Chapter
Applications of Derivatives

Topic-1: Rate of Change of Quantities

Topic-2: Increasing & Decreasing Functions

1 MCQs with One Correct Answer

1. Let the function $g:(-\infty,\infty)\to\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$ be given by $g(u)=2\tan^{-1}(e^u)-\frac{\pi}{2}$. Then, g is **[2008]**
(a) even and is strictly increasing in $(0,\infty)$
(b) odd and is strictly decreasing in $(-\infty,\infty)$
(c) odd and is strictly increasing in $(-\infty,\infty)$
(d) neither even nor odd, but is strictly increasing in $(-\infty,\infty)$

2. If $f(x)=xe^{x(1-x)}$, then $f(x)$ is **[2001S]**
(a) increasing on $[-1/2, 1]$ (b) decreasing on R
(c) increasing on R (d) decreasing on $[-1/2, 1]$

3. For all $x\in(0,1)$ **[2000S]**
(a) $e^x<1+x$ (b) $\log_e(1+x)<x$
(c) $\sin x>x$ (d) $\log_e x>x$

4. If the normal to the curve $y=f(x)$ at the point (3,4) makes an angle $\frac{3\pi}{4}$ with the positive x-axis, then $f'(3)=$ **[2000S]**
(a) -1 (b) $-\frac{3}{4}$ (c) $\frac{4}{3}$ (d) 1

5. Let $f(x)=\int e^x(x-1)(x-2)dx$. Then f decreases in the interval **[2000S]**
(a) $(-\infty,-2)$ (b) $(-2,-1)$ (c) $(1,2)$ (d) $(2,+\infty)$

6. Consider the following statments in S and R **[2000S]**
S: Both $\sin x$ and $\cos x$ are decreasing functions in the interval $\left(\frac{\pi}{2},\pi\right)$
R: If a differentiable function decreases in an interval (a, b), then its derivative also decreases in (a, b).
Which of the following is true ?
(a) Both S and R are wrong
(b) Both S and R are correct, but R is not the correct explanation of S
(c) S is correct and R is the correct explanation for S
(d) S is correct and R is wrong

7. The function $f(x)=\sin^4 x+\cos^4 x$ increases if **[1999 - 2 Marks]**
(a) $0<x<\pi/8$ (b) $\pi/4<x<3\pi/8$
(c) $3\pi/8<x<5\pi/8$ (d) $5\pi/8<x<3\pi/4$

8. The slope of the tangent to a curve $y=f(x)$ at $[x, f(x)]$ is $2x+1$. If the curve passes through the point (1, 2), then the area bounded by the curve, the x-axis and the line $x=1$ is **[1995S]**
(a) $\frac{5}{6}$ (b) $\frac{6}{5}$ (c) $\frac{1}{6}$ (d) 6

9. The function $f(x)=\frac{\ln(\pi+x)}{\ln(e+x)}$ is **[1995S]**
(a) increasing on $(0,\infty)$
(b) decreasing on $(0,\infty)$
(c) increasing on $(0,\pi/e)$, decreasing on $(\pi/e,\infty)$
(d) decreasing on $0,\pi/e)$, increasing on $(\pi/e,\infty)$

10. The function defined by $f(x)=(x+2)e^{-x}$ is **[1994]**
(a) decreasing for all x
(b) decreasing in $(-\infty,-1)$ and increasing in $(-1,\infty)$
(c) increasing for all x
(d) decreasing in $(-1,\infty)$ and increasing in $(-\infty,-1)$

11. If $a+b+c=0$, then the quadratic equation $3ax^2+2bx+c=0$ has **[1983 - 1 Mark]**
(a) at least one root in [0, 1]
(b) one root in [2, 3] and the other in $[-2,-1]$
(c) imaginary roots
(d) none of these

2 Integer Value Answer/ Non-Negative Integer

12. Let the function $f: \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \frac{\sin x}{e^{\pi x}} \frac{\left(x^{2023} + 2024x + 2025\right)}{(x^2 - x + 3)} + \frac{2}{e^{\pi x}} \frac{\left(x^{2023} + 2024x + 2025\right)}{(x^2 - x + 3)}$$

Then the number of solutions of $f(x) = 0$ in $\mathbb{R}$ is ______. **[Adv. 2024]**

13. A vertical line passing through the point (h, 0) intersects the ellipse $\frac{x^2}{4} + \frac{y^2}{3} = 1$ at the points P and Q. Let the tangents to the ellipse at P and Q meet at the point R. If $\Delta(h)$ = area of the triangle PQR, $\Delta_1 = \max\limits_{1/2 \le h \le 1} \Delta(h)$ and $\Delta_2 = \min\limits_{1/2 \le h \le 1} \Delta(h)$, then $\frac{8}{\sqrt{5}}\Delta_1 - 8\Delta_2 =$ **[Adv. 2013]**

4 Fill in the Blanks

14. The set of all x for which $ln(1 + x) \le x$ is equal to .... **[1987 - 2 Marks]**

15. The function $y = 2x^2 - \ln|x|$ is monotonically increasing for values of $x (\neq 0)$ satisfying the inequalities .... and monotonically decreasing for values of x satisfying the inequalities .... **[1983 - 2 Marks]**

5 True / False

16. If $x - r$ is a factor of the polynomial $f(x) = a_n x^4 + ... + a_0$, repeated m times $(1 < m \le n)$, then r is a root of $f'(x) = 0$ repeated m times. **[1983 - 1 Mark]**

6 MCQs with One or More Than One Correct

17. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = \frac{x^2 - 3x - 6}{x^2 + 2x + 4}$

Then which of the following statements is (are) TRUE ? **[Adv. 2021]**

(a) f is decreasing in the interval $(-2, -1)$
(b) f is increasing in the interval $(1, 2)$
(c) f is onto
(d) Range of f is $\left[-\frac{3}{2}, 2\right]$

18. If $f : R \to R$ is a differentiable function such that $f'(x) > 2f(x)$ for all $x \in R$, and $f(0) = 1$, then **[Adv. 2017]**

(a) $f(x)$ is increasing in $(0,\infty)$
(b) $f(x)$ is decreasing in $(0, \infty)$
(c) $f(x) > e^{2x}$ in $(0, \infty)$
(d) $f'(x) < e^{2x}$ in $(0, \infty)$

19. For the function, $f(x) = x \cos \frac{1}{x}, x \ge 1$, **[2009]**

(a) for at least one x in the interval $[1, \infty)$, $f(x+2) - f(x) < 2$
(b) $\lim\limits_{x \to \infty} f'(x) = 1$
(c) for all x in the interval $[1, \infty)$, $f(x+2) - f(x) > 2$
(d) $f'(x)$ is strictly decreasing in the interval $[1, \infty)$

20. Let $h(x) = f(x) - (f(x))^2 + (f(x))^3$ for every real number x. Then **[1998 - 2 Marks]**

(a) h is increasing whenever f is increasing
(b) h is increasing whenever f is decreasing
(c) h is decreasing whenever f is decreasing
(d) nothing can be said in general.

21. If $f(x) = \begin{cases} 3x^2 + 12x - 1, & -1 \le x \le 2 \\ 37 - x & 2 < x \le 3 \end{cases}$ then: **[1993 - 2 Marks]**

(a) $f(x)$ is increasing on $[-1, 2]$
(b) $f(x)$ is continues on $[-1, 3]$
(c) $f'(2)$ does not exist
(d) $f(x)$ has the maximum value at $x = 2$

22. Let f and g be increasing and decreasing functions, respectively from $[0, \infty)$ to $[0, \infty)$. Let $h(x) = f(g(x))$. If $h(0) = 0$, then $h(x) - h(1)$ is **[1987 - 2 Marks]**

(a) always zero
(b) always negative
(c) always positive
(d) strictly increasing
(e) None of these.

7 Match the Following

(Qs. 23-25) : By appropriately matching the information given in the three columns of the following table.

Let $f(x) = x + \log_e x - x\log_e x, x \in (0, \infty)$

Column 1 contains information about zeros of $f(x)$, $f'(x)$ and $f''(x)$.

Column 2 contains information about the limiting behaviour of $f(x)$, $f'(x)$ and $f''(x)$ at infinity.

Column 3 contains information about increasing/decreasing nature of $f(x)$ and $f'(x)$.

Column 1	Column 2	Column 3
(I) $f(x) = 0$ for some $x \in (1, e^2)$	(i) $\lim_{x\to\infty} f(x) = 0$	(P) f is increasing in (0, 1)
(II) $f'(x) = 0$ for some $x \in (1, e)$	(ii) $\lim_{x\to\infty} f(x) = -\infty$	(Q) f is increasing in (e, e^2)
(III) $f'(x) = 0$ for some $x \in (0, 1)$	(iii) $\lim_{x\to\infty} f'(x) = -\infty$	(R) f' is increasing in (0, 1)
(IV) $f''(x) = 0$ for some $x \in (1, e)$	(iv) $\lim_{x\to\infty} f''(x) = 0$	(S) f' is decreasing in (e, e^2)

23. Which of the following options is the only correct combination? **[Adv. 2017]**
(a) (I) (i) (P) (b) (II) (ii) (Q) (c) (III) (iii) (R) (d) (IV) (iv) (S)

24. Which of the following options is the only correct combination? **[Adv. 2017]**
(a) (I) (ii) (R) (b) (II) (iii) (S) (c) (III) (iv) (P) (d) (IV) (i) (S)

25. Which of the following options is the only incorrect combination? **[Adv. 2017]**
(a) (I) (iii) (P) (b) (II) (iv) (Q) (c) (III) (i) (R) (d) (II) (iii) (P)

26. In this questions there are entries in columns I and II. Each entry in column I is related to exactly one entry in column II. Write the correct letter from column II against the entry number in column I in your answer book.

Let the functions defined in column I have domain $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ **[1992 - 2 Marks]**

Column I	Column II
(A) $x + \sin x$	(p) increasing
(B) $\sec x$	(q) decreasing
	(r) neither increasing nor decreasing

10 Subjective Problems

27. If $P(1) = 0$ and $\frac{dP(x)}{dx} > P(x)$ for all $x \geq 1$ then prove that $P(x) > 0$ for all $x > 1$. **[2003 - 4 Marks]**

28. If the function $f: [0,4] \to R$ is differentiable then show that
(i) For $a, b \in (0,4)$, $(f(4))^2 - (f(0))^2 = 8f'(a)f(b)$
(ii) $\int_0^4 f(t)dt = 2[\alpha f(\alpha^2) + \beta f(\beta^2)] \forall\ 0 < \alpha, \beta < 2$
[2003 - 4 Marks]

29. Using the relation $2(1 - \cos x) < x^2$, $x \neq 0$ or otherwise, prove that $\sin(\tan x) \geq x$, $\forall\ x \in \left[0, \frac{\pi}{4}\right]$ **[2003 - 4 Marks]**

30. Let $-1 \leq p \leq 1$. Show that the equation $4x^3 - 3x - p = 0$ has a unique root in the interval $[1/2, 1]$ and identify it. **[2001 - 5 Marks]**

31. Suppose $p(x) = a_0 + a_1x + a_2x^2 + \ldots\ldots + a_nx^n$. If $|p(x)| \leq \left|e^{x-1} - 1\right|$ for all $x \geq 0$, prove that $|a_1 + 2a_2 + \ldots.. + na_n| \leq 1$. **[2000 - 5 Marks]**

32. Let $f(x) = \begin{cases} xe^{ax}, & x \leq 0 \\ x + ax^2 - x^3, & x > 0 \end{cases}$ **[1996 - 3 Marks]**
Where a is a positive constant. Find the interval in which $f'(x)$ is increasing.

33. Let $f(x) = \begin{cases} -x^3 + \dfrac{(b^3 - b^2 + b - 1)}{(b^2 + 3b + 2)}, & 0 \leq x < 1 \\ 2x - 3, & 1 \leq x \leq 3 \end{cases}$
[1993 - 5 Marks]
Find all possible real values of b such that $f(x)$ has the smallest value at $x = 1$.

34. Show that $2\sin x + \tan x \geq 3x$ where $0 \leq x < \frac{\pi}{2}$. **[1990 - 4 Marks]**

35. Show that $1 + x\ln(x + \sqrt{x^2 + 1}) \geq \sqrt{1 + x^2}$ for all $x \geq 0$ **[1983 - 2 Marks]**

Topic-3: Tangents & Normals

1 MCQs with One Correct Answer

1. Let $P(6, 3)$ be a point on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. If the normal at the point P intersects the x-axis at $(9, 0)$, then the eccentricity of the hyperbola is **[2011]**

(a) $\sqrt{\frac{5}{2}}$ (b) $\sqrt{\frac{3}{2}}$

(c) $\sqrt{2}$ (d) $\sqrt{3}$

2. Consider the two curves $C_1 : y^2 = 4x$, $C_2 : x^2 + y^2 - 6x + 1 = 0$. Then, **[2008]**

(a) C_1 and C_2 touch each other only at one point.

(b) C_1 and C_2 touch each other exactly at two points

(c) C_1 and C_2 intersect (but do not touch) at exactly two points

(d) C_1 and C_2 neither intersect nor touch each other

3. The tangent to the curve $y = e^x$ drawn at the point (c, e^c) intersects the line joining the points $(c-1, e^{c-1})$ and $(c+1, e^{c+1})$ **[2007 -3 marks]**

(a) on the left of $x = c$ (b) on the right of $x = c$

(c) at no point (d) at all points

4. If $P(x)$ is a polynomial of degree less than or equal to 2 and S is the set of all such polynomials so that $P(0) = 0$, $P(1) = 1$ and $P'(x) > 0 \ \forall x \in [0, 1]$, then **[2005S]**

(a) $S = \phi$

(b) $S = ax + (1-a)x^2 \ \forall a \in (0, 2)$

(c) $S = ax + (1-a)x^2 \ \forall a \in (0, \infty)$

(d) $S = ax + (1-a)x^2 \ \forall a \in (0, 1)$

5. If $f(x) = x^\alpha \log x$ and $f(0) = 0$, then the value of α for which Rolle's theorem can be applied in $[0, 1]$ is **[2004S]**

(a) -2 (b) -1 (c) 0 (d) $1/2$

6. In [0,1] Lagranges Mean Value theorem is NOT applicable to **[2003S]**

(a) $f(x) = \begin{cases} \frac{1}{2} - x & x < \frac{1}{2} \\ \left(\frac{1}{2} - x\right)^2 & x \geq \frac{1}{2} \end{cases}$

(b) $f(x) = \begin{cases} \frac{\sin x}{x}, & x \neq 0 \\ 1, & x = 0 \end{cases}$

(c) $f(x) = x|x|$

(d) $f(x) = |x|$

7. The point(s) on the curve $y^3 + 3x^2 = 12y$ where the tangent is vertical, is (are) **[2002S]**

(a) $\left(\pm\frac{4}{\sqrt{3}}, -2\right)$ (b) $\left(\pm\sqrt{\frac{11}{3}}, 1\right)$

(c) $(0, 0)$ (d) $\left(\pm\frac{4}{\sqrt{3}}, 2\right)$

8. The triangle formed by the tangent to the curve $f(x) = x^2 + bx - b$ at the point $(1, 1)$ and the coordinate axes, lies in the first quadrant. If its area is 2, then the value of b is **[2001S]**

(a) -1 (b) 3

(c) -3 (d) 1

9. Which one of the following curves cut the parabola $y^2 = 4ax$ at right angles? **[1994]**

(a) $x^2 + y^2 = a^2$ (b) $y = e^{-x/2a}$

(c) $y = ax$ (d) $x^2 = 4ay$

10. The normal to the curve $x = a(\cos\theta + \theta\sin\theta)$, $y = a(\sin\theta - \theta\cos\theta)$ at any point 'θ' is such that **[1983 - 1 Mark]**

(a) it makes a constant angle with the x– axis

(b) it passes through the origin

(c) it is at a constant distance from the origin

(d) none of these

2 Integer Value Answer/ Non-Negative Integer

11. The slope of the tangent to the curve $(y - x^5)^2 = x(1 + x^2)^2$ at the point $(1, 3)$ is **[Adv. 2014]**

4 Fill in the Blanks

12. Let C be the curve $y^3 - 3xy + 2 = 0$. If H is the set of points on the curve C where the tangent is horizontal and V is the set of the point on the curve C where the tangent is vertical then H=............ and V=............ **[1994 - 2 Marks]**

6 MCQs with One or More Than One Correct

13. Let f, g: $[-1, 2] \to \mathbb{R}$ be continuous functions which are twice differentiable on the interval $(-1, 2)$. Let the values of f and g at the points -1, 0 and 2 be as given in the following table:

	$x=-1$	$x=0$	$x=2$
$f(x)$	3	6	0
$g(x)$	0	1	–1

In each of the intervals (–1, 0) and (0, 2) the function (f – 3g)" never vanishes. Then the correct statement(s) is(are) [Adv. 2015]

(a) $f'(x)-3g'(x)=0$ has exactly three solutions in $(-1,0)\cup(0,2)$

(b) $f'(x)-3g'(x)=0$ has exactly one solution in $(-1,0)$

(c) $f'(x)-3g'(x)=0$ has exactly one solution in $(0,2)$

(d) $f'(x)-3g'(x)=0$ has exactly two solutions in $(-1,0)$ and exactly two solutions in $(0,2)$

14. If the line $ax+by+c=0$ is a normal to the curve $xy=1$, then [1986 - 2 Marks]

(a) $a>0, b>0$ (b) $a>0, b<0$
(c) $a<0, b>0$ (d) $a<0, b<0$
(e) none of these.

8 Comprehension/Passage Based Questions

If a continuous function f defined on the real line R, assumes positive and negative values in R then the equation $f(x)=0$ has a root in R. For example, if it is known that a continuous function f on R is positive at some point and its minimum value is negative then the equation $f(x)=0$ has a root in R. [2007 - 4 marks]

Consider $f(x)=ke^x-x$ for all real x where k is a real constant.

15. The line $y=x$ meets $y=ke^x$ for $k\le 0$ at

(a) no point (b) one point
(c) two points (d) more than two points

16. The positive value of k for which $ke^x-x=0$ has only one root is

(a) $\frac{1}{e}$ (b) 1 (c) e (d) $\log_e 2$

17. For $k>0$, the set of all values of k for which $ke^x-x=0$ has two distinct roots is

(a) $\left(0,\frac{1}{e}\right)$ (b) $\left(\frac{1}{e},1\right)$ (c) $\left(\frac{1}{e},\infty\right)$ (d) $(0,1)$

10 Subjective Problems

18. If $|f(x_1)-f(x_2)|<(x_1-x_2)^2$, for all $x_1, x_2\in R$. Find the equation of tangent to the cuve $y=f(x)$ at the point (1, 2). [2005 - 2 Marks]

19. Using Rolle's theorem, prove that there is at least one root in $(45^{1/100}, 46)$ of the polynomial $P(x)=51x^{101}-2323(x)^{100}-45x+1035$. [2004 - 2 Marks]

20. Tangent is drawn to parabola $y^2-2y-4x+5=0$ at a point P which cuts the directrix at the point Q. A point R is such that it divides QP externally in the ratio 1/2 : 1. Find the locus of point R. [2004 - 4 Marks]

21. A curve C has the property that if the tangent drawn at any point P on C meets the co-ordinate axes at A and B, then P is the mid-point of AB. The curve passes through the point (1, 1). Determine the equation of the curve. [1998 - 8 Marks]

22. The curve $y=ax^3+bx^2+cx+5$, touches the x-axis at $P(-2, 0)$ and cuts the y axis at a point Q, where its gradient is 3. Find a, b, c. [1994 - 5 Marks]

23. Find the equation of the normal to the curve

$y=(1+x)^y+\sin^{-1}(\sin^2 x)$ at $x=0$ [1993 - 3 Marks]

24. What normal to the curve $y=x^2$ forms the shortest chord? [1992 - 6 Marks]

25. Find all the tangents to the curve

$y=\cos(x+y),\ -2\pi\le x\le 2\pi$, that are parallel to the line $x+2y=0$. [1985 - 5 Marks]

26. Find the coordinates of the point on the curve $y=\frac{x}{1+x^2}$ where the tangent to the curve has the greatest slope. [1984 - 4 Marks]

27. Find the shortest distance of the point (0, c) from the parabola $y=x^2$ where $0\le c\le 5$. [1982 - 2 Marks]

28. If $f(x)$ and $g(x)$ are differentiable function for $0\le x\le 1$ such that $f(0)=2, g(0)=0, f(1)=6; g(1)=2$, then show that there exist c satisfying $0<c<1$ and $f'(c)=2g'(c)$. [1982 - 2 Marks]

Topic-4: Approximations, Maxima & Minima

1 MCQs with One Correct Answer

1. Consider all rectangles lying in the region

$$\left\{(x,y)\in\mathbb{R}\times\mathbb{R}: 0\le x\le\frac{\pi}{2} \text{ and } 0\le y\le 2\sin(2x)\right\}$$

and having one side on the x-axis. The area of the rectangle which has the maximum perimeter among all such rectangles, is [Adv. 2020]

(a) $\frac{3\pi}{2}$ (b) π (c) $\frac{\pi}{2\sqrt{3}}$ (d) $\frac{\pi\sqrt{3}}{2}$

2. If f : R → R is a twice differentiable function such that $f''(x)>0$ for all $x\in R$, and $f\left(\frac{1}{2}\right)=\frac{1}{2}$, $f(1)=1$, then [Adv. 2017]

(a) $f'(1)\le 0$ (b) $0<f'(1)\le\frac{1}{2}$

(c) $\frac{1}{2} < f'(1) \le 1$ (d) $f'(1) > 1$

3. The least value of $a \in \mathbb{R}$ for which $4\alpha x^2 + \frac{1}{x} \ge 1$, for all $x > 0$, is [Adv. 2016]

(a) $\frac{1}{64}$ (b) $\frac{1}{32}$ (c) $\frac{1}{27}$ (d) $\frac{1}{25}$

4. The total number of local maxima and local minima of the function $f(x) = \begin{cases} (2+x)^3, & -3 < x \le -1 \\ x^{2/3}, & -1 < x < 2 \end{cases}$ is [2008]

(a) 0 (b) 1 (c) 2 (d) 3

5. If $f(x) = x^3 + bx^2 + cx + d$ and $0 < b^2 < c$, then in $(-\infty, \infty)$

(a) $f(x)$ is a strictly increasing function [2004S]
(b) $f(x)$ has a local maxima
(c) $f(x)$ is a strictly decreasing function
(d) $f(x)$ is bounded

6. Tangent is drawn to ellipse

$\frac{x^2}{27} + y^2 = 1$ at $\left(3\sqrt{3}\cos\theta, \sin\theta\right)$ (where $\theta \in (0, \pi/2)$).

