Presidency Public School, class teacher of XIIth class teaches the topic of definite integration.



If $f(x)$ is the continuous function, integral of $f(x)$ over the interval $[a, b]$ is denoted by $\int_a^b f(x)\,dx$ and

$\int_a^b f(x)\,dx = [F(x)]_a^b = [F(b) - F(a)]$.

Based on the above information, solve the following questions:

Q 1. $\int_4^7 x^2 dx$ **is equal to:**

a. $\frac{7}{3}$ b. $\frac{278}{3}$ c. 93 d. $\frac{407}{3}$

Q 2. $\int_1^{\sqrt{3}} \frac{1}{1+x^2} dx$ **is equal to:**

a. $\frac{\pi}{3}$ b. $\frac{2\pi}{3}$ c. $\frac{\pi}{12}$ d. $\frac{\pi}{6}$

Q 3. $\int_{-1}^{1} (x+3)\,dx$ **is equal to:**

a. 2 b. 6 c. −1 d. −6

Q 4. $\int_4^6 e^x dx$ **is equal to:**

a. 1 b. $e^6 - e^4$ c. $e^5 - 1$ d. $e^5 - e^3$

Q 5. $\int_2^3 \frac{1}{x} dx$ **is equal to:**

a. $\log\frac{2}{3}$ b. $\log 3$ c. $\log\frac{3}{2}$ d. $\log 2$

Solutions

1. $\int_4^7 x^2\,dx = \left[\frac{x^3}{3}\right]_4^7 = \left[\frac{(7)^3}{3} - \frac{(4)^3}{3}\right]$

$= \left[\frac{343}{3} - \frac{64}{3}\right] = \frac{279}{3} = 93$

So, option (c) is correct.

2. $\int_1^{\sqrt{3}} \frac{dx}{1+x^2} = [\tan^{-1}x]_1^{\sqrt{3}}$

$= \tan^{-1}\sqrt{3} - \tan^{-1}1 = \frac{\pi}{3} - \frac{\pi}{4} = \frac{\pi}{12}$

So, option (c) is correct.

3. $\int_{-1}^{1} (x+3)\,dx$

$= \left[\frac{x^2}{2} + 3x\right]_{-1}^{1} = \left[\frac{1}{2}+3\right] - \left[\frac{1}{2}-3\right] = 6$

So, option (b) is correct.

4. $\int_4^6 e^x\,dx = [e^x]_4^6 = e^6 - e^4$

So, option (b) is correct.

5. $\int_2^3 \frac{1}{x}dx = [\log|x|]_2^3 = [\log 3 - \log 2]$

$= \log\frac{3}{2}$ $\left[\because \log m - \log n = \log\frac{m}{n}\right]$

So, option (c) is correct.

Case Study 3

For any function $f(x)$, we have

$\int_a^b f(x)\,dx = \int_a^{c_1} f(x)\,dx + \int_{c_1}^{c_2} f(x)\,dx$

$+ \ldots.. + \int_{c_n}^{b} f(x)\,dx$

where, $a < c_1 < c_2 < c_3 \ldots.. < c_{n-1} < c_n < b$.

Based on the above information, solve the following questions:

Q 1. $\int_0^{3/2} |4x - 5|\,dx =$

a. $\frac{13}{10}$ b. $\frac{13}{4}$ c. $\frac{11}{10}$ d. $\frac{11}{4}$

Q 2. $\int_0^{2\pi} |\cos x|\,dx =$

a. 1 b. 2 c. 3 d. 4

Q 3. $\int_{-1}^{2} e^{|x|}\, dx =$

a. $e-2$ b. $2e-2$ c. e^2+e-2 d. $2e^2-1$

Q 4. $\int_{0}^{5} [x]\, dx =$

a. 10 b. 14 c. 17 d. 20

Q 5. $\int_{-2}^{1} f(x)\, dx =$

where, $f(x) = \begin{cases} 3-5x, & x<0 \\ 4+3x, & x \ge 0 \end{cases}$

a. $\frac{23}{2}$ b. $\frac{35}{2}$ c. $\frac{43}{2}$ d. $\frac{47}{2}$

Solutions

1. Integrand $f(x) = |4x-5|$ can be defined as:

$$|4x-5| = \begin{cases} -(4x-5), & x \le \frac{5}{4} \\ 4x-5, & x > \frac{5}{4} \end{cases}$$

Let $I = \int_0^{3/2} |4x-5|\, dx$

$$= \int_0^{5/4} -(4x-5)\, dx + \int_{5/4}^{3/2} (4x-5)\, dx$$
$$= \int_0^{5/4} (5-4x)\, dx + \int_{5/4}^{3/2} (4x-5)\, dx$$
$$= [5x-2x^2]_0^{5/4} + [2x^2-5x]_{5/4}^{3/2}$$
$$= 5\left(\frac{5}{4}-0\right) - 2\left\{\left(\frac{5}{4}\right)^2 - 0\right\} + 2\left\{\left(\frac{3}{2}\right)^2 - \left(\frac{5}{4}\right)^2\right\} - 5\left(\frac{3}{2}-\frac{5}{4}\right)$$
$$= \frac{25}{4} - 2\times\frac{25}{16} + 2\left(\frac{3}{2}+\frac{5}{4}\right)\left(\frac{3}{2}-\frac{5}{4}\right) - 5\times\frac{1}{4}$$
$$= \frac{25}{4} - \frac{25}{8} + 2\times\frac{11}{4}\times\frac{1}{4} - \frac{5}{4}$$
$$= \frac{(100-50+22-20)}{16} = \frac{52}{16} = \frac{13}{4}$$

So, option (b) is correct.

2. Integrand $f(x) = |\cos x|$ can be defined as:

$$|\cos x| = \begin{cases} \cos x, & 0 \le x \le \frac{\pi}{2} \\ -\cos x, & \frac{\pi}{2} \le x \le \frac{3\pi}{2} \\ \cos x, & \frac{3\pi}{2} \le x \le 2\pi \end{cases}$$

Let $I = \int_0^{2\pi} |\cos x|\, dx$

$$= \int_0^{\pi/2} \cos x\, dx + \int_{\pi/2}^{3\pi/2} (-\cos x)\, dx + \int_{3\pi/2}^{2\pi} \cos x\, dx$$
$$= [\sin x]_0^{\pi/2} - [\sin x]_{\pi/2}^{3\pi/2} + [\sin x]_{3\pi/2}^{2\pi}$$
$$= \left(\sin\frac{\pi}{2} - \sin 0\right) - \left(\sin\frac{3\pi}{2} - \sin\frac{\pi}{2}\right) + \left(\sin 2\pi - \sin\frac{3\pi}{2}\right)$$
$$= (1-0) - (-1-1) + [0-(-1)] = 1+2+1 = 4$$

So, option (d) is correct.

3. Integrand $f(x) = e^{|x|}$ can be defined as:

$$e^{|x|} = \begin{cases} x, & x \ge 0 \\ -x, & x < 0 \end{cases}$$

Let $I = \int_{-1}^{2} e^{|x|}\, dx = \int_{-1}^{0} e^{-x} dx + \int_0^2 e^x\, dx$

$$= [-e^{-x}]_{-1}^{0} + [e^x]_0^2$$
$$= -(e^0 - e^1) + (e^2 - e^0)$$
$$= -(1-e) + (e^2-1)$$
$$= -1 + e + e^2 - 1 = e^2 + e - 2$$

So, option (c) is correct.

4. Integrand $f(x) = [x]$ can be defined as:

$$[x] = \begin{cases} 0, & 0 \le x < 1 \\ 1, & 1 \le x < 2 \\ 2, & 2 \le x < 3 \\ 3, & 3 \le x < 4 \\ 4, & 4 \le x < 5 \end{cases}$$

Let $I = \int_0^5 [x]\, dx$

$$= \int_0^1 0\, dx + \int_1^2 1\, dx + \int_2^3 2\, dx + \int_3^4 3\, dx + \int_4^5 4\, dx$$
$$= 0 + [x]_1^2 + 2[x]_2^3 + 3[x]_3^4 + 4[x]_4^5$$
$$= (2-1) + 2(3-2) + 3(4-3) + 4(5-4)$$
$$= 1 + 2\times 1 + 3\times 1 + 4\times 1$$
$$= 1+2+3+4 = 10$$

So, option (a) is correct.

5. Let $I = \int_{-2}^{1} f(x)\, dx = \int_{-2}^{0} f(x)\, dx + \int_0^1 f(x)\, dx$

$$= \int_{-2}^{0} (3-5x)\, dx + \int_0^1 (4+3x)\, dx$$
$$= \left[3x - \frac{5}{2}x^2\right]_{-2}^{0} + \left[4x + \frac{3}{2}x^2\right]_0^1$$
$$= 3(0+2) - \frac{5}{2}(0-4) + 4(1-0) + \frac{3}{2}(1-0)$$
$$= 6 + 10 + 4 + \frac{3}{2} = 20 + \frac{3}{2} = \frac{43}{2}$$

So, option (c) is correct.

Case Study 4

If $f(x)$ is a continuous function defined on $[-a, a]$, then

$$\int_{-a}^{a} f(x) dx = \begin{cases} 2\int_0^a f(x)\, dx, & \text{if } f(x) \text{ is an even function} \\ 0, & \text{if } f(x) \text{ is an odd function} \end{cases}$$

A function $f(x)$ is even, when $f(-x) = f(x)$ and odd when $f(-x) = -f(x)$.

Based on the above information, solve the following questions:

Q 1. If $f(x)$ is an odd function, then the value of $\int_{-1}^{1} \{f(x) + f(-x)\}\, dx$ is:

a. 0 b. 1 c. $\frac{1}{2}$ d. -1

Q 2. If $f(x)$ is even function, then the value of $\int_{-c}^{c} \{f(x) - f(-x)\}\, dx$ is:

a. -1 b. 0 c. $\frac{1}{2}$ d. 1

Q 3. $\int_{-2}^{2} \log\left(\frac{4-x}{4+x}\right) dx =$

a. 0 b. –1 c. $\frac{1}{2}$ d. 2

Q 4. $\int_{-\pi/2}^{\pi/2} x \sin x \, dx =$

a. π b. $\frac{\pi}{2}$ c. 2 d. $\frac{1}{2}$

Q 5. $\int_{-\pi/4}^{\pi/4} \frac{dx}{1+\cos 2x}$ **is equal to:**

a. 1 b. 2 c. 3 d. 4

Solutions

1. Given that, $f(x)$ is odd when

$$f(-x) = -f(x)$$

$$\therefore \int_{-1}^{1} \{f(x) + f(-x)\} dx = \int_{-1}^{1} \{f(x) - f(x)\} dx$$

$$= \int_{-1}^{1} 0 \, dx = 0$$

So, option (a) is correct.

2. Given that, $f(x)$ is even when

$$f(-x) = f(x)$$

$$\therefore \int_{-c}^{c} \{f(x) - f(-x)\} dx = \int_{-c}^{c} \{f(x) - f(x)\} dx$$

$$= \int_{-c}^{c} 0 \, dx = 0$$

So, option (b) is correct.

3. Let the integrand $f(x) = \log\left(\frac{4-x}{4+x}\right)$

$$\therefore \quad f(-x) = \log\left(\frac{4+x}{4-x}\right) = \log\left(\frac{4-x}{4+x}\right)^{-1}$$

$$= -\log\left(\frac{4-x}{4+x}\right)$$

$$= -f(x)$$

$\Rightarrow f(x)$ is an odd function.

$$\therefore \quad \int_{-2}^{2} \log\left(\frac{4-x}{4+x}\right) dx = 0$$

So, option (a) is correct.

4. Let the integrand $f(x) = x \sin x$

$\therefore \; f(-x) = (-x)\sin(-x) = (-x)(-\sin x) = x \sin x = f(x)$

$\Rightarrow \; f(x)$ is an even function.

$$\therefore \int_{-\pi/2}^{\pi/2} x \sin x \, dx = 2\int_{0}^{\pi/2} x \sin x \, dx$$

$$= 2\left(x \int \sin x \, dx\right)_0^{\pi/2} - 2\left[\int\left\{\frac{d}{dx}(x) \int \sin x \, dx\right\} dx\right]_0^{\pi/2}$$

$$= 2\left(-x \cos x\right)_0^{\frac{\pi}{2}} - 2\left(\int -\cos x \, dx\right)_0^{\frac{\pi}{2}}$$

$$= -2\left(x \cos x\right)_0^{\pi/2} + 2\left(\sin x\right)_0^{\pi/2}$$

$$= -2\left[\left(\frac{\pi}{2}\right)\cos\left(\frac{\pi}{2}\right) - 0 \cdot \cos 0\right] + 2\left[\sin\frac{\pi}{2} - \sin 0\right]$$

$$= -2\left[\frac{\pi}{2} \times 0 - 0 \times 1\right] + 2(1 - 0)$$

$$= -2 \times 0 + 2 \times 1 = 0 + 2 = 2$$

So, option (c) is correct.

5. Let the integrand $f(x) = \frac{1}{1+\cos 2x}$.

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$\cos 2x = 2\cos^2 x - 1$

$$\Rightarrow \quad f(x) = \frac{1}{2\cos^2 x}$$

$$\because \quad f(-x) = \frac{1}{2} \cdot \frac{1}{\cos^2(-x)} = \frac{1}{2\cos^2 x} = f(x)$$

$\Rightarrow \; f(x)$ is an even function.

$$\because \quad \int_{-\pi/4}^{\pi/4} \frac{dx}{1+\cos 2x} = 2\int_{0}^{\pi/4} \frac{dx}{1+\cos 2x}$$

$$= 2\int_{0}^{\pi/4} \frac{dx}{2\cos^2 x}$$

$$= \int_{0}^{\pi/4} \sec^2 x \, dx = (\tan x)_0^{\pi/4}$$

$$= \tan\frac{\pi}{4} - \tan 0$$

$$= 1 - 0 = 1$$

So, option (a) is correct.

Case Study 5

The Mathematics teacher teaches the following type of integration.

In this type of integral, integrand is the product of two functions. One is in exponential form and second function is the sum of two functions in which one is derivative of other function. Then, to evaluate such integrals, we directly use the following formula

$$\int e^x [f(x) + f'(x)] dx = e^x f(x) + C$$

Based on the above information, solve the following questions:

Q 1. **Evaluate** $\int e^x (\sin x + \cos x) dx$.

Q 2. **Evaluate** $\int [\sin(\log x) + \cos(\log x)] dx$. *(NCERT)*

Q 3. **Evaluate** $\int \frac{x-3}{(x-1)^3} e^x \, dx$.

Solutions

1. Let $\quad I = \int e^x (\sin x + \cos x) dx$

and $\; f(x) = \sin x$, then $f'(x) = \cos x$

So, the given integral is of the form

$$I = \int e^x [f(x) + f'(x)] dx$$

We know that,

$$\int e^x [f(x) + f'(x)] dx = e^x f(x) + C$$

$$\therefore \quad I = e^x \sin x + C$$

2. Let $I = \int [\sin(\log x) + \cos(\log x)] dx$

Put $\quad \log x = t \;\Rightarrow\; x = e^t$

$$\Rightarrow \quad dx = e^t dt$$

$\therefore \; I = \int e^t (\sin t + \cos t) dt$

$[\because f(x) = \sin t$ and $f'(x) = \cos t]$

So, the given integral is of the form

$$I = \int e^x [f(x) + f'(x)] dx$$

$$\therefore \quad I = e^t \sin t + C = x \sin(\log x) + C$$

3. Let $I = \int \frac{x-3}{(x-1)^3} e^x\, dx = \int \frac{x-1-2}{(x-1)^3} e^x\, dx$

$$= \int e^x \left(\frac{x-1}{(x-1)^3} - \frac{2}{(x-1)^3} \right) dx$$

$$= \int e^x \left(\frac{1}{(x-1)^2} - \frac{2}{(x-1)^3} \right) dx \quad ...(1)$$

Now, let $f(x) = \frac{1}{(x-1)^2}$

$\Rightarrow \quad f'(x) = \frac{-2}{(x-1)^3}$

Then, eq. (1) becomes of the form

$$I = \int e^x [f(x) + f'(x)]\, dx$$

Also, we know that,

$$\int e^x [f(x) + f'(x)]\, dx = e^x f(x) + C$$

Hence, $I = \frac{e^x}{(x-1)^2} + C.$

Case Study 6

Mr. Rohan Gupta of Nalanda Public School is teaching the integration by parts to his student in the classroom.

Let $f(x)$ and $g(x)$ be the differentiable function, then

$$\int f(x)\, g(x)\, dx = f(x) \int g(x)\, dx - \int \left[\frac{d}{dx} f(x) \cdot \int g(x)\, dx \right] dx$$

If $f(x) = u$ and $g(x) = v$, then

$$\int uv\, dx = u \int v\, dx - \int \left[\frac{du}{dx} \cdot \int v\, dx \right] dx$$

Based on the above information, solve the following questions: (CBSE 2020)

Q 1. Evaluate $\int \left(1 + x - \frac{1}{x}\right) e^{x+\frac{1}{x}}\, dx$.

Q 2. Evaluate $\int \frac{e^x}{x+1} [1 + (x+1) \log (x+1)]\, dx$.

Solutions

1. Let $I = \int \left(1 + x - \frac{1}{x}\right) e^{x+\frac{1}{x}}\, dx$

$$= \int e^{x+\frac{1}{x}}\, dx + \int \underset{\text{I}}{x} \left[\underset{\text{II}}{\left(1 - \frac{1}{x^2}\right)} e^{x+\frac{1}{x}} \right] dx$$

$$= \int e^{x+\frac{1}{x}}\, dx + x e^{x+\frac{1}{x}} - \int 1 \times e^{x+\frac{1}{x}}\, dx$$

$\left[\because \int u \cdot v\, dx = u \int v\, dx - \int \left\{ \frac{d}{dx}(u) \int v\, dx \right\} dx \right.$

$I_1 = \int \left(1 - \frac{1}{x^2}\right) e^{x+\frac{1}{x}}\, dx$

put $x + \frac{1}{x} = t \Rightarrow \left(1 - \frac{1}{x^2}\right) dx = dt$

$\left. \therefore \int e^t\, dt = e^t = e^{x+\frac{1}{x}} \right]$

$$= x e^{x+\frac{1}{x}} + C$$

2. Let $I = \int \frac{e^x}{x+1} [1 + (x+1) \log (x+1)]\, dx$

$$= \int e^x \left[\frac{1}{x+1} + \log (x+1) \right] dx$$

$$= \int e^x \left(\frac{1}{x+1} \right) dx + \int \underset{\text{II}}{e^x} \underset{\text{I}}{\log (x+1)}\, dx$$

$$= \int e^x \left(\frac{1}{x+1} \right) dx + [\log (x+1) \int e^x\, dx - \int \left[\left(\frac{d}{dx} \{(\log (x+1)\} \int e^x\, dx \right) dx \right]$$

[Using integration by parts]

$$= \int \frac{e^x}{x+1} dx + e^x \log (x+1) - \int \frac{e^x}{x+1} dx$$

$$= e^x \log (x+1) + C$$

Case Study 7

If $f(x)$ is a continuous function defined on $[0, a]$, then $\int_0^a f(x)\, dx = \int_0^a f(a-x)\, dx$.

Based on the above information, solve the following questions:

Q 1. Evaluate $\int_0^a \frac{f(x)}{f(x) + f(a-x)} dx$.

Q 2. If $f(x) = \frac{\sin x - \cos x}{1 + \sin x \cos x}$, then find the value of $\int_0^{\pi/2} f(x)\, dx$.

Q 3. If $g(x) = \log (1 + \tan x)$, then find the value of $\int_0^{\pi/4} g(x)\, dx$.

Solutions

1. Let $I = \int_0^a \frac{f(x)}{f(x) + f(a-x)} dx \quad ...(1)$

$\Rightarrow \quad I = \int_0^a \frac{f(a-x)}{f(a-x) + f(a-(a-x))} dx$

$\Rightarrow \quad I = \int_0^a \frac{f(a-x)}{f(a-x) + f(x)} dx \quad ...(2)$

On adding eqs. (1) and (2), we get

$$2I = \int_0^a 1\, dx = [x]_0^a = a \Rightarrow I = \frac{1}{2} a$$

2. Let $I = \int_0^{\pi/2} f(x)\, dx = \int_0^{\pi/2} \frac{\sin x - \cos x}{1 + \sin x \cos x} dx \quad ...(1)$

$\therefore \quad I = \int_0^{\pi/2} \frac{\sin\left(\frac{\pi}{2} - x\right) - \cos\left(\frac{\pi}{2} - x\right)}{1 + \sin\left(\frac{\pi}{2} - x\right) \cos\left(\frac{\pi}{2} - x\right)} dx$

$\Rightarrow \quad I = \int_0^{\pi/2} \frac{\cos x - \sin x}{1 + \cos x \sin x} dx$

$\Rightarrow \quad I = -\int_0^{\pi/2} \frac{\sin x - \cos x}{1 + \sin x \cos x} dx = -I$ [From eq. (1)]

$\Rightarrow \quad 2I = 0$

$\therefore \quad I = \int_0^{\pi/2} f(x)\, dx = 0$

3. Let $I = \int_0^{\pi/4} g(x)\,dx = \int_0^{\pi/4} \log(1+\tan x)\,dx$ ...(1)

$$I = \int_0^{\pi/4} \log\left[1+\tan\left(\frac{\pi}{4}-x\right)\right] dx$$

$$= \int_0^{\pi/4} \log\left[1+\frac{\tan\frac{\pi}{4}-\tan x}{1+\tan\frac{\pi}{4}\tan x}\right] dx$$

$$= \int_0^{\pi/4} \log\left[1+\frac{1-\tan x}{1+\tan x}\right] dx$$

$$= \int_0^{\pi/4} \log\left[\frac{2}{1+\tan x}\right] dx$$

$$= \int_0^{\pi/4} \log 2 - \log(1+\tan x)\,dx$$

$$= \int_0^{\pi/4} \log 2\,dx - \int_0^{\pi/4} g(x)\,dx$$

$\Rightarrow$ $I = \log 2\left(\frac{\pi}{4}-0\right) - I$

$\Rightarrow$ $2I = \frac{\pi}{4}\log 2$

$\Rightarrow$ $I = \int_0^{\pi/4} g(x)\,dx = \frac{\pi}{8}\log 2$

Case Study 8

If $f(x)$ is a continuous function defined on $[a, b]$, then $\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$.

Based on the above information, solve the following questions:

Q 1. Evaluate $\int_a^b \frac{f(x)}{f(x)+f(a+b-x)}\,dx$.

Q 2. Evaluate $\int_{\pi/6}^{\pi/3} \log \tan x\,dx$.

Q 3. If $g(x) = \frac{x^{1/n}}{x^{1/n}+(a+b-x)^{1/n}}$, then find the value of $\int_a^b g(x)\,dx$.

