Chapter – 3

Recording of Transactions-I

Ex 3.1

Question 1.
Identify the quadrant in which an angle of each given measure lies
(i) 25°
(ii) 825°
(iii) -55°
(iv) 328°

Solution:
(i) 25° = I quadrant
(ii) 825° = 105° (90° + 15°) = II quadrant
(iii) -55° = IV quadrant
(iv) 328° = IV quadrant (270° + 58°)

360) 825 (2
720
105

(v) -230° = 360° – 230° = 130° = (90° + 40°) II quadrant

Question 2.
For each given angle, find a coterminal angle with measure of θ such that θ° < θ < 360°
(i) 395°
(ii) 525°
(iii) 1150°
(iv) -270°
(v) -450°

Solution:
(i) 395° = 360° + 35°
∴ coterminal angle = 35°

(ii) 525° – 360°= 165°
coterminal angle = 165°

(iii) 1150° = 360 × 3 + 70° = 70°
coterminal angle = 70°

(iv) -270° = coterminal angle =+90° {270° + 90° = 360°}

(v) -450° = -360° – 90° = -90°
∴ coterminal angle = 360° – 90° = 270°

Question 3.

If $a \cos\theta - b \sin\theta = c$, show that $a \sin\theta + b \cos\theta = \pm\sqrt{a^2 + b^2 - c^2}$.

Solution:
$a \cos\theta - b \sin\theta = c$
$\Rightarrow (a \cos\theta - b \sin\theta)^2 = c^2$
(i.e) $a^2 \cos^2\theta + b^2 \sin^2\theta - 2ab \sin\theta \cos\theta = c^2$
(i.e) $a^2 (1 - \sin^2\theta) + b^2 (1 - \cos^2\theta) - 2ab \sin\theta \cos\theta = c^2$
$a^2 - a^2 \sin^2\theta + b^2 - b^2 \cos^2\theta - 2ab \sin\theta \cos\theta = c^2$
$a^2 + b^2 - c^2 = a^2 \sin^2\theta + b^2 \cos^2\theta + 2ab \sin\theta \cos\theta$
(i.e) $(a \sin\theta + b \cos\theta)^2 = a^2 + b^2 - c^2$
$\Rightarrow \quad a \sin\theta + b \cos\theta = \pm\sqrt{a^2 + b^2 - c^2}$

Question 4.

If $\sin\theta + \cos\theta = m$, show that $\cos^6\theta + \sin^6\theta = \frac{4 - 3(m^2 - 1)^2}{4}$, where $m^2 \le 2$.

Solution:

Given, $m = \sin\theta + \cos\theta$
$m^2 - 1 = (\sin\theta + \cos\theta)^2 - 1 = \sin^2\theta + \cos^2\theta + 2\sin\theta\cos\theta - 1$
$= 1 + 2\sin\theta \cos\theta - 1 = 2\sin\theta\cos\theta$

$$(m^2-1)^2 = (2\sin\theta\cos\theta)^2 = 4\sin^2\theta\cos^2\theta$$

$$\text{R.H.S.} = \frac{4-3(m^2-1)^2}{4} = \frac{4-3(4\sin^2\theta\cos^2\theta)}{4}$$

$$= 4\left[\frac{1-3\sin^2\theta\cos^2\theta}{4}\right] = 1-3\sin^2\theta\cos^2\theta \qquad ...(1)$$

$$\text{L.H.S} = \cos^6\theta + \sin^6\theta = (\cos^2\theta)^3 + (\sin^2\theta)^3 \; [a^3+b^3 = (a+b)(a^2-ab+b^2)]$$

$$= (\cos^2\theta + \sin^2\theta)[\cos^4\theta + \sin^4\theta - \cos^2\theta\sin^2\theta]$$

$$= 1[(\cos^2\theta + \sin^2\theta)^2 - 2\sin^2\theta\cos^2\theta - \sin^2\theta\cos^2\theta]$$

$$= 1 - 3\sin^2\theta\cos^2\theta \qquad ...(2)$$

$(1) = (2) \Rightarrow$ LHS = RHS

Question 5.

If $\frac{\cos^4\alpha}{\cos^2\beta} + \frac{\sin^4\alpha}{\sin^2\beta} = 1$, prove that

(*i*) $\sin^4\alpha + \sin^4\beta = 2\sin^2\alpha\sin^2\beta$ (*ii*) $\frac{\cos^4\beta}{\cos^2\alpha} + \frac{\sin^4\beta}{\sin^2\alpha} = 1$

Solution:

(*i*) Given $\frac{\cos^4\alpha}{\cos^2\beta} + \frac{\sin^4\alpha}{\sin^2\beta} = 1$

$$\Rightarrow \frac{\cos^4\alpha}{\cos^2\beta} = 1 - \frac{\sin^4\alpha}{\sin^2\beta} = \frac{\sin^2\beta - \sin^4\alpha}{\sin^2\beta}$$

$$\Rightarrow \text{Given } \frac{\cos^4\alpha}{\cos^2\beta} = \frac{\sin^2\beta - \sin^4\alpha}{\sin^2\beta} \Rightarrow \cos^4\alpha\sin^2\beta = (\sin^2\beta - \sin^4\alpha)\cos^2\beta$$

$$\Rightarrow (1-\sin^2\alpha)\sin^2\beta = (\sin^2\beta - \sin^4\alpha)(1-\sin^2\beta)$$

$$(1+\sin^4\alpha - 2\sin^2\alpha)\sin^2\beta = \sin^2\beta - \sin^4\alpha - \sin^4\beta + \sin^4\alpha\sin^2\beta$$

$$\Rightarrow \sin^2\beta + \sin^4\alpha\sin^2\beta - 2\sin^2\alpha\sin^2\beta = \sin^2\beta - \sin^4\alpha - \sin^4\beta + \sin^4\alpha\sin^2\beta$$