Then the value of θ such that sum of intercepts on axes made by this tangent is minimum, is [2003S]

(a) $\pi/3$ (b) $\pi/6$ (c) $\pi/8$ (d) $\pi/4$

7. The length of a longest interval in which the function $3\sin x - 4\sin^3 x$ is increasing, is [2002S]

(a) $\frac{\pi}{3}$ (b) $\frac{\pi}{2}$ (c) $\frac{3\pi}{2}$ (d) π

8. Let $f(x) = (1 + b^2)x^2 + 2bx + 1$ and let $m(b)$ be the minimum value of $f(x)$. As b varies, the range of $m(b)$ is [2001S]

(a) [0, 1] (b) (0, 1/2] (c) [1/2, 1] (d) (0, 1]

9. Let $f(x) = \begin{cases} |x|, & \text{for } 0 < |x| \le 2 \\ 1, & \text{for } x = 0 \end{cases}$ then at $x = 0$, f has [2000S]

(a) a local maximum (b) no local maximum
(c) a local minimum (d) no extremum

10. On the interval [0, 1] the function $x^{25}(1-x)^{75}$ takes its maximum value at the point [1995S]

(a) 0 (b) $\frac{1}{4}$ (c) $\frac{1}{2}$ (d) $\frac{1}{3}$

11. If $y = a\ln x + bx^2 + x$ has its extremum values at $x = -1$ and $x = 2$, then [1983 - 1 Mark]

(a) $a = 2, b = -1$ (b) $a = 2, b = -\frac{1}{2}$

(c) $a = -2, b = \frac{1}{2}$ (d) none of these

12. AB is a diameter of a circle and C is any point on the circumference of the circle. Then [1983 - 1 Mark]

(a) the area of ΔABC is maximum when it is isosceles
(b) the area of ΔABC is minimum when it is isosceles
(c) the perimeter of ΔABC is minimum when it is isosceles
(d) none of these

2 Integer Value Answer/ Non-Negative Integer

13. Let [Adv. 2022]

$$\alpha = \sum_{k=1}^{\infty} \sin^{2k}\left(\frac{\pi}{6}\right).$$

Let $g : [0, 1] \to \mathbb{R}$ be the function defined by

$$g(x) = 2^{\alpha x} + 2^{\alpha(1-x)}.$$

Then, which of the following statements is/are TRUE ?

(a) The minimum value of $g(x)$ is $2^{\frac{7}{6}}$

(b) The maximum value of $g(x)$ is $1 + 2^{\frac{1}{3}}$

(c) The function $g(x)$ attains its maximum at more than one point

(d) The function $g(x)$ attains its minimum at more than one point

14. A cylindrical container is to be made from certain solid material with the following constraints: It has a fixed inner volume of V mm^3, has a 2 mm thick solid wall and is open at the top. The bottom of the container is a solid circular disc of thickness 2 mm and is of radius equal to the outer radius of the container.

If the volume of the material used to make the container is minimum when the inner radius of the container is 10 mm, then the value of $\frac{V}{250\pi}$ is [Adv. 2015]

15. Let $p(x)$ be a real polynomial of least degree which has a local maximum at $x = 1$ and a local minimum at $x = 3$. If $p(1) = 6$ and $p(3) = 2$, then $p'(0)$ is [2012]

16. let $f : IR \to IR$ be defined as $f(x) = |x| + |x^2 - 1|$. The total number of points at which f attains either a local maximum or a local minimum is [2012]

17. Let f be a function defined on **R** (the set of all real numbers) such that $f'(x) = 2010\,(x-2009)\,(x-2010)^2\,(x-2011)^3\,(x-2012)^4$ for all $x \in \mathbf{R}$.

If g is a function defined on **R** with values in the interval $(0, \infty)$ such that $f(x) = \ln(g(x))$, for all $x \in R$ then the number of points in R at which g has a local maximum is [2010]

18. Let f be a real-valued differentiable function on **R** (the set of all real numbers) such that $f(1) = 1$. If the y-intercept of the tangent at any point $P(x, y)$ on the curve $y = f(x)$ is equal to the cube of the abscissa of P, then find the value of $f(-3)$ [2010]

19. Let $p(x)$ be a polynomial of degree 4 having extremum at $x = 1, 2$ and $\lim_{x \to 0}\left(1 + \frac{p(x)}{x^2}\right) = 2$.

Then the value of $p(2)$ is [2009]

20. The maximum value of the function $f(x) = 2x^3 - 15x^2 + 36x - 48$ on the set

$A = \{x \mid x^2 + 20 \le 9x\}$ is **[2009]**

3 Numeric/ New Stem Based Questions

21. Let the function $f : (0, \pi) \to \mathbb{R}$ be defined by

$f(\theta) = (\sin\theta + \cos\theta)^2 + (\sin\theta - \cos\theta)^4$.

Suppose the function f has a local minimum at θ precisely when $\theta \in \{\lambda_1 \pi, ..., \lambda_r \pi\}$, where $0 < \lambda_1 < ... < \lambda_r < 1$. Then the value of $\lambda_1 + ... + \lambda_r$ is ______ **[Adv. 2020]**

4 Fill in the Blanks

22. Let P be a variable point on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ with foci F_1 and F_2. If A is the area of the triangle PF_1F_2 then the maximum value of A is .... **[1994 - 2 Marks]**

23. The larger of $\cos(\ln\theta)$ and $\ln(\cos\theta)$ if $e^{-\pi/2} < \theta < \frac{\pi}{2}$ is .... **[1983 - 1 Mark]**

5 True / False

24. For $0 < a < x$, the minimum value of the function $\log_a x + \log_x a$ is 2. **[1984 - 1 Mark]**

6 MCQs with One or More Than One Correct

25. Let $f(x) = \frac{\sin \pi x}{x^2}, x > 0$

Let $x_1 < x_2 < x_3 < ..... < x_n < .....$ be all the points of local maximum of f and $y_1 < y_2 < y_3 < ..... < y_n < .....$ be all the points of local minimum of f.

Then which of the following options is/are correct? **[Adv. 2019]**

(a) $x_{n+1} - x_n > 2$

(b) $x_n \in \left(2n, 2n + \frac{1}{2}\right)$ for every n

(c) $|x_n - y_n| > 1$ for every n

(d) $x_1 < y_1$

26. Define the collections $\{E_1, E_2, E_3, ...\}$ of ellipses and $\{R_1, R_2, R_3, ...\}$ of rectangles as follows :

$E_1 : \frac{x^2}{9} + \frac{y^2}{4} = 1$;

R_1 : rectangle of largest area, with sides parallel to the axes, inscribed in E_1;

E_n : ellipse $\frac{x^2}{a_n^2} + \frac{y^2}{b_n^2} = 1$ of largest area inscribed in R_{n-1}, $n > 1$;

R_n : rectangle of largest area, with sides parallel to the axes, inscribed in E_n, $n > 1$.

Then which of the following options is/are correct? **[Adv. 2019]**

(a) The eccentricities of E_{18} and E_{19} are NOT equal

(b) The length of latus rectum of E_9 is $\frac{1}{6}$

(c) $\sum_{n=1}^{N}$ (area of R_n) < 24, for each positive integer N

(d) The distance of a focus from the centre in E_9 is $\frac{\sqrt{5}}{32}$

27. If $f(x) = \begin{vmatrix} \cos(2x) & \cos(2x) & \sin(2x) \\ -\cos x & \cos x & -\sin x \\ \sin x & \sin x & \cos x \end{vmatrix}$, then **[Adv. 2017]**

(a) $f'(x) = 0$ at exactly three points in $(-\pi, \pi)$

(b) $f'(x) = 0$ at more than three points in $(-\pi, \pi)$

(c) $f(x)$ attains its maximum at $x = 0$

(d) $f(x)$ attains its minimum at $x = 0$

28. Let $f : \mathbb{R} \to (0, \infty)$ and $g : \mathbb{R} \to \mathbb{R}$ be twice differentiable functions such that f'' and g'' are continuous functions on $\mathbb{R}$. Suppose $f'(2) = g(2) = 0$, $f''(2) \ne 0$ and $g'(2) \ne 0$. If $\lim_{x \to 2} \frac{f(x)g(x)}{f'(x)g'(x)} = 1$, then **[Adv. 2016]**

(a) f has a local minimum at $x=2$

(b) f has a local maximum at $x=2$

(c) $f''(2) > f(2)$

(d) $f(x) - f''(x) = 0$ for at least one $x \in \mathbb{R}$

29. A rectangular sheet of fixed perimeter with sides having their lengths in the ratio 8 : 15 is converted into an open rectangular box by folding after removing squares of equal area from all four corners. If the total area of removed squares is 100, the resulting box has maximum volume. Then the lengths of the sides of the rectangular sheet are **[Adv. 2013]**

(a) 24 (b) 32 (c) 45 (d) 60

30. If $f(x) = \int_0^x e^{t^2}(t-2)(t-3)\,dt$ for all $x \in (0, \infty)$, then **[2012]**

(a) f has a local maximum at $x = 2$

(b) f is decreasing on $(2, 3)$

(c) there exists some $c \in (0, \infty)$, such that $f''(c) = 0$

(d) f has a local minimum at $x = 3$

31. $f(x)$ is cubic polynomial with $f(2) = 18$ and $f(1) = -1$. Also $f(x)$ has local maxima at $x = -1$ and $f'(x)$ has local minima at $x = 0$, then **[2006 - 5M, -1]**

(a) the distance between $(-1, 2)$ and $(a\, f(a))$, where $x = a$ is the point of local minima is $2\sqrt{5}$

(b) $f(x)$ is increasing for $x \in [1, 2\sqrt{5}]$

(c) $f(x)$ has local minima at $x = 1$

(d) the value of $f(0) = 15$

32. The number of values of x where the function $f(x) = \cos x + \cos(\sqrt{2}x)$ attains its maximum is **[1998 - 2 Marks]**

(a) 0 (b) 1 (c) 2 (d) infinite

33. If $f(x) = \frac{x^2 - 1}{x^2 + 1}$, for every real number x, then the minimum value of f **[1998 - 2 Marks]**

(a) does not exist because f is unbounded

(b) is not attained even though f is bounded

(c) is equal to 1

(d) is equal to -1

34. The smallest positive root of the equation, $\tan x - x = 0$ lies in [1987 - 2 Marks]

(a) $\left(0, \frac{\pi}{2}\right)$ (b) $\left(\frac{\pi}{2}, \pi\right)$

(c) $\left(\pi, \frac{3\pi}{2}\right)$ (d) $\left(\frac{3\pi}{2}, 2\pi\right)$

(e) None of these

35. Let $P(x) = a_0 + a_1x^2 + a_2x^4 + ... + a_nx^{2n}$ be a polynomial in a real variable x with

$0 < a_0 < a_1 < a_2 < ..... < a_n$. The function $P(x)$ has [1986 - 2 Marks]

(a) neither a maximum nor a minimum
(b) only one maximum
(c) only one minimum
(d) only one maximum and only one minimum
(e) none of these.

7 Match the Following

36. A line $L : y = mx + 3$ meets y – axis at $E(0, 3)$ and the arc of the parabola $y^2 = 16x, 0 \le y \le 6$ at the point $F(x_0, y_0)$. The tangent to the parabola at $F(x_0, y_0)$ intersects the y-axis at $G(0, y_1)$. The slope m of the line L is chosen such that the area of the triangle EFG has a local maximum. [Adv. 2013]

Match List I with List II and select the correct answer using the code given below the lists :

List I	List II
P. $m =$	1. $\frac{1}{2}$
Q. Maximum area of ΔEFG is	2. 4
R. $y_0 =$	3. 2
S. $y_1 =$	4. 1

Codes:

	P	Q	R	S
(a)	4	1	2	3
(b)	3	4	1	2
(c)	1	3	2	4
(d)	1	3	4	2

8 Comprehension/Passage Based Questions

Let f : [0, 1] → ℝ (the set of all real numbers) be a function. Suppose the function f is twice differentiable, $f(0) = f(1) = 0$ and satisfies $f''(x) - 2f'(x) + f(x) \ge e^x, x \in [0, 1]$. [Adv. 2013]

37. Which of the following is true for $0 < x < 1$?

(a) $0 < f(x) < \infty$ (b) $-\frac{1}{2} < f(x) < \frac{1}{2}$

(c) $-\frac{1}{4} < f(x) < 1$ (d) $-\infty < f(x) < 0$

38. If the function $e^{-x} f(x)$ assumes its minimum in the interval [0, 1] at $x = \frac{1}{4}$, which of the following is true?

[Adv. 2013]

(a) $f'(x) < f(x), \frac{1}{4} < x < \frac{3}{4}$ (b) $f'(x) > f(x), 0 < x < \frac{1}{4}$

(c) $f'(x) < f(x), 0 < x < \frac{1}{4}$ (d) $f'(x) < f(x), \frac{3}{4} < x < 1$

10 Subjective Problems

39. For a twice differentiable function $f(x)$, $g(x)$ is defined as $g(x) = (f'(x)^2 + f''(x)) f(x)$ on $[a, e]$. If for $a < b < c < d < e$, $f(a) = 0, f(b) = 2, f(c) = -1, f(d) = 2, f(e) = 0$ then find the minimum number of zeros of $g(x)$. [2006 - 6M]

40. If $p(x)$ be a polynomial of degree 3 satisfying $p(-1) = 10$, $p(1) = -6$ and $p(x)$ has maxima at $x = -1$ and $p'(x)$ has minima at $x = 1$. Find the distance between the local maxima and local minima of the curve. [2005 - 4 Marks]

41. Prove that for $x \in \left[0, \frac{\pi}{2}\right]$, $\sin x + 2x \ge \frac{3x(x+1)}{\pi}$. Explain the identity if any used in the proof. [2004 - 4 Marks]

42. Find a point on the curve $x^2 + 2y^2 = 6$ whose distance from the line $x + y = 7$, is minimum. [2003 - 2 Marks]

43. Find the co-ordinates of all the points P on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, for which the area of the triangle PON is maximum, where O denotes the origin and N, the foot of the perpendicular from O to the tangent at P.

[1999 - 10 Marks]

44. Suppose $f(x)$ is a function satisfying the following conditions [1998 - 8 Marks]

(a) $f(0) = 2, f(1) = 1$,

(b) f has a minimum value at $x = 5/2$, and

(c) for all x,

$$f'(x) = \begin{vmatrix} 2ax & 2ax-1 & 2ax+b+1 \\ b & b+1 & -1 \\ 2(ax+b) & 2ax+2b+1 & 2ax+b \end{vmatrix}$$

where a, b are some constants. Determine the constants a, b and the function $f(x)$.

45. Determine the points of maxima and minima of the function $f(x) = \frac{1}{8} \ell n\, x - bx + x^2, x > 0$, where $b \ge 0$ is a constant.

[1996 - 5 Marks]

46. Let (h, k) be a fixed point, where $h > 0, k > 0$. A straight line passing through this point cuts the positive direction of the coordinate axes at the points P and Q. Find the minimum area of the triangle OPQ, O being the origin.

[1995 - 5 Marks]

47. The circle $x^2 + y^2 = 1$ cuts the x-axis at P and Q. Another circle with centre at Q and variable radius intersects the first circle at R above the x-axis and the line segment PQ at

S. Find the maximum area of the triangle QSR. [1994 - 5 Marks]

48. A window of perimeter P (including the base of the arch) is in the form of a rectangle surmounded by a semi circle. The semi-circular portion is fitted with coloured glass while the rectangular part is fitted with clear glass transmits three times as much light per square meter as the coloured glass does.

What is the ratio for the sides of the rectangle so that the window transmits the maximum light? [1991 - 4 Marks]

49. A point P is given on the circumference of a circle of radius r. Chord QR is parallel to the tangent at P. Determine the maximum possible area of the triangle PQR. [1990 - 4 Marks]

50. Find the point on the curve $4x^2 + a^2y^2 = 4a^2$, $4 < a^2 < 8$ that is farthest from the point $(0, -2)$. [1987 - 4 Marks]

51. Let $f(x) = \sin^3 x + \lambda \sin^2 x$, $-\frac{\pi}{2} < x < \frac{\pi}{2}$. Find the intervals in which λ should lie in order that $f(x)$ has exactly one minimum and exactly one maximum. [1985 - 5 Marks]

52. If $ax^2 + \frac{b}{x} \geq c$ for all positive x where $a > 0$ and $b > 0$ show that $27ab^2 \geq 4c^3$. [1982 - 2 Marks]

53. Use the function $f(x) = x^{1/x}$, $x > 0$. to determine the bigger of the two numbers e^{π} and π^{e} [1981 - 4 Marks]

54. Let x and y be two real variables such that $x > 0$ and $xy = 1$. Find the minimum value of $x+y$. [1981 - 2 Marks]

55. Prove that the minimum value of $\frac{(a+x)(b+x)}{(c+x)}$, $a, b > c, x > -c$ is $(\sqrt{a-c} + \sqrt{b-c})^2$. [1979]

Answer Key

Topic-1 : Rate of Change of Quantities

Topic-2 : Increasing & Decreasing Functions

1. (c) 2. (a) 3. (b) 4. (d) 5. (c) 6. (d) 7. (b) 8. (a) 9. (b) 10. (d)
11. (a) 12. (01) 13. (9) 14. $x \geq 0$ 15. $x \in \left(\frac{-1}{2}, 0\right) \cup \left(\frac{1}{2}, \infty\right); \left(-\infty, \frac{-1}{2}\right) \cup \left(0, \frac{1}{2}\right)$ 16. (False) 17. (a, b)
18. (a, c) 19. (b, c, d) 20. (a, b, c) 21. (a, c) 22. (a) 23. (b) 24. (b) 25. (c) 26. (A)→(P), (B)→(r)

Topic-3 : Tangents & Normals

1. (b) 2. (b) 3. (a) 4. (b) 5. (d) 6. (a) 7. (d) 8. (c) 9. (d) 10. (c)
11. (8) 12. ϕ, {(1, 1)} 13. (b, c) 14. (b, c) 15. (b) 16. (a) 17. (a)

Topic-4 : Approximations, Maxima & Minima

1. (c) 2. (d) 3. (c) 4. (c) 5. (a) 6. (c) 7. (a) 8. (d) 9. (d) 10. (b)
11. (b) 12. (a) 13. (a, b, c) 14. (4) 15. (9) 16. (5) 17. (1) 18. (9) 19. (0) 20. (7)
21. (0.5) 22. (abe) 23. cos(lnθ) 24. (False) 25. (a, b, c) 26. (b, c) 27. (b, c) 28. (a, d)
29. (a, c) 30. (a, b, c, d) 31. (b, c) 32. (b) 33. (d) 34. (c) 35. (c) 36. (a) 37. (d)
38. (c)

Hints & Solutions

Topic-2: Increasing & Decreasing Functions

1. (c) Given : $g(u) = 2\tan^{-1}(e^u) - \frac{\pi}{2}$

$\therefore\ g(-u) = 2\tan^{-1}(e^{-u}) - \frac{\pi}{2} = 2\tan^{-1}\left(\frac{1}{e^u}\right) - \frac{\pi}{2}$

$= 2\cot^{-1}(e^u) - \frac{\pi}{2} = 2\left[\frac{\pi}{2} - \tan^{-1}(e^u)\right] - \frac{\pi}{2}$

$= \frac{\pi}{2} - 2\tan^{-1}(e^u) = -g(u)$, $\therefore$ g is an odd function.