Solutions

1. Let $I = \int_a^b \frac{f(x)}{f(x)+f(a+b-x)}\,dx$ ...(1)

$\Rightarrow$ $I = \int_a^b \frac{f(a+b-x)}{f(a+b-x)+f[a+b-(a+b-x)]}\,dx$

$\Rightarrow$ $I = \int_a^b \frac{f(a+b-x)}{f(a+b-x)+f(x)}\,dx$ ...(2)

On adding eqs. (1) and (2), we get

$$2I = \int_a^b 1\,dx = [x]_a^b = b-a$$

$\Rightarrow$ $I = \frac{1}{2}(b-a)$

2. Let $I = \int_{\pi/6}^{\pi/3} \log \tan x\,dx$

$\Rightarrow$ $I = \int_{\pi/6}^{\pi/3} \log \tan\left(\frac{\pi}{3}+\frac{\pi}{6}-x\right) dx$

$\Rightarrow$ $I = \int_{\pi/6}^{\pi/3} \log \cot x\,dx = -\int_{\pi/6}^{\pi/3} \log \tan x\,dx$

$\Rightarrow$ $I = -I$

$\Rightarrow$ $2I = 0 \Rightarrow I = 0$

3. Let $I = \int_a^b g(x)\,dx = \int_a^b \frac{x^{1/n}}{x^{1/n}+(a+b-x)^{1/n}}\,dx$ ...(1)

$$I = \int_a^b \frac{(a+b-x)^{1/n}}{(a+b-x)^{1/n}+[a+b-(a+b-x)]^{1/n}}\,dx$$

$$= \int_a^b \frac{(a+b-x)^{1/n}}{(a+b-x)^{1/n}+x^{1/n}}\,dx \quad ...(2)$$

On adding eqs. (1) and (2), we get

$$2I = \int_a^b 1\,dx = [x]_a^b = b-a$$

$\Rightarrow$ $I = \frac{1}{2}(b-a)$

Very Short Answer Type Questions

Q 1. Evaluate $\int \frac{\cos(\log x)}{x}\,dx$.

Q 2. Evaluate $\int \frac{\cos x}{\sin^2 x}\,dx$.

Q 3. Evaluate $\int x^2 \sin x^3\,dx$.

Q 4. Evaluate $\int \frac{(1+\log x)^2}{x}\,dx$. *(NCERT EXERCISE)*

Q 5. Evaluate $\int \frac{\sin\sqrt{x}}{\sqrt{x}}\,dx$. *(NCERT EXERCISE)*

Q 6. Evaluate $\int \frac{\tan(\sin^{-1} x)}{\sqrt{1-x^2}}\,dx$. *(NCERT EXERCISE)*

Q 7. Evaluate $\int \frac{e^x(1+x)}{\cos^2(e^x \cdot x)}\,dx$. *(NCERT EXERCISE)*

Q 8. Evaluate $\int \frac{x^2 \tan^{-1} x^3}{1+x^6}\,dx$. *(NCERT EXERCISE)*

Q 9. Evaluate $\int \frac{dx}{(e^x-1)(1-e^{-x})}$. *(NCERT EXERCISE)*

Q 10. Find $\int \frac{dx}{x^2-6x+13}$. *(CBSE 2022 Term-2)*

Q 11. Evaluate $\int \frac{dx}{1+\cos x}$. *(NCERT EXEMPLAR)*

Q 12. Find $\int \frac{\sin 2x}{\sqrt{9-\cos^4 x}}\,dx$ *(CBSE SQP 2022 Term-2)*

Q 13. Evaluate $\int_2^3 3^x\,dx$. *(CBSE 2017)*

Q 14. Find the value of $\int_1^4 |x-5|\,dx$. *(CBSE 2020)*

Q 15. Evaluate $\int_0^{2\pi} \cos^5 x\,dx$. *(CBSE 2017)*

Short Answer Type-I Questions

Q 1. Evaluate $\frac{1}{1-\cot x}dx$ or $\int \frac{\sin x}{\sin x - \cos x}dx$. *(NCERT EXERCISE)*

Q 2. Evaluate $\int \frac{x \sin^{-1}(x^2)}{\sqrt{1-x^4}}dx$. *(CBSE 2020)*

Q 3. Find $\int \sqrt{1-\sin 2x}\,dx, \frac{\pi}{4} < x < \frac{\pi}{2}$. *(NCERT EXEMPLAR; CBSE 2019)*

Q 4. Evaluate $\int \frac{\cos 2x + 2\sin^2 x}{\cos^2 x}dx$. *(CBSE 2018)*

Q 5. Evaluate $\int \sin^3 x \cos^2 x\, dx$. *(NCERT EXERCISE)*

Q 6. Find $\int \frac{dx}{x^2+4x+8}$. *(CBSE 2017)*

Q 7. Find $\int \frac{dx}{5-8x-x^2}$. *(CBSE 2017)*

Q 8. Find $\int \frac{\sec^2 x}{\sqrt{\tan^2 x + 4}}dx$. *(CBSE 2019)*

Q 9. Find $\int \frac{\tan^2 x \sec^2 x}{1-\tan^6 x}dx$. *(CBSE 2019)*

Q 10. Evaluate $\int \frac{x^3}{x^2+1}dx$. *(NCERT EXEMPLAR)*

Q 11. Find $\int \frac{x+1}{x(1-2x)}dx$. *(CBSE 2020)*

Q 12. Find $\int \frac{x+1}{(x+2)(x+3)}dx$. *(CBSE 2020)*

Q 13. Find $\int \frac{x}{x^2+3x+2}dx$. *(CBSE 2020)*

Q 14. Evaluate $\int \frac{\cos x}{(1+\sin x)(2+\sin x)}dx$. *(NCERT EXERCISE; CBSE 2019)*

Q 15. Evaluate $\int \log x\, dx$. *(NCERT EXERCISE)*

Q 16. Find $\int \frac{\log x}{(1+\log x)^2}dx$. *(CBSE SQP 2022 Term-2)*

Q 17. Find $\int \frac{\log x - 3}{(\log x)^4}dx$. *(CBSE 2022 Term-2)*

Q 18. Evaluate $\int e^x\left(\tan^{-1} x + \frac{1}{1+x^2}\right)dx$. *(NCERT EXERCISE)*

Q 19. Find $\int \sin x \cdot \log \cos x\, dx$. *(CBSE 2019)*

Q 20. Evaluate $\int_{-1}^{1} \log_e\left(\frac{2-x}{2+x}\right)dx$. *(CBSE SQP 2022-23)*

Q 21. Find $\int \sin^{-1}(2x)\,dx$. *(CBSE 2019)*

Q 22. Evaluate $\int_{-2}^{1} \sqrt{5-4x-x^2}\,dx$. *(CBSE 2022 Term-2)*

Q 23. Evaluate $\int_{-1}^{2} \frac{|x|}{x}dx$. *(CBSE 2019)*

Q 24. Find the value of $\int_0^a \frac{\sqrt{x}}{\sqrt{x}+\sqrt{a-x}}dx$.

Q 25. Evaluate $\int_{-\pi}^{\pi} (1-x^2)\sin x \cos^2 x\, dx$. *(CBSE 2019)*

Q 26. Evaluate, using properties $\int_{-\pi}^{\pi} (3\sin x - 2)^2 dx$. *(CBSE 2022 Term-2)*

Short Answer Type-II Questions

Q 1. If $\frac{d}{dx}\{F(x)\} = \frac{\sec^4 x}{\operatorname{cosec}^4 x}$ and $F\left(\frac{\pi}{4}\right) = \frac{\pi}{4}$, then find $F(x)$. *(CBSE 2022 Term-2)*

Q 2. Find the value of the integral $\int \frac{\sin 2x}{a\cos^2 x + b\sin^2 x}dx$, where a, b are constants.

Q 3. Evaluate $\int \frac{dx}{x(x^{3/2}+1)}$.

Q 4. Find $\int \frac{dx}{\sqrt{3-2x-x^2}}$. *(CBSE SQP 2022-23, CBSE 2017)*

Q 5. Evaluate $\int \frac{dx}{x^{1/2}+x^{1/3}}$. *(NCERT EXERCISE)*

Q 6. Find $\int \frac{e^x}{\sqrt{5-4e^x-e^{2x}}}dx$. *(CBSE 2023)*

Q 7. Evaluate $\int \frac{\sec^2 2x}{(\cot x - \tan x)^2}dx$.

Q 8. Evaluate $\int \frac{\sin^8 x - \cos^8 x}{1-2\sin^2 x \cos^2 x}dx$. *(NCERT EXERCISE)*

Q 9. Evaluate $\int \frac{1}{(\log x - 1)(\log x + 1)}\frac{dx}{x}$.

Q 10. Evaluate $\int \frac{dx}{1+3\cos^2 x}$.

Q 11. Evaluate $\int \sqrt{\frac{x}{1-x^3}}dx; x \in (0, 1)$. *(CBSE SQP 2023-24)*

Q 12. Evaluate $\int \frac{x^2-1}{x\sqrt{x^4+x^2+1}}dx$.

Q 13. Find the value of $\int \frac{2x+1}{\sqrt{2x^2+x-3}}dx$.

Q 14. Find $\int \frac{x^3+x}{x^4-9}dx$. *(CBSE 2022 Term-2)*

Q 15. Find $\int \frac{x^2}{(x^2+1)(3x^2+4)}dx$. *(CBSE 2022 Term-2)*

Q 16. Find $\int \frac{x^4}{(x-1)(x^2+1)}dx$. *(CBSE 2023)*

Q 17. Find $\int \frac{x^2+x+1}{(x+1)^2(x+2)}dx$. *(CBSE 2023)*

Q 18. Find $\int \frac{2\cos x}{(1-\sin x)(1+\sin^2 x)}dx.$ *(CBSE 2018)*

Or $\int \frac{2}{(1-x)(1+x^2)}dx.$ *(CBSE 2023)*

Q 19. Find $\int \frac{e^x}{(e^x-1)^2(e^x+2)}dx.$ *(CBSE 2017)*

Q 20. Find $\int \frac{e^x}{(2+e^x)(4+e^{2x})}dx.$ *(CBSE 2017)*

Q 21. Find $\int \frac{2x}{(x^2+1)(x^2+2)^2}dx.$ *(CBSE 2017)*

Q 22. Evaluate $\int \frac{2x^2+3}{x^2(x^2+9)}dx; x \neq 0.$ *(CBSE SQP 2023-24)*

Q 23. Find $\int \frac{\sin\theta}{(4+\cos^2\theta)(2-\sin^2\theta)}d\theta.$ *(CBSE 2017)*

Q 24. Find $\int \frac{\cos\theta}{(4+\sin^2\theta)(5-4\cos^2\theta)}d\theta.$ *(CBSE 2017)*

Q 25. Find $\int \frac{(3\sin x-2)\cos x}{13-\cos^2 x-7\sin x}dx.$ *(CBSE 2017)*

Q 26. Find $\int \frac{3x+5}{x^2+3x-18}dx.$ *(CBSE 2019)*

Q 27. Evaluate $\int \cos^{-1} x\, dx.$

Q 28. Evaluate $\int e^{\tan^{-1}x}\left(\frac{1+x+x^2}{1+x^2}\right)dx.$ *(NCERT EXEMPLAR)*

Q 29. Evaluate $\int_{-2}^{2} \frac{x^2}{1+5^x}dx.$ *(CBSE 2023)*

Q 30. Evaluate $\int_{\pi/4}^{\pi/2} e^{2x}\left(\frac{1-\sin 2x}{1-\cos 2x}\right)dx.$ *(CBSE 2023)*

Q 31. Prove that $\int_0^{\pi/2} \log(\sin^3 x \cdot \cos^4 x)\, dx = -\frac{7\pi}{2}\log 2.$

Q 32. Evaluate $\int_{-\pi/4}^{\pi/4} \frac{\cos 2x}{1+\cos 2x}dx.$ *(CBSE 2023)*

Q 33. Evaluate $\int_0^4 |x-1|dx.$ *(CBSE SQP 2022-23)*

Q 34. Evaluate $\int_0^{3/2} |x\sin \pi x|\, dx.$ *(CBSE 2017)*

Q 35. Evaluate $\int_{\pi/6}^{\pi/3} \frac{dx}{1+\sqrt{\tan x}}.$ *(CBSE SQP 2022-23)*

Q 36. Prove that $\int_0^{\pi/2} \frac{dx}{1+\sqrt{\cot x}} = \int_0^{\pi/2} \frac{\sqrt{\tan x}}{\sqrt{\tan x}+1}dx$

$$= \int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}dx = \frac{\pi}{4}.$$

Or Prove that $\int_0^{\pi/2} \frac{dx}{1+\sqrt{\tan x}} = \int_0^{\pi/2} \frac{\sqrt{\cot x}}{\sqrt{\cot x}+1}dx$

$$= \int_0^{\pi/2} \frac{\sqrt{\cos x}}{\sqrt{\cos x}+\sqrt{\sin x}} = \frac{\pi}{4}$$ *(NCERT EXERCISE)*

Q 37. Evaluate $\int_0^{2\pi} \frac{dx}{1+e^{\sin x}}.$ *(CBSE 2023, 22 Term-2)*

Q 38. Evaluate $\int_0^{\pi/2} \sqrt{\sin x}\cos^5 x\, dx.$ *(CBSE 2023)*

Q 39. Evaluate $\int_{1/3}^{1} \frac{(x-x^3)^{1/3}}{x^4}dx.$ *(CBSE 2023)*

Q 40. Evaluate $\int_1^3 \{|(x-1)|+|(x-2)|\}dx.$ *(CBSE 2023)*

Long Answer Type Questions

Q 1. Evaluate $\int \frac{dx}{1+x+x^2+x^3}.$

Q 2. Find $\int \frac{(x^3+x+1)}{(x^2-1)}dx.$ *(CBSE SQP 2022-23)*

Q 3. Evaluate $\int \frac{dx}{4+5\sin x}.$

Q 4. Evaluate $\int \left[\log(\log x)+\frac{1}{(\log x)^2}\right]dx.$ *(NCERT EXERCISE)*

Q 5. Evaluate $\int_0^1 x\sqrt{\frac{1-x^2}{1+x^2}}\,dx.$

Q 6. Prove that $\int_0^{\pi/2} \frac{dx}{1+2\cos x} = \frac{1}{\sqrt{3}}\log(2+\sqrt{3}).$

Q 7. Evaluate $\int_0^{\pi/4} \frac{\sin x+\cos x}{16+9\sin 2x}dx.$ *(CBSE 2018)*

Q 8. Evaluate $\int_0^{\pi/4} (\sqrt{\tan x}+\sqrt{\cot x})\,dx.$

Or Evaluate $\int (\sqrt{\tan x}+\sqrt{\cot x})\,dx.$ *(NCERT EXERCISE)*

Q 9. Evaluate $\int_{\pi/4}^{\pi/2} \cos 2x \cdot \log \sin x\, dx.$

Q 10. Evaluate $\int_{-1}^{2} |x^3-x|dx.$ *(CBSE 2022 Term-2)*

Q 11. Evaluate $\int_{-1}^{2} |x^3-3x^2+2x|\, dx.$ *(CBSE SQP 2022 Term-2)*

Q 12. Evaluate $\int_{-\pi/2}^{\pi/2} (\sin|x|+\cos|x|)\, dx.$ *(CBSE 2022 Term-2)*

Q 13. Prove that $\int_a^b f(x)\, dx = \int_a^b f(a+b-x)\, dx$ and hence find the value of $\int_{\pi/8}^{3\pi/8} \frac{\tan^2 x}{\tan^2 x+\cot^2 x}dx.$

Q 14. Evaluate $\int_0^{\pi} \frac{x}{1+\sin x}dx.$ *(CBSE 2023, 22 Term-2)*

Q 15. Evaluate $\int_0^{\pi/2} \frac{x}{\sin x+\cos x}dx.$

Q 16. Show that

$$\int_0^{\pi/2} \frac{\sin^2 x}{\sin x+\cos x}dx = \frac{1}{\sqrt{2}}\log(\sqrt{2}+1).$$ *(NCERT EXEMPLAR)*

Q 17. Prove that $\int_0^1 \frac{\log(1+x)}{1+x^2}dx = \frac{\pi}{8}\log 2.$

Or Evaluate $\int_0^{\pi/4} \log(1+\tan x)\,dx.$

(NCERT EXERCISE, CBSE SQP 2023-24)

Or Prove that $\int_0^{\pi/4} \log\left[\frac{\sin x+\cos x}{\cos x}\right]dx = \frac{\pi}{8}\log 2.$

Q 18. Prove that

$$\int_0^{\pi/2} \log\sin x\,dx = \int_0^{\pi/2} \log\cos x\,dx = -\frac{\pi}{2}\log 2.$$

(NCERT EXERCISE)

Q 19. Evaluate $\int_0^{\pi/2} \sin 2x \tan^{-1}(\sin x)\,dx.$ *(CBSE 2023)*

Q 20. Prove that $\int_0^{\infty} \log\left[x+\frac{1}{x}\right]\cdot\frac{dx}{1+x^2} = \pi\log 2.$

(NCERT EXEMPLAR)

Q 21. Prove that $\int_0^{\pi} \frac{x}{a^2\cos^2 x + b^2\sin^2 x}dx = \frac{\pi^2}{2ab}.$

(NCERT EXERCISE)

Q 22. Prove that $\int_0^{\pi} \frac{x}{1+\cos^2 x}dx = \frac{\pi^2}{2\sqrt{2}}.$

Solutions

Very Short Answer Type Questions

1. Put $t = \log x \Rightarrow dt = \frac{1}{x}dx$

$\therefore \int \frac{\cos(\log x)}{x}dx = \int \cos t\,dt = \sin t + C$

$= \sin(\log x) + C$

2. Put $t = \sin x \Rightarrow dt = \cos x\,dx$

$\therefore \int \frac{\cos x}{\sin^2 x}dx = \int \frac{dt}{t^2} = \left[-\frac{1}{t}\right] + C$

$= -\frac{1}{\sin x} + C = -\text{cosec}\,x + C$

3. Put $x^3 = t$, then $3x^2\,dx = dt$

$\therefore \int x^2 \sin x^3\,dx = \frac{1}{3}\int \sin x^3 (3x^2dx)$

$= \frac{1}{3}\int \sin t\,dt = \frac{1}{3}(-\cos t) + C$

$= -\frac{1}{3}\cos x^3 + C$

4. Put $t = 1 + \log x$, then $dt = \frac{1}{x}dx$

$\therefore \int \frac{(1+\log x)^2}{x}dx = \int t^2 dt$

$= \frac{t^3}{3} + C = \frac{1}{3}(1+\log x)^3 + C$

5. Put $t = \sqrt{x} = x^{1/2}$, then $dt = \frac{1}{2}x^{-1/2}dx \Rightarrow 2dt = \frac{1}{\sqrt{x}}dx$

$\therefore \int \frac{\sin\sqrt{x}}{\sqrt{x}}dx = \int \sin t\cdot 2dt$

$= -2\cos t + C = -2\cos\sqrt{x} + C$

6. Put $t = \sin^{-1} x$, then $dt = \frac{1}{\sqrt{1-x^2}}dx$

$\therefore \int \frac{\tan(\sin^{-1} x)}{\sqrt{1-x^2}}dx = \int \tan t\,dt$

$= \log|\sec t| + C$

$= \log|\sec(\sin^{-1} x)| + C$

7. Put $e^x \cdot x = t \Rightarrow e^x + xe^x = \frac{dt}{dx} \Rightarrow e^x(1+x)\,dx = dt$

$\therefore \int \frac{e^x(1+x)}{\cos^2(e^x\cdot x)}dx = \int \frac{1}{\cos^2 t}dt$

$= \int \sec^2 t\,dt = \tan t + C$

$= \tan(e^x\cdot x) + C$

8. Put $t = \tan^{-1} x^3$, then $dt = \frac{3x^2dx}{1+x^6} \Rightarrow \frac{1}{3}dt = \frac{x^2}{1+x^6}dt$

$\therefore \int \frac{x^2\tan^{-1} x^3}{1+x^6}dx = \int t\times\frac{1}{3}dt = \frac{1}{3}\times\frac{t^2}{2} + C$

$= \frac{1}{6}(\tan^{-1} x^3)^2 + C$

9. $\int \frac{dx}{(e^x-1)(1-e^{-x})} = \int \frac{e^x}{(e^x-1)(e^x-1)}dx$

$= \int \frac{e^x}{(e^x-1)^2}dx = \int \frac{1}{t^2}dt$ $\left[\begin{array}{l}\text{Put } t = e^x - 1\\ \Rightarrow dt = e^x dx\end{array}\right]$

$= \int t^{-2}dt = \frac{t^{-1}}{-1} + C = -\frac{1}{t} + C = -\frac{1}{e^x-1} + C$

10. $\int \frac{dx}{x^2-6x+13} = \int \frac{dx}{(x^2-6x+9)+4} = \int \frac{dx}{(x-3)^2+(2)^2}$

$= \int \frac{dt}{t^2+(2)^2}$ Put $t = x-3 \Rightarrow dt = dx$

$= \frac{1}{2}\tan^{-1}\frac{t}{2} + C = \frac{1}{2}\tan^{-1}\frac{(x-3)}{2} + C$

11. $\int \frac{dx}{1+\cos x} = \int \frac{dx}{1+2\cos^2\frac{x}{2}-1} = \int \frac{dx}{2\cos^2\frac{x}{2}}$

TR!CK

$\cos\theta = 2\cos^2\frac{\theta}{2} - 1$

$= \frac{1}{2}\int \sec^2\frac{x}{2}dx = \frac{1}{2}\times 2\int \sec^2 t\,dt$ (Put $t = \frac{x}{2} \Rightarrow dx = 2dt$)

$= \tan t + C = \tan\frac{x}{2} + C$

12. $\int \frac{\sin 2x}{\sqrt{9-\cos^4 x}}dx = \int \frac{\sin 2x}{\sqrt{(3)^2-(\cos^2 x)^2}}dx$

$$= \int \frac{-dt}{\sqrt{(3)^2-t^2}} \quad \left[\begin{array}{l} \text{Put} \quad t=\cos^2 x \\ \Rightarrow \quad dt = -2\cos x \cdot \sin x\, dx \\ \Rightarrow \quad -dt = \sin 2x\, dx \end{array}\right]$$

$$= -\sin^{-1}\left(\frac{t}{3}\right)+C$$

$$= -\sin^{-1}\left(\frac{1}{3}\cos^2 x\right)+C$$

13. $\int_2^3 3^x dx = \left[\frac{3^x}{\log_e 3}\right]_2^3$

TR!CK

$\int a^x dx = \frac{a^x}{\log_e a}+C$

$$= \frac{1}{\log_e 3}[3^3-3^2] = \frac{1}{\log_e 3}\times(27-9) = \frac{18}{\log_e 3}$$

14. Let $I = \int_1^4 |x-5|\,dx$

The given integrand can be redefine as below:

$$|x-5| = -(x-5); 1<x<4 = 5-x$$

$$I = \int_1^4 (5-x)\,dx = \left[5x-\frac{x^2}{2}\right]_1^4$$

$$= 5(4-1)-\frac{1}{2}\{(4)^2-(1)^2\}$$

$$= 5\times 3 - \frac{1}{2}(16-1) = 15-\frac{15}{2} = \frac{15}{2}$$

15. We have, integrand $f(x) = \cos^5 x$

Now, $f(2\pi - x) = \cos^5(2\pi - x) = \cos^5 x = f(x)$

$\therefore \int_0^{2\pi}\cos^5 x\,dx = 2\int_0^{\pi}\cos^5 x\,dx$

Again, $f(\pi - x) = \cos^5(\pi - x) = [-\cos x]^5$

$= -\cos^5 x = -f(x)$

$\therefore 2\int_0^{\pi}\cos^5 x\,dx = 2\times 0 = 0$

TR!CK

By definite integral property,

$$\int_0^{2a} f(x)\,dx = \begin{cases} 2\int_0^a f(x)\,dx, & \text{if } f(2a-x) = f(x) \\ 0, & \text{if } f(2a-x) = -f(x) \end{cases}$$

Short Answer Type-I Questions

1. $\int \frac{1}{1-\cot x}dx = \int \frac{1}{1-\frac{\cos x}{\sin x}}dx = \int \frac{\sin x}{\sin x - \cos x}dx$

$$= \frac{1}{2}\int \frac{(\sin x - \cos x)+(\sin x + \cos x)}{\sin x - \cos x}dx$$

$$= \frac{1}{2}\int\left[1+\frac{\sin x + \cos x}{\sin x - \cos x}\right]dx$$

TR!CK

$\int \frac{f'(x)}{f(x)}dx = \log|f(x)|+C$

$$= \frac{1}{2}\int\left[1+\frac{\frac{d}{dx}(\sin x - \cos x)}{\sin x - \cos x}\right]dx$$

$$= \frac{1}{2}[x+\log|\sin x - \cos x|]+C$$

COMMON ERR!R

Some students split the numerator and apply integration by parts.