$\Rightarrow \quad \sin^4 \alpha + \sin^4\beta = 2 \sin^2 \alpha \sin^2 \beta$

(*ii*) $\sin^4 \alpha + \sin^4 \beta = 2 \sin^2 \alpha \sin^2\beta$

$\Rightarrow \quad \sin^4 \alpha + \sin^4 \beta - 2 \sin^2 \alpha \sin^2 \beta = 0$

$(\sin^2 \alpha - \sin^2 \beta)^2 = 0$

$\Rightarrow \quad \sin^2 \alpha = \sin^2 \beta$

So, $1 - \sin^2 \alpha = 1 - \sin^2 \beta$

(i.e.) $\cos^2 \alpha = \cos^2 \beta$

So $\dfrac{\cos^4 \beta}{\cos^2 \alpha} = \dfrac{(\cos^2 \alpha)^2}{\cos^2 \alpha} = \cos^2 \alpha$

and $\dfrac{\sin^4 \beta}{\sin^2 \alpha} = \dfrac{(\sin^2 \alpha)^2}{\sin^2 \alpha} = \sin^2 \alpha$

$\therefore \quad$ LHS $= \cos^2 \alpha + \sin^2 \alpha = 1 =$ RHS

Question 6.

If $y = \dfrac{2\sin\alpha}{1+\cos\alpha+\sin\alpha}$, then prove that $\dfrac{1-\cos\alpha+\sin\alpha}{1+\sin\alpha} = y$

Solution:

To Prove $\dfrac{2\sin\alpha}{1+\cos\alpha+\sin\alpha} = \dfrac{1-\cos\alpha+\sin\alpha}{1+\sin\alpha}$

$$\text{LHS} = \frac{2\sin\alpha}{1+\cos\alpha+\sin\alpha} = \frac{2\sin\alpha}{1+\cos\alpha+\sin\alpha} \times \frac{(1+\sin\alpha)-\cos\alpha}{(1+\sin\alpha)-\cos\alpha}$$

$$= \frac{2\sin\alpha[(1+\sin\alpha-\cos\alpha)}{(1+\sin\alpha)^2-\cos^2\alpha} \qquad \text{....(1)}$$

$$\begin{aligned} \text{Dr} &= 1 + \sin^2 \alpha + 2\sin \alpha - \cos^2 \alpha \\ &= \sin^2 \alpha + 2\sin \alpha + (1 - \cos^2 \alpha) \\ &= \sin^2 \alpha + 2\sin \alpha + \sin^2 \alpha \end{aligned}$$

$$= 2\sin^2\alpha + 2\sin\alpha$$
$$= 2\sin\alpha\,(1 + \sin\alpha)$$
$$\therefore (1) \Rightarrow \frac{2\sin\alpha[1+\sin\alpha-\cos\alpha]}{2\sin\alpha(1+\sin\alpha)} = \frac{1+\sin\alpha-\cos\alpha}{1+\sin\alpha} = \text{RHS}$$

Question 7.

If $x = \sum_{n=0}^{\infty}\cos^{2n}\theta$, $y = \sum_{n=0}^{\infty}\sin^{2n}\theta$ and $z = \sum_{n=0}^{\infty}\cos^{2n}\theta\sin^{2n}\theta$, $0 < \theta < \frac{\pi}{2}$, then show that $xyz = x + y + z$.

[Hint: Use the formula $1 + x + x^2 + x^3 + \ldots = \frac{1}{1-x}$, where $|x| < 1$].

Solution:

$$x = 1 + \cos^2\theta + \cos^4\theta + \cos^6\theta + \ldots = \frac{1}{1-\cos^2\theta} = \frac{1}{\sin^2\theta}$$
$$y = 1 + \sin^2\theta + \sin^4\theta + \sin^6\theta + \ldots = \frac{1}{1-\sin^2\theta} = \frac{1}{\cos^2\theta}$$
$$z = 1 + \cos^2\theta\sin^2\theta + \cos^4\theta\sin^4\theta + \ldots = \frac{1}{1-\cos^2\theta\sin^2\theta}$$

$$\text{LHS} = xyz = \frac{1}{\sin^2\theta}\cdot\frac{1}{\cos^2\theta}\cdot\frac{1}{1-\sin^2\theta\cos^2\theta}$$
$$= \frac{1}{\sin^2\theta\cos^2\theta(1-\sin^2\theta\cos^2\theta)} \qquad \ldots(1)$$

$$\text{RHS} = x + y + z = \frac{1}{\sin^2\theta} + \frac{1}{\cos^2\theta} + \frac{1}{1-\sin^2\theta\cos^2\theta}$$
$$= \left(\frac{1}{\sin^2\theta} + \frac{1}{\cos^2\theta}\right) + \left(\frac{1}{1-\sin^2\theta\cos^2\theta}\right)$$
$$= \frac{\cos^2\theta + \sin^2\theta}{\sin^2\theta\cos^2\theta} + \frac{1}{1-\sin^2\theta\cos^2\theta}$$

$$= \frac{1}{\sin^2\theta\cos^2\theta} + \frac{1}{1-\sin^2\theta\cos^2\theta}$$

$$= \frac{1-\sin^2\theta\cos^2\theta+\sin^2\theta\cos^2\theta}{(\sin^2\theta\cos^2\theta)(1-\sin^2\theta\cos^2\theta)}$$

$$= \frac{1}{(\sin^2\theta\cos^2\theta)(1-\sin^2\theta\cos^2\theta)} \quad ...(2)$$

(1) = (2) $\Rightarrow$ LHS = RHS (i.e) $xyz = x + y + z$

Question 8.