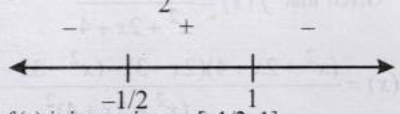
Also $g'(u) = \frac{2e^u}{1+e^{2u}} > 0$, $\forall u \in (-\infty, \infty)$

$\therefore$ g is strictly increasing on $(-\infty, \infty)$.

2. (a) $f(x) = xe^{x(1-x)}$

$\Rightarrow f'(x) = e^{x(1-x)} + (1-2x)\,x\,e^{x(1-x)} = -e^{x(1-x)}(2x+1)(x-1)$

Critical point are $x = -\frac{1}{2}$ and 1



Hence, $f(x)$ is increasing on $[-1/2, 1]$.

3. (b) Let $f(x) = e^x - 1 - x$

$\Rightarrow f'(x) = e^x - 1 > 0$ for $x \in (0,1)$

$\therefore$ $f(x)$ is an increasing function.

$\therefore$ $f(x) > f(0), \forall x \in (0,1)$

$\Rightarrow e^x - 1 - x > 0 \Rightarrow e^x > 1 + x$

$\therefore$ (a) is not correct.

(b) Let $g_e(x) = \log_e(1+x) - x$

$\Rightarrow g'(x) = \frac{1}{1+x} - 1 = -\frac{x}{1+x} < 0, \forall x \in (0,1)$

Hence, $g(x)$ is decreasing on $(0, 1)$

$\Rightarrow g(x) < g(0)$

$\Rightarrow \log_e(1+x) - x < 0 \Rightarrow \log_e(1+x) < x$

$\therefore$ (b) is correct.

Similarly it can be shown that (c) and (d) do not correct.

4. (d) Slope of tangent $y = f(x)$ is $\frac{dy}{dx} = f'(x)_{(3,4)}$

$\therefore$ Slope of normal $= -\frac{1}{f'(x)_{(3,4)}} = -\frac{1}{f'(3)}$

Now $-\frac{1}{f'(3)} = \tan\left(\frac{3\pi}{4}\right)$ (given)

$\Rightarrow -\frac{1}{f'(3)} = \tan\left(\frac{\pi}{2} + \frac{\pi}{4}\right) = -1 \Rightarrow f'(3) = 1$

5. (c) $f(x) = \int e^x(x-1)(x-2)\,dx$

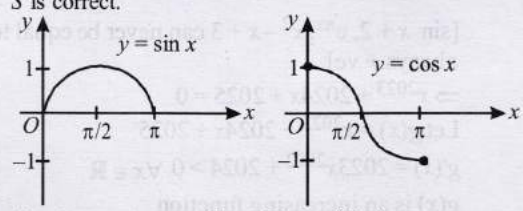
For decreasing function, $f'(x) < 0$

$\Rightarrow e^x(x-1)(x-2) < 0 \Rightarrow (x-1)(x-2) < 0$

$\Rightarrow 1 < x < 2$, $[\because\ e^x > 0\ \forall x \in R]$

6. (d) From graph it is clear that both $\sin x$ and $\cos x$ in the interval $(\pi/2, \pi)$ are decreasing function.

$\therefore$ S is correct.



Consider $f(x) = \sin x$ on $(0, \pi/2) \Rightarrow f'(x) = \cos x$

From graph it is clear that $f(x)$ is increasing on $(0, \pi/2)$ but $f'(x)$ is decreasing on $(0, \pi/2)$

$\therefore$ R is wrong.

7. (b) Given : $f(x) = \sin^4 x + \cos^4 x$

$\Rightarrow f'(x) = 4\sin^3 x\cos x - 4\cos^3 x\sin x$

$= -2.\ \sin 2x \cos 2x = -\sin 4x$

Now for $f(x)$ to be increasing function

$f'(x) > 0 \Rightarrow -\sin 4x > 0 \Rightarrow \sin 4x < 0$

$\Rightarrow \pi < 4x < 2\pi \Rightarrow \frac{\pi}{4} < x < \frac{\pi}{2}$

Hence, $f(x)$ increasing on $(\pi/4, \pi/2)$

Now, $\frac{\pi}{2} = \frac{4\pi}{8} > \frac{3\pi}{8}$

$\therefore$ $f(x)$ will be increasing on $(\pi/4, 3\pi/8)$.

8. (a) Given that slope of tangent at $(x, f(x))$ is $2x + 1$

$\Rightarrow f'(x) = 2x + 1 \Rightarrow f(x) = x^2 + x + c$

Since the curve passes through $(1, 2)$, $\therefore f(1) = 2$

$\Rightarrow 2 = 1 + 1 + c \Rightarrow c = 0$, $\therefore f(x) = x^2 + x$

The graph of $f(x)$ is an upward parabola, which touches the x-axis the origin.

$\therefore$ Required area $= \int_0^1 (x^2 + x)\,dx$

$= \left[\frac{x^3}{3} + \frac{x^2}{2}\right]_0^1 = \frac{1}{3} + \frac{1}{2} = \frac{5}{6}$

9. (b) Given : $f(x) = \frac{\ln(\pi + x)}{\ln(e + x)}$

$\Rightarrow f'(x) = \frac{\left(\frac{1}{\pi + x}\right)\ln(e+x) - \frac{1}{(e+x)}\ln(\pi + x)}{[\ln(e+x)]^2}$

$= \frac{(e+x)\ln(e+x) - (\pi+x)\ln(\pi+x)}{(e+x)(\pi+x)(\ln(e+x))^2} < 0$

On $(0, \infty)$ since $1 < e < \pi$

$\therefore$ $f(x)$ decreases on $(0, \infty)$.

10. (d) Given : $f(x) = (x+2)\,e^{-x}$

$\therefore\ f'(x) = -(x+2)e^{-x} + e^{-x} = -(x+1)e^{-x}$

Put $f'(x) = 0 \Rightarrow x = -1$

For $x \in (-\infty, -1), f'(x) > 0$

and for $x \in (-1, \infty), f'(x) < 0$

$\therefore f(x)$ is increasing on $(-\infty, -1)$ and decreasing on $(-1, -\infty)$.

11. **(a)** Consider the function $f(x) = ax^3 + bx^2 + cx$ on [0, 1] Since $f(x)$ is a polynomial. $\therefore$ $f(x)$ is continuous on [0, 1] and hence, differentiable on (0, 1)
Now $f(0) = f(1) = 0$ $\quad [\because a + b + c = 0]$
$\therefore$ By Rolle's theorem there exists $x \in (0, 1)$ such that
$f'(x) = 0 \Rightarrow 3ax^2 + 2bx + c = 0$
Hence, equation $3ax^2 + 2bx + c = 0$ has at least one root in [0, 1].

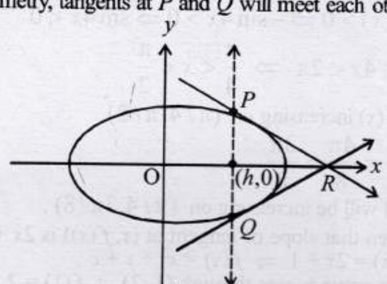
12. **(01)** $f(x) = 0$

$$\Rightarrow \frac{x^{2023} + 2024x + 2025}{(x^2 - x + 3)}\left[\frac{\sin x + 2}{e^{\pi x}}\right] = 0$$

[$\sin x + 2, e^{\pi x}, x^2 - x + 3$ can never be equal to zero *ab* always + ve]
$\Rightarrow x^{2023} + 2024x + 2025 = 0$
Let $g(x) = x^{2023} + 2024x + 2025$
$g'(x) = 2023x^{2022} + 2024 > 0 \ \forall x \in \mathbb{R}$
$g(x)$ is an increasing function
$\therefore f(x) = 0$ has only one solution

13. **(9)** Vertical line $x = h$, meets the ellipse $\frac{x^2}{4} + \frac{y^2}{3} = 1$ at $P\left(h, \frac{\sqrt{3}}{2}\sqrt{4 - h^2}\right)$ and $Q\left(h, \frac{-\sqrt{3}}{2}\sqrt{4 - h^2}\right)$

By symmetry, tangents at P and Q will meet each other at x-axis.



Tangent at P is $\frac{xh}{4} + \frac{y\sqrt{3}}{6}\sqrt{4 - h^2} = 1$

which meets x-axis at $R\left(\frac{4}{h}, 0\right)$

Area of $\Delta PQR = \frac{1}{2} \times \sqrt{3}\sqrt{4 - h^2} \times \left(\frac{4}{h} - h\right)$

i.e., $\Delta(h) = \frac{\sqrt{3}}{2}\frac{(4 - h^2)^{3/2}}{h}$

$$\frac{d\Delta}{dh} = -\sqrt{3}\left[\frac{\sqrt{4 - h^2}(h^2 + 2)}{h^2}\right] < 0$$

$\Rightarrow$ $\Delta(h)$ is a decreasing function.

Hence, $\frac{1}{2} \le h \le 1 \Rightarrow \Delta_{max} = \Delta\left(\frac{1}{2}\right)$ and $\Delta_{min} = \Delta(1)$

$$\therefore \quad \Delta_1 = \frac{\sqrt{3}}{2}\frac{\left(4 - \frac{1}{4}\right)^{3/2}}{\frac{1}{2}} = \frac{45}{8}\sqrt{5}$$

and $\Delta_2 = \frac{\sqrt{3}}{2}\frac{3\sqrt{3}}{1} = \frac{9}{2}$

Hence, $\frac{8}{\sqrt{5}}\Delta_1 - 8\Delta_2 = 45 - 36 = 9$

14. Let $f(x) = \log(1 + x) - x$ for $x > -1$

$$f'(x) = \frac{1}{1 + x} - 1 = \frac{-x}{1 + x}$$

Now $f'(x) > 0$ if $-1 < x < 0$ and $f'(x) < 0$ if $x > 0$
Hence, f increases in $(-1, 0)$ and decreases in $(0, \infty)$.
Now, $f(0) = \log 1 - 0 = 0$, $\therefore$ $x \ge 0 \Rightarrow f(x) \le f(0)$
$\Rightarrow$ $\log(1 + x) - x \le 0 \Rightarrow \log(1 + x) \le x$
Hence, $\log(1 + x) \le x, \forall x \ge 0$

15. $y = 2x^2 - \ln|x|$

$$\Rightarrow \frac{dy}{dx} = 4x - \frac{1}{x} = \frac{(2x + 1)(2x - 1)}{x}, x \ne 0$$

Critical points are 1/2 and –1/2

Clearly $f(x)$ is increasing on $\left(-\frac{1}{2}, 0\right) \cup \left(\frac{1}{2}, \infty\right)$ and

decreasing on $\left(-\infty, -\frac{1}{2}\right) \cup \left(0, \frac{1}{2}\right)$.

16. **(False)**
If $(x - r)$ is a factor of $f(x)$ repeated m times then $f'(x)$ is a polynomial with $(x - r)$ as factor repeated $(m - 1)$ times.
$\Rightarrow$ r is a root of $f'(x)$ repeated $(m - 1)$ times.
$\therefore$ The given statement is false.

17. **(a, b)** Given that $f(x) = \frac{x^2 - 3x - 6}{x^2 + 2x + 4}$

$$f'(x) = \frac{(x^2 + 2x + 4)(2x - 3) - (x^2 - 3x - 6)(2x + 2)}{(x^2 + 2x + 4)^2}$$

$$f'(x) = \frac{5x(x + 4)}{(x^2 + 2x + 4)^2}$$

$f'(x)$: $+$ (–4) $-$ (0) $+$

$\therefore$ Options (a) and (b) are ture.

$f(-4) = \frac{11}{6}$, $f(0) = -\frac{3}{2}$,

Also $f(x)$ is maximum at – 4 and minimum at 0.

Range: $\left[-\frac{3}{2}, \frac{11}{6}\right]$, clearly $f(x)$ is into

18. **(a, c)** Given, $f : R \to R$ is a differential function such that $f'(x) > 2f(x) \ \forall x \in R$

$\Rightarrow \frac{f'(x)}{f(x)} > 2$

Integrating both sides, we get

$\int \frac{f'(x)}{f(x)} dx > \int 2\, dx$

$\Rightarrow \log(f(x)) + C > 2x$
$\Rightarrow f(x) + C > e^{2x}$
Since, $f(0) = 1$
$\Rightarrow f(0) + C > e^0 \Rightarrow c > 0$
$\Rightarrow f(x) > e^{2x}$
$\therefore$ (c) is correct
Since, e^{2x} increases in $(0, \infty)$
Thus $f(x)$ will increase in $(0, \infty)$
Hence options (a) and (c) are correct.

19. **(b, c, d)** Given : $f(x) = x\cos\frac{1}{x},\ x \ge 1$

$\therefore\quad f'(x) = \cos\frac{1}{x} + \frac{1}{x}\sin\frac{1}{x}$

$\lim\limits_{x\to\infty} f'(x) = 1$

Also $f''(x) = \frac{-1}{x^3}\cos\frac{1}{x} < 0,\ \forall x \in [1, \infty)$

$\Rightarrow f'(x)$ is strictly decreasing in $[1, \infty)$

$\therefore\quad f'(x) > \lim\limits_{x\to\infty} f'(x) \Rightarrow \frac{f(x+2) - f(x)}{(x+2)-x} > 1$

$\Rightarrow f(x+2) - f(x) > 2$

20. **(a, c)** $h(x) = f(x) - (f(x))^2 + (f(x))^3\ \forall x \in R$

$h'(x) = f'(x)[1 - 2f(x) + 3(f(x))^2]$

$= 3f'(x)\left[(f(x))^2 - \frac{2}{3}f(x) + \frac{1}{3}\right]$

$= 3f'(x)[\{f(x) - 1/3\}^2 + 2/9]$

Here $h'(x) < 0$ whenever $f'(x) < 0$ and $h'(x) > 0$ whenever $f'(x) > 0$.

Hence $h(x)$ increases (decreases) whenever $f(x)$ increases (decreases).

21. **(a, b, c)** $f(x) = \begin{cases} 3x^2 + 12x - 1, & -1 \le x \le 2 \\ 37 - x, & 2 < x \le 3 \end{cases}$

Hence on $[-1, 2]$, $f'(x) = 6x + 12$

For $-1 \le x \le 2, -6 \le 6x \le 12$

$\Rightarrow 6 \le 6x + 12 \le 24 \Rightarrow f'(x) > 0, \forall x \in [-1, 2]$

$\therefore$ f is increasing on $[-1, 2]$

Also $f(x)$ being polynomial for $x \in [-1, 2) \cup (2, 3]$

$f(x)$ is cont. on $[-1, 3]$ except possibly at $x = 2$

At $x = 2$,

L.H.L. $= \lim\limits_{h\to 0} f(2-h) = \lim\limits_{h\to 0} 3(2-h)^2 + 12(2-h) - 1 = 35$

R.H.L. $= \lim\limits_{h\to 0} f(2+h) = \lim\limits_{h\to 0} 37 - (2+h) = 35$

and $f(2) = 3.2^2 + 12.2 - 1 = 35$

Thus, L.H.L. = R.H.L. $= f(2) \Rightarrow f(x)$ is continuous at $x = 2$

Hence, $f(x)$ is continuous on $[-1, 3]$

Again at $x = 2$

$RD = \lim\limits_{h\to 0} \frac{f(2+h) - f(2)}{h} = \lim\limits_{h\to 0} \frac{37 - (2+h) - 35}{h} = 1$

$LD = \lim\limits_{h\to 0} \frac{f(2) - f(2-h)}{h}$

$= \lim\limits_{h\to 0} \frac{35 - 3(2-h)^2 - 12(2-h) + 1}{h} = 24$

Thus, $LD \ne RD \Rightarrow f'(2)$ does not exist.

Hence, $f(x)$ can not have maximum value at $x = 2$.

22. **(a)** Since g is decreasing in $[0, \infty)$

$\therefore$ For $x \ge y, g(x) \le g(y)$

Also $g(x), g(y) \in [0, \infty)$ and f is increasing from $[0, \infty)$ to $[0, \infty)$.

Now, $g(x), g(y) \in [0, \infty)$ and $g(x) \le g(y)$

$\Rightarrow f(g(x)) \le f(g(y))$, where $x \ge y$, $\therefore h(x) \le h(y)$

$\Rightarrow h$ is decreasing function from $[0, \infty)$ to $[0, \infty)$

$\therefore\quad h(x) \le h(0),\ \forall\ x \ge 0$ But $h(0) = 0$ (given)

$\therefore\quad h(x) \le 0 \forall\ x \ge 0$ ...(i)

Also $h(x) \ge 0 \forall\ x \ge 0$ [as $h(x) \in [0, \infty)$] ...(ii)

From (i) and (ii), we get $h(x) = 0,\ \forall\ x \ge 0$

Hence, $h(x) - h(1) = 0 - 0 = 0 \forall\ x \ge 0$

(For questions 23-25) : $f(x) = x + \log_e x - x\log_e x,\ x \in (0, \infty)$

$\Rightarrow\quad f'(x) = \frac{1}{x} - \log_e x$ and $f''(x) = -\frac{1+x}{x^2}$

$f(1) = 1 > 0$ and $f(e^2) = e^2 + 2 - 2e^2 = 2 - e^2 < 0$

$\Rightarrow\quad f(x) = 0$ for some $x \in (1, e^2)$

Hence, (I) is true.

$f'(1) = 1 > 0$ and $f'(e) = \frac{1}{e} - 1 < 0$

$\Rightarrow\quad f'(x) = 0$ for some $x \in (1, e)$

Hence, (II) is true.

If $x \in (0, 1)$, $\frac{1}{x} > 0$ and $\log_e x < 0$

$\Rightarrow\quad f'(x) = \frac{1}{x} - \log_e x > 0 \Rightarrow f$ is increasing on (0, 1)

$\Rightarrow\quad f'(x) \ne 0$ for some $x \in (0, 1)$

Hence, (III) is false.

If $x \in (1, e)$, $f''(x) < 0 \Rightarrow f'$ is decreasing on (1, e)

$\Rightarrow\quad f''(x) \ne 0$ for some $x \in (1, e)$

Hence, (IV) is false.

Also $\lim\limits_{x\to\infty} f(x) = \lim\limits_{x\to\infty} x + (1-x)\log_e x = -\infty$

Therefore, (i) is false and (ii) is true.

$\lim\limits_{x\to\infty} f'(x) = \lim\limits_{x\to\infty} \frac{1}{x} - \log_e x = -\infty$

Hence, (iii) is true

$\lim\limits_{x\to\infty} f''(x) = \lim\limits_{x\to\infty} -\frac{1}{x^2} - \frac{1}{x} = 0$

Hence, (iv) is true.

f is increasing on (0, 1) already discussed.

Hence, (P) is true.

If $x \in (e, e^2)$ then $f'(x) = \frac{1}{x} - \log_e x < 0$

$\Rightarrow\quad$ f is decreasing in (e, e^2)

Hence, (Q) is true.

For $x \in (0, 1)$, $f''(x) < 0 \Rightarrow f'$ is decreasing in (0, 1)

Hence, R is false.

For $x \in (e, e^2)$, $f''(x) < 0 \Rightarrow f'$ decreasing in (e, e^2)

Hence, (S) is true.

23. **(b)** The only correct combination is (II), (ii), (Q)

24. **(b)** The only correct combination is (II), (iii), (S)

25. **(c)** The only incorrect combination is (III), (i), (R).

26. **(A) → (p), (B) → (r)**

(A) $f(x) = x + \sin x$ on $(-\pi/2, \pi/2) \Rightarrow f'(x) = 1 + \cos x$

Since $0 \le \cos x \le 1$ for $x \in (-\pi/2, \pi/2)$

Hence, $f'(x) > 0$ on $(-\pi/2, \pi/2)$

(A) → p

(B) $f(x) = \sec x \Rightarrow f'(x) = \sec x \tan x$.

Clearly $f'(x) < 0$ in $(-\pi/2, 0)$ and $f'(x) > 0$ in $(0, \pi/2)$

$\therefore$ On $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ $f(x)$ is neither increasing nor decreasing.

(B) → r

27. Given : $\frac{dP(x)}{dx} > P(x), \forall, x \ge 1$ and $P(1) = 0$

$\Rightarrow\quad \frac{dP(x)}{dx} - P(x) > 0$

Multiplying by e^{-x} on both sides, we get

$e^{-x}\frac{dP(x)}{dx} - e^{-x}P(x) > 0 \Rightarrow \frac{d}{dx}\left[e^{-x}P(x)\right] > 0$

$\Rightarrow e^{-x}P(x)$ is an increasing function.