2. Let $I = \int \frac{x\sin^{-1}(x^2)}{\sqrt{1-x^4}}dx = \int \frac{x\sin^{-1}(x^2)}{\sqrt{1-(x^2)^2}}dx$

Put $t = x^2 \Rightarrow dt = 2x\,dx$

$$\therefore \quad I = \int \frac{\sin^{-1}t}{\sqrt{1-t^2}}\cdot\frac{dt}{2} = \frac{1}{2}\int\frac{\sin^{-1}t}{\sqrt{1-t^2}}dt$$

Again put, $z = \sin^{-1}t \Rightarrow dz = \frac{dt}{\sqrt{1-t^2}}$

$$\therefore I = \frac{1}{2}\int z\,dz = \frac{z^2}{4}+C = \frac{1}{4}(\sin^{-1}t)^2+C \qquad [\because z = \sin^{-1}t]$$

$$= \frac{1}{4}\{\sin^{-1}(x^2)\}^2+C \qquad [\because t = x^2]$$

3. $\int\sqrt{1-\sin 2x}\,dx, \frac{\pi}{4}<x<\frac{\pi}{2}$

$$= \int\sqrt{\sin^2 x + \cos^2 x - 2\sin x\cdot\cos x\,dx}$$

$$\left[\begin{array}{l}\because \sin^2\theta+\cos^2\theta = 1 \\ \text{and } \sin 2\theta = 2\sin\theta\cdot\cos\theta\end{array}\right]$$

TR!CK

$\because$ *$\sin x > \cos x$ as $x \in \left(\frac{\pi}{4},\frac{\pi}{2}\right)$*

$\therefore (\sin x - \cos x)^2 = \sin^2 x + \cos^2 x - 2\sin x\cdot\cos x$

$$= \int\sqrt{(\sin x - \cos x)^2}\,dx = \int(\sin x - \cos x)\,dx$$

$$= \int \sin x\,dx - \int\cos x\,dx = -\cos x - \sin x + C$$

4. $\int\frac{\cos 2x + 2\sin^2 x}{\cos^2 x}dx = \int\frac{1-2\sin^2 x + 2\sin^2 x}{\cos^2 x}dx$

$$[\because \cos 2x = 1-2\sin^2 x]$$

$$= \int\frac{1}{\cos^2 x}dx = \int\sec^2 x\,dx = \tan x + C$$

5. $\int\sin^3 x\cos^2 x\,dx = \int\sin^2 x\cos^2 x\cdot\sin x\,dx$

$$= \int(1-\cos^2 x)\cos^2 x\cdot\sin x\,dx$$

TR!CK

$\sin^2\theta+\cos^2\theta = 1\ \forall\ \theta\in R$

$$= \int(1-t^2)t^2(-dt) = \int(t^4-t^2)\,dt$$

$$[\text{Put} \cos x = t \Rightarrow \sin x\,dx = -dt]$$

$$= \frac{t^5}{5}-\frac{t^3}{3}+C$$

$$= \frac{1}{5}\cos^5 x - \frac{1}{3}\cos^3 x + C$$

6. $\int \frac{dx}{x^2+4x+8} = \int \frac{dx}{(x+2)^2+4} = \int \frac{dx}{(x+2)^2+(2)^2}$

$= \int \frac{dt}{(2)^2+t^2}$ [Put $t = x+2 \Rightarrow dt = dx$]

TR!CK

$\int \frac{dx}{a^2+x^2} = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right) + C$

$= \frac{1}{2}\tan^{-1}\left(\frac{t}{2}\right) + C = \frac{1}{2}\tan^{-1}\left(\frac{x+2}{2}\right) + C$

$= \frac{1}{2}\tan^{-1}\left(1+\frac{x}{2}\right) + C$

7. Let $I = \int \frac{dx}{5-8x-x^2} = -\int \frac{dx}{x^2+8x-5}$

$= -\int \frac{dx}{x^2+8x+16-16-5}$

TiP

When denominator is given in quadratic form, make it perfect square then apply the property of ingegral.

$= -\int \frac{dx}{(x+4)^2-21} = \int \frac{dx}{(\sqrt{21})^2-(x+4)^2}$

Put $t = x+4 \Rightarrow dt = dx$

TR!CK

$\int \frac{dx}{a^2-x^2} = \frac{1}{2a} \log\left|\frac{a+x}{a-x}\right| + C$

$\therefore$ $I = \int \frac{dt}{(\sqrt{21})^2-t^2} = \frac{1}{2\sqrt{21}}\log\left|\frac{\sqrt{21}+t}{\sqrt{21}-t}\right| + C$

$= \frac{1}{2\sqrt{21}}\log\left|\frac{\sqrt{21}+x+4}{\sqrt{21}-x-4}\right| + C$ [$\because t = x+4$]

COMMON ERROR

Some students make errors in simplifying the expression before integration.

8.

TiP

Learn all the special integrals thoroughly.

Put $t = \tan x \Rightarrow dt = \sec^2 x\, dx$

$\therefore$ $\int \frac{\sec^2 x}{\sqrt{\tan^2 x+4}}dx = \int \frac{dt}{\sqrt{4+t^2}}$

$= \int \frac{dt}{\sqrt{2^2+t^2}}$

TR!CK

$\int \frac{dx}{\sqrt{a^2+x^2}} = \log\left|x+\sqrt{a^2+x^2}\right| + C$

$= \log|t+\sqrt{2^2+t^2}| + C$

$= \log|\tan x+\sqrt{4+\tan^2 x}| + C$

COMMON ERROR

Sometimes students attempt some other methods of integration and could not reach the proper result.

9.

TiP

Practice more problems to identify the right substitution.

Let $I = \int \frac{\tan^2 x \sec^2 x}{1-\tan^6 x}dx = \int \frac{\tan^2 x\sec^2 x}{1-(\tan^3 x)^2}dx$

Put $t = \tan^3 x \Rightarrow dt = 3\tan^2 x \cdot \sec^2 x\, dx$

$= \int \frac{1}{1-t^2}\cdot\frac{dt}{3} = \frac{1}{3}\int\frac{dt}{(1)^2-t^2}$

TR!CK

$\int \frac{dx}{a^2-x^2} = \frac{1}{2a}\log\left|\frac{a+x}{a-x}\right| + C$

$\therefore$ $I = \frac{1}{3}\cdot\frac{1}{2\times 1}\log\left|\frac{1+t}{1-t}\right| + C$ [$\because t = \tan^3 x$]

$= \frac{1}{6}\log\left|\frac{1+\tan^3 x}{1-\tan^3 x}\right| + C$

COMMON ERROR

Some students put $\tan x = t$ and make it more complicated.

10. By division method,

$$\begin{array}{r|l} x^2+1 \,\big)\, x^3 \,\big(\, x \\ \pm x^3 \pm x \\ \hline -x \end{array}$$

$\therefore$ $\frac{x^3}{x^2+1} = x - \frac{x}{x^2+1}$ ...(1)

Therefore, $\int \frac{x^3}{x^2+1}dx = \int\left\{x-\frac{x}{x^2+1}\right\}dx$

$= \int x\,dx - \int\frac{x}{x^2+1}dx$

TR!CK

$\int \frac{f'(x)}{f(x)}dx = \log|f(x)| + C$

$= \int x\,dx - \frac{1}{2}\int\frac{2x}{x^2+1}dx$

$= \frac{x^2}{2} - \frac{1}{2}\log|x^2+1| + C$

11. By partial fraction,

$\frac{x+1}{x(1-2x)} = \frac{A}{x} + \frac{B}{1-2x}$

$\Rightarrow$ $(x+1) = A(1-2x) + Bx$

$\Rightarrow$ $x+1 = (-2A+B)x + A$

On comparing the like powers of x, we get

$-2A+B = 1$ and $A = 1$

$\therefore$ $-2\times 1 + B = 1$ $\Rightarrow$ $B = 3$

So, $\frac{x+1}{x(1-2x)} = \frac{1}{x} + \frac{3}{1-2x}$

$\therefore$ $\int\frac{x+1}{x(1-2x)}dx = \int\frac{1}{x}dx + 3\int\frac{1}{1-2x}dx$

$= \log|x| - \frac{3}{2}\log|1-2x| + C$

12. By partial fraction,

$$\frac{(x+1)}{(x+2)(x+3)} = \frac{A}{(x+2)} + \frac{B}{(x+3)}$$

$$\Rightarrow \quad x+1 = A(x+3) + B(x+2)$$

$$\Rightarrow \quad x+1 = (A+B)x + (3A+2B)$$

On comparing the like powers of x, we get

$$A+B=1 \text{ and } 3A+2B=1$$

$$\therefore \quad 3A+2(1-A)=1 \Rightarrow 3A+2-2A=1$$

$$\Rightarrow \quad A=-1 \text{ and } B=1-(-1)=1+1=2$$

So; $\frac{x+1}{(x+2)(x+3)} = \frac{-1}{(x+2)} + \frac{2}{(x+3)}$

$$\therefore \int \frac{x+1}{(x+2)(x+3)}dx = -\int \frac{dx}{x+2} + 2\int \frac{dx}{x+3}$$

$$= -\log|x+2| + 2\log|x+3| + C$$

$$= \log(x+3)^2 - \log|x+2| + C$$

$$= \log\frac{(x+3)^2}{|x+2|} + C$$

13. $\int \frac{x}{x^2+3x+2}dx = \int \frac{x}{(x+1)(x+2)}dx$ ...(1)

By partial fraction,

$$\frac{x}{(x+1)(x+2)} = \frac{A}{x+1} + \frac{B}{x+2}$$

$$\Rightarrow \quad x = A(x+2) + B(x+1)$$

$$\Rightarrow \quad x = (A+B)x + (2A+B)$$

On comparing the like powers of x, we get

$$A+B=1 \text{ and } 2A+B=0$$

On solving, we get

$$2A+1-A=0$$

$$\Rightarrow \quad A+1=0 \Rightarrow A=-1$$

and $\quad -1+B=1 \Rightarrow B=2$

$$\therefore \quad \frac{x}{(x+1)(x+2)} = \frac{-1}{(x+1)} + \frac{2}{(x+2)}$$

From eq. (1),

$$\int \frac{x}{(x+1)(x+2)}dx = -\int \frac{dx}{x+1} + 2\int \frac{dx}{x+2}$$

$$= -\log|x+1| + 2\log|x+2| + C$$

$$= \log(x+2)^2 - \log|x+1| + C$$

$$= \log\frac{(x+2)^2}{|x+1|} + C$$

14. Put $\sin x = t$ then, $\cos x\, dx = dt$

TIP

If the denominator has the derivative in numerator, then put it as 't' and reduce into partial fraction.

$$\therefore \int \frac{\cos x}{(1+\sin x)(2+\sin x)}dx = \int \frac{1}{(1+t)(2+t)}dt$$

$$= \int \left[\frac{1}{(1+t)} - \frac{1}{(2+t)}\right]dt$$

[Converting into partial fraction]

$$= \log|1+t| - \log|2+t|$$

$$= \log\left|\frac{1+t}{2+t}\right| = \log\left|\frac{1+\sin x}{2+\sin x}\right| + C$$

COMMON ERROR

Many students apply incorrect substitution and find it difficult to reduce in partial fraction.

15. $\int \log x\, dx = \int 1 \cdot \log x\, dx$

TRICK

Integral by parts,

$$\int \underset{\text{I}}{u} \cdot \underset{\text{II}}{v}dx = u\int v dx - \int\left\{\frac{d}{dx}u\int v dx\right\}dx,$$

where u and v are the functions of x.

$$= \log x \cdot \int 1 dx - \int\left(\frac{d}{dx}\log x\right)\left(\int 1 dx\right)dx$$

$$= (\log x)\cdot x - \int \frac{1}{x}\cdot x\, dx$$

$$= x\log x - x + C = x(\log x - 1) + C$$

16. Let $\quad I = \int \frac{\log x}{(1+\log x)^2}dx$

Put $\quad t = 1+\log x \Rightarrow \log x = t-1$

$$\Rightarrow \quad dt = \frac{1}{x}dx$$

$$\therefore \quad I = \int \frac{t-1}{t^2}\cdot e^{t-1}dt$$

$$= \frac{1}{e}\int\left\{\frac{e^t}{t} - \frac{e^t}{t^2}\right\}dt$$

$$= \frac{1}{e}\left[\frac{1}{t}\int e^t dt - \int\left\{\frac{d}{dt}\frac{1}{t}\int e^t dt\right\}dt\right] - \frac{1}{e}\int \frac{e^t}{t^2}dt$$

$$= \frac{1}{e}\left[\frac{e^t}{t} + \int \frac{e^t}{t^2}dt\right] - \frac{1}{e}\int \frac{e^t}{t^2}dt$$

$$= \frac{1}{e}\cdot\frac{e^t}{t} + \frac{1}{e}\int \frac{e^t}{t^2}dt - \frac{1}{e}\int \frac{e^t}{t^2}dt$$

$$= \frac{e^{t-1}}{t} + C = \frac{x}{1+\log x} + C$$

17. $\int \frac{\log x - 3}{(\log x)^4}dx = \int \frac{dx}{(\log x)^3} - 3\int \frac{dx}{(\log x)^4}$

$$= \int \frac{e^t dt}{t^3} - 3\int \frac{dx}{(\log x)^4}$$

$$\left[\text{Let } t = \log x \Rightarrow dt = \frac{dx}{x} \Rightarrow dx = e^t dt\right]$$

$$= \frac{1}{t^3}\int e^t dt - \int\left\{\frac{d}{dt}\frac{1}{t^3}\int e^t dt\right\}dt - 3\int \frac{e^t dt}{t^4}$$

$$= \frac{e^t}{t^3} - \int \frac{-3}{t^4}\cdot e^t dt - 3\int \frac{e^t}{t^4}dt$$

$$= \frac{e^t}{t^3} + 3\int \frac{e^t}{t^4}dt - 3\int \frac{e^t}{t^4}dt$$

$$= \frac{x}{(\log x)^3} + C$$

18. Let $I = \int e^x \left(\tan^{-1} x + \frac{1}{1+x^2}\right) dx$

TiP

Use the correct formula in the right place.

TR!CK

$\int e^x [f(x) + f'(x)]\, dx = e^x f(x) + C$

Here $f(x) = \tan^{-1} x$ and $f'(x) = \frac{1}{1+x^2}$

$$I = \int e^x \left\{\tan^{-1} x + \frac{d}{dx}(\tan^{-1} x)\right\} dx$$

$$= e^x \tan^{-1} x + C$$

COMMON ERR(!)R

Some students use the formula for integration by parts and make it lengthy.

19. Put $t = \cos x \Rightarrow dt = -\sin x\, dx \Rightarrow \sin x\, dx = -dt$

$\therefore \int \sin x \cdot \log \cos x\, dx = \int \log \cos x \cdot \sin x\, dx$

$$= \int \log t(-dt) = -\int \underset{II}{1} \cdot \underset{I}{\log t}\,(dt)$$

TR!CK

Integral by parts

$$\int \underset{I}{u} \cdot \underset{II}{v}\, dx = u\int v dx - \int \left\{\frac{d}{dx}(u)\int v dx\right\} dx$$

where u and v are the functions of x.

$$= -\log t \cdot \int 1 dt + \left[\int \left\{\frac{d}{dt}\log t \int 1 dt\right\} dt\right]$$

$$= -(\log t)t + \int \frac{1}{t} \cdot t\, dt$$

$$= -t\log t + \int 1 dt = -t\log t + t + C$$

$$= t(1 - \log t) + C = \cos x(1 - \log \cos x) + C$$

20. Let $I = \int_{-1}^{1} \log_a\left(\frac{2-x}{2+x}\right) dx$

and $f(x) = \log_a\left(\frac{2-x}{2+x}\right)$

$\therefore \quad f(-x) = \log_a\left(\frac{2+x}{2-x}\right) = \log_a\left(\frac{2-x}{2+x}\right)^{-1}$

$$= -\log_a\left(\frac{2-x}{2+x}\right) = -f(x)$$

So, $f(x)$ is an odd function.

$$\left[\because \int_{-a}^{a} f(x)dx = \begin{cases} 0, \text{ if } f(x) \text{ is odd function} \\ 2\int_0^a f(x)dx, \text{ if } f(x) \text{ is even function}\end{cases}\right]$$

$\therefore \ I = \int_{-1}^{1} \log_a\left(\frac{2-x}{2+x}\right) dx = 0$

21. Put $2x = t \Rightarrow 2\, dx = dt \Rightarrow dx = \frac{1}{2}dt$

TR!CK

Integral by parts

$$\int \underset{I}{u} \cdot \underset{II}{v} dx = u\int v dx - \int \left\{\frac{d}{dx}u\int v dx\right\} dx$$

where u and v are the functions of x.

Let $I = \int \sin^{-1}(2x)\, dx = \int \sin^{-1} t \cdot \frac{dt}{2} = \frac{1}{2}\int \underset{I}{\sin^{-1} t} \cdot \underset{II}{1}\, dt$

$$= \frac{1}{2}\left[\sin^{-1} t \int 1 dt - \int\left\{\frac{d}{dt}\sin^{-1} t \int 1 dt\right\} dt\right]$$

[Integral by parts]

$$= \frac{1}{2}\left[(\sin^{-1} t)\cdot t - \int \frac{1}{\sqrt{1-t^2}} \cdot t\, dt\right]$$

$$= \frac{1}{2} t\sin^{-1} t - \frac{1}{2}\int \frac{t}{\sqrt{1-t^2}} dt$$

Again put, $z^2 = 1 - t^2 \Rightarrow 2z\, dz = -2t\, dt$

$\Rightarrow \quad t\, dt = -z\, dz$

$\therefore \ I = \frac{1}{2} t\sin^{-1} t - \frac{1}{2}\int \frac{-zdz}{\sqrt{z^2}} = \frac{1}{2} t\sin^{-1} t + \frac{1}{2}\int \frac{z}{z} dz$

$$= \frac{t}{2}\sin^{-1} t + \frac{1}{2}\int 1 dz = \frac{t}{2}\sin^{-1} t + \frac{1}{2}z + C$$

$$= \frac{t}{2}\sin^{-1} t + \frac{1}{2}\sqrt{1-t^2} + C \qquad [\because z = \sqrt{1-t^2}]$$

$$= \frac{2x}{2}\sin^{-1}(2x) + \frac{1}{2}\sqrt{1-(2x)^2} + C \qquad [\because t = 2x]$$

$$= x\sin^{-1}(2x) + \frac{1}{2}\sqrt{1-4x^2} + C$$

22. Let $\int_{-2}^{1} \sqrt{5-4x-x^2}dx = \int_{-2}^{1} \sqrt{-(x^2+4x+4-4-5)}dx$

$$= \int_{-2}^{1} \sqrt{9-(x+2)^2}dx$$

Put $t = x + 2 \Rightarrow dt = dx$

U.L. = 3 and L.L. = 0

$\therefore \quad I = \int_0^3 \sqrt{(3)^2 - t^2}dt$

$$= \left[\frac{t}{2}\sqrt{9-t^2} + \frac{9}{2}\sin^{-1}\left(\frac{t}{3}\right)\right]_0^3$$

$$= 0 + \frac{9}{2}\sin^{-1}\left(\frac{3}{3}\right) - 0 - \frac{9}{2}\sin^{-1}(0)$$

$$= \frac{9}{2} \times \sin^{-1}(1) - \frac{9}{2} \times 0 = \frac{9}{2} \times \frac{\pi}{2} - 0 = \frac{9\pi}{4}$$

23. Let $I = \int_{-1}^{2} \frac{|x|}{x} dx$

Here, integrand $f(x)$ can be redefine in following manner:

$$f(x) = \begin{cases} \frac{-x}{x}, & -1 < x < 0 \\ \frac{x}{x}, & 0 < x < 2 \end{cases} = \begin{cases} -1 & -1 < x < 0 \\ 1 & 0 < x < 2 \end{cases}$$

TR!CK

$$\int_a^c f(x)\, dx = \int_a^b f(x)\, dx + \int_b^c f(x)\, dx,$$

where $a < b < c$.

$\therefore \quad I = \int_{-1}^{0} (-1)\, dx + \int_0^2 (1)\, dx$

$$= -[x]_{-1}^{0} + [x]_0^2 = -(0-(-1)) + (2-0) = -1 + 2 = 1$$

24. Let $I=\int_0^a \frac{\sqrt{x}}{\sqrt{x}+\sqrt{a-x}}\,dx$ ...(1)

Then, $I=\int_0^a \frac{\sqrt{a-x}}{\sqrt{a-x}+\sqrt{a-a+x}}\,dx$

$\Rightarrow$ $I=\int_0^a \frac{\sqrt{a-x}}{\sqrt{a-x}+\sqrt{x}}\,dx$ ...(2)

TR!CK

$\int_0^a f(x)\,dx=\int_0^a f(a-x)\,dx$

Adding eqs. (1) and (2),

$$2I=\int_0^a \frac{\sqrt{x}+\sqrt{a-x}}{\sqrt{a-x}+\sqrt{x}}\,dx=\int_0^a 1\,dx$$

$\Rightarrow$ $2I=[x]_0^a=(a-0)\Rightarrow I=\frac{a}{2}$

25. Let $I=\int_{-\pi}^{\pi}(1-x^2)\sin x\cos^2 x\,dx$

TR!CK

$\int_{-a}^{a} f(x)\,dx$

$=\begin{cases}2\int_0^a f(x)\,dx, & \text{if } f(-x)=f(x) \text{ i.e., } f(x) \text{ is even.}\\ 0, & \text{if } f(-x)=-f(x) \text{ i.e., } f(x) \text{ is odd.}\end{cases}$

Here, integrand $f(x)=(1-x^2)\sin x\cdot\cos^2 x$

$\therefore$ $f(-x)=(1-x^2)\sin(-x)\cdot\cos^2(-x)$

$=-(1-x^2)\sin x\cdot\cos^2 x=-f(x)$

i.e., $f(x)$ is odd function.