If $\tan^2\theta = 1 - k^2$, show that $\sec\theta + \tan^3\theta \operatorname{cosec}\theta = (2 - k^2)^{3/2}$. Also, find the values of k for which this result holds.

Solution:

$\tan^2\theta = 1 - k^2$

$-k^2 = \tan^2\theta - 1$

$2 - k^2 = \tan^2\theta - 1 + 2 = 1 + \tan^2\theta = \sec^2\theta$

$2 - k^2 = \sec^2\theta$

RHS $= (2 - k^2)^{3/2} = (\sec^2\theta)^{3/2} = \sec^3\theta$ ...(1)

Now LHS $= \sec\theta + \tan^3\theta \operatorname{cosec}\theta$

$$= \frac{1}{\cos\theta} + \frac{\sin^3\theta}{\cos^3\theta}.\frac{1}{\sin\theta} = \frac{1}{\cos\theta} + \frac{\sin^2\theta}{\cos^3\theta}$$

$$= \frac{\cos^2\theta+\sin^2\theta}{\cos^3\theta} = \frac{1}{\cos^3\theta} = \sec^3\theta \quad ...(2)$$

(1) = (2) $\Rightarrow$ LHS = RHS

$2 - k^2 \geq 0 \Rightarrow -k^2 \geq -2 \Rightarrow \therefore k^2 \leq 2 \Rightarrow k = \sqrt{2}$

$\therefore k \in (-1, 1)$

Question 9.
If sec θ + tan θ = p, obtain the values of sec θ, tan θ and sin θ in terms of p.

Solution:
Given, sec θ + tan θ = p
we know $\sec^2\theta - \tan^2\theta = 1$
(i.e) (sec θ + tan θ) (sec θ – tan θ) = 1

$$\Rightarrow \sec\theta - \tan\theta = \frac{1}{\sec\theta + \tan\theta} = \frac{1}{p}$$

$$\text{Now} \quad \sec\theta + \tan\theta = p \quad \ldots(1)$$

$$\sec\theta - \tan\theta = \frac{1}{p} \quad \ldots(2)$$

$$(1) + (2) \Rightarrow \quad 2\sec\theta = p + \frac{1}{p} = \frac{p^2+1}{p}$$

$$\Rightarrow \quad \sec\theta = \frac{p^2+1}{2p}$$

$$(1) - (2) \Rightarrow \quad 2\tan\theta = p - \frac{1}{p} = \frac{p^2-1}{p}$$

$$\Rightarrow \quad \tan\theta = \frac{p^2-1}{2p}$$

$$\text{Now } \frac{\tan\theta}{\sec\theta} = \frac{\sin\theta}{\cos\theta} \times \cos\theta = \sin\theta$$

$$\therefore \sin\theta = \frac{\tan\theta}{\sec\theta} = \frac{p^2-1}{2p} \bigg/ \frac{p^2+1}{2p} = \frac{p^2-1}{p^2+1}$$

$$\text{So, } \sec\theta = \frac{p^2+1}{2p};\ \tan\theta = \frac{p^2-1}{2p};\ \sin\theta = \frac{p^2-1}{p^2+1}$$

Question 10.
If cot θ (1 + sin θ) = 4m and cot θ (1 – sin θ) = 4n, then prove that $(m^2 - n^2)^2 = mn$.

Solution:
cot θ (1 + sin θ) = 4m

$$\Rightarrow \quad \frac{\cos\theta}{\sin\theta}(1+\sin\theta) = 4m$$

$$\Rightarrow \quad \cot\theta + \cos\theta = 4m$$

$$\Rightarrow \quad m = \frac{\cot\theta + \cos\theta}{4} \qquad ...(1)$$

$$\cot\theta\,(1 - \sin\theta) = 4n$$

$$\Rightarrow \quad \frac{\cos\theta}{\sin\theta}(1 - \sin\theta) = 4n$$

$$\Rightarrow \quad \cot\theta - \cos\theta = 4n$$

$$\Rightarrow \quad n = \frac{\cot\theta - \cos\theta}{4} \qquad ...(2)$$

To prove $(m^2 - n^2)^2 = mn$

$$\text{Now RHS} = mn = \left(\frac{\cot\theta + \cos\theta}{4}\right)\left(\frac{\cot\theta - \cos\theta}{4}\right)$$

$$= \frac{\cot^2\theta - \cos^2\theta}{16} \qquad ...(3)$$

Now $(m^2 - n^2) = [(m + n)(m - n)]$

$$m + n = \frac{\cot\theta + \cos\theta + \cot\theta - \cos\theta}{} = \frac{2\cot\theta}{4} = \frac{\cot\theta}{2}$$

$$m - n = \frac{\cot\theta + \cos\theta - \cot\theta + \cos\theta}{4} = \frac{2\cos\theta}{4} \quad \frac{\cos\theta}{2}$$