$\therefore \quad \forall x > 1, e^{-x}P(x) > e^{-1}P(1) = 0 \qquad [\because P(1) = 0]$

$\Rightarrow \quad e^{-x}P(x) > 0, \forall x > 1$

$\because \quad e^{-x} > 0 \ \therefore \ P(x) > 0, \forall x > 1$

28. (i) From Lagrange's mean value theorem

$$\frac{f(4) - f(0)}{4 - 0} = f'(a) \text{ for } a \in (0, 4) \qquad \text{......(i)}$$

Also from Intermediate mean value theorem

$$\frac{f(4) + f(0)}{2} = f(b) \text{ for } b \in (0, 4) \qquad \text{......(ii)}$$

From (i) and (ii), we get

$$\frac{(f(4))^2 - (f(0))^2}{8} = f'(a) f(b) \text{ for } a, b \in (0, 4)$$

(ii) $\displaystyle\int_0^4 f(t)dt = 2\int_0^2 x f(x^2)dx, t = x^2$

$$= 2\int_0^1 x f(x^2)dx + 2\int_1^2 x f(x^2)dx = 2I_1 + 2I_2 \text{ (say)} \quad \text{.... (i)}$$

Let $F_1(x) = \displaystyle\int_0^x t f(t^2)dt, 0 \le x \le 1$

and $F_2(x) = \displaystyle\int_1^x t f(t^2)dt, 1 \le x \le 2$

$F_1(x)$ and $F_2(x)$ are differentiable on [0, 2]

By Langranges mean value theorem $\exists \alpha \in (0,1)$

and $\beta \in (1, 2)$ such that

$$F_1'(\alpha) = \frac{F_1(1) - F_1(0)}{1 - 0} = \alpha f(\alpha^2)$$

$$F_2'(\beta) = \frac{F_2(2) - F_2(1)}{2 - 1} = \beta f(\beta^2)$$

Since $F_1(0) = 0$, $F_2(1) = 0$,

we get $I_1 = \alpha f(\alpha^2)$ and $I_2 = \beta f(\beta^2)$

$\therefore$ From (i), $\displaystyle\int_0^4 f(t)dt = 2\left[\alpha f(\alpha^2) + \beta f(\beta^2)\right]$

29. Given : $2(1 - \cos x) < x^2, \ x \ne 0$

To prove $\sin(\tan x) \ge x, \ x \in [0, \pi/4)$.

Let us consider $f(x) = \sin(\tan x) - x$

$$\Rightarrow \quad f'(x) = \cos(\tan x)\sec^2 x - 1 = \frac{\cos(\tan x) - \cos^2 x}{\cos^2 x}$$

As given $2(1 - \cos x) < x^2, \ x \ne 0 \Rightarrow \cos x > 1 - \dfrac{x^2}{2}$

Similary, $\cos(\tan x) > 1 - \dfrac{\tan^2 x}{2}$

$$\therefore \ f'(x) > \frac{1 - \frac{1}{2}\tan^2 x - \cos^2 x}{\cos^2 x} = \frac{\sin^2 x\left[1 - \frac{1}{2\cos^2 x}\right]}{\cos^2 x}$$

$$= \frac{\sin^2 x(\cos 2x)}{2\cos^4 x} > 0, \forall x \in [0, \pi/4)$$

$\therefore \quad f'(x) > 0 \Rightarrow f(x)$ is an increasing function.

$\therefore$ For $x \in [0, \pi/4)$, $x \ge 0 \Rightarrow f(x) \ge f(0)$

$\Rightarrow \quad \sin(\tan x) - x \ge \sin(\tan 0) - 0$

$\Rightarrow \quad \sin(\tan x) - x \ge 0 \Rightarrow \sin(\tan x) \ge x$ (proved)

30. Given that $-1 \le p \le 1$.

Consider $f(x) = 4x^3 - 3x - p = 0$

$$\therefore \quad f(1/2) = \frac{1}{2} - \frac{3}{2} - p = -1 - p \le 0 \text{ as } (-1 \le p)$$

and $f(1) = 4 - 3 - p = 1 - p \ge 0$ as $(p \le 1)$

$\Rightarrow \ f(x)$ has at least one real root between [1/2, 1].

Also $f'(x) = 12x^2 - 3 > 0$ on [1/2, 1]

$\Rightarrow f$ is increasing on [1/2, 1]

$\Rightarrow f$ has only one real root between [1/2, 1]

To find the root, we observe $f(x)$ contains $4x^3 - 3x$ which is multipe angle formula of $\cos 3\theta$ if we put $x = \cos\theta$.

$\therefore$ Let the required root be $\cos\theta$, then $4\cos^3\theta - 3\cos\theta - p = 0$

$$\Rightarrow \cos 3\theta = p \Rightarrow 3\theta = \cos^{-1} p \Rightarrow \theta = \frac{1}{3}\cos^{-1}(p)$$

$$\therefore \quad \text{Root is } \cos\left(\frac{1}{3}\cos^{-1}(p)\right)$$

31. Given : $p(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n \quad \ldots$ (i)

and $\quad |p(x)| \le |e^{x-1} - 1|, \ \forall \ x \ge 0 \qquad \ldots$ (ii)

To prove that $|a_1 + 2a_2 + \ldots + n a_n| \le 1$

It can be clearly seen that in order to prove the result it is sufficient to prove that $|p'(1)| \le 1$

We know that $|p'(1)| = \displaystyle\lim_{h \to 0}\left|\frac{p(1+h) - p(1)}{h}\right|$

$$\le \lim_{h \to 0}\frac{|p(1+h)| + |p(1)|}{|h|}$$

[using $|x - y| \le |x| + |y|$]

But $|p(1)| \le |e^0 - 1|$ [using equation (ii) for $x = 1$]

$\Rightarrow \ |p(1)| \le 0$

But being absolute value, $|p(1)| \ge 0$

Thus we must have $|p(1)| = 0$

Now $|p(1+h)| \le |e^h - 1|$ (using eqn. (ii) for $x = 1 + h$)

$$\therefore \quad |p'(1)| \le \lim_{h \to 0}\frac{|e^h - 1|}{|h|} = 1$$

or $\quad |p'(1)| \le 1 \ \Rightarrow \ |a_1 + 2a_2 + \ldots + na^n| \le 1$

32. Given that, $f(x) = \begin{cases} xe^{ax}, & x \le 0 \\ x + ax^2 - x^3, & x > 0 \end{cases}$

Differentiating both sides, we have

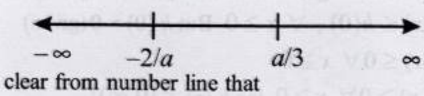
$$f'(x) = \begin{cases} axe^{ax} + e^{ax}, & x \le 0 \\ 1 + 2ax - 3x^2, & x > 0 \end{cases}$$

Again differentiating both sides, we have

$$f''(x) = \begin{cases} 2ae^{ax} + a^2 xe^{ax}; & x \le 0 \\ 2a - 6x; & x > 0 \end{cases}$$

For critical points, we put $f''(x) = 0$

$$\Rightarrow \quad x = \begin{cases} -\dfrac{2}{a}, \text{ if } x \le 0 \\ \dfrac{a}{3}, \text{ if } x > 0 \end{cases}$$



It is clear from number line that

$f''(x)$ is +ve on $\left(-\dfrac{2}{a}, \dfrac{a}{3}\right)$

$\Rightarrow$ $f'(x)$ increases on $\left(-\dfrac{2}{a}, \dfrac{a}{3}\right)$

33. $f(x)=\begin{cases}-x^3+\dfrac{b^3-b^2+b-1}{b^2+3b+2}, & 0\le x<1\\ 2x-3, & 1\le x\le 3\end{cases}$

$\therefore\ f(1)=2(1)-3=-1$
Also $f(x)$ is increasing on $[1, 3]$, $f'(x)$ being $2>0$.
$\therefore\ f(1)=-1$ is the smallest value of $f(x)$
Now $f'(x)=-3x^2$ for $x\in[0,1]$ such that $f'(x)<0$
$\Rightarrow f(x)$ is decreasing on $[0, 1]$
$\therefore$ For fixed value of b, its smallest occur when $x\to 1$

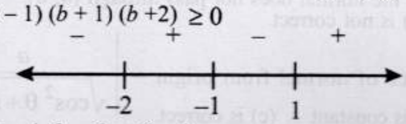
i.e., $\lim\limits_{h\to 0} f(1-h)=\lim\limits_{h\to 0}\left[-(1-h)^3+\dfrac{b^3-b^2+b-1}{b^2+3b+2}\right]$

$$=-1+\frac{b^3-b^2+b-1}{b^2+3b+2}$$

As given that the smallest value of $f(x)$ occur at $x=1$
$\therefore$ Any other smallest value $\ge f(1)$

$$\Rightarrow -1+\frac{b^3-b^2+b-1}{b^2+3b+2}\ge -1 \Rightarrow \frac{b^3-b^2+b-1}{b^2+3b+2}\ge 0$$

$\Rightarrow (b-1)(b+1)(b+2)\ge 0$



$\therefore\ b\in(-2,-1)\cup(1,\infty)$.

34. Let $f(x)=2\sin x+\tan x-3x$ on $0\le x<\pi/2$
$\Rightarrow f'(x)=2\cos x+\sec^2 x-3$
and $f''(x)=-2\sin x+2\sec^2 x\tan x=2\sin x\,[\sec^3 x-1]$
for $0\le x<\pi/2$, $f''(x)\ge 0$
$\Rightarrow f'(x)$ is an increasing function on $0\le x<\pi/2$.
Hence, for $x\ge 0$, $f'(x)\ge f'(0)$
$\therefore\ f'(x)\ge 0$ for $0\le x<\pi/2$
$\Rightarrow f(x)$ is an increasing function on $0\le x<\pi/2$
Hence, for $x\ge 0$, $f(x)\ge f(0)$
$\Rightarrow 2\sin x+\tan x-3x\ge 0,\ 0\le x<\pi/2$
$\Rightarrow 2\sin x+\tan x\ge 3x,\ 0\le x<\pi/2$ (proved)

35. To show: $1+x\ln(x+\sqrt{x^2+1})\ge\sqrt{1+x^2}$ for $\forall\, x\ge 0$

Consider $f(x)=1+x\ln(x+\sqrt{x^2+1})-\sqrt{1+x^2}$

Now, $f'(x)=\ln(x+\sqrt{x^2+1})+\dfrac{x}{x+\sqrt{x^2+1}}\left[1+\dfrac{x}{\sqrt{x^2+1}}\right]-\dfrac{x}{\sqrt{1+x^2}}$

$=\ln(x+\sqrt{x^2+1})$

Since $x+\sqrt{x^2+1}\ge 1$ for $x\ge 1$

$\Rightarrow\ \ln(x+\sqrt{x^2+1})\ge 0 \Rightarrow f'(x)\ge 0, \forall\, x\ge 0$
Therefore, $f(x)$ is increasing function.
Now for $x\ge 0 \Rightarrow f(x)\ge f(0)$

$\Rightarrow 1+x\ln(x+\sqrt{x^2+1})-\sqrt{1+x^2}\ge 0$

$\Rightarrow\ 1+x\ln(x+\sqrt{x^2+1})\ge\sqrt{1+x^2}$

Topic-3: Tangents & Normals

1. **(b)** Given hyperbola is $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1$

$$\Rightarrow \frac{2x}{a^2}-\frac{2y}{b^2}\frac{dy}{dx}=0\Rightarrow \frac{dy}{dx}=\frac{b^2x}{a^2y}$$

$\therefore$ Slope of normal at P (6, 3) $=-\dfrac{1}{\left(\dfrac{dy}{dx}\right)_{(6,3)}}=-\dfrac{3a^2}{6b^2}$

$\therefore$ Equation of normal is $\dfrac{y-3}{x-6}=-\dfrac{3a^2}{6b^2}$

As it intersects x-axis at (9, 0)

$\therefore\ \dfrac{0-3}{9-6}=\dfrac{-3a^2}{6b^2}\Rightarrow a^2=2b^2$ ...(i)

Now for hyperbola, $b^2=a^2(e^2-1)$
$\Rightarrow\ b^2=2b^2(e^2-1)$ [using (i)]

$\Rightarrow\ \dfrac{1}{2}=e^2-1 \Rightarrow\ e=\sqrt{\dfrac{3}{2}}$

2. **(b)** Given curves are $C_1: y^2=4x$ ...(i)
and $C_2: x^2+y^2-6x+1=0$ ...(ii)
On solving (i) and (ii), we get
$x=1$ and $y=2$ or -2
$\therefore$ Points of intersection of the two curves are (1, 2) and (1, −2).

For C_1, $\dfrac{dy}{dx}=\dfrac{2}{y}$

$\therefore\ \left(\dfrac{dy}{dx}\right)_{(1,2)}=1=m_1$ and $\left(\dfrac{dy}{dx}\right)_{(1,-2)}=-1=m_1'$

For C_2, $\dfrac{dy}{dx}=\dfrac{3-x}{y}$

$\therefore\ \left(\dfrac{dy}{dx}\right)_{(1,2)}=1=m_2$ and $\left(\dfrac{dy}{dx}\right)_{(1,-2)}=-1=m_2'$

$\because\ m_1=m_2$ and $m_1'=m_2'$,
$\therefore$ C_1 and C_2 touch each other at two points.

3. **(a)** Given curve $y=e^x\Rightarrow \dfrac{dy}{dx}=e^x$

Slope of the tangent to the given curve at $(c, e^c)=e^c$
The equation of tangent to the curve $y=e^x$ at (c, e^c) is

$$y-e^c=e^c(x-c)\quad ...(i)$$

Now, equation of line joining $(c-1, e^{c-1})$ and $(c+1, e^{c+1})$ is

$$y-e^{c-1}=\frac{e^{c+1}-e^{c-1}}{(c+1)-(c-1)}[x-(c-1)]$$

$$\Rightarrow\ y-e^{c-1}=\frac{e^c(e-e^{-1})}{2}[x-c+1]\quad ...(ii)$$

Subtracting equation (i) from (ii), we get

$$e^c-e^{c-1}=e^c(x-c)\left[\frac{e-e^{-1}-2}{2}\right]+e^c\left(\frac{e-e^{-1}}{2}\right)$$

$$\Rightarrow\ x-c=\frac{\left[1-e^{-1}-\left(\dfrac{e-e^{-1}}{2}\right)\right]}{\dfrac{e-e^{-1}-2}{2}}=\frac{2-e-e^{-1}}{e-e^{-1}-2}$$

$$= \frac{e+e^{-1}-2}{2-(e-e^{-1})} = \frac{\frac{e+e^{-1}}{2}-1}{1-\frac{e-e^{-1}}{2}} = \frac{+ve}{-ve} = -ve$$

$\Rightarrow x-c<0 \Rightarrow x<c$

Hence, the two lines meet on the left of line $x = c$.

4. **(b)** Let the polynominal be $P(x) = ax^2 + bx + c$

Given $P(0) = 0$ and $P(1) = 1$

$\Rightarrow c = 0$ and $a + b = 1 \Rightarrow a = 1 - b$

$\therefore P(x) = (1-b)x^2 + bx, \Rightarrow P'(x) = 2(1-b)x + b$

Given $P'(x) > 0, \forall x \in [0,1] \Rightarrow 2(1-b)x + b > 0$

$\Rightarrow$ When $x = 0$, $b > 0$ and when $x = 1$, $b < 2 \Rightarrow 0 < b < 2$

Hence, $S = \left\{(1-a)x^2 + ax, a \in (0,2)\right\}$

5. **(d)** Given : $f(x) = x^\alpha \log x$ and $f(0) = 0$

For Rolle's theorem in $[a, b]$

$f(a) = f(b)$, In $[0, 1] \Rightarrow f(0) = f(1) = 0$

Since, the function has to be continuous in $[0, 1]$

$$\therefore \quad f(0) = \lim_{x\to 0^+} f(x) = 0 \Rightarrow \lim_{x\to 0} x^\alpha \log x = 0$$

$$\Rightarrow \lim_{x\to 0}\frac{\log x}{x^{-\alpha}} = 0 \Rightarrow \lim_{x\to 0}\frac{1/x}{-\alpha x^{-\alpha-1}} = 0 \quad \text{[L Rule]}$$

$$\Rightarrow \lim_{x\to 0}\frac{-x^\alpha}{\alpha} = 0 \Rightarrow \alpha > 0$$

6. **(a)** There is only one function in option (a) whose critical point $\frac{1}{2} \in (0,1)$. For rest of the parts critical point $0 \notin (0, 1)$. It can be easily seen that functions in options (b), (c) and (d) are continuous on $[0, 1]$ and differentiable in $(0, 1)$.

$$\text{Now for } f(x) = \begin{cases} \frac{1}{2}-x, & x < 1/2 \\ \left(\frac{1}{2}-x\right)^2, & x \ge 1/2 \end{cases}$$

$$\text{Here } f'\left(\frac{1^-}{2}\right) = -1 \text{ and } f'\left(\frac{1^+}{2}\right) = -2\left(\frac{1}{2}-\frac{1}{2}\right) = 0$$

$$\therefore \quad f'\left(\frac{1^-}{2}\right) \ne f'\frac{1^+}{2}$$

Thus f is not differentiable at $1/2 \in (0,1)$

Hence, LMV is not applicable for this function in $[0, 1]$

7. **(d)** Given curve : $y^3 + 3x^2 = 12y$

$$\Rightarrow 3y^2\frac{dy}{dx} + 6x = 12\frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{2x}{4-y^2}$$

For vertical tangents $\frac{dy}{dx} = \frac{1}{0} \Rightarrow 4 - y^2 = 0 \Rightarrow y = \pm 2$

For $y = 2$, $x^2 = \frac{24-8}{3} = \frac{16}{3} \Rightarrow x = \pm\frac{4}{\sqrt{3}}$

For $y = -2$, $x^2 = \frac{-24+8}{3} = -ve$ (not possible)

Hence, required. points are $(\pm 4/\sqrt{3}, 2)$.

8. **(c)** Equation of tangent to $y = x^2 + bx - b$ at $(1, 1)$ is

$$\frac{y+1}{2} = x\cdot 1 + b\left(\frac{x+1}{2}\right) - b \Rightarrow (b+2)x - y = b+1$$

Its x-intercept $= \frac{b+1}{b+2}$ and y-intercept $= -(b+1)$

Given area $(\Delta) = 2 \Rightarrow \frac{1}{2}\left(\frac{b+1}{b+2}\right)[-(b+1)] = 2$

$\Rightarrow (b+3)^2 = 0 \Rightarrow b = -3$

9. **(d)** For $y^2 = 4ax$, y-axis is tangent at $(0, 0)$, while for $x^2 = 4ay$, x-axis is tangent at $(0, 0)$. Hence, the two curves $y^2 = 4ax$ and $x^2 = 4ay$ cut each other at right angles.

10. **(c)** Given: $x = a(\cos\theta + \theta\sin\theta)$ and $y = a(\sin\theta - \theta\cos\theta)$

$$\therefore \quad \frac{dx}{d\theta} = a(-\sin\theta + \sin\theta + \theta\cos\theta) = a\theta\cos\theta$$

$$\text{and } \frac{dy}{d\theta} = a(\cos\theta - \cos\theta + \theta\sin\theta) = a\theta\sin\theta$$

Dividing (2) by (1), we get

$$\text{Now, } \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \tan\theta = \text{Slope of tangent}$$

$\therefore$ Slope of normal $= -\cot\theta$ $\therefore$ Equation of normal is

$$y - a(\sin\theta - \theta\cos\theta) = -\frac{\cos\theta}{\sin\theta}(x - a(\cos\theta + \theta\sin\theta))$$

$\Rightarrow x\cos\theta + y\sin\theta = a$

As θ varies inclination of normal is not constant.

$\therefore$ (a) is not correct.

Clearly the normal does not pass through $(0, 0)$.

$\therefore$ (b) is not correct.

Distance of normal from origin $= \left|\frac{a}{\sqrt{\cos^2\theta + \sin^2\theta}}\right| = a$ which is constant $\therefore$ (c) is correct.

11. **(8)** $(y - x^5)^2 = x(1+x^2)^2$

$$2(y-x^5)\cdot\left(\frac{dy}{dx} - 5x^4\right) = (1+x^2)^2 + 2x(1+x^2)\cdot 2x$$

At point $(1, 3)$

$$2(3-1)\left(\frac{dy}{dx} - 5\right) = (1+1)^2 + 2(1+1).2 \Rightarrow \frac{dy}{dx} = 8$$

12. Given curve is $C : y^3 - 3xy + 2 = 0$

On differentiating it with respect to x, we get

$$3y^2\frac{dy}{dx} - 3x\frac{dy}{dx} - 3y = 0 \Rightarrow \frac{dy}{dx} = \frac{y}{-x+y^2}$$

$\therefore$ Slope of tangent to C at point (x_1, y_1) is

$$\frac{dy}{dx} = \frac{y_1}{-x_1 + y_1^2}$$

For horizontal tangent, $\frac{dy}{dx} = 0 \Rightarrow y_1 = 0$

For $y_1 = 0$ in C, we get no value of x_1

$\therefore$ There is no point on C at which tangent is horizontal.