$\therefore$ $I=0$

26. $\int_{-\pi}^{\pi}(3\sin x-2)^2dx$

$$=\int_{-\pi}^{\pi}(9\sin^2 x+4-12\sin x)dx$$

$\left[\because \int_{-a}^{a} f(x)dx=\begin{cases}2\int_0^a f(x)\,dx; \text{ if } f(x) \text{ is an even function.}\\ 0; \quad \text{if } f(x) \text{ is an odd function}\end{cases}\right]$

$=2\int_0^{\pi}(9\sin^2 x)\,dx+4\times 2\int_0^{\pi}1\,dx-12\times 0$

$=9\int_0^{\pi}(1-\cos 2x)\,dx+8[x]_0^{\pi}-0$

$=9\left[x-\frac{\sin 2x}{2}\right]_0^{\pi}+8(\pi-0)=9\left\{\pi-\frac{\sin 2\pi}{2}-0\right\}+8\pi$

$=9\pi-\frac{9}{2}\times 0+8\pi=17\pi$ **Hence proved.**

Short Answer Type-II Questions

1. Given, $\frac{d}{dx}F(x)=\frac{\sec^4 x}{\mathrm{cosec}^4 x}$

$\Rightarrow$ $F(x)=\int\frac{\sec^4 x}{\mathrm{cosec}^4 x}dx+C=\int\frac{1}{\cos^4 x}\times\sin^4 x\,dx+C$

$=\int\tan^4 x\,dx+C=\int\tan^2 x\cdot\tan^2 x\,dx+C$

$=\int\tan^2 x(\sec^2 x-1)dx+C$

$=\int\tan^2 x\sec^2 x\,dx-\int\tan^2 x\,dx+C$

[Let $t=\tan x \Rightarrow dt=\sec^2 x\,dx$]

$=\int t^2dt-\int(\sec^2 x-1)\,dx+C$

$=\frac{t^3}{3}-(\tan x-x)+C=\frac{1}{3}\tan^3 x-\tan x+x+C$ ...(1)

Given: $F\left(\frac{\pi}{4}\right)=\frac{\pi}{4}$

$\Rightarrow$ $\frac{1}{3}\tan^3\frac{\pi}{4}-\tan\frac{\pi}{4}+\frac{\pi}{4}+C=\frac{\pi}{4}$

$\Rightarrow$ $\frac{1}{3}(1)^3-1+C=0 \Rightarrow C=\frac{2}{3}$

From eq. (1), we get

$$F(x)=\frac{\tan^3 x}{3}-\tan x+x+\frac{2}{3}$$

2. Let $I=\int\frac{\sin 2x}{a\cos^2 x+b\sin^2 x}dx$

$\left\{\begin{array}{l}\text{Put } a\cos^2 x+b\sin^2 x=t\\ \Rightarrow \{2a\cos x(-\sin x)\\ \quad +2b\sin x\cdot\cos x\}\,dx=dt\\ \Rightarrow (b-a)\cdot 2\sin x\cdot\cos x\,dx=dt\\ \Rightarrow \sin 2x\,dx=\frac{dt}{b-a}\end{array}\right.$

$\therefore$ $I=\int\frac{1}{t}\cdot\frac{dt}{(b-a)}=\frac{1}{b-a}\int\frac{dt}{t}$

$=\frac{1}{b-a}\log|t|+C$

$=\frac{1}{b-a}\cdot\log|a\cos^2 x+b\sin^2 x|+C$

3.

TiP

Practice more problems based on substitution.

Let $I=\int\frac{dx}{x(x^{3/2}+1)}=\int\frac{dx}{x^{5/2}\left(1+\frac{1}{x^{3/2}}\right)}$

$=\int\frac{x^{-5/2}}{1+x^{-3/2}}dx$

$\left\{\begin{array}{l}\text{Put } t=1+x^{-3/2}\\ \Rightarrow dt=-\frac{3}{2}x^{-5/2}dx\end{array}\right.$

$\therefore$ $I=\int\frac{-2}{3t}dt=-\frac{2}{3}\int\frac{dt}{t}=-\frac{2}{3}\log|t|+C$

$=-\frac{2}{3}\log|1+x^{-3/2}|+C$

COMMON ERROR

Mostly students find difficulty in finding the correct substitution.

4. Let $I=\int\frac{dx}{\sqrt{3-2x-x^2}}=\int\frac{dx}{\sqrt{-(x^2+2x-3)}}$

$=\int\frac{dx}{\sqrt{\{4-(x+1)^2\}}}$

Put $t=x+1\Rightarrow dt=dx$

TR!CK

$\int\frac{1}{\sqrt{a^2-x^2}}dx=\sin^{-1}\left(\frac{x}{a}\right)+C$

$\therefore$ $I=\int\frac{dt}{\sqrt{(2)^2-t^2}}=\sin^{-1}\left(\frac{t}{2}\right)+C=\sin^{-1}\left(\frac{x+1}{2}\right)+C$

5.

TiP

Practice more problems based on substitution.

Let $I=\int \frac{dx}{x^{1/2}+x^{1/3}}=\int \frac{dx}{x^{1/3}(1+x^{1/6})}$

Put $x^{1/6}=t \Rightarrow x=t^6$

$\Rightarrow$ $dx=6t^5dt$

$\therefore$ $I=\int \frac{6t^5dt}{t^2(1+t)}=6\int \frac{t^3dt}{1+t}$

$=6\int \frac{-1+(t^3+1)}{(t+1)}dt$

$=-6\int \frac{dt}{t+1}+6\int \frac{t^3+1}{t+1}dt$

TR!CK

$a^3+b^3=(a+b)(a^2-ab+b^2)$

$=-6\log|t+1|+6\int \frac{(t+1)(t^2-t+1)}{(t+1)}dt$

$=-6\log|t+1|+6\int (t^2-t+1)\,dt$

$=-6\log|t+1|+6\left\{\frac{t^3}{3}-\frac{t^2}{2}+t\right\}+C$

$=-6\log|1+x^{1/6}|+6\left(\frac{x^{1/2}}{3}-\frac{x^{1/3}}{2}+x^{1/6}\right)+C$

$=2x^{1/2}-3x^{1/3}+6x^{1/6}-6\log|1+x^{1/6}|+C$

COMMON ERROR

Mostly students find difficulty in finding the correct substitution.

6. Let $I=\int \frac{e^x}{\sqrt{5-4e^x-e^{2x}}}dx$

Put $e^x=t \Rightarrow e^x dx=dt$

$\therefore$ $I=\int \frac{dt}{\sqrt{5-4t-t^2}}$

$=\int \frac{dt}{\sqrt{-(t^2+4t+4)+5+4}}$

$=\int \frac{dt}{\sqrt{(3)^2-(t+2)^2}}$

$=\sin^{-1}\frac{t+2}{3}+C$

$=\sin^{-1}\frac{e^x+2}{3}+C$

7. $\cot x-\tan x=\frac{\cos x}{\sin x}-\frac{\sin x}{\cos x}=\frac{\cos^2 x-\sin^2 x}{\sin x\cdot\cos x}$

TR!CKS

- $\cos 2\theta=\cos^2\theta-\sin^2\theta$,
- $\sin 2\theta=2\sin\theta\cdot\cos\theta$

$=\frac{2(\cos^2 x-\sin^2 x)}{2\sin x\cdot\cos x}=\frac{2\cos 2x}{\sin 2x}=2\cot 2x$

Now, let $I=\int \frac{\sec^2 2x}{(\cot x-\tan x)^2}dx=\int \frac{\sec^2 2x}{(2\cot 2x)^2}dx$

$=\frac{1}{4}\int \frac{\sec^2 2x}{\cot^2 2x}dx=\frac{1}{4}\int \tan^2 2x\cdot\sec^2 2x\,dx$

Put $t=\tan 2x \Rightarrow 2\sec^2 2x\,dx=dt \Rightarrow \sec^2 2x\,dx=\frac{dt}{2}$

$\therefore$ $I=\frac{1}{4}\int t^2\cdot\frac{dt}{2}=\frac{1}{8}\int t^2\,dt$

$=\frac{1}{8}\left(\frac{t^3}{3}\right)+C=\frac{1}{24}\tan^3 2x+C$

8. $\sin^8 x-\cos^8 x=(\sin^4 x+\cos^4 x)(\sin^4 x-\cos^4 x)$

$=(\sin^4 x+\cos^4 x+2\sin^2 x\cos^2 x-2\sin^2 x\cos^2 x)$

$(\sin^2 x+\cos^2 x)(\sin^2 x-\cos^2 x)$

$[\because a^4-b^4=(a^2+b^2)(a^2-b^2)]$

$=[(\sin^2 x+\cos^2 x)^2-2\sin^2 x\cos^2 x]\cdot 1\cdot(-\cos 2x)$

TR!CKS

- $\sin^2\theta+\cos^2\theta=1, \forall\ \theta\in R$
- $\cos^2\theta-\sin^2\theta=\cos 2\theta$

$=(1-2\sin^2 x\cos^2 x)(-\cos 2x)$

$\Rightarrow \frac{\sin^8 x-\cos^8 x}{1-2\sin^2 x\cos^2 x}=-\cos 2x$

$\therefore \int \frac{\sin^8 x-\cos^8 x}{1-2\sin^2 x\cos^2 x}dx=-\int \cos 2x\,dx$

$=\frac{-\sin 2x}{2}+C$

$=-\sin x\cdot\cos x+C$

9. Put $t=\log x \Rightarrow dt=\frac{1}{x}dx$

$\therefore \int \frac{1}{(\log x-1)(\log x+1)}\cdot\frac{dx}{x}=\int \frac{1}{(t-1)(t+1)}dt$

$=\int \frac{1}{t^2-1}dt=\frac{1}{2\times 1}\cdot\log\left|\frac{t-1}{t+1}\right|+C$

TR!CK

$\int \frac{dx}{x^2-a^2}=\frac{1}{2a}\log\left|\frac{x-a}{x+a}\right|+C$

$=\frac{1}{2}\log\left|\frac{\log x-1}{\log x+1}\right|+C$

10. Let $I=\int \frac{dx}{1+3\cos^2 x}=\int \frac{\sec^2 x}{\sec^2 x+3}dx$

TR!CKS

- $1+\tan^2\theta=\sec^2\theta$
- $\int \frac{dx}{a^2+x^2}=\frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)+C$

$=\int \frac{\sec^2 x}{4+\tan^2 x}dx$

Put $t=\tan x \Rightarrow dt=\sec^2 x\,dx$

$\therefore$ $I=\int \frac{dt}{(2)^2+t^2}=\frac{1}{2}\tan^{-1}\left(\frac{t}{2}\right)+C$

$=\frac{1}{2}\tan^{-1}\left(\frac{1}{2}\tan x\right)+C$

11. Let $I=\int\sqrt{\frac{x}{1-x^3}}\,dx$

$$=\int\sqrt{\frac{x}{x^3\left(\frac{1}{x^3}-1\right)}}\,dx=\int\frac{1}{x}\sqrt{\frac{1}{\left(\frac{1}{x^3}-1\right)}}\,dx$$

Put $\frac{1}{x^3}-1=t^2\Rightarrow-\frac{3}{x^4}dx=2t\,dt$

$$\Rightarrow\quad -\frac{3}{2}\frac{dx}{x}(t^2+1)=t\,dt$$

$$\Rightarrow\quad \frac{dx}{x}=-\frac{2}{3}\left(\frac{t}{t^2+1}\right)dt$$

$$\therefore\ I=\int-\frac{1}{3}\left(\frac{2t}{t^2+1}\right)\times\sqrt{\frac{1}{t^2}}\,dt$$

$$=-\frac{2}{3}\int\frac{1}{(t^2+1)}dt=-\frac{2}{3}\tan^{-1}t+C$$

$$=-\frac{2}{3}\tan^{-1}\left(\sqrt{\frac{1}{x^3}-1}\right)+C$$

Alternate Method

$$\int\sqrt{\frac{x}{1-x^3}}\,dx=\int\frac{\sqrt{x}}{\sqrt{1-(x^{3/2})^2}}dx$$

$$\left[\text{put } t=x^{3/2}\Rightarrow dt=\frac{3}{2}\sqrt{x}\,dx\Rightarrow\sqrt{x}\,dx=\frac{2}{3}dt\right]$$

$$=\frac{2}{3}\int\frac{dt}{\sqrt{1-t^2}}=\frac{2}{3}\sin^{-1}t+C=\frac{2}{3}\sin^{-1}(x^{3/2})+C$$

12. Let $I=\int\frac{x^2-1}{x\sqrt{x^4+x^2+1}}dx=\int\frac{x^2-1}{x^2\sqrt{x^2+1+\frac{1}{x^2}}}dx$

$$=\int\frac{\left(1-\frac{1}{x^2}\right)}{\sqrt{\left(x+\frac{1}{x}\right)^2-1}}dx$$

Put $t=x+\frac{1}{x}\Rightarrow dt=\left(1-\frac{1}{x^2}\right)dx$

$$\therefore\quad I=\int\frac{dt}{\sqrt{t^2-1}}=\log\left|t+\sqrt{t^2-1}\right|+C$$

$$=\log\left|x+\frac{1}{x}+\sqrt{\left(x+\frac{1}{x}\right)^2-1}\right|+C$$

13. We have,

$$2x+1=A\frac{d}{dx}(2x^2+x-3)+B$$

$$\Rightarrow\quad 2x+1=A(4x+1)+B$$

$$\Rightarrow\quad 2x+1=(4A)x+(A+B)$$

On comparing the coefficients of like power of x.

$$4A=2\ \Rightarrow\ A=\frac{1}{2}\ \text{and}\ A+B=1$$

$$\Rightarrow\ \frac{1}{2}+B=1\ \Rightarrow\ B=\frac{1}{2}$$

Let $I=\int\frac{2x+1}{\sqrt{2x^2+x-3}}dx=\int\frac{\frac{1}{2}(4x+1)+\frac{1}{2}}{\sqrt{2x^2+x-3}}dx$

$$=\frac{1}{2}\int\frac{4x+1}{\sqrt{2x^2+x-3}}dx+\frac{1}{2}\int\frac{dx}{\sqrt{2x^2+x-3}}$$

Put $2x^2+x-3=t^2\Rightarrow(4x+1)\,dx=2t\,dt$

$$=\frac{1}{2}\int\frac{2t}{t}dt+\frac{1}{2\sqrt{2}}\int\frac{dx}{\sqrt{x^2+\frac{x}{2}-\frac{3}{2}}}$$

$$\therefore\quad I=\int1dt+\frac{1}{2\sqrt{2}}\int\frac{dx}{\sqrt{\left(x^2+\frac{x}{2}+\frac{1}{16}\right)-\frac{1}{16}-\frac{3}{2}}}$$

$$=t+\frac{1}{2\sqrt{2}}\int\frac{dx}{\sqrt{\left(x+\frac{1}{4}\right)^2-\frac{25}{16}}}$$

$$=t+\frac{1}{2\sqrt{2}}\int\frac{dx}{\sqrt{\left(x+\frac{1}{4}\right)^2-\left(\frac{5}{4}\right)^2}}$$

TR!CKS

- $\int\frac{f'(x)}{\sqrt{f(x)}}dx=2\sqrt{f(x)}+C$
- $\int\frac{dx}{\sqrt{x^2+a^2}}=\log|x+\sqrt{x^2+a^2}|+C$

$$=t+\frac{1}{2\sqrt{2}}\log\left|x+\frac{1}{4}+\sqrt{\left(x+\frac{1}{4}\right)^2-\left(\frac{5}{4}\right)^2}\right|+C_1$$

$$=t+\frac{1}{2\sqrt{2}}\log\left|\frac{4x+1}{4}+\sqrt{x^2+\frac{x}{2}-\frac{3}{2}}\right|+C_1$$

$$=t+\frac{1}{2\sqrt{2}}\log\left|\frac{4x+1}{4}+\frac{1}{\sqrt{2}}\sqrt{2x^2+x-3}\right|+C_1$$

$$=t+\frac{1}{2\sqrt{2}}\log\left|\frac{4x+1+2\sqrt{4x^2+2x-6}}{4}\right|+C_1$$

$$=t+\frac{1}{2\sqrt{2}}\log|4x+1+2\sqrt{4x^2+2x-6}|-\frac{1}{2\sqrt{2}}\log4+C_1$$

$$=\sqrt{2x^2+x-3}+\frac{1}{2\sqrt{2}}\log|4x+1+2\sqrt{4x^2+2x-6}|+C$$

where, $\left[C=C_1-\frac{1}{2\sqrt{2}}\log4\right]$

14. $\int\frac{x^3+x}{x^4-9}dx=\int\frac{x^3}{x^4-9}dx+\int\frac{x}{x^4-9}dx$

$$=\frac{1}{4}\int\frac{4x^3}{x^4-9}dx+\int\frac{x}{x^4-9}dx$$

$$=\frac{1}{4}\log\left|x^4-9\right|+\int\frac{x}{(x^2-3)(x^2+3)}dx$$

[Let $t=x^2\Rightarrow dt=2x\,dx$]

$$=\frac{1}{4}\log\left|x^4-9\right|+\frac{1}{2}\int\frac{dt}{(t-3)(t+3)}$$

$$=\frac{1}{4}\log\left|x^4-9\right|+\frac{1}{2}\left\{\int\frac{1}{6(t-3)}-\frac{1}{6(t+3)}\right\}dt$$

[By partial fraction]

$$=\frac{1}{4}\log\left|x^4-9\right|+\frac{1}{12}\{\log|t-3|-\log|t+3|\}+C$$

$$=\frac{1}{4}\log\left|x^4-9\right|+\frac{1}{12}\log\left|\frac{t-3}{t+3}\right|+C$$

$$=\frac{1}{4}\log\left|x^4-9\right|+\frac{1}{12}\log\left|\frac{x^2-3}{x^2+3}\right|+C$$

15. $\int \frac{x^2}{(x^2+1)(3x^2+4)}dx$ Let $t = x^2 \Rightarrow dt = 2x\,dx$

By partial fraction,

$$\frac{t}{(t+1)(3t+4)} = \frac{A}{t+1} + \frac{B}{3t+4}$$

$$\Rightarrow \quad t = A(3t+4) + B(t+1)$$

$$\Rightarrow \quad t = (3A+B)t + (4A+B)$$

On comparing the like powers of t, we get

$3A + B = 1$ and $4A + B = 0$

On solving both equations, we get

$3A - 4A = 1 \Rightarrow A = -1$ and $B = 4$

$$\therefore \int \frac{x^2}{(x^2+1)(3x^2+4)}dx = \int \left\{\frac{-1}{x^2+1} + \frac{4}{3x^2+4}\right\}dx$$

$$= -\int \frac{dx}{x^2+1} + \frac{4}{3}\int \frac{dx}{x^2 + \left(\frac{4}{3}\right)} = -\tan^{-1}x$$

$$+\frac{4}{3} \times \frac{1}{(2/\sqrt{3})}\tan^{-1}\left(\frac{x}{2/\sqrt{3}}\right) + C$$

$$= -\tan^{-1}x + \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{x\sqrt{3}}{2}\right) + C$$

16. Let $I = \int \frac{x^4}{(x-1)(x^2+1)}dx$

$$= \int \frac{(x^4-1)+1}{(x-1)(x^2+1)}dx$$

$$= \int \frac{(x-1)(x+1)(x^2+1)}{(x-1)(x^2+1)}dx + \int \frac{1}{(x-1)(x^2+1)}dx$$

$$= \int (x+1)dx + \int \frac{1}{(x-1)(x^2+1)}dx$$

Using partial fraction in second integrand,

Let $\frac{1}{(x-1)(x^2+1)} = \frac{A}{(x-1)} + \frac{Bx+C}{(x^2+1)}$

$\Rightarrow 1 = A(x^2+1) + (Bx+C)(x-1)$

$\Rightarrow 1 = (A+B)x^2 + (-B+C)x + (A-C)$ ...(1)

On comparing the coefficients of like powers of x,

$A + B = 0, -B + C = 0$ and $A - C = 1$

$\Rightarrow A = \frac{1}{2}, B = -\frac{1}{2}$ and $C = -\frac{1}{2}$

Put these values in eq. (1) we get

$$I = \int (x+1)dx + \int \left[\frac{1}{2(x-1)} + \frac{-\frac{1}{2}x - \frac{1}{2}}{(x^2+1)}\right]dx$$

$$= \int (x+1)dx + \frac{1}{2}\int \frac{1}{x-1}dx - \frac{1}{2}\int \frac{x}{x^2+1}dx - \frac{1}{2}\int \frac{1}{x^2+1}dx$$

$$\left[\because \int \frac{f'(x)}{f(x)}dx = \log|f(x)| + C\right]$$

$$= \left(\frac{x^2}{2} + x\right) + \frac{1}{2}\log|x-1| - \frac{1}{4}\log|x^2+1| - \frac{1}{2}\tan^{-1}x + C$$

$$= \left(\frac{x^2}{2} + x\right) + \frac{1}{2}\log\left|\frac{x-1}{\sqrt{x^2+1}}\right| - \frac{1}{2}\tan^{-1}x + C$$

17. We have, $\int \frac{x^2+x+1}{(x+1)^2(x+2)}dx$

By partial fraction,

$$\frac{x^2+x+1}{(x+1)^2(x+2)} = \frac{A}{(x+1)} + \frac{B}{(x+1)^2} + \frac{C}{x+2}$$

$$\Rightarrow x^2+x+1 = A(x+1)(x+2) + B(x+2) + C(x+1)^2$$

$$\Rightarrow x^2+x+1 = (A+C)x^2 + (3A+B+2C)x + 2A + 2B + C$$

On comparing the coefficients of like powers of x,

$A + C = 1, 3A + B + 2C = 1$ and $2A + 2B + C = 1$

$\Rightarrow A = -2, B = 1$ and $C = 3$

$$\therefore \frac{x^2+x+1}{(x+1)^2(x+2)} = \frac{-2}{(x+1)} + \frac{1}{(x+1)^2} + \frac{3}{(x+2)}$$

$$\therefore \int \frac{x^2+x+1}{(x+1)^2(x+2)}dx$$

$$= \int -\frac{2}{x+1}dx + \int \frac{1}{(x+1)^2}dx + \int \frac{3}{(x+2)}dx$$

$$= -2\log|x+1| - \frac{1}{(x+1)} + 3\log|x+2| + C$$

18. Let $I = \int \frac{2\cos x}{(1-\sin x)(1+\sin^2 x)}dx$

Put $t = \sin x \Rightarrow dt = \cos x\,dx$

$$\therefore \quad I = \int \frac{2dt}{(1-t)(1+t^2)} \quad ...(1)$$

By partial fraction,

$$\frac{1}{(1-t)(1+t^2)} = \frac{A}{(1-t)} + \frac{Bt+C}{1+t^2}$$

$$\Rightarrow \quad 1 = A(1+t^2) + (1-t)(Bt+C)$$

$$\Rightarrow \quad 1 = A + At^2 + Bt + C - Bt^2 - Ct$$

$$\Rightarrow \quad 1 = (A+C) + (B-C)t + (A-B)t^2$$

On comparing the coefficient of like powers of t, we get

$A + C = 1$ ...(2)