$$\therefore (m + n)(m - n) = \frac{\cot\theta}{2}\,\frac{\cos\theta}{2} = \frac{\cot\theta\cos\theta}{4}$$

$$\text{So LHS} = (m^2 - n^2)^2 = [(m - n)(m + n)]^2$$

$$= \left(\frac{\cot\theta\cos\theta}{4}\right)^2 = \frac{\cot^2\theta\cos^2\theta}{16}$$

$$\text{RHS} = mn = \frac{\cot^2\theta - \cos^2\theta}{16}$$

$$= \frac{1}{16}\left[\frac{\cos^2\theta}{\sin^2\theta} - \cos^2\theta\right] = \frac{\cos^2\theta}{16\sin^2\theta}[1-\sin^2\theta]$$

$$= \frac{1}{16}\cot^2\theta\cos^2\theta = \text{LHS}$$

$$\therefore \text{LHS} = \text{RHS} \Rightarrow (m^2 - n^2)^2 = mn$$

Question 11.
If cosec θ – sin θ = a^3 and sec θ – cos θ = b^3, then prove that $a^2b^2(a^2 + b^2) = 1$.

Solution:

$$\text{cosec}\,\theta - \sin\theta = a^3$$

$$\Rightarrow \frac{1}{\sin\theta} - \sin\theta = a^3$$

$$\text{(i.e)} \quad \frac{1-\sin^2\theta}{\sin\theta} = a^3$$

$$\Rightarrow a^3 = \frac{\cos^2\theta}{\sin\theta} \Rightarrow a = \frac{(\cos\theta)^{2/3}}{(\sin\theta)^{1/3}}$$

Similary, $\sec\theta - \cos\theta = b^3$

$$\Rightarrow \frac{1}{\cos\theta} - \cos\theta = b^3$$

$$\frac{1-\cos^2\theta}{\cos\theta} = b^3$$

$$\Rightarrow \frac{\sin^2\theta}{\cos\theta} = b^3 \Rightarrow b = \frac{(\sin\theta)^{2/3}}{(\cos\theta)^{1/3}}$$

$$\text{Now } a^3 = \frac{\cos^2\theta}{\sin\theta}; \; b^3 = \frac{\sin^2\theta}{\cos\theta}$$

$$\therefore a^3b^3 = \frac{\cos^2\theta}{\sin\theta}.\frac{\sin^2\theta}{\cos\theta} = \sin\theta\cos\theta \Rightarrow ab = (\sin\theta\cos\theta)^{1/3}$$

$$\Rightarrow \quad a^2b^2 = \{(\sin\theta\cos\theta)^{1/3}\}^2 = (\sin\theta\cos\theta)^{2/3}$$

$$= \sin^{2/3}\theta\cos^{2/3}\theta$$

To prove $a^2b^2(a^2+b^2) = 1$

$$a^2+b^2 = \left\{\left[\frac{(\cos\theta)^{2/3}}{(\sin\theta)^{1/3}}\right]^2 + \left[\frac{(\sin\theta)^{2/3}}{(\cos\theta)^{1/3}}\right]^2\right\}$$

$$= \frac{(\cos\theta)^{4/3}}{(\sin\theta)^{2/3}} + \frac{(\sin\theta)^{4/3}}{(\cos\theta)^{2/3}}$$

$$= \frac{\cos^2\theta + \sin^2\theta}{(\sin\theta\cos\theta)^{2/3}} = \frac{1}{(\sin\theta\cos\theta)^{2/3}}$$

$$\text{LHS} = a^2b^2(a^2+b^2) = (\sin\theta\cos\theta)^{2/3} \times \frac{1}{(\sin\theta\cos\theta)^{2/3}}$$

$$= 1 = \text{RHS}$$

Question 12.

Eliminate θ from the equations a sec θ – c tan θ = b and b sec θ + d tan θ = c.

Solution:

Taking sec θ = X and tan θ = Y we get the equations as

$$aX - cY = b \quad \ldots(1)$$

$$bX + dY = c \quad \ldots(2)$$

$$(1) \times d \Rightarrow \quad adX - cdY = bd \quad \ldots(3)$$

$$(2) \times c \Rightarrow \quad bcX + cdY = c^2 \quad \ldots(4)$$

$$(3) + (4) \Rightarrow \quad X(ad + bc) = bd + c^2$$

$$\therefore \qquad X = \frac{bd + c^2}{ad + bc}$$

$$\text{(i.e)} \qquad \sec\theta = \frac{bd \quad c}{ad \quad bc}$$

$$(2) \times a \Rightarrow \qquad abX + adY = ac \qquad \ldots(5)$$

$$\text{Now } (1) \times b \Rightarrow \qquad abX - bcY = b^2 \qquad \ldots(6)$$

$$(5) - (6) \Rightarrow \qquad Y(ad + bc) = ac - b^2$$

$$\Rightarrow \qquad Y = \frac{ac - b^2}{ad + bc}$$

$$\text{(i.e)} \qquad \tan\theta = \frac{ac - b^2}{ad + bc}$$

$$\therefore \sec\theta = \frac{bd + c^2}{ad + bc} \text{ and } \tan\theta = \frac{ac - b^2}{ad + bc}$$