$\therefore$ $H = \phi$

For vertical tangent $\frac{dy}{dx} = \frac{1}{0} \Rightarrow -x_1 + y_1^2 = 0 \Rightarrow x_1 = y_1^2$

From C, $y_1^3 - 3y_1^3 + 2 = 0 \Rightarrow y_1 = 1 \Rightarrow x_1 = 1$

$\therefore$ There is only one point $(1, 1)$ on C at which vertical tangent can be drawn

$\therefore$ $V = \{(1, 1)\}$

13. **(b, c)** Let $h(x) = f(x) - 3g(x)$

$h(-1) = h(0) = h(2) = 3$

$\therefore$ By Rolle's theorem $h'(x) = 0$ has atleast one solution in $(-1, 0)$ and atleast one solution in $(0, 2)$ But $h''(x)$ never vanishes in $(-1, 0)$ and $(0, 2)$ therefore $h'(x) = 0$ should have exactly one solution in each of the two intervals $(-1, 0)$ and $(0, 2)$.

14. **(b, c)** Let the line $ax + by + c = 0$ be normal to the curve $xy = 1$ at the point (x', y'), then $x'y' = 1$ ...(i)

On differentiating the curve $xy = 1$ w.r.t. x, we get

$$y + x\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{y}{x}$$

$$\Rightarrow \left.\frac{dy}{dx}\right)_{(x',y')} = \frac{-y'}{x'}, \quad \therefore \text{ Slope of normal } = \frac{x'}{y'}$$

$ax + by + c = 0$ be also the normal, therefore slope of normal

$$= \frac{-a}{b}, \therefore \frac{x'}{y'} = -\frac{a}{b} \quad ...(ii)$$

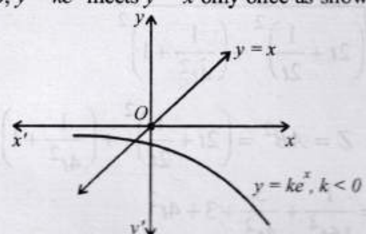
From eq. (i), $x'y' > 0 \Rightarrow x', y'$ are of same sign.

$$\Rightarrow \frac{x'}{y'} = +ve \Rightarrow -\frac{a}{b} = +ve \Rightarrow \frac{a}{b} = -ve$$

$\Rightarrow$ a and b are of opposite sign.

$\Rightarrow$ Either $a < 0$ and $b > 0$ or $a > 0$ and $b < 0$.

15. **(b)** For $k = 0$, line $y = x$ meets $y = 0$, i.e., x-axis only at one point. For $k < 0$, $y = ke^x$ meets $y = x$ only once as shown in the graph.



16. **(a)** Let $f(x) = ke^x - x$

Now for $f(x) = 0$ to have only one root means the line $y = x$ must be tangential to the curve $y = ke^x$.

Let it be so at (x_1, y_1), then

$$\left(\frac{dy}{dx}\right)_{\text{curve1}} = \left(\frac{dy}{dx}\right)_{\text{curve2}} \Rightarrow 1 = ke^{x_1} \Rightarrow e^{x_1} = \frac{1}{k}$$

Also $y_1 = ke^{x_1}$ and $y_1 = x_1$

$\Rightarrow x_1 = 1 \Rightarrow 1 = ke \Rightarrow k = 1/e$

17. **(a)** For $y = x$ to be tangent to the curve $y = ke^x$, $k = 1/e$

$\therefore$ For $y = ke^x$ to meet $y = x$ at two points we should have

$$k < \frac{1}{e} \Rightarrow k \in \left(0, \frac{1}{e}\right) \text{ as } k > 0.$$

18. Given : $|f(x_1) - f(x_2)| < (x_1 - x_2)^2$, $x_1, x_2 \in R$

Let $x_1 = x + h$ and $x_2 = x$

$\therefore$ $|f(x+h) - f(x)| < h^2 \Rightarrow |f(x+h) - f(x)| < |h|^2$

$$\Rightarrow \left|\frac{f(x+h) - f(x)}{h}\right| < |h|$$

Taking limit as $h \to 0$ on both sides, we get

$$\lim_{h\to 0}\left|\frac{f(x+h) - f(x)}{h}\right| < \delta \text{ (a small +ve number)}$$

$\Rightarrow |f'(x)| < \delta \Rightarrow f'(x) = 0$

$\Rightarrow f(x)$ is a constant function. Let $f(x) = k$ i.e., $y = k$

Since $f(x)$ passes through $(1, 2)$, $\therefore y = 2$

Hence, equation of tangent at $(1, 2)$ is

$y - 2 = 0\,(x - 1)$ i.e. $y = 2$

19. Given : $P(x) = 51x^{101} - 2323x^{100} - 45x + 1035$

To show that at least one root of $P(x)$ lies in $(45^{1/100}, 46)$, using Rolle's theorem, we consider antiderivative of $P(x)$ i.e.

$$F(x) = \frac{x^{102}}{2} - \frac{2323x^{101}}{101} - \frac{45x^2}{2} + 1035x$$

Since $F(x)$ a polynominal function, $F(x)$ is continuous and differentiable.

$$\text{Now, } F(45^{1/100}) = \frac{(45)^{\frac{102}{100}}}{2} - \frac{2323(45)^{\frac{101}{100}}}{101} - \frac{45.(45)^{\frac{2}{100}}}{2} + 1035(45)^{\frac{1}{100}}$$

$$= \frac{45}{2}(45)^{\frac{2}{100}} - 23 \times 45(45)^{\frac{1}{100}} - \frac{45.(45)^{\frac{2}{100}}}{2} + 1035(45)^{\frac{1}{100}} = 0$$

$$\text{And } F(46) = \frac{(46)^{102}}{2} - \frac{2323(46)^{101}}{101} - \frac{45(46)^2}{2} + 1035(46)$$

$= 23\,(46)^{101} - 23\,(46)^{101} - 23 \times 45 \times 46 + 1035 \times 46 = 0$

$\therefore$ $F(45^{\frac{1}{100}}) = F(46) = 0$

$\therefore$ Rolle's theorem is applicable.

Hence, there must exist at least one root of $F'(x) = 0$

i.e. $P(x) = 0$ in the interval $\left(45^{\frac{1}{100}}, 46\right)$

20. Given equation of parabola is

$y^2 - 2y - 4x + 5 = 0$ ...(i)

$\Rightarrow$ $(y - 1)^2 = 4\,(x - 1)$

Any parametric point on this parabola is $P\,(t^2 + 1, 2t + 1)$

On differentiating equation (i) w.r. to x, we get

$$2y\frac{dy}{dx} - 2\frac{dy}{dx} - 4 = 0 \qquad \Rightarrow \frac{dy}{dx} = \frac{2}{y-1}$$

$\therefore$ Slope of tangent to parabola (i) at P $(t^2 + 1, 2t + 1)$ is

$$m = \frac{2}{2t} = \frac{1}{t}$$

$\therefore$ Equation of tangent at P $(t^2 + 1, 2t + 1)$ is

$$y - (2t + 1) = \frac{1}{t}(x - t^2 - 1)$$

$\Rightarrow$ $yt - 2t^2 - t = x - t^2 - 1$

$\Rightarrow$ $x - yt + (t^2 + t - 1) = 0$ ...(ii)

Now directrix of given parabola is $(x - 1) = -1 \Rightarrow x = 0$

Tangent (ii) meets directix at $Q\left(0, \frac{t^2 + t - 1}{t}\right)$

Low let point R be (h, k).

Since R divides the line segment joining Q and P in the ratio $\frac{1}{2} : 1$ i.e., $1 : 2$ externally.

$$\therefore \quad (h, k) = \left(\frac{1(1 + t^2) - 0}{-1}, \frac{t + 2t^2 - 2t^2 - 2t + 2}{-t}\right)$$

$$\Rightarrow \quad h = -(1 + t^2) \text{ and } k = \frac{t - 2}{t} \Rightarrow t^2 = -1 - h \text{ and } t = \frac{2}{1 - k}$$

On eliminating t, we get $\left(\frac{2}{1-k}\right)^2 = -1 - h$

$\Rightarrow$ $4 = -(1 - k)^2\,(1 - h) \Rightarrow (h - 1)\,(k - 1)^2 + 4 = 0$

$\therefore$ Locus of R (h, k) is $(x - 1)\,(y - 1)^2 + 4 = 0$

21. Equation of the tangent at point (x, y) on the curve is
$Y - y = dy/dx\,(X - x)$
This meets axes in A$(x - y\,dx/dy, 0)$ and B$(0, y - x\,dy/dx)$ mid point of AB is

$$\left[\frac{1}{2}\left(x - y\frac{dx}{dy}\right), \frac{1}{2}\left(y - x\frac{dy}{dx}\right)\right]$$

Given $\frac{1}{2}\left(x - y\frac{dx}{dy}\right) = x$ and $\frac{1}{2}\left(y - x\frac{dy}{dx}\right) = y$

$$\frac{xdy}{dx} = -y \text{ or } \frac{dy}{y} = -\frac{dx}{x}$$

Integrating, we get $\log y = -\log x + a$
Putting $x = 1, y = 1$ we get
$\log 1 = -\log 1 + a \Rightarrow a = 0$
$\therefore \log y + \log x = 0 \Rightarrow \log(xy) = 0$
$xy = e^0 = 1$
which is a rectangular hyperbola.

22. Given that $y = ax^3 + bx^2 + cx + 5$ touches the x-axis at $P(-2, 0)$

$\Rightarrow \left(\frac{dy}{dx}\right)_{x=-2} = 0$ and $P(-2, 0)$ lies on curve

$\Rightarrow 3ax^2 + 2bx + c]_{x=-2} = 0 \Rightarrow 12a - 4b + c = 0$ ...(i)
and $-8a + 4b - 2c + 5 = 0$ ...(ii)
Also the curve cuts the y-axis at Q
Now, for $x = 0, y = 5 \;\therefore Q(0, 5)$
At Q gradient of the curve is 3

$\Rightarrow \left.\frac{dy}{dx}\right|_{x=0} = 3 \Rightarrow 3ax^2 + 2bx + c]_{x=0} = 3$

$\Rightarrow c = 3$ ...(iii)
On solving (i), (ii) and (iii), we get
$a = -1/2, b = -3/4$ and $c = 3$

23. The given curve is $y = (1 + x)^y + \sin^{-1}(\sin^2 x)$
Here at $x = 0, y = (1 + 0)^y + \sin^{-1}(0) \Rightarrow y = 1$
$\therefore$ Point at which normal has been drawn is (0, 1).
For slope of normal we need to find dy/dx, and for that we consider the curve as $y = u + v \Rightarrow \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$
where $u = (1 + x)^y$ ...(i)
and $v = \sin^{-1}(\sin^2 x)$ ...(ii)
On taking log on both sides of (i) we get
$\log u = y \log(1 + x)$

$$\Rightarrow \frac{1}{u}\frac{du}{dx} = \frac{y}{1+x} + \log(1+x).\frac{dy}{dx}$$

$$\Rightarrow \frac{du}{dx} = (1+x)^y\left[\frac{y}{1+x} + \log(1+x)\frac{dy}{dx}\right]$$

Now, $v = \sin^{-1}(\sin^2 x)$

$$\therefore \frac{dv}{dx} = \frac{1}{\sqrt{1 - \sin^4 x}}.2\sin x\cos x = \frac{2\sin x}{\sqrt{1 + \sin^2 x}}.$$

$$\therefore \frac{dy}{dx} = (1+x)^y\left[\frac{y}{1+x} + \log(1+x)\frac{dy}{dx}\right] + \frac{2\sin x}{\sqrt{1 + \sin^2 x}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{y(1+x)^{y-1} + \frac{2\sin x}{\sqrt{1+\sin^2 x}}}{1 - (1+x)^y \log(1+x)}$$

$\left.\frac{dy}{dx}\right|_{(0,1)} = 1$, $\therefore$ Slope of normal $= -1$

$\therefore$ Equation of normal to given curve at (0, 1) is
$y - 1 = -1(x - 0) \Rightarrow x + y = 1.$

24. The given curve is $y = x^2$ ...(i)
Consider any point $A(t, t^2)$ on (i) at which normal chord drawn is shortest.
Then eq. of normal to (i) at $A(t, t^2)$ is

$$y - t^2 = -\frac{1}{\left(\frac{dy}{dx}\right)_{(t,t^2)}}(x - t)$$

$$\Rightarrow y - t^2 = -\frac{1}{2t}(x - t) \Rightarrow x + 2ty = t + 2t^3 \quad ...(ii)$$

This normal meets the curve again at point B which can be obtained by solving (i) and (ii) as follows :
On putting $y = x^2$ in (ii), we get
$2tx^2 + x - (t + 2t^3) = 0,$
$D = 1 + 8t(t + 2t^3) = 1 + 8t^2 + 16t^4 = (1 + 4t^2)^2$

$$\therefore x = \frac{-1 + 1 + 4t^2}{4t}, \frac{-1 - 1 - 4t^2}{4t} = t, -\frac{1}{2t} - t$$

$$\therefore y = t^2, t^2 + \frac{1}{4t^2} + 1$$

Hence, $B\left(-t - \frac{1}{2t}, t^2 + \frac{1}{4t^2} + 1\right)$

$\therefore$ Length of normal chord

$$AB = \sqrt{\left(2t + \frac{1}{2t}\right)^2 + \left(\frac{1}{4t^2} + 1\right)^2}$$

Consider $Z = AB^2 = \left(2t + \frac{1}{2t}\right)^2 + \left(\frac{1}{4t^2} + 1\right)^2$

$$\Rightarrow Z = \frac{1}{16t^4} + \frac{3}{4t^2} + 3 + 4t^2$$

For shortest chord, Z should be minimum and for that $\frac{dZ}{dt} = 0$

$$\Rightarrow -\frac{1}{4t^5} - \frac{3}{2t^3} + 8t = 0 \Rightarrow (2t^2 - 1)(16t^4 + 8t^2 + 1) = 0$$

$\Rightarrow t^2 = \frac{1}{2}$ (leaving –ve values of t^2) $\Rightarrow t = \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}},$

$$\frac{d^2Z}{dt^2} = \frac{5}{4t^6} + \frac{9}{2t^4} + 8$$

$\left.\frac{d^2Z}{dt^2}\right|_{t=\frac{1}{\sqrt{2}}} = +ve$ also $\left.\frac{d^2Z}{dt^2}\right|_{t=-\frac{1}{\sqrt{2}}} = +ve$

$\therefore$ Z is minimum at $t = \frac{1}{\sqrt{2}}$ or $-\frac{1}{\sqrt{2}}$

For $t = \frac{1}{\sqrt{2}}$, normal chord is [from (ii)] $x + \sqrt{2}y = \sqrt{2}$

For $t = -\frac{1}{\sqrt{2}}$, normal chord is [from (ii)] $x - \sqrt{2}y = -\sqrt{2}$

25. Given, $y = \cos(x + y)$

$$\Rightarrow \left(\frac{dy}{dx}\right) = -\sin(x+y).\left(1 + \frac{dy}{dx}\right) \quad ...(i)$$

Since, tangent is parallel to $x + 2y = 0$,
then slope $\frac{dy}{dx} = -\frac{1}{2}$

From Eq. (i), $-\frac{1}{2} = -\sin(x+y).\left(1 - \frac{1}{2}\right)$

$\Rightarrow \sin(x + y) = 1$, which shows $\cos(x + y) = 0$.
$\therefore y = 0$

$\Rightarrow \quad x+y=\frac{\pi}{2}$ or $-\frac{3\pi}{2}$ $\therefore \quad x=\frac{\pi}{2}$ or $-\frac{3\pi}{2}$

Thus, required points are $\left(\frac{\pi}{2},0\right)$ and $\left(-\frac{3\pi}{2},0\right)$

$\therefore$ Equation of tangents are

$$\frac{y-0}{x-\pi/2}=-\frac{1}{2}$$

and $\frac{y-0}{x+3\pi/2}=-\frac{1}{2} \Rightarrow 2y=-x+\frac{\pi}{2}$

and $2y=-x-\frac{3\pi}{2} \Rightarrow x+2y=\frac{\pi}{2}$

and $x+2y=-\frac{3\pi}{2}$

are the required equation of tangenets.

26. Equation of the curve is $y=\frac{x}{1+x^2}$

$$\Rightarrow \quad \frac{dy}{dx}=\frac{1+x^2-x(2x)}{(1+x^2)^2}=\frac{1-x^2}{(1+x^2)^2}$$

Now let $f(x)=\frac{dy}{dx}=\frac{1-x^2}{(1+x^2)^2}$

$$\Rightarrow \quad f'(x)=\frac{(1+x^2)^2(-2x)-(1-x^2)2(1+x^2)(2x)}{(1+x^2)^4}$$

$$=\frac{(1+x^2)(-2x)-(1-x^2)2.2x}{(1+x^2)^3}=\frac{x(2x^2-6)}{(1+x^2)^3}$$

For the greatest value of slope, we have

$$f'(x)=\frac{x(2x^2-6)}{(1+x^2)^3}=0 \Rightarrow x=0,\pm\sqrt{3}$$

Now, $f''(x)=\frac{12x^2(3-x^2)}{(1+x^2)^4}-\frac{6(1-x^2)}{(1+x^2)^3}$

$\therefore \quad f''(0)=-6$ and $f''(\pm\sqrt{3})=\frac{3}{16}$

Thus, second order derivative at $x=0$ is negative and second order derivative at $x=\pm\sqrt{3}$ is positive.

Therefore, the tangent to the curve has maximum slope at (0, 0).

27. A parabola, $y=x^2$, a point $(0,c), 0\le c\le 5$.
Any point on parabola is (x, x^2)
Distance between (x, x^2) and (0, 1) is

$$D=\sqrt{x^2+(x^2-c)^2}$$

To minimum D we consider

$$D^2=x^4-(2c-1)x^2+c^2=\left(x^2-\frac{2c-1}{2}\right)^2+c-\frac{1}{4},$$

which is minimum when $x^2-\frac{2c-1}{2}=0 \Rightarrow x^2=\frac{2c-1}{2}$

$$\Rightarrow \quad D_{\min}=\sqrt{c-\frac{1}{4}}$$

28. Given : $f(x)$ and $g(x)$ are differentiable for $x\in[0,1]$ such that $f(0)=2$; $f(1)=6$, $g(0)=0$; $g(1)=2$

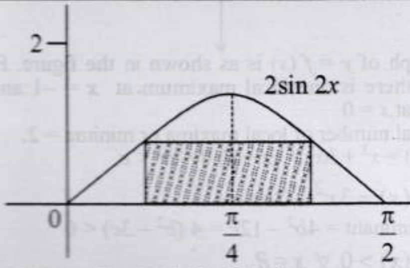
To show that there exists $c\in(0,1)$ such that $f'(c)=2g'(c)$
Let us consider $h(x)=f(x)-2g(x)$
Then $h(x)$ is continuous on [0, 1] and differentiable on (0, 1)

Now, $h(0)=f(0)-2g(0)=2-2\times 0=2$
and $h(1)=f(1)-2g(1)=6-2\times 2=2$
Hence, $h(0)=h(1)$
$\therefore$ All the conditions of Rolle's theorem are satisfied for $h(x)$ on [0, 1]

Therefore, there exists $c\in(0,1)$ such that $h'(c)=0$
$\Rightarrow f'(c)-2g'(c)=0 \Rightarrow f'(c)=2g'(c)$

Topic-4: Approximations, Maxima & Minima

1. (c)



Let perimeter of rectangle be P(x) then

$$P(x)=2\left(\frac{\pi}{2}-2x\right)+4\sin 2x=\pi-4x+4\sin 2x$$

Now, $\frac{dP}{dx}=0-4+8\cos 2x$

$$\frac{dP}{dx}=0 \Rightarrow x=\frac{\pi}{6};\ \frac{d^2P}{dx^2}=-16\sin 2x$$

$$\left(\frac{d^2P}{dx^2}\right)_{\left(x=\frac{\pi}{6}\right)}=-16\sin\frac{\pi}{3}<0$$

$\therefore \quad x=\frac{\pi}{6}$ is point of local maxima

$\therefore$ Max. length of one side of rectangle $=2\sin 2\frac{\pi}{3}=\sqrt{3}$

Length of other side $=\frac{\pi}{6}$

Hence, required area of rectangle $=\frac{\pi}{6}\times\sqrt{3}=\frac{\pi}{2\sqrt{3}}$

2. **(d)** $f''(x)>0\ \forall\ x\in R$, $f\left(\frac{1}{2}\right)=\frac{1}{2}$, $f(1)=1$

$\therefore$ f '(x) is an increasing function on R.
By Lagrange's Mean Value theorem.

$$f'(\alpha)=\frac{f(1)-f\left(\frac{1}{2}\right)}{1-\frac{1}{2}},\text{ for some } \alpha\in\left(\frac{1}{2},1\right)$$

$\Rightarrow \quad f'(\alpha)=1$ for some $\alpha\in\left(\frac{1}{2},1\right)$ $\therefore \quad f'(1)>1$

3. **(c)** Let $f(x)=4\alpha x^2+\frac{1}{x} \Rightarrow f'(x)=8\alpha x-\frac{1}{x^2}$

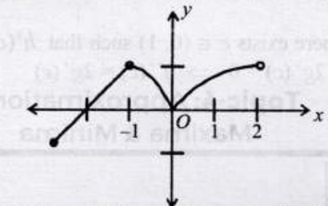
Put $f'(x)=0 \Rightarrow x=\frac{1}{2\alpha^{1/3}}$

For x > 0

$$f_{\min}=1 \Rightarrow 4\alpha\left(\frac{1}{2\alpha^{1/3}}\right)^2+2\alpha^{1/3}=1$$

$$\Rightarrow \quad 3\alpha^{\frac{1}{3}}=1 \text{ or } \alpha=\frac{1}{27}$$

4. **(c)** Given function : $f(x)=\begin{cases}(2+x)^3, & -3<x\le -1\\ x^{2/3}, & -1<x<2\end{cases}$



The graph of $y = f(x)$ is as shown in the figure. From graph, clearly, there is one local maximum at $x = -1$ and one local minima at $x = 0$

$\therefore$ Total number of local maxima or minima = 2.

