Matrices

- **Matrix** : A matrix is a rectangular arrangement of numbers or functions arranged into a fixed number of rows and columns. A matrix is written inside brackets [ ]. Each entry in a matrix is called an element of the matrix.

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}_{m \times n}$$

First row (R_1), Second row (R_2); (C_1) First column, Second column (C_2)

$m \times n$ → order of Matrix (No. of rows x No. of columns)

We shall write $A = [a_{ij}]_{m\times n}$ → order of Matrix; i-th row, j-th column

- **Column Matrix** : A matrix is said to be a column matrix if it has only one column. Ex $\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}_{4\times 1}$ column

- **Row Matrix** : A matrix is said to be a row matrix if it has only one row. Ex $[a\ b\ c\ d]_{1\times 4}$ row

- **Square Matrix** : No. of rows (m) = No. of colums (n) Ex $\begin{array}{c|c} \begin{bmatrix} 3 & 5 & 9 \\ 4 & 7 & 4 \\ 1 & 0 & 2 \end{bmatrix} & \begin{matrix} R_1 \\ R_2 \\ R_3 \end{matrix} \\ \begin{matrix} C_1 & C_2 & C_3 \end{matrix} & \end{array}$ (3×3) → m, n

- **Diagonal Matrix** : A square matrix is said to be a diagonal matrix if all its non diagonal elements are zero. Ex $\begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}$ $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$ non-diagonal elements zero.

- **Scalar Matrix** : A diagonal matrix is said to be a scalar matrix if its diagonal elements are equal. Ex $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ $\begin{bmatrix} \sqrt{3} & 0 & 0 \\ 0 & \sqrt{3} & 0 \\ 0 & 0 & \sqrt{3} \end{bmatrix}$ diagonal elements equal

- **Identity Matrix** : A square matrix in which all diagonal elements are 1 and rest are all zero. Ex $[1]$ $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ diagonal elements (1)

- **Null OR Zero Matrix** : If all its elements are zero. We denote zero matrix by O. Ex $[0]$, $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

- **Equal Matrix** : Two matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ are said to be equal if –
  (i) they are of the same order
  (ii) Each Element of A is equal to the corresponding element of B i.e. $a_{ij} = b_{ij}$ for all i and j

- **Upper triangular Matrix** : An upper triangular matrix, if $a_{ij} = 0\ \forall\ i > j$, i.e. all entries below principal diagonal are zero. Example : $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ $\begin{bmatrix} 1 & 4 & 5 \\ 0 & 2 & 6 \\ 0 & 0 & 3 \end{bmatrix}$ Upper triangular

- **Lower triangular Matrix** : A lower triangular matrix, if $a_{ij} = 0\ \forall\ i < j$, i.e. all entries above principal diagonal are zero. Example : $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ $\begin{bmatrix} 1 & 0 & 0 \\ 4 & 2 & 0 \\ 5 & 6 & 3 \end{bmatrix}$ lower triangular

- **Transpose of a matrix** Matrix obtained by interchanging rows and columns of A and denoted by A' OR A^T

- **Properties** (i) $\boxed{(A^T)^T = A}$ (ii) $\boxed{(kA)^T = kA^T}$ K is any constant (iii) $\boxed{(A+B)^T = A^T + B^T}$ (iv) $\boxed{(AB)^T = B^T A^T}$

- **Symmetric Matrices** $\boxed{A' = A}$
- **Skew-Symmetric matrices** $\boxed{A' = -A}$

- Diagonal Elements of a skew symmetric matrice are zero.
- For any square matrix A with real entries than $(A + A^T)$ is a symmetric and $(A - A^T)$ is skew symmetric.
- Any square matrix can be expressed as the sum of a symmetric and a skew symmetric matrix.

$$\boxed{A = \frac{1}{2}(A + A') + \frac{1}{2}(A - A')}$$

Addition of Matrices If $A = [a_{ij}]_{m\times n}$ and $B = [b_{ij}]_{m\times n}$ two matrices of the same order $m\times n$, then their sum $A+B$ is $m\times n$ matrix such that,