$B - C = 0 \Rightarrow B = C$ ...(3)

and $A - B = 0 \Rightarrow A = B$ ...(4)

From eqs. (3) and (4), we get

$A = C$ ...(5)

From eqs. (2) and (5), we get

$A + A = 1$

$\Rightarrow \quad A = \frac{1}{2}$ and $C = \frac{1}{2} = B$

$$\therefore \frac{1}{(1-t)(1+t^2)} = \frac{1}{2(1-t)} + \frac{\frac{1}{2}t + \frac{1}{2}}{1+t^2}$$

From eq. (1), we get

$$\int \frac{2\,dt}{(1-t)(1+t^2)} = 2\int \left\{\frac{1}{2(1-t)} + \frac{1}{2}\cdot\frac{t+1}{1+t^2}\right\}dt$$

$$= \int \frac{dt}{1-t} + \int \frac{1+t}{1+t^2}dt$$

$$= \int \frac{1}{1-t}dt + \int \frac{1}{1+t^2}dt + \frac{1}{2}\int \frac{2t}{1+t^2}dt$$

TR!CKS

- $\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)+C$
- $\int \frac{f'(x)}{f(x)}dx = \log|f(x)|+C$

$$= -\log|1-t| + \tan^{-1}t + \frac{1}{2}\log|1+t^2| + C$$

$$= \log\left|\frac{\sqrt{1+t^2}}{1-t}\right| + \tan^{-1}t + C \qquad [\because t = \sin x]$$

$$= \log\left|\frac{\sqrt{1+\sin^2 x}}{1-\sin x}\right| + \tan^{-1}(\sin x) + C$$

19. $\int \frac{e^x}{(e^x-1)^2(e^x+2)}dx = \int \frac{dt}{(t+2)(t-1)^2} \qquad ...(1)$

[Put $t = e^x \Rightarrow dt = e^x dx$]

By partial fraction,

$$\frac{1}{(t+2)(t-1)^2} = \frac{A}{(t+2)} + \frac{B}{(t-1)} + \frac{C}{(t-1)^2}$$

$\Rightarrow \quad 1 = A(t-1)^2 + B(t+2)(t-1) + C(t+2)$

$\Rightarrow \quad 1 = A(t^2-2t+1) + B(t^2+t-2) + C(t+2)$

$\Rightarrow \quad 1 = (A-2B+2C) + (-2A+B+C)t + (A+B)t^2$

On comparing coefficients of the like powers of t, we get

$A - 2B + 2C = 1 \qquad ...(2)$

$-2A + B + C = 0 \qquad ...(3)$

and $\quad A + B = 0 \Rightarrow A = -B \qquad ...(4)$

Put the value of A from eq. (4) in eqs. (2) and (3), we get

$-3B + 2C = 1$ and $3B + C = 0$

$\Rightarrow \quad 3C = 1 \Rightarrow C = \frac{1}{3}$

Put the value of C in eq. (2), we get

$A - 2B + \frac{2}{3} = 1 \Rightarrow A - 2B = \frac{1}{3} \qquad ...(5)$

From eqs. (4) and (5), we get

$-3B = \frac{1}{3}$

$\Rightarrow \quad B = -\frac{1}{9}$ and $A = \frac{1}{9}$

$$\therefore \quad \frac{1}{(t+2)(t-1)^2} = \frac{1}{9(t+2)} - \frac{1}{9(t-1)} + \frac{1}{3(t-1)^2}$$

From eq. (1),

$$\int \frac{dt}{(t+2)(t-1)^2} = \int\left\{\frac{1}{9(t+2)} - \frac{1}{9(t-1)} + \frac{1}{3(t-1)^2}\right\}dt$$

$$= \frac{1}{9}\int\frac{dt}{t+2} - \frac{1}{9}\int\frac{dt}{t-1} + \frac{1}{3}\int(t-1)^{-2}dt$$

$$= \frac{1}{9}\log|t+2| - \frac{1}{9}\log|t-1| + \frac{1}{3}\frac{(t-1)^{-2+1}}{-2+1} + C$$

$$= \frac{1}{9}\{\log|t+2| - \log|t-1|\} - \frac{1}{3(t-1)} + C$$

$$= \frac{1}{9}\log\left|\frac{t+2}{t-1}\right| - \frac{1}{3(t-1)} + C$$

$$= \frac{1}{9}\log\left|\frac{e^x+2}{e^x-1}\right| - \frac{1}{3(e^x-1)} + C \qquad [\because t = e^x]$$

20. Let $I = \int \frac{e^x}{(2+e^x)(4+e^{2x})}dx = \int \frac{e^x}{(2+e^x)\{4+(e^x)^2\}}dx$

Put $\quad t = e^x \Rightarrow dt = e^x dx$

$\therefore \quad I = \int \frac{dt}{(2+t)(4+t^2)} \qquad ...(1)$

By partial fraction,

$$\frac{1}{(2+t)(4+t^2)} = \frac{A}{2+t} + \frac{Bt+C}{4+t^2}$$

$\Rightarrow \quad 1 = A(4+t^2) + (Bt+C)(2+t)$

$\Rightarrow \quad 1 = 4A + At^2 + 2Bt + 2C + Bt^2 + Ct$

$\Rightarrow \quad 1 = (4A+2C) + (2B+C)t + (A+B)t^2$

On comparing the coefficient of like powers of t, we get

$4A + 2C = 1 \qquad ...(2)$

$2B + C = 0 \qquad ...(3)$

and $\quad A + B = 0 \qquad ...(4)$

From eqs. (3) and (4), we get

$-2A + C = 0$

$\Rightarrow \quad C = 2A$

From eq. (2), we get

$4A + 4A = 1$

$\Rightarrow \quad 8A = 1$

$\Rightarrow \quad A = \frac{1}{8}$ and $C = \frac{1}{4}$

From eq. (4), we get $B = -\frac{1}{8}$

$$\therefore \quad \frac{1}{(2+t)(4+t^2)} = \frac{1}{8(2+t)} + \frac{-\frac{t}{8}+\frac{1}{4}}{(4+t^2)}$$

From eq. (1),

$$I = \int \frac{dt}{(2+t)(4+t^2)}$$

$$= \frac{1}{8}\int\frac{dt}{2+t} - \frac{1}{16}\int\frac{2t}{4+t^2}dt + \frac{1}{4}\int\frac{dt}{4+t^2}$$

TR!CKS

- $\int \frac{f'(x)}{f(x)}dx = \log|f(x)|+C$
- $\int \frac{1}{a^2+x^2}dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)+C$

$$= \frac{1}{8}\log|2+t| - \frac{1}{16}\log|4+t^2| + \frac{1}{4}\times\frac{1}{2}\tan^{-1}\frac{t}{2} + C$$

$$= \frac{1}{8}\log|2+e^x| - \frac{1}{16}\log|4+e^{2x}| + \frac{1}{8}\tan^{-1}\left(\frac{e^x}{2}\right) + C$$

$$\left[\because \int\frac{f'(x)}{f(x)}dx = \log|f(x)|+C\right]$$

$[\because t = e^x]$

$$= \frac{1}{8}\log|2+e^x| - \frac{1}{8}\log|\sqrt{4+e^{2x}}| + \frac{1}{8}\tan^{-1}\left(\frac{e^x}{2}\right) + C$$

$$= \frac{1}{8}\log\left|\frac{2+e^x}{\sqrt{4+e^{2x}}}\right| + \frac{1}{8}\tan^{-1}\left(\frac{e^x}{2}\right) + C$$

21. $\int \frac{2x}{(x^2+1)(x^2+2)^2}dx = \int \frac{dt}{(t+1)(t+2)^2}$ ...(1)

[Let, $x^2 = t \Rightarrow 2x\,dx = dt$]

By partial fraction,

$$\frac{1}{(t+1)(t+2)^2} = \frac{A}{(t+1)} + \frac{B}{(t+2)} + \frac{C}{(t+2)^2}$$

$\Rightarrow \quad 1 = A(t+2)^2 + B(t+1)(t+2) + C(t+1)$

$\Rightarrow \quad 1 = A(t^2+4+4t) + B(t^2+3t+2) + C(t+1)$

$\Rightarrow \quad 1 = (4A+2B+C) + (4A+3B+C)t + (A+B)t^2$

On comparing the coefficient of like powers of 't', we get

$4A+2B+C = 1$ ...(2)

$4A+3B+C = 0$ ...(3)

$A+B = 0 \Rightarrow B = -A$ ...(4)

Subtract eq. (2) from eq. (3), we get

$B = -1$

So, $A = 1$ [from eq. (4)]

From eq. (2), we get

$4(1)+2(-1)+C = 1$

$\Rightarrow \quad 4-2+C = 1$

$\Rightarrow \quad C = -1$

$\therefore \quad \frac{1}{(t+1)(t+2)^2} = \frac{1}{(t+1)} - \frac{1}{(t+2)} - \frac{1}{(t+2)^2}$

From eq. (1), we get

$$\int \frac{dt}{(t+1)(t+2)^2} = \int \left\{\frac{1}{(t+1)} - \frac{1}{(t+2)} - \frac{1}{(t+2)^2}\right\}dt$$

$$= \int \frac{dt}{(t+1)} - \int \frac{dt}{(t+2)} - \int (t+2)^{-2}dt$$

$$= \log|t+1| - \log|t+2| - \frac{(t+2)^{-2+1}}{-2+1} + C$$

$$= \log\left|\frac{t+1}{t+2}\right| + \frac{1}{t+2} + C$$

$$= \log\left(\frac{x^2+1}{x^2+2}\right) + \frac{1}{x^2+2} + C \qquad [\because t = x^2]$$

22. Let $I = \int \frac{2x^2+3}{x^2(x^2+9)}dx$

By partial fraction,

$$\frac{2x^2+3}{x^2(x^2+9)} = \frac{Ax+B}{x^2} + \frac{Cx+D}{x^2+9}$$

$\Rightarrow \quad 2x^2+3 = (Ax+B)(x^2+9) + (Cx+D)x^2$

$\Rightarrow \quad 2x^2+3 = (A+C)x^3 + (B+D)x^2 + 9Ax + 9B$

On comparing the coefficient of like powers of x, we get

$A+C = 0, B+D = 2, 9A = 0$ and $9B = 3$

$\Rightarrow \quad A = 0, B = \frac{1}{3}, C = 0$ and $D = \frac{5}{3}$

$\therefore \quad \frac{2x^2+3}{x^2(x^2+9)} = \frac{1}{3x^2} + \frac{5}{3(x^2+9)}$

$\therefore \quad I = \int \frac{1}{3x^2}dx + \frac{5}{3}\int \frac{1}{x^2+9}dx$

$$= -\frac{1}{3x} + \frac{5}{3} \times \frac{1}{3}\tan^{-1}\left(\frac{x}{3}\right) + C$$

$$= -\frac{1}{3x} + \frac{5}{9}\tan^{-1}\left(\frac{x}{3}\right) + C$$

23. $\int \frac{\sin\theta}{(4+\cos^2\theta)(2-\sin^2\theta)}d\theta$

TiP

Reduce the given expression in proper rational function using substitution method and then apply partial fraction method.

$$= \int \frac{\sin\theta}{(4+\cos^2\theta)\{2-(1-\cos^2\theta)\}}d\theta \quad [\because \sin^2\theta + \cos^2\theta = 1]$$

$$= \int \frac{\sin\theta}{(4+\cos^2\theta)(1+\cos^2\theta)}d\theta$$

Let $t = \cos\theta \Rightarrow dt = -\sin\theta\,d\theta$

$\therefore \quad I = -\int \frac{dt}{(4+t^2)(1+t^2)}$ ...(1)

By partial fraction,

$$\frac{1}{(4+t^2)(1+t^2)} = \frac{At+B}{4+t^2} + \frac{Ct+D}{1+t^2}$$

$\Rightarrow \quad 1 = (At+B)(1+t^2) + (Ct+D)(4+t^2)$

$\Rightarrow \quad 1 = At + B + At^3 + Bt^2 + 4Ct + 4D + Ct^3 + Dt^2$

$\Rightarrow \quad 1 = (B+4D) + (A+4C)t + (B+D)t^2 + (A+C)t^3$

On comparing the coefficient of like powers of t, we get

$B+4D = 1 \Rightarrow B = 1-4D$ ...(2)

$A+4C = 0 \Rightarrow A = -4C$ ...(3)

$B+D = 0 \Rightarrow B = -D$ ...(4)

and $A+C = 0 \Rightarrow A = -C$ ...(5)

From eqs. (3) and (5), we get

$A = 0$ and $C = 0$

From eqs. (2) and (4), we get

$-D = 1-4D \Rightarrow 3D = 1 \Rightarrow D = \frac{1}{3}$

$\therefore \quad B = -\frac{1}{3}$

So, $\frac{1}{(4+t^2)(1+t^2)} = \frac{0-\frac{1}{3}}{(4+t^2)} + \frac{0+\frac{1}{3}}{(1+t^2)}$

From eq. (1),

$$I = -\int \frac{dt}{(4+t^2)(1+t^2)} = -\int \left\{-\frac{1}{3(4+t^2)} + \frac{1}{3(1+t^2)}\right\}dt$$

$$= \frac{1}{3}\int \frac{dt}{2^2+t^2} - \frac{1}{3}\int \frac{dt}{1^2+t^2} = \frac{1}{3} \times \frac{1}{2}\tan^{-1}\frac{t}{2} - \frac{1}{3}\cdot\tan^{-1}t + C$$

TR!CK

$$\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$$

$$= \frac{1}{6}\tan^{-1}\left(\frac{1}{2}\cos\theta\right) - \frac{1}{3}\tan^{-1}(\cos\theta) + C$$

COMMON ERR!R

Several students commit the mistake of ignoring the constant of integration and use of partial fraction.

24. $\int \frac{\cos\theta}{(4+\sin^2\theta)(5-4\cos^2\theta)}d\theta$

$$= \int \frac{\cos\theta}{(4+\sin^2\theta)\{(5-4(1-\sin^2\theta))\}}d\theta$$

$[\because \sin^2\theta + \cos^2\theta = 1]$

$$= \int \frac{\cos\theta}{(4+\sin^2\theta)(1+4\sin^2\theta)}d\theta$$

Put $t=\sin\theta \Rightarrow dt = \cos\theta\, d\theta$

$$\therefore \quad I = \int \frac{dt}{(4+t^2)(1+4t^2)}$$

By partial fraction,

$$\frac{1}{(4+t^2)(1+4t^2)} = \frac{At+B}{4+t^2} + \frac{Ct+D}{1+4t^2}$$

$\Rightarrow \quad 1 = (At+B)(1+4t^2) + (Ct+D)(4+t^2)$

$\Rightarrow \quad 1 = At + B + 4At^3 + 4Bt^2 + 4Ct + 4D + Ct^3 + Dt^2$

$\Rightarrow \quad 1 = (4A+C)t^3 + (4B+D)t^2 + (A+4C)t + (B+4D)$

On comparing the coefficient of like powers of t, we get

$4A + C = 0 \quad \Rightarrow \quad C = -4A$ ...(1)

$4B + D = 0 \quad \Rightarrow \quad D = -4B$ ...(2)

$A + 4C = 0 \quad \Rightarrow \quad A = -4C$ ...(3)

and $\quad B + 4D = 1$ ...(4)

From eqs. (1) and (3), we get

$C = -4(-4C) \Rightarrow C = 16C \Rightarrow C = 0$ and $A = 0$

From eqs. (2) and (4), we get

$B + 4(-4B) = 1 \Rightarrow B - 16B = 1 \Rightarrow -15B = 1 \Rightarrow B = -\frac{1}{15}$

and $\quad D = -4\left(-\frac{1}{15}\right) = \frac{4}{15}$

$$\therefore \quad \frac{1}{(4+t^2)(1+4t^2)} = \frac{0-\frac{1}{15}}{4+t^2} + \frac{0+\frac{4}{15}}{1+4t^2}$$

$$\therefore \quad I = \int\left\{-\frac{1}{15(4+t^2)} + \frac{4}{15(1+4t^2)}\right\}dt$$

$$= \frac{-1}{15}\int \frac{dt}{4+t^2} + \frac{4}{15}\int \frac{dt}{1+4t^2}$$

TR!CK

$$\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + C$$

$$= -\frac{1}{15}\int \frac{dt}{(2)^2+t^2} + \frac{1}{15}\int \frac{dt}{\left(\frac{1}{2}\right)^2 + t^2}$$

$$= -\frac{1}{15}\cdot\frac{1}{2}\cdot\tan^{-1}\frac{t}{2} + \frac{1}{15}\cdot\frac{1}{1/2}\cdot\tan^{-1}\frac{t}{1/2} + C$$

$$= -\frac{1}{30}\cdot\tan^{-1}\left(\frac{1}{2}\sin\theta\right) + \frac{2}{15}\cdot\tan^{-1}(2\sin\theta) + C$$

25. Let $I = \int \frac{(3\sin x - 2)\cos x}{13 - \cos^2 x - 7\sin x}dx$

$$= \int \frac{(3\sin x - 2)\cos x}{13 - (1-\sin^2 x) - 7\sin x}dx \quad [\because \sin^2 x + \cos^2 x = 1]$$

$$= \int \frac{(3\sin x - 2)\cos x}{12 + \sin^2 x - 7\sin x}dx$$

Put $t = \sin x \Rightarrow dt = \cos x\, dx$

$$\therefore \quad I = \int \frac{(3t-2)}{t^2 - 7t + 12}dt = \int \frac{3t-2}{t^2 - 4t - 3t + 12}dt$$

$$= \int \frac{3t-2}{(t-3)(t-4)}dt \quad \text{...(1)}$$

By partial fraction, $\frac{3t-2}{(t-3)(t-4)} = \frac{A}{t-3} + \frac{B}{t-4}$

$\Rightarrow \quad 3t - 2 = A(t-4) + B(t-3)$

$\Rightarrow \quad 3t - 2 = (A+B)t + (-4A - 3B)$

On comparing the coefficient of like powers of 't' on both sides,

$A + B = 3 \quad \Rightarrow \quad B = 3 - A$ ...(2)

and $\quad -4A - 3B = -2 \Rightarrow 4A + 3B = 2$ ...(3)

From eqs. (2) and (3), we get

$4A + 3(3 - A) = 2$

$\Rightarrow \quad 4A + 9 - 3A = 2$

$\Rightarrow \quad A = -7$

From eq. (2), we get $B = 3 - (-7) = 3 + 7 = 10$

$$\therefore \quad \frac{3t-2}{(t-3)(t-4)} = \frac{-7}{t-3} + \frac{10}{t-4}$$

From eq. (1), we get

$$\int \frac{3t-2}{(t-3)(t-4)}dt = \int\left\{\frac{-7}{t-3} + \frac{10}{t-4}\right\}dt$$

$$= -7\int \frac{dt}{t-3} + 10\int \frac{dt}{t-4}$$

$$= -7\log|t-3| + 10\log|t-4| + C$$

$[\because t = \sin x]$

$$= -7\log|\sin x - 3| + 10\log|\sin x - 4| + C$$

26. $\int \frac{3x+5}{x^2+3x-18}dx$

Numerator $= A\cdot\frac{d}{dx}$(Denominator) $+ B$

$\Rightarrow \quad 3x + 5 = A\frac{d}{dx}(x^2 + 3x - 18) + B$

$\Rightarrow \quad 3x + 5 = A(2x+3) + B$

$\Rightarrow \quad 3x + 5 = 2Ax + (3A + B)$

On comparing the coefficient of like powers of x and constant terms, we get

$2A = 3 \Rightarrow A = \frac{3}{2}$

and $\quad 3A + B = 5$

$\Rightarrow \quad B = 5 - 3\times\frac{3}{2} = \frac{10-9}{2} = \frac{1}{2}$

$$\therefore \quad \int \frac{3x+5}{x^2+3x-18}dx = \int \frac{\frac{3}{2}(2x+3) + \frac{1}{2}}{x^2+3x-18}dx$$

$$= \frac{3}{2}\int \frac{2x+3}{x^2+3x-18}dx + \frac{1}{2}\int \frac{dx}{x^2+3x-18}$$

$$= \frac{3}{2}\int \frac{\frac{d}{dx}(x^2+3x-18)}{x^2+3x-18}dx + \frac{1}{2}\int \frac{dx}{x^2+3x+\frac{9}{4}-\frac{9}{4}-18}$$

$$= \frac{3}{2}\log|x^2+3x-18| + \frac{1}{2}\int \frac{dx}{\left(x+\frac{3}{2}\right)^2 - \frac{81}{4}}$$

TR!CKS

- $\int \frac{f'(x)}{f(x)}dx = \log|f(x)| + C$
- $\int \frac{dx}{x^2 - a^2} = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + C$

$$= \frac{3}{2}\log|x^2+3x-18| + \frac{1}{2}\int \frac{dt}{t^2 - \left(\frac{9}{2}\right)^2}$$

$\left[\text{Put } t = x + \frac{3}{2} \Rightarrow dt = dx\right]$

$$= \frac{3}{2}\log|x^2+3x-18| + \frac{1}{2}\times\frac{1}{2\times\frac{9}{2}}\cdot\log\left|\frac{t-\frac{9}{2}}{t+\frac{9}{2}}\right| + C$$

$$= \frac{3}{2}\log|x^2+3x-18| + \frac{1}{18}\log\left|\frac{x+\frac{3}{2}-\frac{9}{2}}{x+\frac{3}{2}+\frac{9}{2}}\right| + C$$

$$= \frac{3}{2}\log|x^2+3x-18| + \frac{1}{18}\log\left|\frac{x-3}{x+6}\right| + C$$

27. Let $I = \int \cos^{-1}x\,dx = \int \cos^{-1}x\cdot 1\,dx$

TR!CK

Integration by parts,

$\int \underset{\text{I}}{u}\cdot\underset{\text{II}}{v}\,dx = u\int v\,dx - \int\left\{\frac{d}{dx}u\int v\,dx\right\}dx$

where u and v are the functions of x.

$$= \cos^{-1}x\int 1\,dx - \int\left\{\frac{d}{dx}\cos^{-1}x\int 1\cdot dx\right\}dx$$

(Using integration by parts)

$$= x\cdot\cos^{-1}x - \int\frac{-1}{\sqrt{1-x^2}}\cdot x\,dx$$

$$= x\cos^{-1}x + \int\frac{x}{\sqrt{1-x^2}}dx$$

$$= x\cos^{-1}x + \frac{1}{2}\int\frac{2x}{\sqrt{1-x^2}}dx$$

Put $t = 1 - x^2 \Rightarrow dt = -2x\,dx$

$$\therefore\quad I = x\cos^{-1}x - \frac{1}{2}\int\frac{dt}{\sqrt{t}} \quad \left[\because \int\frac{f'(x)}{\sqrt{f(x)}}dx = 2\sqrt{f(x)} + C\right]$$

$$= x\cos^{-1}x - \frac{1}{2}\cdot 2\sqrt{t} + C = x\cos^{-1}x - \sqrt{1-x^2} + C$$