5. **(a)** $f(x) = x^3 + bx^2 + cx + d,\ 0 < b^2 < c$

$\Rightarrow\ f'(x) = 3x^2 + 2bx + c$

Its discriminant $= 4b^2 - 12c = 4(b^2 - 3c) < 0$

$\therefore\ f'(x) > 0\ \forall\ x \in R$

Hence, $f(x)$ is strictly increasing $\forall\ x \in R$

6. **(c)** Equation of tangent to the ellipse $\dfrac{x^2}{27} + y^2 = 1$ at $(3\sqrt{3}\cos\theta, \sin\theta), \theta \in (0, \pi/2)$ is $\dfrac{\sqrt{3}\,x\cos\theta}{9} + y.\sin\theta = 1$

$\therefore$ Intercept on x-axis $= \dfrac{9}{\sqrt{3}\cos\theta}$ and

Intercept on y-axis $= \dfrac{1}{\sin\theta}$

$\therefore$ Sum of intercepts, $S = 3\sqrt{3}\sec\theta + \operatorname{cosec}\theta$

For minimum value of S, $\dfrac{dS}{d\theta} = 0$

$\Rightarrow\ 3\sqrt{3}\sec\theta\tan\theta - \operatorname{cosec}\theta\cot\theta = 0$

$\Rightarrow\ 3\sqrt{3}\sin^3\theta - \cos^3\theta = 0 \Rightarrow \tan^3\theta = \dfrac{1}{3\sqrt{3}} = \left(\dfrac{1}{\sqrt{3}}\right)^3$

$\Rightarrow\ \tan\theta = \dfrac{1}{\sqrt{3}} \Rightarrow \theta = \pi/6$

7. **(a)** $3\sin x - 4\sin^3 x = \sin 3x$ which increases for

$3x \in \left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right) \Rightarrow x \in \left(-\dfrac{\pi}{6}, \dfrac{\pi}{6}\right)$ whose length is $\dfrac{\pi}{3}$.

8. **(d)** $f(x) = (1 + b^2)x^2 + 2bx + 1$

It is a quadratic expression with coefficent of $x^2 = 1 + b^2 > 0$.

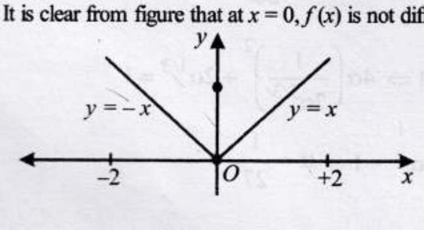
$\therefore\ f(x)$ represents an upward parabola whose minimum value is $\dfrac{-D}{4a}$. Here D is being the discreminant.

$\therefore\ m(b) = -\dfrac{4b^2 - 4(1+b^2)}{4(1+b^2)} = \dfrac{1}{1+b^2}$

Now, $\dfrac{1}{1+b^2} > 0$ and $b^2 \ge 0 \Rightarrow 1 + b^2 \ge 1$

$\Rightarrow\ \dfrac{1}{1+b^2} \le 1$. Hence $m(b) = (0, 1]$

9. **(d)** It is clear from figure that at $x = 0$, $f(x)$ is not differentiable.



$\Rightarrow\ f(x)$ has neither maximum nor minimum at $x = 0$.

10. **(b)** Let $y = x^{25}(1 - x)^{75}$

$\Rightarrow\ \dfrac{dy}{dx} = 25x^{24}(1-x)^{75} - 75x^{25}(1-x)^{74}$
$= 25x^{24}(1 - x)^{74}(1 - 4x)$

Put $\dfrac{dy}{dx} = 0 \Rightarrow x = 0,\ 1,\ 1/4$

Now $x = 1/4 \in (0, 1)$

Also at $x = 0$, $y = 0$, at $x = 1$, $y = 0$, and at $x = 1/4$, $y > 0$

$\therefore$ Maximum value of y occurs at $x = 1/4$

11. **(b)** $y = a\ln x + bx^2 + x$

has its extremum values at $x = -1$ and 2

$\therefore\ \dfrac{dy}{dx} = 0$ at $x = -1$ and 2

$\Rightarrow\ \dfrac{a}{x} + 2bx + 1 = 0$ or $2bx^2 + x + a = 0$ has -1 and 2 as its roots.

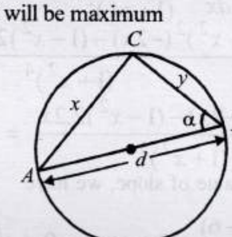
$\therefore\ 2b - 1 + a = 0$ and $8b + 2 + a = 0$

On solving the above two equations, we get $a = 2$, $b = -1/2$.

12. **(a)** In the figure, let AB is the diameter of length 'd'. Also suppose $AC = x$, $BC = y$ and $\angle ABC = \alpha$.

$\therefore$ Area of $\Delta ABC = \dfrac{1}{2} \times d\cos\alpha \times d\sin\alpha = \dfrac{d^2}{4}\sin 2\alpha$

Area of ΔABC will be maximum



when $\sin 2\alpha = 1$ i.e. $\alpha = 45°$

$\therefore\ \Delta ABC$ is an isosceles triangle.

13. **(a, b, c)** $\alpha = \displaystyle\sum_{k=1}^{\infty}\left(\frac{1}{2}\right)^{2k} = \sum_{k=1}^{\infty}\left(\frac{1}{4}\right)^{k} = \frac{1/4}{1-\frac{1}{4}} = \frac{1}{3}\ \left[\because \sin\frac{\pi}{6} = \frac{1}{2}\right]$

Hence, $g(x) = 2^{x/3} + 2^{(1-x)/3}$

Now, $g'(x) = 2^{x/3}.\dfrac{1}{3}\ln 2 - \dfrac{1}{3}2^{\frac{1}{3}}2^{\frac{-x}{3}}.\ln 2 = \dfrac{\ln 2(2^{2x/3} - 2^{1/3})}{3.2^{x/3}}$

$g'(x) = 0$ at $x = \dfrac{1}{2}$

And, $g'(x)$ changes sign from negative to positive at $x = \dfrac{1}{2}$ hence $x = \dfrac{1}{2}$ is point of local minimum as well as absolute minimum of $g(x)$ for $x \in [0, 1]$. Hence, minimum value of $g(x) =$

$g\left(\dfrac{1}{2}\right) = 2^{1/6} + 2^{1/6} = 2^{7/6}\ \Rightarrow$ Option (a) is correct

Maximum value of $g(x)$ is either equal to $g(0)$ or $g(1)$.

$g(0) = 1 + 2^{1/3}$; $g(1) = 2^{1/3} + 1$

Hence (b) and (c) are also correct

14. **(4)** Let r be the internal radius, R be the external radius and h be the internal height of the cylinder.

Now, $V = \pi r^2 h \Rightarrow h = \dfrac{V}{\pi r^2}$

Also Vol. of material $= M = \pi[(r + 2)^2 - r^2]h + \pi(r + 2)^2 \times 2$

$= 4\pi(r + 1).\dfrac{V}{\pi r^2} + 2\pi(r + 2)^2$

$$\Rightarrow \quad M = 4V\left[\frac{1}{r}+\frac{1}{r^2}\right] + 2\pi(r+2)^2$$

$$\Rightarrow \quad \frac{dM}{dr} = 4V\left[\frac{-1}{r^2}-\frac{2}{r^3}\right] + 4\pi(r+2)$$

For min. value of M, put $\frac{dM}{dr} = 0$

$$\Rightarrow \quad \frac{-4V}{r^3}(r+2) + 4\pi(r+2) = 0$$

$$\Rightarrow \quad \frac{4V}{r^3} = 4\pi \text{ or } r^3 = \frac{V}{\pi} = 1000 \Rightarrow \frac{V}{250\pi} = 4$$

15. **(9)** Since, $p(x)$ has a local maximum at $x = 1$ and a local minimum at $x = 3$ and $p(x)$ is a real polynomial of least degree
Hence, let $p'(x) = k(x-1)(x-3) = k(x^2 - 4x + 3)$

$$\Rightarrow \quad p(x) = k\left(\frac{x^3}{3} - 2x^2 + 3x\right) + c$$

Now, $p(1) = 6$ and $p(3) = 2$

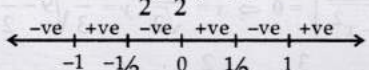
$$\Rightarrow \quad \frac{4}{3}k + C = 6 \text{ and } 0 + C = 2 \Rightarrow k = 3$$

$\therefore \quad p'(x) = 3(x-1)(x-3) \Rightarrow p'(0) = 9$

16. **(5)** $$f(x) = |x| + |x^2 - 1| = \begin{cases} -x + x^2 - 1, & x < -1 \\ -x - x^2 + 1, & -1 \le x \le 0 \\ x - x^2 + 1, & 0 < x < 1 \\ x^2 + x - 1, & x \ge 1 \end{cases}$$

$$\therefore \quad f'(x) = \begin{bmatrix} 2x - 1, & x < -1 \\ -2x - 1, & -1 \le x \le 0 \\ -2x + 1, & 0 < x < 1 \\ 2x + 1, & x > 1 \end{bmatrix}$$

Critical points are $\frac{1}{2}, \frac{-1}{2}, -1, 0$ and 1.



We observe at five points $f'(x)$ changes its sign
$\therefore$ There are 5 points at which either local maximum or local minimum.

17. **(1)** Let $g(x) = e^{f(x)}, \forall x \in R$

$\Rightarrow g'(x) = e^{f(x)} \cdot f'(x)$

$\Rightarrow f'(x)$ changes its sign from positive to negative in the neighbourhood of $x = 2009$

$\Rightarrow f(x)$ has local maxima at $x = 2009$.

So, the number of local maximum is one.

18. **(9)** The equation of tangent to the curve $y = f(x)$ at the point $P(x, y)$ is

$$(X - x)\frac{dy}{dx} - (Y - y) = 0 \Rightarrow X\frac{dy}{dx} - Y = x\frac{dy}{dx} - y$$

Its y-intercept $= y - x\frac{dy}{dx} = x^3 \Rightarrow \frac{dy}{dx} - \frac{y}{x} = -x^2$

$$\text{I.F.} = e^{\int -\frac{1}{x}dx} = \frac{1}{x}$$

$$\therefore \quad y \cdot \frac{1}{x} = \int (-x^2)\frac{1}{x}\,dx = \frac{-x^2}{2} + c, \; y = \frac{-x^3}{2} + cx$$

Since, $f(1) = 1 \Rightarrow$ At $x = 1, y = 1$

$$\therefore \quad 1 = \frac{-1}{2} + c \Rightarrow c = 3/2, \; \therefore \; y = \frac{x^3}{2} + \frac{3x}{2}$$

At $x = -3$, $y = \frac{27}{2} - \frac{9}{2} = 9$, $\therefore f(-3) = 9$.

19. **(0)** Let $p(x) = ax^4 + bx^3 + cx^2 + dx + e$

Now $\lim\limits_{x\to 0}\left[1 + \frac{p(x)}{x^2}\right] = 2 \Rightarrow \lim\limits_{x\to 0} \frac{p(x)}{x^2} = 1$ ...(i)

$\Rightarrow p(0) = 0 \Rightarrow e = 0$

On applying L'Hospital rule to eqn. (i), we get

$$\lim_{x\to 0} \frac{p'(x)}{2x} = 1 \Rightarrow p'(0) = 0 \Rightarrow d = 0$$

Again on applying L 'H rule, we get

$$\lim_{x\to 0} \frac{p''(x)}{2} = 1 \Rightarrow p''(0) = 2 \Rightarrow c = 1$$

$\therefore \quad p(x) = ax^4 + bx^3 + x^2$

$\Rightarrow \quad p'(x) = 4ax^3 + 3bx^2 + 2x$

As $p(x)$ has extremum at $x = 1$ and 2

$\therefore \quad p'(1) = 0$ and $p'(2) = 0$

$\Rightarrow \quad 4a + 3b + 2 = 0$ ...(i)

and $8a + 3b + 1 = 0$ ...(ii)

On solving (i) and (ii), we get $a = \frac{1}{4}$ and $b = -1$

$$\therefore \quad p(x) = \frac{1}{4}x^4 - x^3 + x^2, \; \therefore \; p(2) = 0$$

20. **(7)** Given $f(x) = 2x^3 - 15x^2 + 36x - 48$

and $A = \{x \mid x^2 + 20 \le 9x\}$

$\Rightarrow \quad A = \{x \mid x^2 - 9x + 20 \le 0\}$

$\Rightarrow \quad A = \{x \mid (x-4)(x-5) \le 0\} \Rightarrow A = [4, 5]$

Also $f'(x) = 6x^2 - 30x + 36 = 6(x^2 - 5x + 6) = 6(x-2)(x-3)$

Clearly $\forall \, x \in A, \; f'(x) > 0$

$\therefore$ f is strictly increasing function on A.

$\therefore$ Maximum value of f on $A = f(5)$
$= 2 \times 5^3 - 15 \times 5^2 + 36 \times 5 - 48 = 7$

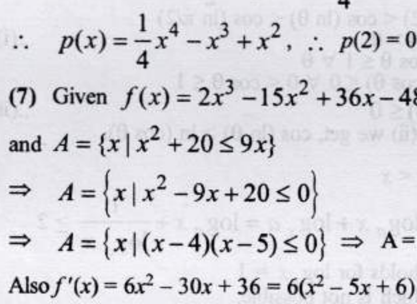
21. **(0.50)** $f(\theta) = (\sin\theta + \cos\theta)^2 + (\sin\theta - \cos\theta)^4$
$= (1 + \sin 2\theta) + (1 - \sin 2\theta)^2$

$f(\theta) = \sin^2 2\theta - \sin 2\theta + 2$

$f'(\theta) = 2(\sin 2\theta).(2\cos 2\theta) - 2\cos 2\theta = 2\cos 2\theta(2\sin 2\theta - 1) = 0$

$$\cos 2\theta = 0 \Rightarrow \theta = \frac{\pi}{4}, \frac{3\pi}{4}$$

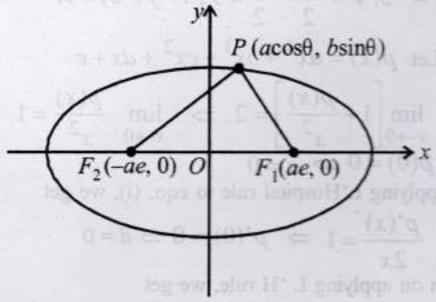
$$\sin 2\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{12}, \frac{5\pi}{12}$$



so, $f(\theta)$ is minimum at $\theta = \frac{\pi}{12}, \frac{5\pi}{12}$

$$\therefore \lambda_1 + \lambda_2 = \frac{1}{12} + \frac{5}{12} = \frac{6}{12} = \frac{1}{2}$$

22. Let $P(a\cos\theta, b\sin\theta)$ be any point on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ with foci $F_1\ (ae, 0)$ and $F_2\ (-ae, 0)$



Now, area of ΔPF_1F_2 *i.e.*, $A=\frac{1}{2}\begin{vmatrix} a\cos\theta & b\sin\theta & 1 \\ ae & 0 & 1 \\ -ae & 0 & 1\end{vmatrix}$

$\Rightarrow\ A=\frac{1}{2}|-b\sin\theta(ae+ae)|=abe\,|\sin\theta|$

$\therefore\ A_{\max}=abe \qquad [\because |\sin\theta|\le 1]$

23. Given $e^{-\pi/2}<\theta<\pi/2 \Rightarrow -\frac{\pi}{2}<\ln\theta<\ln\pi/2$

$\Rightarrow\ \cos(-\pi/2)<\cos(\ln\theta)<\cos(\ln\pi/2)$

$\Rightarrow\ \cos(\ln\theta)>0$ ...(i)

Now, $-1\le\cos\theta\le 1\ \forall\ \theta$

$\therefore\ -1\le\ln(\cos\theta)\le 0\ \forall\ 0<\cos\theta\le 1$

$\Rightarrow\ \ln(\cos\theta)\le 0$ ...(ii)

From (i) and (ii) we get, $\cos(\ln\theta)>\ln(\cos\theta)$.

24. **(False)**

Given : $0<a<x$

Let $f(x)=\log_a x+\log_x a=\log_a x+\frac{1}{\log_a x}\ge 2$

But equality holds for $\log_a x=1$

$\Rightarrow x=a$ which is not possible.

$\therefore\ f(x)>2,\ \therefore\ f_{\min}$ cannot be 2.

$\therefore$ The given statement is false.

25. **(a, b, c)** $f(x)=\frac{\sin\pi x}{x^2}, x>0$

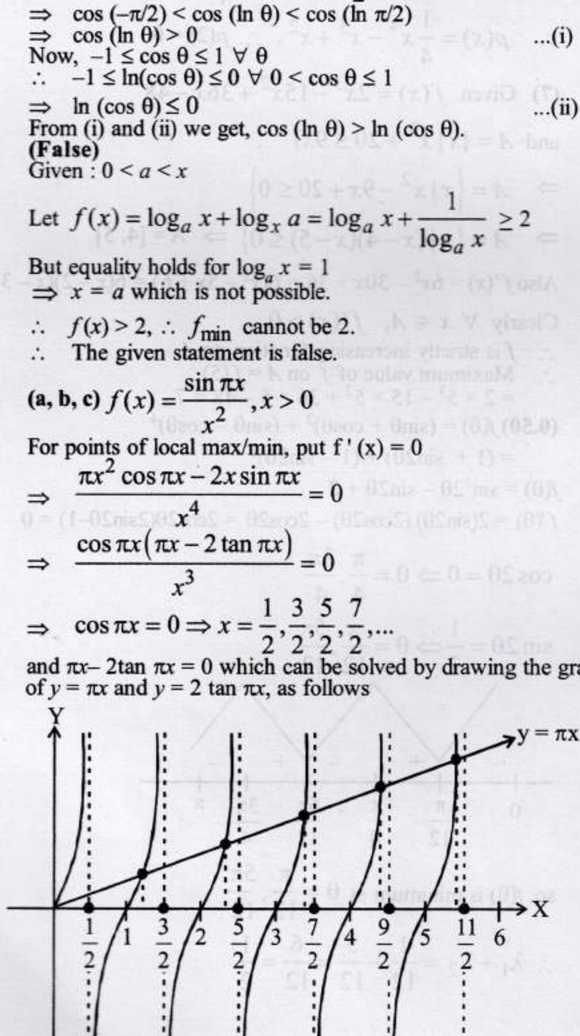
For points of local max/min, put $f'(x)=0$

$\Rightarrow\ \frac{\pi x^2\cos\pi x-2x\sin\pi x}{x^4}=0$

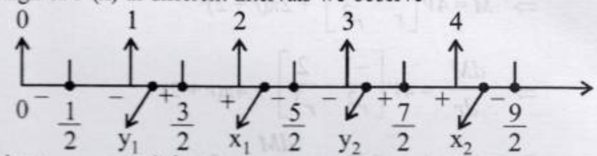
$\Rightarrow\ \frac{\cos\pi x(\pi x-2\tan\pi x)}{x^3}=0$

$\Rightarrow\ \cos\pi x=0\Rightarrow x=\frac{1}{2},\frac{3}{2},\frac{5}{2},\frac{7}{2},\ldots$

and $\pi x-2\tan\pi x=0$ which can be solved by drawing the graphs of $y=\pi x$ and $y=2\tan\pi x$, as follows



Plotting the stationary points on number line and finding the sign of f '(x) in different intervals we observe



i.e. $x_{n+1}-x_n>2$ for every n

$x_n\in\left(2n, 2n+\frac{1}{2}\right)$ for every n

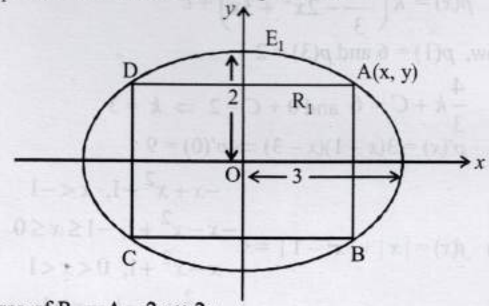
$|x_n-y_n|>1$ for every n

$x_1>y_1$

Hence (a), (b), (c) are correct, but option (d) is incorrect.