We know $\sec^2\theta - \tan^2\theta = 1$

$$\Rightarrow \qquad \frac{(bd + c^2)^2}{(ad + bc)^2} - \frac{(ac - b^2)^2}{(ad + bc)^2} = 1$$

$$\Rightarrow \qquad \frac{(bd + c^2)^2 - (ac - b^2)^2}{(ad + bc)^2} = 1$$

$$\Rightarrow \qquad (bd + c^2)^2 - (ac - b^2)^2 = (ad + bc)^2$$

$$\Rightarrow \qquad (bd + c^2)^2 = (ad + bc)^2 + (ac - b^2)^2$$

Ex 3.2

Question 1.
Express each of the following angles in radian measure:
(i) 30°
(ii) 135°
(iii) -205°
(iv) 150°
(v) 330°

Solution:

(*i*) **30°** $= 30 \times \frac{\pi}{180} = \frac{\pi}{6}$ radians

(*ii*) **135°** $= 135 \times \frac{\pi}{180} = \frac{3\pi}{4}$ radians

(*iii*) **–205°** $= -205 \times \frac{\pi}{180} = \frac{-41\pi}{36}$ radians

(*iv*) **150°** $= 150 \times \frac{\pi}{180} = \frac{5\pi}{6}$ radians

(*v*) **330°** $= 330 \times \frac{\pi}{180} = \frac{11\pi}{6}$ radians

Question 2.
Find the degree measure corresponding to the following radian measures

(*i*) $\frac{\pi}{3}$ (*ii*) $\frac{\pi}{9}$ (*iii*) $\frac{2\pi}{5}$ (*iv*) $\frac{7\pi}{3}$ (*v*) $\frac{10\pi}{9}$

Solution:

(*i*) $\frac{\pi}{3} = \frac{\pi}{3} \times \frac{180}{\pi} = 60°$

(*ii*) $\frac{\pi}{9} = \frac{\pi}{9} \times \frac{180}{\pi} = 20°$

(*iii*) $\frac{2\pi}{5} = \frac{2\pi}{5} \times \frac{180}{\pi} = 72°$

(*iv*) $\frac{7\pi}{3} = \frac{7\pi}{3} \times \frac{180}{\pi} = 420°$

(*v*) $\frac{10\pi}{9} = \frac{10\pi}{9} \times \frac{180}{\pi} = 200°$

Question 3.
What must be the radius of a circular running path, around which an athlete must run 5 times in order to describe 1 km?

Solution:
Distance travelled in 5 rounds = 1 km = 1000 m
Distance travelled in 1 round = 1000/5 = 200 m
Let the radius of the circular path be r metre

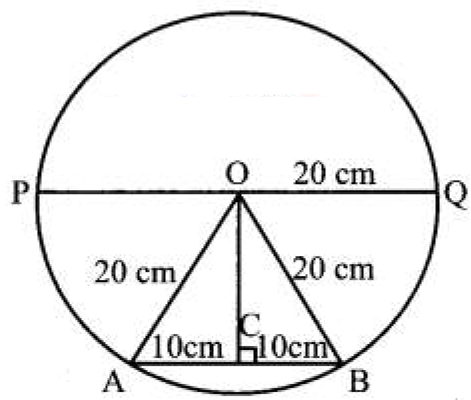
So $2\pi r = 200$

(i.e) $2\times\frac{22}{7}\times r = 200$

$\therefore r = \frac{200\times7}{44} = 31.82$ m

Question 4.
In a circle of diameter 40 cm, a chord is of length 20 cm. Find the length of the minor arc of the chord.

Solution:
O = centre of the circle
PQ = diameter = 40 cm
∴ OQ = 20 cm
radius = 20 cm
⇒ OA = OB = 20 cm
chord AB = 20 cm
OC ⊥ r AB
∴ AC = CB = 10 cm
Now from the right angled triangle OCB



$\sin\theta = \frac{10}{20} = \frac{1}{2} \Rightarrow \theta = \pi/6$

So $\angle AOB = 2\times\frac{\pi}{6} = \frac{\pi}{3} = 60^\circ$

Now $r = 20$ cm, $\theta = \pi/3$

So arc AB $= \frac{60}{360}2\pi r$

$= \frac{60}{360}\times2\times\pi\times20 = \frac{20\pi}{3}$ cm (or) 20.95 cm.

Question 5.
Find the degree measure of the angle subtended at the centre of circle of radius 100 cm by an arc of length 22 cm.

Solution:
r = 100 cm; arc length = 22 cm

$$\text{Arc length} = \frac{\theta}{360} \times 2\pi r$$

$$\Rightarrow \frac{\theta}{360} \times 2 \times \frac{22}{7} \times 100 = 22 \quad \Rightarrow \theta = \frac{7 \times 22 \times 360}{44 \times 100} \quad \frac{126}{10} = 12° \quad \frac{6}{10} \times 60 = 12°36'$$

Question 6.
What is the length of the arc intercepted by a central angle of measure 41° in a circle of radius 10 ft?

Solution:
θ = 41°, r = 10 ft

$$\text{Arc length} = \frac{\theta}{360} \times 2\pi r$$

$$= \frac{41}{360} \times 2 \times \frac{22}{7} \times 10$$

$$= \frac{451}{63} = 7.16\text{ft}$$

Question 7.
If in two circles, arcs of the same length subtend angles 60° and 75° at the centre, find the ratio of their radii.