$(A+B)_{ij} = a_{ij} + b_{ij} \ \forall \ i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$

Properties of matrix addition

(i) Commutativity $\boxed{A+B = B+A}$

(ii) Associativity $\boxed{(A+B)+C = A+(B+C)}$

(iii) Existence of identity $\boxed{A+O = A+O = A}$

(iv) Existence of inverse $\boxed{A+(-A) = O = (-A)+A}$

(v) Cancellation laws $\boxed{A+B = A+C \Rightarrow B = C \text{ and } B+A = C+A \Rightarrow B = C}$

Scalar Multiplication of a matrix Let $A = [a_{ij}]_{m\times n}$ be a matrix and k is a scalar. Then the matrix obtained by multiplying each element of matrix A by k and is denoted by kA or Ak.

Properties of scalar Multiplication of a matrix If $A = [a_{ij}]$ and $B = [b_{ij}]$ be two matrices of the same order, say $m\times n$, and k and l are scalars, then (i) $\boxed{k(A+B) = kA + kB}$ (ii) $\boxed{(k+l)A = kA + lA}$

Multiplication of Matrices Two matrices A and B are said to be defined for multiplication, if the number of columns of A (pre-multiplier) is equal to the number of rows of B (post multiplier).

$$\boxed{A_{m\times n} \times B_{n\times p} = AB_{m\times p}}$$

equal

Properties of Multiplication of Matrices

(i) Associative law $\boxed{(AB)C = A(BC)}$

(ii) Distributive law (a) $\boxed{A(B+C) = AB + AC}$ (b) $\boxed{(A+B)C = AC + BC}$

(iii) Existence of multiplicative identity $\boxed{IA = AI = A}$

Elementary operation (Transformation) of a matrix : There are six operations (transformation) on a matrix 3 of which due to row and 3 of due to column, called Elementary Transformation.

(i) The interchange of any two rows or two column.

(ii) The multiplication of the elements of any row or column by a non-zero number.

(iii) The addition to the elements of any row or column, the corresponding elements of any other row or column multiplied by any non-zero number.

Invertible matrix If A is square matrix of order $m\times n$ and if there exist another square matrix B of the same order such that $AB = BA = I_m$

The A is invertible and B is called inverse of A.

Inverse of a square matrix, if it exists, is unique.

If A and B are invertible matrices of the same order, then $(AB)^{-1} = B^{-1}A^{-1}$

Inverse of a matrix by elementary operations

Let X, A and B be matrices of, the same order such that $X = AB$. In order to apply sequence of elementary row operations on the matrix equation $X = AB$, we will apply these row operations simultaneously on X and on the first matrix A of the product on RHS.

Similarly, in order to apply a sequence of elementary column operations on matrix eq. $X = AB$, we will apply, these operations simultaneously on X and on second matrix B of the product AB on RHS.

In view of the above discussion, we conclude that if A is a matrix such that A^{-1} exists, then to find A^{-1} using elementary row operations, write $A = IA$ and apply sequence of row operation on $A = IA$ till we get, $I = BA$. The matrix B will be the inverse of A. Similarly, if we wish to find A^{-1} using column operations, then write $A = AI$ and apply a sequence of column operations on $A = AI$ till we get, $I = AB$.

Remark In case, after applying one or more elementary row (column) operations on $A = IA$ ($A = AI$) if we obtain all zeros in one or more rows of the matrix A on L.H.S. then A^{-1} does not exist.

Determinants

Determinant : To every square matrix $A = [a_{ij}]$ of order n, we can associate a no. (real or complex) called determinant of the square matrix A, where $a_{ij} = (i, j)^{th}$ element of A.
Denoted as : det A or $|A|$.

Determinant of matrix of order one : Let $A = [a]$ be the matrix of order 1, then determinant of A is defined to be equal to a.