26. **(b, c)** $E_1:\ \frac{x^2}{9}+\frac{y^2}{4}=1$

R_1 : rectangle ABCD with largest area.



Area of $R_1=A=2x\times 2y$

$\Rightarrow\ A=4x\times\frac{2}{3}\sqrt{9-x^2}=\frac{8}{3}x\sqrt{9-x^2}$

$\therefore\ \frac{dA}{dx}=\frac{8}{3}\left[\sqrt{9-x^2}-\frac{x^2}{\sqrt{9-x^2}}\right]$

$\Rightarrow\ \frac{8}{3}\left[\frac{9-2x^2}{\sqrt{9-x^2}}\right]=0\Rightarrow x=\frac{3}{\sqrt{2}}, y=\frac{2}{3}\sqrt{9-\frac{9}{2}}=\frac{2}{\sqrt{2}}$

$\Rightarrow$ For E_2 : $a=\frac{3}{\sqrt{2}}, b=\frac{2}{\sqrt{2}}$

Similarly for E_3 : $a=\frac{3}{(\sqrt{2})^2}, b=\frac{2}{(\sqrt{2})^2}$ and so on.

Now eccentricity depends on $\frac{b}{a}$ which is same for all E_n, therefore eccentricity for all the E_n's will remain $\sqrt{1-\frac{4}{9}}=\frac{\sqrt{5}}{3}$.

$\therefore$ (a) is incorrect.

For E_9 : $a=\frac{3}{(\sqrt{2})^8}, b=\frac{2}{(\sqrt{2})^8}$

$\Rightarrow$ Length of latus rectum $=\frac{2b^2}{a}=\frac{2\times\frac{4}{256}}{\frac{3}{16}}=\frac{1}{6}$

Hence (b) is correct.

Area of $R_1=4\times\frac{3}{\sqrt{2}}\times\frac{2}{\sqrt{2}}=\frac{24}{2}$

Area of $R_2=4\times\frac{3}{(\sqrt{2})^2}\times\frac{2}{(\sqrt{2})^2}=\frac{24}{2^2}$

Area of $R_3 = 4 \times \frac{3}{(\sqrt{2})^3} \times \frac{2}{(\sqrt{2})^3} = \frac{2}{(2)^3}$ and so on

$$\sum_{n=1}^{\infty} \text{area of } R_n = \frac{24}{2} + \frac{24}{2^2} + \frac{24}{2^3} + \ldots = \frac{12}{1-\frac{1}{2}} = 24$$

$\Rightarrow \sum_{n=1}^{N} (\text{area of } R_n) < 24$ for each positive integer N.

Hence, (c) is correct.

For $E_9 : a = \frac{3}{(\sqrt{2})^8}, b = \frac{2}{(\sqrt{2})^8}, e = \frac{\sqrt{5}}{3}$

Thus, focus $= (ae, 0) = \left(\frac{\sqrt{5}}{16}, 0\right)$

Hence, distance of focus from centre $= \frac{\sqrt{5}}{16}$

$\therefore$ (d) is incorrect.

27. **(b, c)** $f(x) = \begin{vmatrix} \cos 2x & \cos 2x & \sin 2x \\ -\cos x & \cos x & -\sin x \\ \sin x & \sin x & \cos x \end{vmatrix}$

$$\Rightarrow f(x) = \begin{vmatrix} 0 & \cos 2x & \sin 2x \\ -2\cos x & \cos x & -\sin x \\ 0 & \sin x & \cos x \end{vmatrix} \quad [C_1 \to C_1 - C_2]$$

$\Rightarrow f(x) = 2\cos 3x \cos x \Rightarrow f(x) = \cos 4x + \cos 2x$

$\therefore f_{max} = 2$ at $x = 0$

$f'(x) = -4\sin 4x - 2\sin 2x = -2\sin 2x\,[4\cos 2x + 1]$

$f'(x) = 0 \Rightarrow \sin 2x = 0$ or $\cos 2x = \frac{-1}{4}$

$\Rightarrow x = -\frac{\pi}{2}, 0, \frac{\pi}{2}$ which is true for some $x \in (-\pi, \pi)$

$\therefore f'(x) = 0$ at more than three points in $(-\pi, \pi)$

28. **(a, d)** $\lim_{x\to 2} \frac{f(x)g(x)}{f'(x)g'(x)} = 1 \left[\frac{0}{0}\text{form}\right]$

$\Rightarrow \lim_{x\to 2} \frac{f'(x)g(x) + f(x)g'(x)}{f''(x)g'(x) + f'(x)g''(x)} = 1$ [LH Rule]

$\Rightarrow \frac{g'(2)f(2)}{f''(2)g'(2)} = 1 \Rightarrow f(2) = f''(2)$

$\therefore f(x) - f''(x) = 0$ for atleast one $x \in R$.

$\because$ Range of $f(x)$ is $(0, \infty)$

$\therefore f(x) > 0, \forall x \in R$

$\Rightarrow f(2) > 0 \Rightarrow f''(2) > 0$

$\Rightarrow$ f has a local minimum at $x = 2$

29. **(a, c)** Let length and breadth of rectangular sheet be $8x$, and $15x$ respectively. Also let y be the length of square cut off from each corner.

$\therefore$ Volume of box $= (8x - 2y)(15x - 2y)y$

$\Rightarrow V = 120x^2y - 46xy^2 + 4y^3$

$\frac{dV}{dy} = 120x^2 - 92xy + 12y^2$

Now, $4y^2 = 100 \Rightarrow y = 5$

Put $\frac{dV}{dy} = 0$ for maximum value of V.

$\Rightarrow 30x^2 - 115x + 75 = 0 \Rightarrow x = 3, \frac{5}{6}$

For $x = 3$, sides are 45 and 24.

30. **(a, b, c, d)** $f(x) = \int_0^{x^2} e^{t^2}(t-2)(t-3)dt$

$\Rightarrow f'(x) = e^{x^2}.(x-2)(x-3)$

Put $f'(x) = 0 \Rightarrow x = 2, 3$

$f''(x) = e^{x^2}.2x(x^2 - 5x + 6) + e^{x^2}(2x - 5)$

$f''(2) = -$ ve and $f''(3) = +$ ve

$\therefore$ $x = 2$ is a point of local maxima.

and $x = 3$ is a point of local minima.

Also for $x \in (2, 3)$, $f'(x) < 0$

$\Rightarrow$ f is decreasing on $(2, 3)$.

Also we observe $f''(0) < 0$ and $f''(1) > 0$

$\therefore$ There exists some $C \in (0, 1)$ such that $f''(C) = 0$

Hence all the options are correct.

31. **(b, c)** Let $f(x) = ax^3 + bx^2 + cx + d$

Now, $f(2) = 18 \Rightarrow 8a + 4b + 2c + d = 18$ ... (i)

and $f(1) = -1 \Rightarrow a + b + c + d = -1$ ... (ii)

$\because$ $f(x)$ has local max. at $x = -1$

$\therefore$ $3a - 2b + c = 0$ ... (iii)

Also $f'(x)$ has local min. at $x = 0 \therefore b = 0$ ... (iv)

On solving (i), (ii), (iii) and (iv), we get

$$f(x) = \frac{1}{4}(19x^3 - 57x + 34) \therefore f(0) = \frac{17}{2}$$

Also $f'(x) = \frac{57}{4}(x^2 - 1) > 0, \forall x > 1$

and $f'(x) = 0 \Rightarrow x = 1, -1$

Now, $f''(-1) < 0, f''(1) > 0 \Rightarrow x = -1$ is a point of local max. and $x = 1$ is a point of local min.

Distance between $(-1, 2)$ and $(1, f(1))$, i.e. $(1, -1)$ is $\sqrt{13} \neq 2\sqrt{5}$

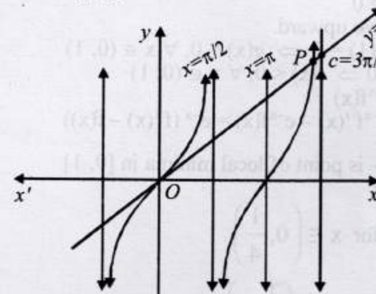
32. **(b)** The maximum value of $f(x) = \cos x + \cos(\sqrt{2x})$ is 2 which occurs at $x = 0$. Also, there is no other value of x for which this value will be attained again.

33. **(d)** $f(x) = \frac{x^2 - 1}{x^2 + 1} = \frac{(x^2 + 1) - 2}{x^2 + 1} = 1 - \frac{2}{x^2 + 1}$

For $f(x)$ to be minimum $\frac{2}{x^2 + 1}$ should be maximum, which is so if $x^2 + 1$ is minimum. And $x^2 + 1$ is minimum at $x = 0$.

$\therefore f_{\min} = \frac{0 - 1}{0 + 1} = -1$

34. **(c)**



It is clear from the graph, curves $y = \tan x$ and $y = x$ intersect at P in $(\pi, 3\pi/2)$.

Hence, smallest positive root of $\tan x - x = 0$ lies in $(\pi, 3\pi/2)$.

35. **(c)** $P(x) = a_0 + a_1x^2 + a_2x^4 + ... + a_nx^{2n}$

$\Rightarrow P'(x) = 2a_1x + 4a_2x^3 + ... + 2na_nx^{2n-1}$

$P'(x) = 0 \Rightarrow x = 0$

$P''(x) = 2a_1 + 12a_2x^2 + ... + 2n(2n-1)a_nx^{2n-2}$

$\therefore$ $P''(0) > 0$ as $a_1 > 0$

$\therefore$ $P(x)$ has only one minimum at $x = 0$.

36. **(a)** Equation of tangent to $y^2 = 16x$ at F (x_0, y_0) is

$yy_0 = 8(x + x_0) \Rightarrow G\left(0, \dfrac{8x_0}{y_0}\right)$

Area of ΔEFG, $A = \dfrac{1}{2} \times (3 - y_1) \times x_0$

$\Rightarrow A = \dfrac{1}{2}x_0\left(3 - \dfrac{8x_0}{y_0}\right)$

$\Rightarrow A = \dfrac{1}{2} \times \dfrac{y_0^2}{16}\left(3 - \dfrac{y_0}{2}\right) = \dfrac{1}{32}\left(3y_0^2 - \dfrac{y_0^3}{2}\right)$

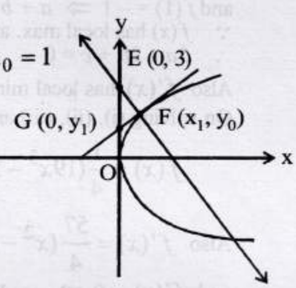
$\Rightarrow \dfrac{dA}{dy_0} = \dfrac{1}{32}\left(6y_0 - \dfrac{3y_0^2}{2}\right)$

$\Rightarrow \dfrac{dA}{dy_0} = 0 \Rightarrow y_0 = 4 \Rightarrow x_0 = 1$

$\therefore$ $y_1 = \dfrac{8 \times 1}{4} = 2$

Also $y_0 = mx_0 + 3$

$\therefore$ $4 = m + 3$ or $m = 1$



$\therefore$ Maximum area of ΔEFG

$= \dfrac{1}{32}\left[3 \times 4^2 - \dfrac{4^3}{2}\right]$

$= \dfrac{1}{32}[48 - 32] = \dfrac{1}{2}$

37. **(d)** $f''(x) - 2f'(x) + f(x) \geq e^x$

$\Rightarrow [f''(x) - f'(x)] - [f'(x) - f(x)] \geq e^x$

$\Rightarrow \left[e^{-x}f''(x) - e^{-x}f'(x)\right] - \left[e^{-x}f'(x) - e^{-x}f(x)\right] \geq 1$

$\Rightarrow \dfrac{d}{dx}\left[e^{-x}f'(x)\right] - \dfrac{d}{dx}\left[e^{-x}f(x)\right] \geq 1$

$\Rightarrow \dfrac{d}{dx}\left[e^{-x}f'(x) - e^{-x}f(x)\right] \geq 1$

$\Rightarrow \dfrac{d}{dx}\left[\dfrac{d}{dx}\left(e^{-x}f(x)\right)\right] \geq 1$

Let $g(x) = e^{-x}f(x)$

$\therefore g''(x) \geq 1 > 0$

$\Rightarrow$ g is concave upward.

Also $g(0) = g(1) = 0 \Rightarrow g(x) < 0, \forall x \in (0, 1)$

$\Rightarrow e^{-x} f(x) < 0 \Rightarrow f(x) < 0, \forall x \in (0, 1)$

38. **(c)** $g(x) = e^{-x}f(x)$

$\Rightarrow g'(x) = e^{-x}f'(x) - e^{-x}f(x) = e^{-x}(f'(x) - f(x))$

Since, $x = \dfrac{1}{4}$ is point of local minima in [0, 1]

$\therefore$ $g'(x) < 0$ for $x \in \left(0, \dfrac{1}{4}\right)$

and $g'(x) > 0$ for $x \in \left(\dfrac{1}{4}, 1\right)$

Hence, In $\left(0, \dfrac{1}{4}\right)$, $g'(x) < 0$

$\Rightarrow e^{-x}(f'(x) - f(x)) < 0 \Rightarrow f'(x) < f(x)$

39. $g(x) = (f'(x))^2 + f''(x)f(x) = \dfrac{d}{dx}(f(x)f'(x))$

Let $h(x) = f(x)f'(x)$

$\therefore$ $f(x) = 0$ has four roots namely a, α, β, e; where $b < \alpha < c$ and $c < \beta < d$.

And $f'(x) = 0$ at three points k_1, k_2, k_3; where $a < k_1 < \alpha$, $\alpha < k_2 < \beta$, $\beta < k_3 < e$

[Since between any two roots of a polynomial function $f(x) = 0$, there lies atleast one root of $f'(x) = 0$]

Hence, there are atleast 7 roots of $f(x) \cdot f'(x) = 0$

Therefore, there are atleast 6 roots of $\dfrac{d}{dx}(f(x)f'(x)) = 0$ i.e. of $g(x) = 0$

40. Let $p(x) = ax^3 + bx^2 + cx + d$

$\because$ $p(-1) = 10$, $\therefore -a + b - c + d = 10$ ...(i)

Also, $p(1) = -6$, $\therefore a + b + c + d = -6$ ...(ii)

$p(x)$ has max. at $x = -1$, $\therefore$ $p'(-1) = 0$

$\Rightarrow$ $3a - 2b + c = 0$ ...(iii)

$p'(x)$ has min. at $x = 1$, $\therefore$ $p''(1) = 0$

$\Rightarrow$ $6a + 2b = 0$ ...(iv)

On solving (i), (ii), (iii) and (iv), we get

From (iv), $b = -3a$

From (iii), $3a + 6a + c = 0 \Rightarrow c = -9a$

From (ii), $a - 3a - 9a + d = -6 \Rightarrow d = 11a - 6$

From (i), $-a - 3a + 9a + 11a - 6 = 10$

$\Rightarrow$ $a = 1 \Rightarrow b = -3, c = -9, d = 5$

$\therefore$ $p(x) = x^3 - 3x^2 - 9x + 5$ put $p'(x) = 0$

$\Rightarrow$ $3x^2 - 6x - 9 = 0 \Rightarrow 3(x + 1)(x - 3) = 0$

$\Rightarrow$ $x = -1$ is a point of max. (given)

and $x = 3$ is a point of min.

[$\because$ max. and min. occur alternatively]

$\therefore$ points of local max. is $(-1, 10)$ and local min. is $(3, -22)$.

And distance between them

$= \sqrt{[3-(-1)^2] + (-22-10)^2} = 4\sqrt{65}$

41. Let $f(x) = \sin x + 2x - \dfrac{3x(x+1)}{\pi}$

$\Rightarrow f'(x) = \cos x + 2 - \dfrac{3}{\pi}(2x+1)$

$\Rightarrow f''(x) = -\sin x - \dfrac{6}{\pi} < 0, \forall x \in [0, \pi/2]$

Hence, $f'(x)$ is a decreasing function. ...(i)

Also $f'(0) = 3 - \dfrac{3}{\pi} > 0$ ...(ii)

and $f'(\pi/2) = 2 - \dfrac{3}{\pi}(\pi + 1) = -1 - \dfrac{3}{\pi} < 0$ ...(iii)

From equations (i), (ii) and (iii), there exists a certain value of $x \in [0, \pi/2]$ for which $f'(x) = 0$ and this point must be a point of maximum for $f(x)$ because the sign of $f'(x)$ changes from +ve to –ve.

Also we can see that $f(0) = 0$ and

$f\left(\dfrac{\pi}{2}\right) = \pi + 1 - \dfrac{3}{2}\left(\dfrac{\pi}{2} + 1\right) = \dfrac{\pi}{4} - \dfrac{1}{2} > 0$

Let $x = p$ be the point at which the max. of $f(x)$ occurs. There will be only one max. point in $[0, \pi/2]$ because $f'(x) = 0$ is only once in the interval.

Consider , $x \in [0, p]$

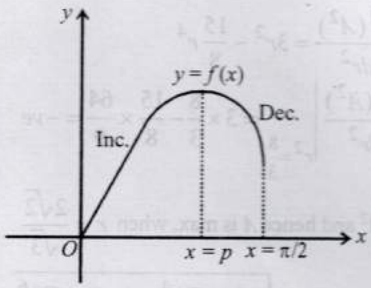
$\Rightarrow f'(x) > 0 \Rightarrow f(x)$ is an increasing function.

$\Rightarrow f(0) \leq f(x)$ [as $0 \leq x$] $\Rightarrow f(x) \geq 0$ ...(iv)

Also for $x \in [p, \pi/2]$

$\Rightarrow f'(x) < 0 \Rightarrow f(x)$ is decreasing function.

$\Rightarrow$ for $x < \pi/2$, $f(x) > f(\pi/2) > 0$ ... (v)



Hence from (iv) and (v), $f(x) \geq 0, \forall x \in [0, \pi/2]$

42. The given curve $\dfrac{x^2}{6} + \dfrac{y^2}{3} = 1$ is an ellipse.

Any parametric point on it is $P(\sqrt{6}\cos\theta, \sqrt{3}\sin\theta)$.
Its distance from line $x + y = 7$ is given by

$$D = \frac{\sqrt{6}\cos\theta + \sqrt{3}\sin\theta - 7}{\sqrt{2}}$$

For min. value of D, put $\dfrac{dD}{d\theta} = 0$

$$\Rightarrow -\sqrt{6}\sin\theta + \sqrt{3}\cos\theta = 0 \Rightarrow \tan\theta = 1/\sqrt{2}$$

$$\Rightarrow \cos\theta = \frac{\sqrt{2}}{\sqrt{3}} \text{ and } \sin\theta = \frac{1}{\sqrt{3}}$$

Hence, required point P is (2, 1)

43. **(a, d)** Let $P(at^2, 2at)$ be any point on the parabola $y^2 = 4ax$.

$\therefore$ Tangent to the parabola at P is $y = \dfrac{x}{t} + at$,
which meets the axis of parabola *i.e* x-axis at $T(-at^2, 0)$.

Also normal to parabola at P is $tx + y = 2at + at^3$
which meets the axis of parabola at $N(2a + at^2, 0)$

Let $G(x, y)$ be the centriod of ΔPTN, then

$$x = \frac{at^2 - at^2 + 2a + at^2}{3} \text{ and } y = \frac{2at}{3}$$

$$\Rightarrow x = \frac{2a + at^2}{3} \quad \text{... (i)} \quad \text{and} \quad y = \frac{2at}{3} \quad \text{... (ii)}$$

Eliminating t from (i) and (ii), we get the locus of centriod G as

$$3x = 2a + a\left(\frac{3y}{2a}\right)^2 \Rightarrow \quad y^2 = \frac{4a}{3}\left(x - \frac{2}{3}a\right),$$

which is a parabola with vertex $\left(\dfrac{2a}{3}, 0\right)$, directrix as

$x - \dfrac{2a}{3} = -\dfrac{a}{3} \Rightarrow x = \dfrac{a}{3}$, latus rectum as $\dfrac{4a}{3}$ and focus as $(a, 0)$.