Solution:
Let the two radii be r_1 and r_2
The central angles arc 60° and 75°
The arc lengths be s_1 and s_2

we are given $s_1 = s_2$

$$\Rightarrow \quad \frac{60}{360}\times 2\pi r_1 = \frac{75}{360}\times 2\pi r_2$$

$$60\, r_1 = 75\, r_2$$

$$\Rightarrow \quad \frac{r_1}{r_2} = \frac{75}{60} = \frac{5}{4}$$

$$r_1 : r_2 = 5 : 4$$

So their radii are in the ratio 5 : 4.

Question 8.
The perimeter of a certain sector of a circle is equal to the length of the arc of a semicircle having the same radius. Express the angle of the sector in degrees, minutes and seconds.

Solution:
Let r be the radius and so perimeter of a sector = l + 2r
Length of arc of the semicircle = πr
we are given l + 2r = πr
(i.e) l = πr – 2r

$$\frac{\theta}{360}\times 2\pi r = \pi r - 2r = r(\pi - 2)$$

$$\Rightarrow \frac{\theta}{360}\times 2\pi = \pi - 2$$

$$\theta = \frac{\pi - 2}{\pi}\times 180 = \frac{\frac{22}{7} - 2}{\frac{22}{7}}\times 180$$

$$= \frac{8}{7}\times\frac{7}{22}\times 180$$

$$= \frac{8\times 90}{11} = \frac{720}{11} = 65\frac{5}{11} = 65^\circ\ \frac{5}{11}\times 60$$

$$= 65^\circ\ \frac{300'}{11} = 65^\circ 27'\frac{3}{11}$$

$$= 65°27' \frac{3\times 60}{11} = 65°27'16"$$

Question 9.
An airplane propeller rotates 1000 times per minute. Find the number of degrees that a point on the edge of the propeller will rotate in 1 second.

Solution:
Number of rotations in 1 min = 1000

So Number of rotations in 1 sec $= \frac{1000}{60} = \frac{50}{3}$

The angle rotated in 1 rotation = 360°

So, the angle rotated in $\frac{50}{3}$ rotation $= \frac{50}{3}\times 360° = 6000°$

Question 10.
A train is moving on a circular track of 1500 m radius at the rate of 66 km / hr. What angle will it turn in 20 seconds?

Solution:
Speed of the train = 66 km/hr

$$= 66\times\frac{5}{18}\text{m / sec}$$

$$= \frac{55}{3}\text{m / sec}$$

(i.e) Distance travelled in sec $= \frac{55}{3}\text{m}$

So, distance travelled in 20 sec $= \frac{55}{3}\times 20 = \frac{1100}{3}\text{m}$

Now radius of the circular track = 1500 m

we are given $\frac{\theta}{360}\times 2\times\frac{22}{7}\times 1500 = \frac{1100}{3}$

$$\Rightarrow \theta = \frac{1100}{3}\times\frac{360\times 7}{2\times 22\times 1500} = 14°$$

Question 11.
A circular metallic plate of radius 8 cm and thickness 6 mm is melted and molded into a pie (a sector of the circle with thickness) of radius 16 cm and thickness 4 mm. Find the angle of the sector.

Solution:
Area of the circular plate melted

$$= \pi(8^2) = 64\pi \text{ cm}^2$$

$$\text{Thickness} = 6 \text{ mm} = \frac{6}{10} \text{ cm}$$

$$\text{So, volume} = \text{area} \times \text{thickness}$$

$$= 64\pi \times \frac{6}{10} = V_1$$

$$\text{Area of the pie made} = \frac{\theta}{360} \times \pi(16^2)$$

$$= (256\pi)\frac{\theta}{360}$$

$$\text{Thickness of the pie made} = 4 \text{ mm}$$

$$= \frac{4}{10} \text{ cm}$$

$$\therefore \text{Volume of the pie made} = (256\pi)\frac{\theta}{360} \times \frac{4}{10} = V_2$$

We are given $V_1 = V_2$

$$\Rightarrow \quad 64\pi\frac{6}{10} = (256\pi)\left(\frac{\theta}{360}\right) \times \frac{4}{10}$$

$$\theta = 64 \times \frac{6}{10} \times \frac{1}{256} \times 360 \times \frac{10}{4}$$

$$= 135° \text{ (i.e) } \theta = \frac{3\pi}{4}$$

Ex 3.3

Question 1.
Find the values of
(i) sin(480°)
(ii) sin(-1110°)
(iii) cos(300°)
(iv) tan(1050°)
(v) cot(660°)
(vi) tan $\left(\frac{19\pi}{3}\right)$
(vii) sin $\left(-\frac{11\pi}{3}\right)$
Solution:

(i) sin 480°

sin 480° = sin (5 × 90° + 30°)

= cos 30° = $\frac{\sqrt{3}}{2}$

(ii) sin (- 1110°)

sin (- 1110°) = sin (1110°)

= – sin(12 × 90 + 30°)

= – sin 30° = $-\frac{1}{2}$

(iii) cos 300°

cos 300° = cos (360° – 60°)

= cos 60° = $\frac{1}{2}$

(iv) tan(1050°) = tan [3(360°) – 30°]

= – tan 30° = $-\frac{1}{\sqrt{3}}$

(v) cot(660°) = cot (360° × 2 – 60°)

$$= -\cot 60° = -\frac{1}{\sqrt{3}}$$

(vi) $\tan\left(\frac{19\pi}{3}\right) = \tan\frac{19}{3}\times 180 = \tan\frac{19}{3}\times\frac{360}{2}$

$$= \tan\frac{19}{6}(360°)$$

$$= \tan 3\frac{1}{6}(360°)$$

$$= \tan\left[3(360)° + \frac{360°}{6}\right] = \tan 60° = \sqrt{3}$$

(vii) $\sin\left(-\frac{11\pi}{3}\right) = -\sin\frac{11\pi}{3} = -\sin\frac{11}{3}\times 180$

$$= -\sin\frac{11}{3}\times\frac{360}{2} = -\sin 360°\left(\frac{11}{6}\right)$$

$$= -\sin 360°\left(2-\frac{1}{6}\right)$$

$$= -\left[\sin(360°\times 2) - \frac{360°}{6}\right]$$

$$= -(-\sin 60°) = \sqrt{3}/2$$

Question 2.