Determinant of matrix of order 2X2 : Let $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}_{2\times2}$ $\Rightarrow$ $\det(A) = |A| = \Delta = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}_{2\times2}$

$= a_{11}a_{22} - a_{21}a_{22}$

Determinant of matrix of order 3X3 : Let $A = \begin{bmatrix} \oplus & \ominus & \oplus \\ a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

ऐसे ही $2^{nd}, 3^{rd}$ row and $1^{st}, 2^{nd}, 3^{rd}$ column के through भी expansion कर सकते हैं। और answer हर बार same आएगा चाहे expansion कैसे भी करो।

Expansion along first row (R_1)

$$\det(A) = |A| = \Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} - \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$= a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= a_{11}(a_{22}a_{33} - a_{32}a_{23}) - a_{12}(a_{21}a_{33} - a_{31}a_{23}) + a_{13}(a_{21}a_{32} - a_{31}a_{22})$$

Note : (i) For matrix A, $|A|$ is read as determinant of A and not modulus of A.
(ii) Only square matrices have determinants.

Properties of Determinants :

(i) The value of the determinant remains unchanged if its rows and columns interchanged.
(ii) If any two rows (or columns) of a determinant are interchanged, then sign of determinant changes.
(iii) If any two rows (or columns) of a determinant are identical, then value of determinant is zero.
(iv) If each element of a row (or a column) of a determinant is multiplied by a constant k, then its value gets multiplied by k.
(V) If some or all elements of a row or column of a determinant is expressed sum of two (or more) terms, then the determinant can be expressed as sum of two (or more) determinants.
(vi) If, to each element of any row or column of a determinant, the equimultiple of corresponding elements of other row (or column) are added, then the value of determinant remains the same i.e. the value of determinant remains same if we apply the operation $R_i \to R_i + kR_j$ or $C_i \to C_i + kC_j$.

Area of Triangle : Area of Triangle with vertices $(x_1, y_1), (x_2, y_2)$ and (x_3, y_3) is,

$$\Delta = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

Note :

(i) Area is a positive quantity, we always take the absolute value of Δ.
(ii) If Area is given, use both positive and negative values of the determinant for calculation.
(iii) The area of the triangle formed by three collinear points is zero.

✓ **Minors** : Minor of an element a_{ij} of the $|A|$ is determinant obtained by deleting i^{th} row and j^{th} column and is denoted by M_{ij}.

✓ **Cofactors** : Cofactor of an element a_{ij}, denoted by A_{ij} is defined by $A_{ij} = (-1)^{i+j} M_{ij}$ minor of a_{ij}

✓ **Adjoint of a matrix** : The adjoint of a square matrix $A = [a_{ij}]$ is defined as the transpose of the matrix $[A_{ij}]_{m \times n}$. Adjoint of the matrix A denoted by adj A.

(the cofactor of the element a_{ij})

✓ **Singular matrix** : A square matrix A is said to be singular if $|A| = 0$

✓ **Non-Singular matrix** : A square matrix A is said to be non-singular if $|A| \neq 0$

✓ **Theorem 1** If A be any given square matrix of order n, then $A(\text{adj}A) = (\text{adj}A)A = |A|I$

✓ **Theorem 2** If A and B are non-singular matrices of the same order, then AB and BA are also non-singular matrices of the same order.

✓ **Theorem 3** $|AB| = |A||B|$ where A and B are square matrices of same order.

✓ **Theorem 4** A square matrix A is invertible if and only if A is non-singular matrix.

✓ **Consistent system** : If system of equation have solution (one or more) exists.

✓ **Incosistent system** : If system has no solution or solution does not exist.

✓ **System of linear equation using inverse of a matrix** :

Consider the equations,

$$a_1x + b_1y + c_1z = d_1$$
$$a_2x + b_2y + c_2z = d_2$$
$$a_3x + b_3y + c_3z = d_3$$

Here, $A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$ $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$

Then the system of equations can be written as, $AX = B \Rightarrow \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$

Case I If A is a non-singular matrix, then its inverse exists. Now $X = A^{-1}B$

Case II If A is a singular matrix, then $|A| = 0$
$(\text{Adj }A)B \neq 0$ sol^n does not exist (incosistent)
$(\text{Adj }A)B = 0$ infinitely many sol^n or no sol^n (consistent or inconsistent)