44. $$f'(x) = \begin{vmatrix} 2ax & 2ax-1 & 2ax+b+1 \\ b & b+1 & -1 \\ 2(ax+b) & 2ax+2b+1 & 2ax+b \end{vmatrix}$$

$$f'(x) = \begin{vmatrix} 2ax & 2ax-1 & 2ax+b+1 \\ b & b+1 & -1 \\ 0 & 0 & 1 \end{vmatrix} \quad [R_3 \to R_3 - R_1 - 2R_2]$$

$$= \begin{vmatrix} 2ax & 2ax-1 \\ b & b+1 \end{vmatrix} = \begin{vmatrix} 2ax & -1 \\ b & 1 \end{vmatrix} \quad [C_2 \to C_2 - C_1]$$

$\therefore \quad f'(x) = 2ax + b$

Integrating, we get $f(x) = ax^2 + bx + c$
where c is an arbitrary constant. Since f has a maximum at $x = 5/2$,

$$f'(5/2) = 0 \Rightarrow 5a + b = 0 \quad \text{...(i)}$$

Also $f(0) = 2 \Rightarrow C = 2$
and $f(1) = 1 \Rightarrow a + b + c = 1$
$\therefore \quad a + b = -1$ ...(ii)

Solving (i) and (ii), we get, $a = 1/4, b = -5/4$

$$\therefore \quad f(x) = \frac{1}{4}x^2 - \frac{5}{4}x + 2.$$

45. $f(x) = \dfrac{1}{8}\ell nx - bx + x^2, x > 0, b \geq 0$

$$\Rightarrow \quad f'(x) = \frac{1}{8x} - b + 2x \quad \text{...(i)}$$

Put $f'(x) = 0$ (for max. or min. $\Rightarrow 16x^2 - 8bx + 1 = 0$

$$\therefore \quad x = \frac{1}{4}\left[b \pm \sqrt{b^2 - 1}\right] \quad \text{...(ii)}$$

Above will give real values of x if $b^2 - 1 \geq 0$ i.e. $b \geq 1$ or $b \leq -1$.
But b is given to be +ve. Hence we choose $b \geq 1$

If $b = 1$ then $x = \dfrac{1}{4}$; If $b > 1$ then $x = \dfrac{1}{4}\left[b \pm \sqrt{b^2 - 1}\right]$

$$f''(x) = -\frac{1}{8x^2} + 2 = \frac{16x^2 - 1}{8x^2}$$

Its sign will depend on $16x^2 - 1$ as $8x^2$ is +ve. We shall consider its sign for $x = \dfrac{1}{4}$ and $x = \dfrac{1}{4}\left[b \pm \sqrt{b^2 - 1}\right]$

$f''(x) = 0$ at $x = 1/4$

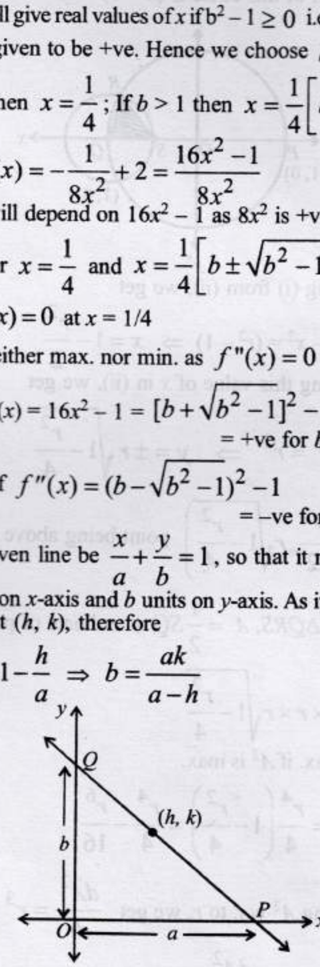
Hence, neither max. nor min. as $f''(x) = 0$

N^r of $f''(x) = 16x^2 - 1 = [b + \sqrt{b^2 - 1}]^2 - 1$
$= +$ve for $b > 1 \; \therefore$ Minima

or N^r of $f''(x) = (b - \sqrt{b^2 - 1})^2 - 1$
$= -$ve for $b > 1 \; \therefore$ Maxima

46. Let the given line be $\dfrac{x}{a} + \dfrac{y}{b} = 1$, so that it makes an intercept of a units on x-axis and b units on y-axis. As it passes through the fixed point (h, k), therefore

$$\Rightarrow \quad \frac{k}{b} = 1 - \frac{h}{a} \Rightarrow b = \frac{ak}{a-h} \quad \text{...(i)}$$



Now Area of $\Delta OPQ = A = \dfrac{1}{2}ab$

$$\therefore \quad A = \frac{k}{2}\left[\frac{a^2}{a-h}\right] \quad \text{[using (i)]}$$

For min. value of A, $\dfrac{dA}{da}=0$

$$\Rightarrow \frac{k}{2}\left[\frac{a^2-2ah}{(a-h)^2}\right]=0 \Rightarrow a=2h$$

Also,

$$\frac{d^2A}{da^2}=\frac{k}{2}\left[\frac{(2a-2h)(a-h)^2-2(a-h)(-1)(a^2-2ah)}{(a-h)^4}\right]$$

$\therefore$

$$\left.\frac{d^2A}{da^2}\right|_{a=2h}=\frac{k}{2}\left[\frac{(2h^3+2h)(0)}{h^4}\right]=\frac{k}{h}>0,\ k[\because h,\ k>0]$$

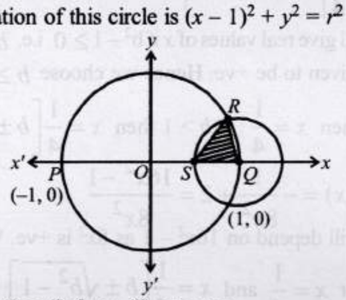
$\Rightarrow$ A is min. when $a=2h$

Hence, $A_{\min}=\dfrac{k}{2}\left[\dfrac{4h^2}{h}\right]=2kh$

47. The given circle is $x^2+y^2=1$, which intersect x-axis at $P(-1, 0)$ and $Q(1, 0)$. ...(i)

Let radius of circle with centre at $Q(1, 0)$ be r, where r is variable.

Then equation of this circle is $(x-1)^2+y^2=r^2$ ...(ii)



On subtracting (i) from (ii), we get

$$(x-1)^2-x^2=(r^2-1) \Rightarrow x=1-\frac{r^2}{2}$$

On substituting this value of x in (ii), we get

$$\frac{r^4}{4}+y^2=r^2 \Rightarrow y=\pm r\sqrt{1-\frac{r^2}{4}}$$

$\therefore$ $R\left(1-\dfrac{r^2}{2}, r\sqrt{1-\dfrac{r^2}{4}}\right)$ point being above x-axis.

$\Rightarrow$ Area of ΔQRS, $A=\dfrac{1}{2}SQ\times$ ordinate of point R

$$\Rightarrow A=\frac{1}{2}\times r\times r\sqrt{1-\frac{r^2}{4}}$$

A will be max. if A^2 is max.

Now, $A^2=\dfrac{r^4}{4}\left(1-\dfrac{r^2}{4}\right)=\dfrac{r^4}{4}-\dfrac{r^6}{16}$

Differentiating A^2 w.r. to r, we get $\dfrac{dA^2}{dr}=r^3-\dfrac{3}{8}r^5$

For A^2 to be max., $\dfrac{dA^2}{dr}=0$

$$\Rightarrow r^3\left(1-\frac{3}{8}r^2\right)=0 \Rightarrow r=\frac{2\sqrt{2}}{\sqrt{3}} \Rightarrow r^2=\frac{8}{3}$$

Now $\dfrac{d^2(A^2)}{dr^2}=3r^2-\dfrac{15}{8}r^4$

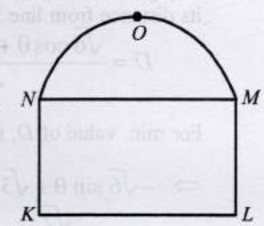
$$\Rightarrow \left.\frac{d^2(A^2)}{dr^2}\right|_{r^2=\frac{8}{3}}=3\times\frac{8}{3}-\frac{15}{8}\times\frac{64}{9}=-ve$$

Hence, A^2 and hence A is max. when $r=\dfrac{2\sqrt{2}}{\sqrt{3}}$

$$\therefore \text{ Max. area}=\sqrt{\frac{1}{4}\left(\frac{2\sqrt{2}}{\sqrt{3}}\right)^4-\frac{1}{16}\left(\frac{2\sqrt{2}}{\sqrt{3}}\right)^6}$$

$$=\frac{4}{3\sqrt{3}}=\frac{4\sqrt{3}}{9} \text{ sq. units.}$$

48. Let $KLMONK$ be the window as shown in the figure and $KL=x$ m and $LM=y$ m



Then its perimeter including the base NM of arch,

$$P=\left(2x+2y+\frac{\pi x}{2}\right)$$

$$=\left(2+\frac{\pi}{2}\right)x+2y \quad \text{...(i)}$$

Now, area of rectangle $KLMN=xy$

and area of arch $NMON=\dfrac{\pi}{2}\left(\dfrac{x}{2}\right)^2$

Let λ be the light transmitted by coloured glass per sq. m. Then 3λ will be the light transmitted by clear glass per sq. m.

Hence the amount of light transmitted,

$$A=3\lambda(xy)+\lambda\left[\frac{\pi}{2}\left(\frac{x}{2}\right)^2\right]=\lambda\left[3xy+\frac{\pi x^2}{8}\right] \quad \text{...(ii)}$$

On substituting the value of y from (i) in (ii), we get

$$A=\lambda\left[3x\frac{1}{2}\left[P-\left(\frac{4+\pi}{2}\right)x\right]+\frac{\pi x^2}{8}\right]$$

$$=\lambda\left[\frac{3Px}{2}-\frac{3(4+\pi)}{4}x^2+\frac{\pi x^2}{8}\right]$$

$$\therefore \frac{dA}{dx}=\lambda\left[\frac{3P}{2}-\frac{3(4+\pi)}{2}x+\frac{\pi x}{4}\right]$$

For A to be maximum, $\dfrac{dA}{dx}=0$

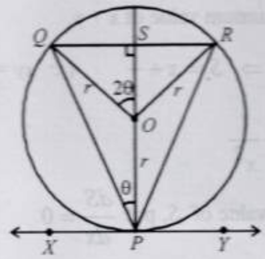
$$\Rightarrow x=\frac{\frac{3P}{2}}{\frac{-\pi}{4}+\left(\frac{12+3\pi}{2}\right)} \Rightarrow x=\frac{6P}{5\pi+24}$$

Now $\dfrac{d^2A}{dx^2}=\lambda\left[\dfrac{-3(4+\pi)}{2}+\dfrac{\pi}{4}\right]<0$

Hence, A is maximum when $x=\dfrac{6P}{5\pi+24} \Rightarrow \dfrac{y}{x}=\dfrac{\pi+6}{6}$

Hence, required ratio $=(6+\pi):6$

49. Here $QR \parallel XY$ and diameter through P is perpendicular QR.



Let $\angle QPS = \theta$, then $\angle QOS = 2\theta$

Now area of ΔPQR is given by $A = \frac{1}{2} QR.SP$

Here $QR = 2.QS = 2r \sin 2\theta$
and $PS = OS + OP = r \cos 2\theta + r$

$\therefore \quad A = \frac{1}{2}.2r \sin 2\theta.(r + r\cos 2\theta)$

$= r^2. 2 \sin\theta \cos\theta . 2\cos^2\theta = 4r^2 \sin\theta \cos^3\theta$

For max. value of area, put $\frac{dA}{d\theta} = 0$

$\Rightarrow \; 4r^2 [\cos^4\theta - 3\sin^2\theta\cos^2\theta] = 0$
$\Rightarrow \; \cos^2\theta(\cos^2\theta - 3\sin^2\theta) = 0$

$\Rightarrow \tan\theta = \frac{1}{\sqrt{3}} \Rightarrow \theta = 30°$

Now, $\frac{d^2A}{d\theta^2} = 4r^2[-4\cos^3\theta\sin\theta - 6\sin\theta\cos^3\theta + 6\sin^3\theta\cos\theta]$

$= 4r^2 [-10 \sin\theta\cos^3\theta + 6\sin^3\theta\cos\theta]$

$$\left.\frac{d^2A}{d\theta^2}\right|_{\theta=30°} = 4r^2\left[-10.\frac{1}{2}.\frac{3\sqrt{3}}{8} + 6.\frac{1}{8}.\frac{\sqrt{3}}{2}\right]$$

$$= 4r^2\left(\frac{-12\sqrt{3}}{8}\right) < 0$$

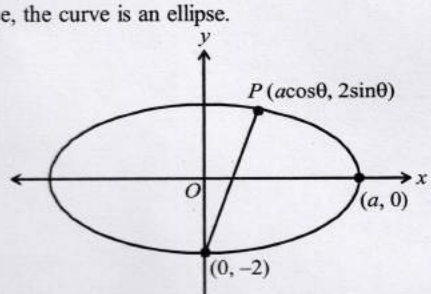
Hence area (A) is maximum at $\theta = 30°$

Now $A_{\max} = 4r^2 \sin 30° \cos^3 30° = \frac{3\sqrt{3}}{4} r^2$

50. Given curve $4x^2 + a^2y^2 = 4a^2$, $4 < a^2 < 8$

$\Rightarrow \quad \frac{x^2}{a^2} + \frac{y^2}{4} = 1, \; 4 < a^2 < 8$

Hence, the curve is an ellipse.



Let us consider a point P ($a\cos\theta$, $2\sin\theta$) on the ellipse.
Let the distance of P ($a\cos\theta$, $2\sin\theta$) from $(0, -2)$ is L.
$\therefore \quad L^2 = (a\cos\theta - 0)^2 + (2\sin\theta + 2)^2$

$\Rightarrow \quad \frac{d(L^2)}{d\theta} = \cos\theta[-2a^2\sin\theta + 8\sin\theta + 8]$

For max. or min. value of L, $\frac{d(L^2)}{d\theta} = 0$

$\Rightarrow \; \cos\theta\,[-2a^2 \sin\theta + 8\sin\theta + 8] = 0$
$\Rightarrow \;$ Either $\cos\theta = 0$ or $(8 - 2a^2)\sin\theta + 8 = 0$

$\Rightarrow \; \theta = \frac{\pi}{2}$ or $\sin\theta = \frac{4}{a^2 - 4}$ Since $a^2 < 8 \Rightarrow a^2 - 4 < 4$

$\Rightarrow \; \frac{4}{a^2-4} > 1 \Rightarrow \sin\theta > 1$ which is not possible

Also $\frac{d^2(L^2)}{d\theta^2} = \cos\theta[-2a^2\cos\theta + 8\cos\theta]$
$+ (-\sin\theta)\,[-2a^2\sin\theta + 8\sin\theta + 8]$

At $\theta = \frac{\pi}{2}, \frac{d^2(L^2)}{d\theta^2} = 0 - [16 - 2a^2] = 2(a^2 - 8) < 0$

as $a^2 < 8$

$\therefore \;$ L is max. at $\theta = \pi/2$ and the farthest point is $(0, 2)$.

51. $f(x) = \sin^3 x + \lambda \sin^2 x$ for $-\pi/2 < x < \pi/2$
$\therefore \; f'(x) = 3\sin^2 x \cos x + 2\lambda \sin x \cos x$

$= \frac{1}{2} \sin 2x (3\sin x + 2\lambda)$

Now put $f'(x) = 0 \Rightarrow x = 0$

or $\lambda = -\frac{3}{2}\sin x$

Also, $f''(x) = \cos 2x\,(3\sin x + 2\lambda) + \frac{3}{2}\sin 2x \cos x$

Now for $\lambda = \frac{-3}{2}\sin x$, we have

$f''(x) = 3\sin x\cos^2 x = -2\lambda\cos^2 x$
When, if $0 < x < \pi/2$, then $-3/2 < \lambda < 0$ and hence $f''(x) > 0$.
$\Rightarrow \; f(x)$ has one minimum for this value of λ.
Now for $x = 0$, we have $f''(0) = 2\lambda < 0$.
Hence, $f(x)$ has a maximum at $x = 0$
Again if $-\pi/2 < x < 0$, then $0 < \lambda < 3/2$ and hence
$f''(x) = -2\lambda\cos^2 x < 0$.
$\therefore \; f(x)$ has a maximum.
Now for $x = 0$, $f''(a) = 2\lambda > 0$ so that $f(x)$ has a minimum.
Hence, for exactly one maximum and minimum value of $f(x)$, λ must lie in the interval
$-3/2 < \lambda < 0$ or $0 < \lambda < 3/2$ i.e., $\lambda \in (-3/2, 0) \cup (0, 3/2)$

52. Given : $ax^2 + \frac{b}{x} \geq c \quad \forall x > 0, a > 0, b > 0$

To show : $27\,ab^2 \geq 4c^3$.
Let us consider the function $f(x) = ax^2 + b/x$

then $f'(x) = 2ax - \frac{b}{x^2}$

Put $f'(x) = 0$

$\Rightarrow \; x^3 = b/2a \Rightarrow x = (b/2a)^{1/3}$ Now, $f''(x) = 2a + \frac{2b}{x^3}$

$\Rightarrow \; f''\left(\left(\frac{b}{2a}\right)^{1/3}\right) = 2a + \frac{2b}{b} \times 2a = 6a > 0$

Hence, f is minimum at $x = \left(\frac{b}{2a}\right)^{1/3}$

Since, $ax^2 + \frac{b}{x} \geq c$ is true $\forall\, x$

Hence for $x = \left(\frac{b}{2a}\right)^{1/3}$

$$\Rightarrow a\left(\frac{b}{2a}\right)^{2/3} + \frac{b}{(b/2a)^{1/3}} \geq c$$

$$\Rightarrow \frac{a\left(\frac{b}{2a}\right)+b}{(b/2a)^{1/3}} \geq c \Rightarrow \frac{3b}{2}\left(\frac{2a}{b}\right)^{1/3} \geq c$$

Since a, b are +ve, hence on cubing both sides we get

$$\frac{27b^3}{8}\cdot\frac{2a}{b} \geq c^3 \Rightarrow 27ab^2 \geq 4c^3 \quad \text{(proved)}$$

53. $f(x) = x^{1/x},\ x > 0$

Let $y = x^{1/x} \Rightarrow \log y = \frac{1}{x}\log x$

Differentiating w.r.t. x we get

$$\frac{1}{y}\frac{dy}{dx} = \frac{\frac{1}{x}.x - 1.\log x}{x^2} \Rightarrow \frac{dy}{dx} = \frac{y(1-\log x)}{x^2}$$

For max/min value put $\frac{dy}{dx} = 0$

$$\Rightarrow \frac{y(1-\log x)}{x^2} = 0 \Rightarrow \log x = 1 \Rightarrow x = e$$

Now, $$\frac{d^2y}{dx^2} = \frac{\left(\frac{dy}{dx}(1-\log x) - \frac{1}{x}y\right)x^2 - 2xy(1-\log x)}{x^4}$$

$$\left.\frac{d^2y}{dx^2}\right|_{x=e} = \left(\frac{-xy}{x^4}\right)_{x=e}$$

$$\left[\because\ \frac{dy}{dx} = 0, 1-\log x = 0\ at\ x = e\right]$$

$$= \frac{-e^{1/e}}{e^3} = -ve \Rightarrow y \text{ is max at } x = e$$

Hence, $f_{max} = e^{1/e} \Rightarrow x^{1/x} < e^{1/e},\ \forall\, x$

$\Rightarrow \pi^{1/\pi} < e^{1/e}$

$\Rightarrow$ Raising to the power πe on both sides we get

$\Rightarrow [(\pi)^{1/\pi}]^{\pi e} < [e^{1/e}] \Rightarrow \pi^e < e^\pi$ or $e^\pi > \pi^e$

54. Given that x and y are two real variables such that $x > 0$ and $xy = 1$.

To find the minimum value of $x + y$.

Let $S = x + y \Rightarrow S = x + \frac{1}{x} \quad (\because\ xy = 1)$

$$\Rightarrow \frac{dS}{dx} = 1 - \frac{1}{x^2}$$

For minimum value of S, put $\frac{dS}{dx} = 0$

$$\Rightarrow 1 - \frac{1}{x^2} = 0 \Rightarrow x = \pm 1$$

But $x > 0$,

$\therefore\ x = 1$ Now $\frac{d^2S}{dx^2} = \frac{2}{x^3}$

$$\Rightarrow \left.\frac{d^2S}{dx^2}\right|_{x=1} = 2 = +ve$$

Hence, S is minimum when $x = 1 \therefore\ S_{min} = 1 + \frac{1}{1} = 2$

55. $f(x) = \frac{(a+x)(b+x)}{(c+x)}, a, b > c, x > -c$

$$= \frac{(a-c+x+c)(b-c+x+c)}{x+c}$$

$$= \frac{(a-c)(b-c)}{x+c} + (x+c) + (a-c) + (b-c)$$

$$\Rightarrow f'(x) = \frac{-(a-c)(b-c)}{(x+c)^2} + 1$$

Put, $f'(x) = 0 \Rightarrow x = -c \pm \sqrt{(a-c)(b-c)}$

$\Rightarrow\ x = -c + \sqrt{(a-c)(b-c)}$ [+ve sign is taken $\because x > -c$]

Now, $f''(x) = \frac{2(a-c)(b-c)}{(x+c)^3} > 0$ for $a, b > c$ and $x > -c$

$\Rightarrow f(x)$ is least at $x = -c + \sqrt{(a-c)(b-c)}$

Hence, $$f_{min} = \frac{(a-c)(b-c)}{\sqrt{(a-c)(b-c)}} + \sqrt{(a-c)(b-c)} + (a-c) + (b-c)$$

$$= (a-c) + (b-c) + 2\sqrt{(a-c)(b-c)} = (\sqrt{a-c} + \sqrt{b-c})^2$$