28. Let $I = \int e^{\tan^{-1}x}\left(\frac{1+x+x^2}{1+x^2}\right)dx = \int e^{\tan^{-1}x}\left(1+\frac{x}{1+x^2}\right)dx$

$$= \int e^{\tan^{-1}x}dx + \int\left(\frac{e^{\tan^{-1}x}}{1+x^2}\right)\cdot x\,dx$$

TR!CK

Integration by parts,

$\int \underset{\text{I}}{u}\cdot\underset{\text{II}}{v}\,dx = u\int v\,dx - \int\left\{\frac{d}{dx}u\int v\,dx\right\}dx$

where u and v are the functions of x.

$$= \int e^{\tan^{-1}x}dx + \left[x\int\frac{e^{\tan^{-1}x}}{1+x^2}dx - \int\left(\frac{d}{dx}x\right)\left(\int\frac{e^{\tan^{-1}x}}{1+x^2}dx\right)dx\right]$$

$$\therefore\ I = \int e^{\tan^{-1}x}dx + [x\cdot e^{\tan^{-1}x} - \int 1\cdot e^{\tan^{-1}x}dx]$$

$\left[\begin{array}{l}\text{Put } t = \tan^{-1}x \Rightarrow dt = \frac{1}{1+x^2}dx \\ \because \int\frac{e^{\tan^{-1}x}}{1+x^2}dx = \int e^t dt = e^t = e^{\tan^{-1}x}\end{array}\right]$

$$= xe^{\tan^{-1}x} + C$$

29. Let $I = \int_{-2}^{2}\frac{x^2}{1+5^x}dx$ ...(1)

$\Rightarrow\ I = \int_{-2}^{2}\frac{(-x)^2}{1+5^{-x}}dx \quad \left[\because \int_a^b f(x)dx = \int_a^b f(a+b-x)dx\right]$

$\Rightarrow\ I = \int_{-2}^{2}\frac{5^x(x^2)}{1+5^x}$ ...(2)

Adding eqs. (1) and (2), we get

$$2I = \int_{-2}^{2}\frac{(1+5^x)x^2}{1+5^x}dx$$

$$\Rightarrow\quad 2I = \int_{-2}^{2}x^2dx$$

$$\Rightarrow\quad 2I = \left[\frac{x^3}{3}\right]_{-2}^{2}$$

$$\Rightarrow\quad 2I = \frac{1}{3}[2^3 - (-2)^3] = \frac{1}{3}[8+8]$$

$$\Rightarrow\quad I = \frac{1}{6}\times 16 = \frac{8}{3}$$

30. Let $I = \int_{\pi/4}^{\pi/2}e^{2x}\left(\frac{1-\sin 2x}{1-\cos 2x}\right)dx$

$$= \int_{\pi/4}^{\pi/2}e^{2x}\frac{(1-2\sin x\cos x)}{2\sin^2 x}dx$$

$$= \int_{\pi/4}^{\pi/2}e^{2x}\left(\frac{1}{2}\text{cosec}^2x - \cot x\right)dx$$

$$= -\int_{\pi/4}^{\pi/2}e^{2x}\left(\cot x - \frac{1}{2}\text{cosec}^2x\right)dx$$

Put $2x = t \Rightarrow 2dx = dt$

U.L. $= \pi$ and L.L $= \pi/2$

$$\therefore\quad I = -\frac{1}{2}\int_{\pi/2}^{\pi}e^t\left(\cot\frac{t}{2} - \frac{1}{2}\text{cosec}^2\frac{t}{2}\right)dt$$

TR!CK

$\int e^t\{f(t) + f'(t)\}dt = e^t f(t)$

Let $f(t) = \cot\left(\frac{t}{2}\right) \Rightarrow f'(t) = -\frac{1}{2}\text{cosec}^2\frac{t}{2}$

$$\therefore\quad I = -\frac{1}{2}\left[e^t\cot\frac{t}{2}\right]_{\pi/2}^{\pi}$$

$$= -\frac{1}{2}\left[e^{\pi}\cot\frac{\pi}{2} - e^{\pi/2}\cot\frac{\pi}{4}\right]$$

$$= -\frac{1}{2}(e^{\pi}(0) - e^{\pi/2}(1))$$

$$= \frac{1}{2}e^{\pi/2}$$

31. Let $I=\int_0^{\pi/2}\log(\sin^3 x\cdot\cos^4 x)\,dx$

$=\int_0^{\pi/2}(\log\sin^3 x+\log\cos^4 x)\,dx$

$=3\int_0^{\pi/2}\log\sin x\,dx+4\int_0^{\pi/2}\log\cos x\,dx$

$=3\left\{\frac{-\pi}{2}\log 2\right\}+4\left\{\frac{-\pi}{2}\log 2\right\}$

[**Note:** Refer solution of L. A. Q - 18.

$\because \int_0^{\pi/2}\log\sin x\,dx=\int_0^{\pi/2}\log\cos x\,dx=\frac{-\pi}{2}\log 2$]

$=-\frac{7\pi}{2}\log 2$

32. Let $I=\int_{-\pi/4}^{\pi/4}\frac{\cos 2x}{1+\cos 2x}dx$

Let $f(x)=\frac{\cos 2x}{1+\cos 2x}$

$\Rightarrow f(-x)=\frac{\cos 2(-x)}{1+\cos 2(-x)}$

$=\frac{\cos 2x}{1+\cos 2x}=f(x)$

which is even function.

$\therefore I=2\int_0^{\pi/4}\frac{\cos 2x}{1+\cos 2x}dx$

$=2\int_0^{\pi/4}\frac{(\cos^2 x-\sin^2 x)}{2\cos^2 x}dx$

$=\int_0^{\pi/4}(1-\tan^2 x)\,dx$

$=\int_0^{\pi/4}(2-\sec^2 x)\,dx$

$=[2x-\tan x]_0^{\pi/4}$

$=\left[\frac{2\pi}{4}-\tan\frac{\pi}{4}-0\right]$

$=\frac{\pi}{2}-1$

33. Let $I=\int_0^4|x-1|\,dx$

The given integrand can be redefine as below:

$$|x-1|=\begin{cases}-(x-1);\ 0<x<1\\ (x-1);\ 1\le x<4\end{cases}$$

$$I=\int_0^1-(x-1)\,dx+\int_1^4(x-1)\,dx$$

$=-\left[\frac{x^2}{2}-x\right]_0^1+\left[\frac{x^2}{2}-x\right]_1^4$

$=-\left[\frac{1}{2}-1-0\right]+\left[\frac{16}{2}-4-\left(\frac{1}{2}-1\right)\right]$

$=-\left(-\frac{1}{2}\right)+\left(8-4+\frac{1}{2}\right)=\frac{1}{2}+\frac{9}{2}=\frac{10}{2}=5$

34. Let $f(x)=|x\sin\pi x|$ Redefine the given function in the limits.

$$\therefore\quad f(x)=\begin{cases}x\sin\pi x>0, & \text{for } 0<x<1\\ -x\sin\pi x<0, & \text{for } 1<x<3/2\end{cases}$$

So, $\int_0^{3/2}|x\sin\pi x|\,dx$

$=\int_0^1 x\sin\pi x\,dx+\int_1^{3/2}(-x\sin\pi x)\,dx$ ...(1)

TR!CK

$\int_a^c f(x)\,dx=\int_a^b f(x)\,dx+\int_b^c f(x)\,dx$

where, $a<b<c$

Let $I=\int x\sin\pi x\,dx$

TR!CK

Integration by parts

$\int \underset{\text{I}}{u}\cdot\underset{\text{II}}{v}\,dx=u\int v\,dx-\int\left\{\frac{d}{dx}u\int v\,dx\right\}dx,$

where u and v are the functions of x.

$=x\int\sin\pi x\,dx-\int\left\{\frac{d}{dx}x\int\sin\pi x\,dx\right\}dx$

$=x\left\{-\frac{\cos\pi x}{\pi}\right\}-\int 1\cdot\left\{-\frac{\cos\pi x}{\pi}\right\}dx$

$=-\frac{x\cos\pi x}{\pi}+\frac{1}{\pi}\int\cos\pi x\,dx$

$=-\frac{x}{\pi}\cos\pi x+\frac{1}{\pi}\cdot\frac{\sin\pi x}{\pi}=-\frac{x}{\pi}\cos\pi x+\frac{1}{\pi^2}\cdot\sin\pi x$

From eq. (1), we get

$\int_0^{3/2}|x\sin\pi x|\,dx=\left[-\frac{x}{\pi}\cos\pi x+\frac{1}{\pi^2}\sin\pi x\right]_0^1$

$-\left[-\frac{x}{\pi}\cos\pi x+\frac{1}{\pi^2}\sin\pi x\right]_1^{3/2}$

$=\left\{-\frac{1}{\pi}\cos\pi+\frac{1}{\pi^2}\sin\pi+\frac{0}{\pi}\cdot\cos 0-\frac{1}{\pi^2}\sin 0\right\}$

$-\left\{-\frac{3/2}{\pi}\cos\frac{3\pi}{2}+\frac{1}{\pi^2}\sin\frac{3\pi}{2}+\frac{1}{\pi}\cos\pi-\frac{1}{\pi^2}\sin\pi\right\}$

$=\left\{-\frac{1}{\pi}(-1)+\frac{1}{\pi^2}\times 0+\frac{0}{\pi}(1)-\frac{1}{\pi^2}\times 0\right\}$

$-\left\{\frac{-3/2}{\pi}\times 0+\frac{1}{\pi^2}(-1)+\frac{1}{\pi}(-1)-\frac{1}{\pi^2}\times 0\right\}$

$=\left\{\frac{1}{\pi}+0+0-0+0+\frac{1}{\pi^2}+\frac{1}{\pi}+0\right\}$

$=\frac{1}{\pi^2}+\frac{2}{\pi}=\frac{1}{\pi^2}(1+2\pi)=\frac{2\pi+1}{\pi^2}$

35. Let $I=\int_{\pi/6}^{\pi/3}\frac{dx}{1+\sqrt{\tan x}}=\int_{\pi/6}^{\pi/3}\frac{\sqrt{\cos x}}{\sqrt{\cos x}+\sqrt{\sin x}}dx$ ...(1)

Now, $I=\int_{\pi/6}^{\pi/3}\frac{\sqrt{\cos\left(\frac{\pi}{6}+\frac{\pi}{3}-x\right)}}{\sqrt{\cos\left(\frac{\pi}{6}+\frac{\pi}{3}-x\right)}+\sqrt{\sin\left(\frac{\pi}{6}+\frac{\pi}{3}-x\right)}}dx$

$\left[\because\int_a^b f(x)\,dx=\int_a^b f(a+b-x)\,dx\right]$

$=\int_{\pi/6}^{\pi/3}\frac{\sqrt{\cos\left(\frac{\pi}{2}-x\right)}}{\sqrt{\cos\left(\frac{\pi}{2}-x\right)}+\sqrt{\sin\left(\frac{\pi}{2}-x\right)}}dx$

$=\int_{\pi/6}^{\pi/3}\frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}dx$ ...(2)

Adding eqs. (1) and (2), we get

$$2I = \int_{\pi/6}^{\pi/3} \frac{\sqrt{\cos x}+\sqrt{\sin x}}{\sqrt{\cos x}+\sqrt{\sin x}}\, dx = \int_{\pi/6}^{\pi/3} 1\, dx = [x]_{\pi/6}^{\pi/3}$$

$$\Rightarrow \qquad 2I = \left(\frac{\pi}{3}-\frac{\pi}{6}\right) = \frac{\pi}{6}$$

Hence, $$I = \int_{\pi/6}^{\pi/3} \frac{dx}{1+\sqrt{\tan x}} = \frac{\pi}{12}$$

36. Let $$I = \int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\cos x}+\sqrt{\sin x}}\, dx \qquad \ldots(1)$$

Then $$I = \int_0^{\pi/2} \frac{\sqrt{\sin\left\{\frac{\pi}{2}-x\right\}}}{\sqrt{\sin\left\{\frac{\pi}{2}-x\right\}}+\sqrt{\cos\left\{\frac{\pi}{2}-x\right\}}}\, dx$$

TR!CK

> $\int_0^a f(x)\, dx = \int_0^a f(a-x)\, dx$

or $$I = \int_0^{\pi/2} \frac{\sqrt{\cos x}}{\sqrt{\cos x}+\sqrt{\sin x}}\, dx \qquad \ldots(2)$$

Adding eqs. (1) and (2), we get

$$2I = \int_0^{\pi/2} \frac{\sqrt{\sin x}+\sqrt{\cos x}}{\sqrt{\sin x}+\sqrt{\cos x}}\, dx$$

$$= \int_0^{\pi/2} 1\, dx = [x]_0^{\pi/2}$$

$$\Rightarrow \; 2I = \frac{\pi}{2} - 0 = \frac{\pi}{2} \;\Rightarrow\; I = \frac{\pi}{2\times 2} = \frac{\pi}{4} \qquad \ldots(3)$$

From eqs. (1), (2) and (3), we get

$$\int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}\, dx = \int_0^{\pi/2} \frac{\sqrt{\cos x}}{\sqrt{\sin x}+\sqrt{\cos x}}\, dx = \frac{\pi}{4}$$

Hence proved.

37. Let $$I = \int_0^{2\pi} \frac{dx}{1+e^{\sin x}} \qquad \ldots(1)$$

Now, $$I = \int_0^{2\pi} \frac{dx}{1+e^{\sin(2\pi - x)}}\, dx = \int_0^{2\pi} \frac{dx}{1+e^{-\sin x}}\, dx$$

$$\left[\because \int_0^a f(x)\, dx = \int_0^a f(a-x)\, dx\right]$$

$$= \int_0^{2\pi} \frac{e^{\sin x}}{e^{\sin x}+1}\, dx \qquad \ldots(2)$$

Adding eqs. (1) and (2), we get

$$2I = \int_0^{2\pi} \frac{1+e^{\sin x}}{e^{\sin x}+1}\, dx = \int_0^{2\pi} 1\, dx = [x]_0^{2\pi} = 2\pi - 0$$

$$\Rightarrow \qquad I = \frac{2\pi}{2} = \pi$$

38. Let $I = \int_0^{\pi/2} \sqrt{\sin x}\cos^5 x\, dx$

$$= \int_0^{\pi/2} \sqrt{\sin x}(1-\sin^2 x)^2 \cos x\, dx$$

Put $\sin x = t \Rightarrow \cos x\, dx = dt$

U.L. = 1 and L.L. = 0

$$\therefore \quad I = \int_0^1 \sqrt{t}(1-t^2)^2\, dt$$

$$= \int_0^1 \sqrt{t}(1+t^4-2t^2)\, dt$$

$$= \int_0^1 [t^{1/2}+t^{9/2}-2t^{5/2}]\, dt$$

$$= \left[\frac{t^{3/2}}{3/2}+\frac{t^{11/2}}{11/2}-2\frac{t^{7/2}}{7/2}\right]_0^1$$

$$= \left[\frac{2}{3}t^{3/2}+\frac{2}{11}t^{11/2}-\frac{4}{7}t^{7/2}\right]_0^1$$

$$= \frac{2}{3}(1)+\frac{2}{11}(1)-\frac{4}{7}(1)-0$$

$$= \frac{154+42-132}{231} = \frac{64}{231}$$

39. Let $I = \int_{1/3}^1 \frac{(x-x^3)^{1/3}}{x^4}\, dx$

$$= \int_{1/3}^1 \frac{x\left(\frac{1}{x^2}-1\right)^{1/3}}{x^4}\, dx$$

$$= \int_{1/3}^1 \frac{\left(\frac{1}{x^2}-1\right)^{1/3}}{x^3}\, dx$$

Put $\frac{1}{x^2}-1 = t \;\Rightarrow\; -\frac{2}{x^3}dx = dt$

U.L. = 0 and L.L. = 8

$$\therefore \; I = \int_8^0 -\frac{t^{1/3}}{2}\, dt = \int_0^8 \frac{t^{1/3}}{2}\, dt = \frac{1}{2}\left[\frac{t^{4/3}}{4/3}\right]_0^8$$

$$\left[\because \int_a^b f(x)\, dx = -\int_b^a f(x)\, dx\right]$$

$$= \frac{3}{8}(8^{4/3}-0) = \frac{3}{8}\times 16 = 6$$

40. Let $I = \int_1^3 \{|x-1|+|x-2|\}\, dx$

Let $f(x) = |x-1|+|x-2|$

$$= \begin{cases} (x-1)-(x-2), & 1 \le x < 2 \\ (x-1)+(x-2), & 2 \le x < 3 \end{cases}$$

$$= \begin{cases} 1 & 1 \le x < 2 \\ 2x-3, & 2 \le x < 3 \end{cases}$$

$$\therefore \; I = \int_1^2 1\, dx + \int_2^3 (2x-3)\, dx = [x]_1^2 + \left[\frac{2x^2}{2}-3x\right]_2^3$$

$$= [2-1]+[9-9-(4-6)]$$

$$= 1+2 = 3$$

Long Answer Type Questions

1. $1+x+x^2+x^3 = (1+x)+x^2(1+x) = (1+x)(1+x^2)$

$$\therefore \quad \frac{1}{1+x+x^2+x^3} = \frac{1}{(1+x)(1+x^2)}$$

Let $$\frac{1}{(1+x)(1+x^2)} = \frac{A}{1+x}+\frac{Bx+C}{(1+x^2)} \qquad \ldots(1)$$

$$\Rightarrow \qquad 1 = A(1+x^2)+(Bx+C)(1+x)$$

$$\Rightarrow \qquad 1 = (A+C)+(A+B)x^2+(B+C)x \qquad \ldots(2)$$

Comparing coefficients of x^2, x and constant terms on both sides, we get

$$A+C = 1 \qquad \ldots(3)$$
$$A+B = 0 \qquad \ldots(4)$$
$$B+C = 0 \qquad \ldots(5)$$

Adding eqs. (3), (4) and (5), we get

$$2(A+B+C) = 1 \quad \Rightarrow \quad A+B+C = 1/2 \qquad \ldots(6)$$

From eqs. (3) and (6), we get

$$1+B = 1/2 \quad \Rightarrow \quad B = -1/2$$

From eqs. (4) and (6), we get

$$0+C = 1/2 \quad \Rightarrow \quad C = 1/2$$

From eqs. (5) and (6), we get

$$A+0 = 1/2 \quad \Rightarrow \quad A = 1/2$$

Therefore, $\frac{1}{1+x+x^2+x^3}=\frac{1}{2(1+x)}+\frac{1-x}{2(1+x^2)}$

$\therefore \quad \int \frac{1}{1+x+x^2+x^3}dx=\frac{1}{2}\int\frac{1}{1+x}dx+\frac{1}{2}\int\frac{1-x}{1+x^2}dx$

TR!CK

$\int \frac{dx}{a^2+x^2}=\frac{1}{a}tan^{-1}\left(\frac{x}{a}\right)+C$

$=\frac{1}{2}\log|1+x|+\frac{1}{2}\int\frac{1}{1+x^2}dx-\frac{1}{2}\times\frac{1}{2}\int\frac{2x}{1+x^2}dx$

$=\frac{1}{2}\log|1+x|+\frac{1}{2}\tan^{-1}x-\frac{1}{4}\log|1+x^2|+C$

2. $\int\frac{(x^3+x+1)}{(x^2-1)}dx=\int\frac{(x^3-x+2x+1)}{(x^2-1)}dx$

$=\int\frac{x(x^2-1)+(2x+1)}{(x^2-1)}dx$

$=\int\left\{x+\frac{2x+1}{x^2-1}\right\}dx$

$=\int x\,dx+\int\frac{2x+1}{x^2-1}dx$

$=\frac{x^2}{2}+\int\frac{2x}{x^2-1}dx+\int\frac{dx}{x^2-1}$

TiP

$\int\frac{dx}{x^2-a^2}=\frac{1}{2a}log\left|\frac{x-a}{x+a}\right|+C$

$=\frac{x^2}{2}+\log|x^2-1|+\frac{1}{2\times1}\log\left|\frac{x-1}{x+1}\right|+C$

$=\frac{x^2}{2}+\frac{1}{2}\log(x^2-1)^2+\frac{1}{2}\log\left|\frac{x-1}{x+1}\right|+C$

$=\frac{x^2}{2}+\frac{1}{2}\log\left|(x^2-1)^2\cdot\frac{(x-1)}{(x+1)}\right|+C$

$=\frac{x^2}{2}+\frac{1}{2}\log\left|(x-1)^2(x+1)^2\cdot\frac{(x-1)}{(x+1)}\right|+C$

$=\frac{x^2}{2}+\frac{1}{2}\log|(x+1)(x-1)^3|+C$

3.

TR!CK

$sin\,\theta=\frac{2tan\frac{\theta}{2}}{1+tan^2\frac{\theta}{2}}$

Let $I=\int\frac{dx}{4+5\sin x}=\int\frac{dx}{4+5\left(\frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}\right)}$

$=\int\frac{\left(1+\tan^2\frac{x}{2}\right)}{4+4\tan^2\frac{x}{2}+10\tan\frac{x}{2}}dx$

$=\int\frac{\sec^2\frac{x}{2}}{4\tan^2\frac{x}{2}+10\tan\frac{x}{2}+4}dx \quad \left[\because 1+\tan^2\frac{\theta}{2}=\sec^2\frac{\theta}{2}\right]$

Let $\quad t=\tan\frac{x}{2}\Rightarrow dt=\frac{1}{2}\sec^2\frac{x}{2}dx$

TR!CK

Product of extreme terms $2\times2=4$

$\because \quad 4=2\times2=4\times1$

$\therefore$ *Here we will take 4 and 1 as a factors of 4.*

So, middle term becomes, $5=4+1$.

$\therefore I=\int\frac{dt}{2t^2+5t+2}=\int\frac{dt}{2t^2+4t+t+2}=\int\frac{dt}{(2t+1)(t+2)}$

$=\int\left[\frac{2}{3(2t+1)}-\frac{1}{3(t+2)}\right]dt$ [By partial fraction]

$=\frac{2}{3}\int\frac{dt}{2t+1}-\frac{1}{3}\int\frac{dt}{t+2}$

$=\frac{2}{3}\times\frac{1}{2}\log|2t+1|-\frac{1}{3}\log|t+2|+\log C$

$=\frac{1}{3}\log\left|2\tan\frac{x}{2}+1\right|-\frac{1}{3}\log\left|\tan\frac{x}{2}+2\right|+\log C$

$=\frac{1}{3}\log\left|\frac{2\tan\frac{x}{2}+1}{\tan\frac{x}{2}+2}\right|+\log C$

4. Let $\quad I=\int\left\{\log(\log x)+\frac{1}{(\log x)^2}\right\}dx$

$=\int\log(\log x)\,dx+\int\frac{dx}{(\log x)^2}$

TR!CK

Integration by parts,

$\int \underset{\text{I}}{u}\cdot\underset{\text{II}}{v}\,dx=u\int v\,dx-\int\left\{\frac{d}{dx}u\int v\,dx\right\}dx$

where u and v are the functions of x.

$=\log(\log x)\int 1dx-\int\left\{\frac{d}{dx}\log(\log x)\int 1dx\right\}dx$

$+\int\frac{dx}{(\log x)^2}$

[Using integration by parts]

$=x\log(\log x)-\int\left\{\frac{1}{\log x}\cdot\frac{1}{x}\cdot x\right\}dx+\int\frac{dx}{(\log x)^2}$

$=x\log(\log x)-\int\frac{dx}{(\log x)}+\int\frac{dx}{(\log x)^2}$

$=x\log(\log x)-\frac{1}{\log x}\int 1dx+\int\left\{\frac{d}{dx}\frac{1}{(\log x)}\int 1dx\right\}$

$+\int\frac{dx}{(\log x)^2}$

[Again using integration by parts]

$=x\log(\log x)-\frac{x}{\log x}-\int\frac{x}{(\log x)^2}\cdot\frac{1}{x}dx+\int\frac{dx}{(\log x)^2}$

$=x\log(\log x)-\frac{x}{\log x}-\int\frac{dx}{(\log x)^2}+\int\frac{dx}{(\log x)^2}$

$=x\log(\log x)-\frac{x}{\log x}+C$

5. Let $I=\int_0^1 x\sqrt{\frac{1-x^2}{1+x^2}}\,dx$

Put $x^2=t$, then $2x\,dx=dt \Rightarrow x\,dx=\frac{1}{2}dt$

When $x=0$ then $t=0^2=0$

When $x=1$ then $t=1^2=1$

∴ $I=\frac{1}{2}\int_0^1\sqrt{\frac{1-t}{1+t}}\,dt$

$=\frac{1}{2}\int_0^1\sqrt{\frac{1-t}{1+t}\times\frac{1-t}{1-t}}\,dt=\frac{1}{2}\int_0^1\frac{1-t}{\sqrt{1-t^2}}\,dt$

TR!CK

$\int\frac{dx}{\sqrt{a^2-x^2}}=\sin^{-1}\left(\frac{x}{a}\right)+C$

$=\frac{1}{2}\int_0^1\frac{1}{\sqrt{1-t^2}}\,dt-\frac{1}{2}\int_0^1\frac{t}{\sqrt{1-t^2}}\,dt$

$=\frac{1}{2}[\sin^{-1}t]_0^1+\frac{1}{2}\int_0^1\frac{-t}{\sqrt{1-t^2}}\,dt$

Put $1-t^2=u^2 \Rightarrow -t\,dt=u\,du$

∴ $I=\frac{1}{2}(\sin^{-1}1-\sin^{-1}0)+\frac{1}{2}\int_1^0\frac{u}{\sqrt{u^2}}\,du$

$=\frac{1}{2}\left[\frac{\pi}{2}-0\right]+\frac{1}{2}\int_1^0\frac{u}{u}\,du$

$=\frac{\pi}{4}+\frac{1}{2}\int_1^0 1\,du=\frac{\pi}{4}+\frac{1}{2}[u]_1^0$

$=\frac{\pi}{4}+\frac{1}{2}(0-1)=\frac{\pi}{4}-\frac{1}{2}=\frac{\pi-2}{4}$

6. Let $I=\int_0^{\pi/2}\frac{dx}{1+2\cos x}$

$=\int_0^{\pi/2}\frac{dx}{1+2\left(\frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}\right)}$

TR!CK

$\cos x=\frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}},\quad 1+\tan^2\frac{\theta}{2}=\sec^2\frac{\theta}{2}$

$=\int_0^{\pi/2}\frac{\left(1+\tan^2\frac{x}{2}\right)}{1+\tan^2\frac{x}{2}+2-2\tan^2\frac{x}{2}}\,dx$

$=\int_0^{\pi/2}\frac{\sec^2\frac{x}{2}}{3-\tan^2\frac{x}{2}}\,dx$

Put $t=\tan\frac{x}{2} \Rightarrow \frac{dt}{dx}=\frac{1}{2}\sec^2\frac{x}{2}$

∴ $I=\int_0^{\pi/2}\frac{2dt}{3-t^2}=2\int_0^{\pi/2}\frac{dt}{(\sqrt{3})^2-t^2}$

$=2\times\frac{1}{2\sqrt{3}}\left[\log\left|\frac{\sqrt{3}+t}{\sqrt{3}-t}\right|\right]_0^{\pi/2}$

$=\frac{1}{\sqrt{3}}\left[\log\left|\frac{\sqrt{3}+\tan\frac{x}{2}}{\sqrt{3}-\tan\frac{x}{2}}\right|\right]_0^{\pi/2}$

$=\frac{1}{\sqrt{3}}\left[\log\left|\frac{\sqrt{3}+\tan\frac{\pi}{4}}{\sqrt{3}-\tan\frac{\pi}{4}}\right|-\log\left|\frac{\sqrt{3}+\tan 0}{\sqrt{3}-\tan 0}\right|\right]$

$=\frac{1}{\sqrt{3}}\left[\log\left|\frac{\sqrt{3}+1}{\sqrt{3}-1}\right|-\log\left|\frac{\sqrt{3}+0}{\sqrt{3}-0}\right|\right]$

$=\frac{1}{\sqrt{3}}\left[\log\left(\frac{\sqrt{3}+1}{\sqrt{3}-1}\times\frac{\sqrt{3}+1}{\sqrt{3}+1}\right)-\log 1\right]$

$=\frac{1}{\sqrt{3}}\left[\log\left(\frac{(\sqrt{3}+1)^2}{3-1}\right)-0\right]$

$=\frac{1}{\sqrt{3}}\log\left[\frac{3+1+2\sqrt{3}}{2}\right]=\frac{1}{\sqrt{3}}\log\left(\frac{4+2\sqrt{3}}{2}\right)$

$=\frac{1}{\sqrt{3}}\log(2+\sqrt{3})$ **Hence proved.**

7. Let $I=\int_0^{\pi/4}\frac{\sin x+\cos x}{16+9\sin 2x}\,dx$

TR!CK

$(-\sin x+\cos x)^2=\sin^2x+\cos^2x-2\sin x\cdot\cos x$
$=1-\sin 2x$

∴ $\sin 2x=1-(-\sin x+\cos x)^2$

∴ $I=\int_0^{\pi/4}\frac{\sin x+\cos x}{16+9\{1-(-\sin x+\cos x)^2\}}\,dx$

$=\int_0^{\pi/4}\frac{\sin x+\cos x}{25-9(\sin x-\cos x)^2}\,dx$

Put $t=-\sin x+\cos x \Rightarrow dt=-(\cos x+\sin x)\,dx$

∴ U.L. = 0 and L.L. = +1

∴ $I=\int_1^0\frac{-dt}{25-9t^2}=-\int_1^0\frac{dt}{25-9t^2}$

$=\frac{1}{9}\int_0^1\frac{dt}{\left(\frac{5}{3}\right)^2-t^2}$

$\left[\because\int_a^b f(x)\,dx=-\int_b^a f(x)\,dx\right]$

TR!CK

$\int\frac{dx}{a^2-x^2}=\frac{1}{2a}\log\left|\frac{a+x}{a-x}\right|+C$

$=\frac{1}{9}\times\frac{1}{2\times\frac{5}{3}}\left[\log\left|\frac{\frac{5}{3}+t}{\frac{5}{3}-t}\right|\right]_0^1$

$=\frac{1}{30}\left\{\log\left(\frac{\frac{5}{3}+1}{\frac{5}{3}-1}\right)-\log\left(\frac{\frac{5}{3}+0}{\frac{5}{3}-0}\right)\right\}$

$=\frac{1}{30}\left\{\log\left(\frac{8}{2}\right)-\log(1)\right\}=\frac{1}{30}\cdot\log 4-0$

$=\frac{1}{30}\times 2\log 2=\frac{1}{15}\log 2$

8. Let $I=\int(\sqrt{\tan x}+\sqrt{\cot x})\,dx=\int\left(\sqrt{\frac{\sin x}{\cos x}}+\sqrt{\frac{\cos x}{\sin x}}\right)dx$

$$=\int\frac{\sin x+\cos x}{\sqrt{\cos x\sin x}}dx=\sqrt{2}\int\frac{\sin x+\cos x}{\sqrt{2\sin x\cos x}}dx$$

$$=\sqrt{2}\int\frac{\sin x+\cos x}{\sqrt{\sin 2x}}dx$$

TiP

Learn to substitute and simplify trigonometric equation.

$$=\sqrt{2}\int\frac{\sin x+\cos x}{\sqrt{1-(\sin x-\cos x)^2}}dx$$

TR!CK

$(\sin x-\cos x)^2=\sin^2 x+\cos^2 x-2\sin x\cdot\cos x$

$\Rightarrow\ (\sin x-\cos x)^2=1-\sin 2x$

$\Rightarrow\ \sin 2x=1-(\sin x-\cos x)^2$

Put $\sin x-\cos x=t\Rightarrow(\cos x+\sin x)\,dx=dt$

$$\therefore\quad I=\sqrt{2}\int\frac{dt}{\sqrt{1-t^2}}\quad\left[\because\int\frac{dx}{\sqrt{a^2-x^2}}=\sin^{-1}\left(\frac{x}{a}\right)+C\right]$$

$$=\sqrt{2}\sin^{-1}t+C$$

$$=\sqrt{2}\sin^{-1}(\sin x-\cos x)+C$$

$$\therefore\ \int_0^{\pi/4}(\sqrt{\tan x}+\sqrt{\cot x})\,dx$$

$$=[\sqrt{2}\sin^{-1}(\sin x-\cos x)]_0^{\pi/4}$$

$$=\sqrt{2}\left[\sin^{-1}\left\{\sin\left(\frac{\pi}{4}\right)-\cos\left(\frac{\pi}{4}\right)\right\}-\sin^{-1}\{\sin(0)-\cos(0)\}\right]$$

$$=\sqrt{2}\left[\sin^{-1}\left(\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}\right)-\sin^{-1}(0-1)\right]$$

$$=\sqrt{2}\,(\sin^{-1}0-\sin^{-1}(-1))$$

$$=\sqrt{2}\,(0+\sin^{-1}1)=\sqrt{2}\times\frac{\pi}{2}=\frac{\pi}{\sqrt{2}}$$

COMMON ERR(!)R

Sometimes students make errors in substitution and simplification which leads errors in further simplification.

9. Let $I=\int_{\pi/4}^{\pi/2}\underset{\text{II}}{\cos 2x}\cdot\underset{\text{I}}{\log\sin x}\,dx$

$$\Rightarrow\ I=[\log\sin x\int\cos 2x\,dx]_{\pi/4}^{\pi/2}-\int_{\pi/4}^{\pi/2}\left\{\frac{d}{dx}\log\sin x\int\cos 2x\,dx\right\}dx$$

[Using integral by parts]

$$=\left[\log\sin x\cdot\frac{1}{2}\sin 2x\right]_{\pi/4}^{\pi/2}-\int_{\pi/4}^{\pi/2}\frac{1}{\sin x}\cdot\cos x\times\frac{1}{2}\sin 2x\,dx$$

TR!CK

$\sin 2\theta=2\sin\theta\cdot\cos\theta,\ \cos 2\theta=2\cos^2\theta-1$

$$=\frac{1}{2}[\sin 2x\cdot\log\sin x]_{\pi/4}^{\pi/2}-\int_{\pi/4}^{\pi/2}\frac{1}{\sin x}\times\cos x\times\frac{1}{2}\times 2\sin x\cdot\cos x\,dx$$

$$=\frac{1}{2}[\sin 2x\cdot\log\sin x]_{\pi/4}^{\pi/2}-\int_{\pi/4}^{\pi/2}\frac{1}{2}\cdot(2\cos^2 x)\,dx$$

$$=\frac{1}{2}[\sin 2x\cdot\log\sin x]_{\pi/4}^{\pi/2}-\frac{1}{2}\int_{\pi/4}^{\pi/2}(1+\cos 2x)\,dx$$

$$=\frac{1}{2}[\sin 2x\cdot\log\sin x]_{\pi/4}^{\pi/2}-\frac{1}{2}\left[x+\frac{1}{2}\sin 2x\right]_{\pi/4}^{\pi/2}$$

$$=\frac{1}{2}\left[\sin\pi\cdot\log\sin\frac{\pi}{2}-\sin\frac{\pi}{2}\cdot\log\sin\frac{\pi}{4}\right]-\frac{1}{2}\left[\frac{\pi}{2}+\frac{1}{2}\sin\pi-\frac{\pi}{4}-\frac{1}{2}\sin\frac{\pi}{2}\right]$$

$$=\frac{1}{2}\left[0\cdot\log(1)-1\cdot\log\left(\frac{1}{\sqrt{2}}\right)\right]-\frac{1}{2}\left[\frac{\pi}{2}+\frac{1}{2}\cdot 0-\frac{\pi}{4}-\frac{1}{2}\cdot 1\right]$$

$$=\frac{1}{2}(0-\log 2^{-1/2})-\frac{1}{2}\left(\frac{\pi}{2}+0-\frac{\pi}{4}-\frac{1}{2}\right)$$

$$=\frac{1}{2}\left(\frac{1}{2}\log 2\right)-\frac{1}{2}\left(\frac{\pi}{4}-\frac{1}{2}\right)$$

$$=\frac{1}{4}\log 2-\frac{\pi}{8}+\frac{1}{4}=\frac{1}{4}(1+\log 2)-\frac{\pi}{8}$$

10. Here, integrand $f(x)=|x^3-x|=|x(x^2-1)|$

$$=|x(x-1)(x+1)|,\ -1<x<2$$

The given integrand can be redefined as below:

$$f(x)=\begin{cases}x(x-1)(x+1), & -1<x<0\\ -x(x-1)(x+1), & 0<x<1\\ x(x-1)(x+1), & 1<x<2\end{cases}$$

$$\therefore\ \int_{-1}^{2}|x^3-x|\,dx=\int_{-1}^{0}f(x)dx+\int_0^1 f(x)dx+\int_1^2 f(x)\,dx$$

$$=\int_{-1}^{0}(x^3-x)dx+\int_0^1(x-x^3)dx+\int_1^2(x^3-x)dx$$

$$=\left[\frac{x^4}{4}-\frac{x^2}{2}\right]_{-1}^{0}+\left[\frac{x^2}{2}-\frac{x^4}{4}\right]_0^1+\left[\frac{x^4}{4}-\frac{x^2}{2}\right]_1^2$$

$$=0-\left\{\frac{1}{4}-\frac{1}{2}\right\}+\left\{\frac{1}{2}-\frac{1}{4}\right\}-0+\left\{\frac{16}{4}-\frac{4}{2}\right\}-\left\{\frac{1}{4}-\frac{1}{2}\right\}$$

$$=-\left(-\frac{1}{4}\right)+\left(\frac{1}{4}\right)+(4-2)-\left(\frac{-1}{4}\right)=\frac{1}{4}+\frac{1}{4}+\frac{1}{4}+2=\frac{11}{4}$$

11. Here, integrand $f(x)=|x^3-3x^2+2x|,\ -1<x<2$

$$=|x(x^2-3x+2)|$$

$$=|x(x-1)(x-2)|$$

The given integrand can be redefined as below:

$$f(x)=\begin{cases}-x(x-1)(x-2), & -1<x<0\\ x(x-1)(x-2), & 0<x<1\\ -x(x-1)(x-2), & 1<x<2\end{cases}$$

$$\therefore\ \int_{-1}^{2}|x^3-3x^2+2x|\,dx$$

$$=\int_{-1}^{0}-x(x-1)(x-2)\,dx+\int_0^1 x(x-1)(x-2)\,dx+\int_1^2-x(x-1)(x-2)\,dx$$

$$=\int_{-1}^{0}(-x^3+3x^2-2x)\,dx+\int_0^1(x^3-3x^2+2x)\,dx+\int_1^2(-x^3+3x^2-2x)\,dx$$

$$= \left[\frac{-x^4}{4} + 3\cdot\frac{x^3}{3} - 2\cdot\frac{x^2}{2} \right]_{-1}^{0} + \left[\frac{x^4}{4} - 3\cdot\frac{x^3}{3} + 2\cdot\frac{x^2}{2} \right]_{0}^{1}$$

$$+ \left[\frac{-x^4}{4} + 3\cdot\frac{x^3}{3} - 2\cdot\frac{x^2}{2} \right]_{1}^{2}$$

$$= 0 - \left[-\frac{(-1)^4}{4} + (-1)^3 - (-1)^2 \right] + \left[\frac{1}{4} - 1 + 1 - 0 \right]$$

$$+ \left[-\frac{1}{4}\cdot(2)^4 + (2)^3 - (2)^2 \right] - \left[\frac{-1}{4} + 1 - 1 \right]$$

$$= -\left(-\frac{1}{4} - 1 - 1 \right) + \frac{1}{4} + (-4 + 8 - 4) - \left(-\frac{1}{4} \right)$$

$$= \frac{9}{4} + \frac{1}{4} + \frac{1}{4} = \frac{11}{4}$$

12.

TR!CK

$$|x| = \begin{cases} x; & x > 0 \\ -x; & x < 0 \end{cases}$$

$$\therefore \int_{-\pi/2}^{\pi/2} (\sin|x| + \cos|x|)\, dx = \int_{-\pi/2}^{0} [\sin(-x) + \cos(-x)]\, dx + \int_{0}^{\pi/2} [\sin(x) + \cos(x)]\, dx$$

$$= \int_{-\pi/2}^{0} (-\sin x + \cos x)\, dx + \int_{0}^{\pi/2} (\sin x + \cos x)\, dx$$

$$= [\cos x + \sin x]_{-\pi/2}^{0} + [-\cos x + \sin x]_{0}^{\pi/2}$$

$$= [\cos 0 + \sin 0] - \left[\cos\left(\frac{-\pi}{2}\right) + \sin\left(-\frac{\pi}{2}\right) \right] + \left[-\cos\frac{\pi}{2} + \sin\frac{\pi}{2} \right] - [-\cos 0 + \sin 0]$$

$$= (1 + 0) - (-0 - 1) + (-0 + 1) - (-1 + 0)$$

$$= 1 + 1 + 1 + 1 = 4$$

13. Let $x = a + b - t$, then $dx = -dt$

When $x = a$ then $a + b - t = a \Rightarrow t = b$

When $x = b$ then $a + b - t = b \Rightarrow t = a$

$$\therefore \quad \text{LHS} = \int_a^b f(x)\, dx = \int_b^a f(a + b - t)(-dt)$$

$$= -\int_b^a f(a + b - t)\, dt$$

$$= \int_a^b f(a + b - t)\, dt$$

$$= \int_a^b f(a + b - x)\, dx = \text{RHS}$$ **Hence proved.**

Now, let $I = \int_{\pi/8}^{3\pi/8} \frac{\tan^2 x}{\tan^2 x + \cot^2 x} dx$ ...(1)

$$I = \int_{\pi/8}^{3\pi/8} \frac{\tan^2\left(\frac{3\pi}{8} + \frac{\pi}{8} - x\right) dx}{\tan^2\left(\frac{3\pi}{8} + \frac{\pi}{8} - x\right) + \cot^2\left(\frac{3\pi}{8} + \frac{\pi}{8} - x\right)}$$

TR!CK

$$\int_a^b f(a + b - x)\, dx = \int_a^b f(x)\, dx$$

$$I = \int_{\pi/8}^{3\pi/8} \frac{\tan^2\left(\frac{\pi}{2} - x\right) dx}{\tan^2\left(\frac{\pi}{2} - x\right) + \cot^2\left(\frac{\pi}{2} - x\right)}$$

$$I = \int_{\pi/8}^{3\pi/8} \frac{\cot^2 x}{\cot^2 x + \tan^2 x} dx \qquad ...(2)$$

Adding eqs. (1) and (2),

$$2I = \int_{\pi/8}^{3\pi/8} \frac{\tan^2 x + \cot^2 x}{\tan^2 x + \cot^2 x} dx = \int_{\pi/8}^{3\pi/8} 1\, dx$$

$$\Rightarrow \quad I = \frac{1}{2}[x]_{\pi/8}^{3\pi/8} = \frac{1}{2}\left(\frac{3\pi}{8} - \frac{\pi}{8} \right) = \frac{1}{2} \times \frac{\pi}{4} = \frac{\pi}{8}$$

Hence $\int_{\pi/8}^{3\pi/8} \frac{\tan^2 x\, dx}{\tan^2 x + \cot^2 x} = \frac{\pi}{8}$

14. Let $I = \int_0^{\pi} \frac{x}{1 + \sin x} dx$ ...(1)

Now, $I = \int_0^{\pi} \frac{(\pi - x)}{1 + \sin(\pi - x)} dx$

$$\left[\because \int_0^a f(x)\, dx = \int_0^a f(a - x)\, dx \right]$$

$$\Rightarrow \quad I = \int_0^{\pi} \frac{(\pi - x)}{1 + \sin x} dx \qquad ...(2)$$

Adding eqs. (1) and (2), we get

$$2I = \int_0^{\pi} \frac{\pi}{1 + \sin x} dx$$

$$\Rightarrow \quad I = \frac{\pi}{2} \int_0^{\pi} \frac{1}{1 + \left(\frac{2\tan x/2}{1 + \tan^2 x/2} \right)} dx$$

$$\Rightarrow \quad I = \frac{\pi}{2} \int_0^{\pi} \frac{1 + \tan^2 x/2}{1 + \tan^2 \frac{x}{2} + 2\tan\frac{x}{2}} dx$$

$$\Rightarrow \quad I = \frac{\pi}{2} \int_0^{\pi} \frac{\sec^2 x/2}{\left(1 + \tan\frac{x}{2}\right)^2} dx$$

Let $t = 1 + \tan\frac{x}{2}$

$$\Rightarrow \quad dt = \frac{1}{2}\sec^2\frac{x}{2} dx$$

U.L. $= \infty$ and L.L $= 1$

$$I = \frac{\pi}{2} \int_1^{\infty} \frac{2dt}{t^2} = \frac{\pi}{2} \times 2\left[\frac{-1}{t} \right]_1^{\infty}$$

$$= -\pi[0 - 1] = \pi$$

15. Let $I = \int_0^{\pi/2} \frac{x}{\sin x + \cos x} dx$ ...(1)

Now, $I = \int_0^{\pi/2} \frac{(\pi/2 - x)}{\sin(\pi/2 - x) + \cos(\pi/2 - x)} dx$

$$= \int_0^{\pi/2} \frac{\left(\frac{\pi}{2} - x\right)}{\cos x + \sin x} dx \qquad ...(2)$$

TR!CK

$$\int_0^a f(x)\, dx = \int_0^a f(a - x)\, dx$$

Adding eqs. (1) and (2), we get

$$2I = \int_0^{\pi/2} \frac{x + \left(\frac{\pi}{2} - x\right) dx}{\sin x + \cos x} = \frac{\pi}{2} \int_0^{\pi/2} \frac{dx}{\sin x + \cos x}$$

$$= \frac{\pi}{2} \int_0^{\pi/2} \frac{dx}{\sqrt{2}\left(\sin x \cdot \frac{1}{\sqrt{2}} + \cos x \cdot \frac{1}{\sqrt{2}} \right)}$$

$$= \frac{\pi}{2\sqrt{2}} \int_0^{\pi/2} \frac{dx}{\left\{\cos x \cdot \cos\frac{\pi}{4} + \sin x \cdot \sin\frac{\pi}{4}\right\}}$$

TR!CK

cos (A – B) = cos A · cos B + sin A · sin B

$$= \frac{\pi}{2\sqrt{2}} \int_0^{\pi/2} \frac{dx}{\cos\left(x - \frac{\pi}{4}\right)} = \frac{\pi}{2\sqrt{2}} \int_0^{\pi/2} \sec\left(x - \frac{\pi}{4}\right) dx$$

TR!CK

$\int \sec x \, dx = \log \left| \tan\left(\frac{x}{2} + \frac{\pi}{4}\right) \right| + C$

$$= \frac{\pi}{2\sqrt{2}} \left[\log \left| \tan\left\{ \frac{1}{2}\left(x - \frac{\pi}{4}\right) + \frac{\pi}{4} \right\} \right| \right]_0^{\pi/2}$$

$$= \frac{\pi}{2\sqrt{2}} \left[\log \left| \tan\left(\frac{x}{2} + \frac{\pi}{8}\right) \right| \right]_0^{\pi/2}$$

$$= \frac{\pi}{2\sqrt{2}} \left[\log \left| \tan\left(\frac{\pi}{4} + \frac{\pi}{8}\right) \right| - \log \left| \tan\left(0 + \frac{\pi}{8}\right) \right| \right]$$

$$= \frac{\pi}{2\sqrt{2}} \left[\log \left| \frac{\tan\frac{3\pi}{8}}{\tan\frac{\pi}{8}} \right| \right]$$

$$\left[\because \tan\frac{\pi}{8} = \sqrt{2} - 1 \text{ and } \tan\frac{3\pi}{8} = \sqrt{2} + 1\right]$$

$$\therefore I = \frac{\pi}{4\sqrt{2}} . \log \left| \frac{\sqrt{2}+1}{\sqrt{2}-1} \right| = \frac{\pi}{4\sqrt{2}} \log \left| \frac{\sqrt{2}+1}{\sqrt{2}-1} \times \frac{\sqrt{2}+1}{\sqrt{2}+1} \right|$$

$$= \frac{\pi}{4\sqrt{2}} \log \left| \frac{(\sqrt{2}+1)^2}{2-1} \right| = \frac{\pi}{4\sqrt{2}} \log \left| \frac{2+1+2\sqrt{2}}{1} \right|$$

$$= \frac{\pi}{4\sqrt{2}} \log (3 + 2\sqrt{2})$$

16. Let $I = \int_0^{\pi/2} \frac{\sin^2 x}{\sin x + \cos x} dx$ ...(1)

$$\Rightarrow I = \int_0^{\pi/2} \frac{\sin^2\left(\frac{\pi}{2} - x\right)}{\sin\left(\frac{\pi}{2} - x\right) + \cos\left(\frac{\pi}{2} - x\right)} dx$$

$$\left[\because \int_0^a f(x)\, dx = \int_0^a f(a-x)\, dx\right]$$

$$= \int_0^{\pi/2} \frac{\cos^2 x}{\cos x + \sin x} dx \quad ...(2)$$

Adding eqs. (1) and (2), we get

$$2I = \int_0^{\pi/2} \frac{\sin^2 x + \cos^2 x}{\sin x + \cos x} dx = \int_0^{\pi/2} \frac{dx}{\sin x + \cos x}$$

$$[\because \sin^2\theta + \cos^2\theta = 1]$$

Note: Refer solution of L.A. Q -15.

$$\because \int_0^{\pi/2} \frac{dx}{\sin x + \cos x} = \frac{1}{\sqrt{2}} \log (3 + 2\sqrt{2})$$

$$= \frac{1}{\sqrt{2}} \log (\sqrt{2}^2 + 1^2 + 2\sqrt{2}) = \frac{1}{\sqrt{2}} \log (\sqrt{2} + 1)^2$$

$$= \frac{2}{\sqrt{2}} \log (\sqrt{2} + 1)$$

From eq. (1),

$$2I = \frac{2}{\sqrt{2}} \log (\sqrt{2} + 1)$$

$$\Rightarrow I = \frac{1}{\sqrt{2}} \log (\sqrt{2} + 1)$$ **Hence proved.**

17. Let $x = \tan t$, then $dx = (\sec^2 t)\, dt$

When $x = 0$ then $t = 0$, when $x = 1$ then $t = \frac{\pi}{4}$

$$\text{Let } I = \int_0^1 \frac{\log(1+x)}{(1+x^2)} dx = \int_0^{\pi/4} \frac{\log(1+\tan t)}{1+\tan^2 t} \cdot (\sec^2 t)\, dt$$

$$= \int_0^{\pi/4} \frac{\log(1+\tan t)}{\sec^2 t} \cdot (\sec^2 t)\, dt$$

TR!CK

$1 + \tan^2\theta = \sec^2\theta, \tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \cdot \tan B}$

$$= \int_0^{\pi/4} [\log(1 + \tan t)]\, dt$$

$$= \int_0^{\pi/4} [\log(1 + \tan x)]\, dx \left[\because \int_a^b f(t)\, dt = \int_a^b f(x)\, dx\right]$$

$$= \int_0^{\pi/4} \log\left(\frac{\sin x + \cos x}{\cos x}\right) dx$$

$$\text{or } I = \int_0^{\pi/4} [\log(1 + \tan x)]\, dx$$

$$= \int_0^{\pi/4} \log\left[1 + \tan\left\{\frac{\pi}{4} - x\right\}\right] dx$$

$$\left[\because \int_0^a f(x)\, dx = \int_0^a f(a-x)\, dx\right]$$

$$= \int_0^{\pi/4} \log\left[1 + \frac{1 - \tan x}{1 + \tan x}\right] dx$$

$$= \int_0^{\pi/4} \log\left[\frac{1 + \tan x + 1 - \tan x}{1 + \tan x}\right] dx$$

$$= \int_0^{\pi/4} \log\left(\frac{2}{1 + \tan x}\right) dx$$

$$= \int_0^{\pi/4} [\log 2 - \log(1 + \tan x)]\, dx$$

$$\left[\because \log\left(\frac{m}{n}\right) = \log m - \log n\right]$$

$$= \log 2 \int_0^{\pi/4} dx - \int_0^{\pi/4} \log(1 + \tan x)\, dx$$

$$\Rightarrow I = \log 2\, [x]_0^{\pi/4} - I$$

$$\Rightarrow 2I = \log 2 \left[\frac{\pi}{4} - 0\right] \Rightarrow I = \frac{\pi}{8} \log 2$$ **Hence proved.**

18. Let $I = \int_0^{\pi/2} \log \sin x\, dx$ ...(1)

$$= \int_0^{\pi/2} \log \sin\left(\frac{\pi}{2} - x\right) dx$$

$$\Rightarrow I = \int_0^{\pi/2} \log \cos x\, dx \quad ...(2)$$

TR!CK

Here integrand f(x) = log sin x

Now f (π – x) = log sin (π – x) = log sin x = f (x)

∴ f(x) is an even function.

Adding eqs. (1) and (2), we get

$$2I = \int_0^{\pi/2} (\log \sin x + \log \cos x)\, dx$$

$$= \int_0^{\pi/2} \log(\sin x \cos x)\, dx$$

$$= \int_0^{\pi/2} \log\left(\frac{\sin 2x}{2}\right) dx \quad [\because \sin 2\theta = 2\sin\theta\cos\theta]$$

$$= \int_0^{\pi/2} (\log \sin 2x - \log 2)\, dx$$

$$= \int_0^{\pi/2} \log \sin 2x\, dx - \int_0^{\pi/2} \log 2\, dx$$

In first integral, let $2x = t$ then $dx = \frac{1}{2}dt$

When $x = 0$, then $t = 0$ and when $x = \frac{\pi}{2}$, then $t = \pi$

$\therefore \quad 2I = \frac{1}{2}\int_0^{\pi}\log \sin t\, dt - \log 2\int_0^{\pi/2} 1 dx$

$= \frac{1}{2}\int_0^{\pi}\log \sin x\, dx - \frac{\pi}{2}\cdot \log 2 \quad \left[\because \int_a^b f(t)\,dt = \int_a^b f(x)\,dx\right]$

$= \frac{1}{2}\times 2\int_0^{\pi/2}\log \sin x\, dx - \frac{\pi}{2}\cdot \log 2$

$\left[\because \int_0^{2a} f(x)\,dx = \begin{cases} 2\int_0^a f(x)\,dx, & \text{if } f(2a-x) = f(x) \\ 0, & \text{if } f(2a-x) = f(x)\end{cases}\right]$

$= \int_0^{\pi/2}\log \sin x\, dx - \frac{\pi}{2}\log 2$

$\therefore \quad 2I = I - \frac{\pi}{2}\log 2 \;\Rightarrow\; I = -\frac{\pi}{2}(\log 2)$

Therefore, $\int_0^{\pi/2}\log \sin x\, dx = \int_0^{\pi/2}\log \cos x\, dx$

$= -\frac{\pi}{2}(\log 2)$ **Hence proved.**

19. Let $I = \int_0^{\pi/2}\sin 2x \tan^{-1}(\sin x)\,dx$

$= \int_0^{\pi/2} 2\sin x \cos x \tan^{-1}(\sin x)\,dx$

Put $\sin x = t \;\Rightarrow\; \cos x\,dx = dt$

When $x = 0$, then $t = \sin 0 = 0$

When $x = \frac{\pi}{2}$, then $t = \sin\frac{\pi}{2} = 1$

$\therefore \quad I = 2\int_0^1 \underset{\text{II}}{t}\cdot \underset{\text{I}}{\tan^{-1} t}\,dt$

$= 2\left[\tan^{-1} t\int t\,dt - \int\left\{\frac{d}{dt}(\tan^{-1} t)\int t\,dt\right\}dt\right]_0^1$

$= 2\left[(\tan^{-1} t)\frac{t^2}{2} - \int \frac{1}{1+t^2}\times\frac{t^2}{2}dt\right]_0^1$

$= 2\left[\frac{t^2}{2}\tan^{-1} t - \frac{1}{2}\int\frac{t^2+1-1}{1+t^2}dt\right]_0^1$

$= \left[t^2 \tan^{-1} t - \int dt + \int\frac{1}{1+t^2}dt\right]_0^1$

$= [t^2\tan^{-1} t - t + \tan^{-1} t]_0^1$

$= [1^2\tan^{-1}(1) - 1 + \tan^{-1}(1) - 0 + 0 - \tan^{-1} 0]$

$= \left[1^2\times\frac{\pi}{4} - 1 + \frac{\pi}{4} - 0\right] = \frac{\pi}{4} - 1 + \frac{\pi}{4} = \frac{\pi}{2} - 1$

20. Let $x = \tan t$, then $dx = \sec^2 t\,dt$

When $x = 0$ then $\tan t = 0 \Rightarrow t = 0$

When $x = \infty$ then $\tan t = \infty \Rightarrow t = \frac{\pi}{2}$

$\therefore$ LHS $= \int_0^{\infty}\log\left[x + \frac{1}{x}\right]\cdot\frac{dx}{1+x^2}$

$= \int_0^{\pi/2}\log\left[\tan t + \frac{1}{\tan t}\right]\cdot\frac{(\sec^2 t)\,dt}{1+\tan^2 t}$

$= \int_0^{\pi/2}\log\left[\frac{\sin t}{\cos t} + \frac{\cos t}{\sin t}\right]\cdot\frac{(\sec^2 t)\,dt}{\sec^2 t}$

TR!CK

$1 + \tan^2\theta = \sec^2\theta$, $\sin^2\theta + \cos^2\theta = 1$

$= \int_0^{\pi/2}\log\left[\frac{\sin^2 t + \cos^2 t}{\cos t\cdot\sin t}\right]dt$

$= \int_0^{\pi/2}\log\left[\frac{1}{\sin t\cdot\cos t}\right]dt$

$= -\int_0^{\pi/2}\{\log(\sin t\cdot\cos t)^{-1}\}\,dt$

$= -\int_0^{\pi/2}\{\log(\sin t\cdot\cos t)\}\,dt \qquad [\because \log m^n = n\log m]$

$= -\int_0^{\pi/2}(\log\sin t + \log\cos t)\,dt$

$= -\int_0^{\pi/2}(\log\sin t)\,dt - \int_0^{\pi/2}(\log\cos t)\,dt$

$= -\int_0^{\pi/2}(\log\sin t)\,dt - \int_0^{\pi/2}\left[\log\cos\left\{\frac{\pi}{2} - t\right\}\right]dt$

$\left[\because \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right]$

$= -\int_0^{\pi/2}(\log\sin t)\,dt - \int_0^{\pi/2}(\log\sin t)\,dt$

$[\because \cos(90° - \theta) = \sin\theta]$

$= -2\int_0^{\pi/2}(\log\sin x)\,dx$

$= -2\left[-\frac{\pi}{2}\cdot\log 2\right] \qquad \left[\because \int_0^{\pi/2}\log\sin x\,dx = -\frac{\pi}{2}\log 2\right]$

$= \pi\log 2 = \text{RHS}$ **Hence proved.**

21. Let $\quad I = \int_0^{\pi}\frac{x}{a^2\cos^2 x + b^2\sin^2 x}dx \quad$ ...(1)

and $\quad I = \int_0^{\pi}\frac{(\pi - x)}{a^2\cos^2(\pi - x) + b^2\sin^2(\pi - x)}dx$

$\left[\because \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right]$

$= \int_0^a\frac{(\pi - x)}{a^2\cos^2 x + b^2\sin^2 x}dx \quad$ ...(2)

Adding eqs. (1) and (2),

$2I = \int_0^{\pi}\frac{\pi}{a^2\cos^2 x + b^2\sin^2 x}dx$

TR!CK

$\int_0^{2a} f(x)\,dx = \begin{cases} 2\int_0^a f(x)\,dx, & \text{if } f(2a-x) = f(x) \\ 0, & \text{if } f(2a-x) = -f(x)\end{cases}$

$\Rightarrow \quad I = \frac{\pi}{2}\cdot 2\int_0^{\pi/2}\frac{dx}{a^2\cos^2 x + b^2\sin^2 x}$

$[\because f(\pi - x) = f(x)]$

$\Rightarrow \quad I = \frac{\pi}{b^2}\int_0^{\pi/2}\frac{\sec^2 x}{\frac{a^2}{b^2} + \tan^2 x}dx$

$= \frac{\pi}{b^2}\int_0^{\pi/2}\frac{dt}{\left(\frac{a}{b}\right)^2 + t^2}$

[Let $t = \tan x \Rightarrow dt = \sec^2 dx$]

$= \frac{\pi}{b^2}\left[\frac{1}{(a/b)}\tan^{-1}\left(\frac{t}{a/b}\right)\right]_0^{\pi/2}$

$= \frac{\pi}{b^2}\times\frac{b}{a}\left[\tan^{-1}\left(\frac{b}{a}\tan x\right)\right]_0^{\pi/2}$

$$= \frac{\pi}{ab}\left[\tan^{-1}\left(\frac{b}{a}\tan\frac{\pi}{2}\right) - \tan^{-1}\left(\frac{b}{a}\tan 0\right)\right]$$

$$= \frac{\pi}{ab}\left[\tan^{-1}(\infty) - \tan^{-1}(0)\right] = \frac{\pi}{ab}\left(\frac{\pi}{2} - 0\right)$$

$$\therefore \quad I = \int_0^{\pi} \frac{x}{a^2\cos^2 x + b^2\sin^2 x}\,dx = \frac{\pi^2}{2ab}$$ **Hence proved.**

22. Let $I = \int_0^{\pi} \frac{x}{1+\cos^2 x}\,dx$ ...(1)

Then $I = \int_0^{\pi} \frac{\pi - x}{1+\cos^2(\pi - x)}\,dx$

or $I = \int_0^{\pi} \frac{\pi - x}{1+(-\cos x)^2}\,dx = \int_0^{\pi} \frac{\pi - x}{1+\cos^2 x}\,dx$ ...(2)

Adding eqs. (1) and (2), we get

$$2I = \int_0^{\pi} \frac{x}{1+\cos^2 x}\,dx + \int_0^{\pi} \frac{\pi - x}{1+\cos^2 x}\,dx$$

$$= \int_0^{\pi} \frac{x + \pi - x}{1+\cos^2 x}\,dx = \pi\int_0^{\pi} \frac{1}{1+\cos^2 x}\,dx$$

TR!CK

Here, integrand $f(x) = \frac{1}{1+\cos^2 x}$

Now, $f(\pi - x) = \frac{1}{1+\cos^2(\pi - x)}$

$$= \frac{1}{1+(-\cos x)^2} = \frac{1}{1+\cos^2 x} = f(x)$$

$$= 2 \times \pi\int_0^{\pi/2} \frac{1}{1+\cos^2 x}\,dx$$

$$\Rightarrow \quad I = \pi\int_0^{\pi/2} \frac{1}{1+\cos^2 x}\,dx$$

$$\left[\because \int_0^{2a} f(x)\,dx = \begin{cases} 2\int_0^a f(x)\,dx, & \text{if } f(2a-x) = f(x) \\ 0, & \text{if } f(2a-x) = -f(x) \end{cases}\right]$$

$$= \pi\int_0^{\pi/2} \frac{1}{(\sin^2 x + \cos^2 x) + \cos^2 x}\,dx$$

$[\because \sin^2\theta + \cos^2\theta = 1]$

$$= \pi\int_0^{\pi/2} \frac{1}{\sin^2 x + 2\cos^2 x}\,dx = \pi\int_0^{\pi/2} \frac{\sec^2 x}{\tan^2 x + 2}\,dx$$

[Divide numerator and denominator by $\cos^2 x$]

Let $\tan x = t$, then $(\sec^2 x)\,dx = dt$

When $x = 0$ then $t = \tan 0 = 0$

When $x = \frac{\pi}{2}$ then $t = \frac{\pi}{2} = \infty$

$$\therefore \quad I = \pi\int_0^{\pi/2} \frac{\sec^2 x}{\tan^2 + 2}\,dx = \pi\int_0^{\infty} \frac{1}{t^2 + (\sqrt{2})^2}\,dt$$

$$= \pi \times \frac{1}{\sqrt{2}}\left[\tan^{-1}\left\{\frac{t}{\sqrt{2}}\right\}\right]_0^{\infty}$$

$$= \frac{\pi}{\sqrt{2}}(\tan^{-1}\infty - \tan^{-1}0)$$

$$= \frac{\pi}{\sqrt{2}}\left[\frac{\pi}{2} - 0\right] = \frac{\pi^2}{2\sqrt{2}}$$ **Hence proved.**

Chapter Test

Multiple Choice Questions

Q 1. Evaluate $\int 2^{2^{2^x}} 2^{2^x} 2^x\,dx$.

a. $\frac{1}{(\log 2)^3} 2^{2^{2^x}} + C$

b. $\frac{1}{(\log 2)^3} 2^{2^x} + C$

c. $\frac{1}{(\log 2)^2} 2^{2^x} + C$

d. $\frac{1}{(\log 2)^4} 2^{2^{2^x}} + C$

Q 2. The value of $\int_{-\pi/2}^{\pi/2} \frac{dx}{e^{\sin x} + 1}$ is equal to:

a. 0

b. 1

c. $-\frac{\pi}{2}$

d. $\frac{\pi}{2}$

Assertion and Reason Type Questions

Directions (Q. Nos. 3-4): *In the following questions, each question contains Assertion (A) and Reason (R). Each question has 4 choices (a), (b), (c) and (d) out of which only one is correct. The choices are:*

a. Both Assertion (A) and Reason (R) are true and Reason (R) is the correct explanation of Assertion (A)

b. Both Assertion (A) and Reason (R) are true but Reason (R) is not the correct explanation of Assertion (A)

c. Assertion (A) is true and Reason (R) is false

d. Assertion (A) is false and Reason (R) is true

Q 3. Assertion (A): $\int_{\pi/6}^{\pi/3} \frac{(\sin x)^{\sqrt{2}}}{(\sin x)^{\sqrt{2}} + (\cos x)^{\sqrt{2}}}\,dx = \frac{\pi}{12}$

Reason (R): $\int_{\pi/6}^{\pi/3} \frac{f(x)}{f(x) + f\left(\frac{\pi}{2} - x\right)}\,dx = \frac{\pi}{12}$

Q 4. Assertion (A): $\int_{-\pi/2}^{\pi/2} |\sin x|\,dx = 2$

Reason (R):

$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$, **where** $c \in (a, b)$.

Case Study Based Questions

Q 5. Case Study 1

For a function $f(x)$, if $f(-x) = f(x)$, then $f(x)$ is an even function and $f(-x) = -f(x)$, then $f(x)$ is an odd function. Again, we have

$$\int_{-a}^{a} f(x)\,dx = \begin{cases} 2\int_0^a f(x)\,dx, & \text{if } f(x) \text{ is even} \\ 0, & \text{if } f(x) \text{ is odd} \end{cases}$$

Based on the given information, solve the following questions:

(i) Show that $f(x) = x^2 \sin x$ is an odd function.

(ii) Find: $\int_{-\pi}^{\pi} f(x)\, dx$.

(iii) If $g(x) = x \sin x$, then find $\int_{-\pi}^{\pi} x \sin x\, dx$.

Or

Find $\int_{-\pi/2}^{\pi/2} \sin^7 x\, dx$.

Q 6. Case Study 2

If $f(x)$ is a continuous function defined on $[0, a]$, then

$$\int_0^a f(x)\, dx = \int_0^a f(a-x)\, dx.$$

Based on the above information, give the answer of the following questions:

(i) Evaluate $\int_0^a \frac{f(x)}{f(x)+f(a-x)} dx$.

(ii) If $f(x) = \frac{\sin x - \cos x}{1 + \sin x \cos x}$, then find $f\left(\frac{\pi}{2} - x\right)$.

(iii) If $g(x) = \log(1 + \tan x)$, then find $g\left(\frac{\pi}{4} - x\right)$.

Very Short Answer Type Questions

Q 7. Evaluate $\int \frac{dx}{x^2 - a^2}$ when $x > a$.

Q 8. Evaluate $\int_0^1 \frac{x}{\sqrt{1+x^2}} dx$.

Short Answer Type-I Questions

Q 9. Evaluate $\int \frac{dx}{1+\sin x}$ or Evaluate $\int \frac{1-\sin x}{\cos^2 x} dx$.

Q 10. Evaluate $\int \frac{\cos x}{\cos(x-\alpha)} dx$.

Short Answer Type-II Questions

Q 11. Evaluate $\int \frac{\sin^3 x \cdot \cos x}{\sqrt{a - \sin^2 x} \cdot \sqrt{a + \sin^2 x}} dx$.

Q 12. Evaluate $\int \frac{x \tan^{-1} x}{(1+x^2)^{3/2}} dx$.

Long Answer Type Questions

Q 13. Evaluate $\int_{-2}^{1} |x^3 - x|\, dx$

Q 14. Evaluate $\int_0^1 \sin^{-1}\left[\frac{2x}{1+x^2}\right] dx$.