$\left(\frac{5}{7}, \frac{2\sqrt{6}}{7}\right)$ is a point on the terminal side of an angle θ is standard position. Determine the trigonometric function values of angle θ .

Solution:

In the diagram $ON = \frac{5}{7}$; $PN = \frac{2\sqrt{6}}{7}$; $ON^2 + NP^2 = OP^2$

(i.e) $\frac{25}{49}+\frac{24}{49}=\frac{49}{49}=OP^2 \Rightarrow OP=1$

$\sin\theta = \frac{PN}{OP}=\frac{2\sqrt{6}/7}{1}=\frac{2\sqrt{6}}{7}$

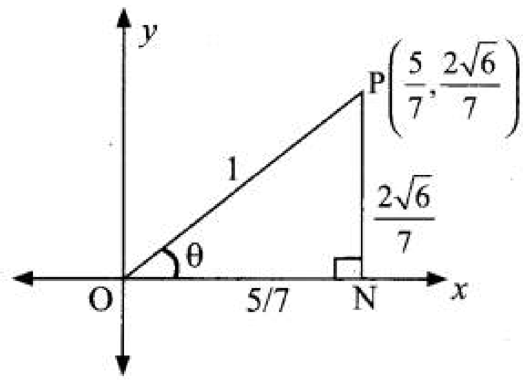
$\cos\theta = \frac{ON}{OP}=\frac{5/7}{1}=\frac{5}{7}$

$\tan\theta = \frac{PN}{ON}=\frac{2\sqrt{6}/7}{5/7}=\frac{2\sqrt{6}}{5}$

$\text{cosec}\,\theta = \frac{1}{\sin\theta}=\frac{7}{2\sqrt{6}}$

$\sec\theta = \frac{1}{\cos\theta}=\frac{7}{5}$

$\cot\theta = \frac{1}{\tan\theta}=\frac{5}{2\sqrt{6}}$



Question 3.
Find the values of the other five trigonometric functions for the following:

(i) cos θ = −1/2; θ lies in the III quadrant.

Solution:

Taking the Numerical values

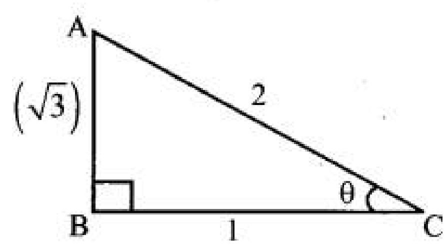
$\cos\theta = -\frac{1}{2}$; θ from ΔABC, AB = $\sqrt{4-1}=\sqrt{3}$

θ is in III quadrant. ∴ tan θ, cot θ are +ve

Now $\sin\theta = -\frac{\sqrt{3}}{2}$; $\text{cosec}\,\theta = -\frac{2}{\sqrt{3}}$

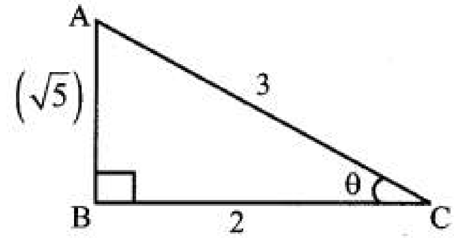
$\cos\theta = -\frac{1}{2}$; $\sec\theta = -\frac{2}{1}=-2$

$\tan\theta = +\frac{\sqrt{3}}{1}=\sqrt{3}$; $\cot\theta = \frac{1}{\sqrt{3}}$



(ii) cos θ = 2/3 ; θ lies in the I quadrant

Solution:



$\cos\theta = \frac{2}{3}$; θ from Δ ABC, AB = $\sqrt{9-4} = \sqrt{5}$

θ is in I quadrant

∴ All trigonometric values are +ve

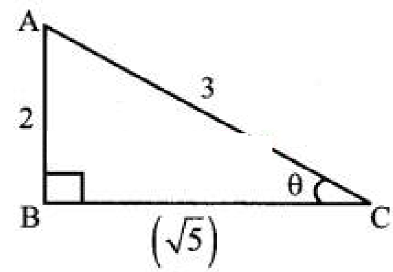
$\sin\theta = \frac{\sqrt{5}}{3}$; $\operatorname{cosec}\theta = \frac{3}{\sqrt{5}}$

$\cos\theta = \frac{2}{3}$; $\sec\theta = \frac{3}{2}$

$\tan\theta = \frac{\sqrt{5}}{2}$; $\cot\theta = \frac{2}{\sqrt{5}}$

(iii) sin θ = –2/3 ; θ lies in the IV quadrant

Solution:



θ from Δ ABC, BC = $\sqrt{9-4} = \sqrt{5}$

θ is in IV quadrant ∴ cos θ, sec θ are +ve

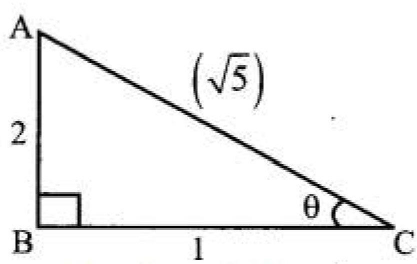
$\sin\theta = -\frac{2}{3}$; $\operatorname{cosec}\theta = \frac{-3}{2}$

$\cos\theta = \frac{\sqrt{5}}{3}$; $\sec\theta = \frac{3}{\sqrt{5}}$

$\tan\theta = -\frac{2}{\sqrt{5}}$; $\cot\theta = -\frac{\sqrt{5}}{2}$

(iv) tan θ = -2; θ lies in the II quadrant

Solution:



$\tan\theta = -2$;

From Δ ABC, BC = $\sqrt{4+1} = \sqrt{5}$

θ is in III quadrant

$\therefore$ $\sin\theta$, $\operatorname{cosec}\theta$ are + ve

$\sin\theta = \frac{2}{\sqrt{5}}$; $\operatorname{cosec}\theta = \frac{\sqrt{5}}{2}$

$\cos\theta = -\frac{1}{\sqrt{5}}$; $\sec\theta = -\frac{\sqrt{5}}{1} = -\sqrt{5}$

$\tan\theta = -\frac{2}{1} = -2$; $\cot\theta = -\frac{1}{2}$

(v) sec θ = 13/5 ; θ lies in the IV quadrant

Solution:

$\sec\theta = 13/5$

From Δ ABC, BC = $\sqrt{13^2 - 5^2} = \sqrt{169-25} = \sqrt{144} = 12$

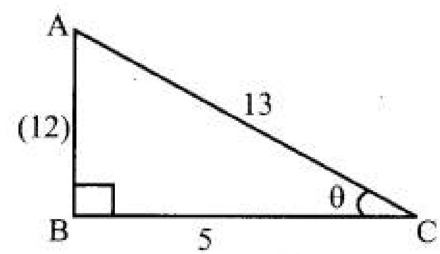
θ is in IV quadrant

$\cos\theta$, $\sec\theta$ are +ve.

$\sin\theta = -\frac{12}{13}$; $\operatorname{cosec}\theta = -\frac{13}{12}$

$\cos\theta = \frac{5}{13}$; $\sec\theta = \frac{13}{5}$

$\tan\theta = -\frac{12}{5}$; $\cot\theta = -\frac{5}{12}$



Question 4.

Prove that $\frac{\cot(180° + \theta)\sin(90° - \theta)\cos(-\theta)}{\sin(270° + \theta)\tan(-\theta)\operatorname{cosec}(360° + \theta)} = \cos^2\theta \cot\theta.$

Solution:

cot(180° + θ) = cot θ

sin (90° – θ) = cos θ

cos(-θ) = cos θ

sin (270 + θ) = – cos θ

tan(-θ) = -tan θ
cosec (360° + θ) = cosec θ

$$\therefore \quad \text{LHS} = \frac{\cot\theta.\cos\theta.\cos\theta}{(-\cos\theta)(-\tan\theta)(\operatorname{cosec}\theta)}$$

$$= \frac{\frac{\cos\theta}{\sin\theta}.\cos\theta.\cos\theta}{\cos\theta.\frac{\sin\theta}{\cos\theta}.\frac{1}{\sin\theta}} = \cot\theta \ \cos^2\theta = \text{RHS}$$

Question 5.
Find all the angles between 0° and 360° which satisfy the equation $\sin^2 \theta = 3/4$

Solution:

$$\sin^2 \theta = \frac{3}{4}$$

$$\Rightarrow \sin \theta = \pm\sqrt{\frac{3}{4}} = \pm\frac{\sqrt{3}}{2}$$

$$\sin \theta = +\frac{\sqrt{3}}{2} \Rightarrow \theta = 60° \text{ (or) } 120°$$

$$\sin \theta = -\frac{\sqrt{3}}{2} \Rightarrow \theta = -60° \text{ (or) } 240° \text{ (or) } 300°$$

Question 6.

Show that $\sin^2 \frac{\pi}{18} + \sin^2 \frac{\pi}{9} + \sin \frac{7\pi}{18} + \sin^2 \frac{4\pi}{9} = 2.$

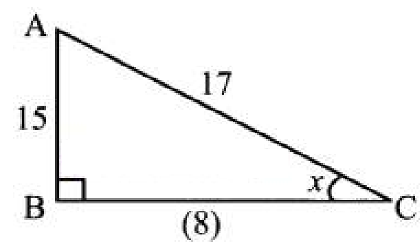
Solution:
LHS = $\sin^2 10° + \sin^2 20° + \sin^2 70° + \sin^2 80°$
= $\sin^2 10° + \sin^2 (90° – 10°) + \sin^2 20° + \sin^2(90° – 20°)$
= $\sin^2 10° + (\cos 10°)^2 + \sin^2 20° + (\cos 20°)^2$
= $(\sin^2 10+ \cos^2 10) + \sin^2 20° + \cos^2 20°$
= 1 + 1 = 2 = RHS

Ex 3.4

Question 1.

If $\sin x = \frac{15}{17}$ and $\cos y = \frac{12}{13}$, $0 < x < \frac{\pi}{2}$, $0 < y < \frac{\pi}{2}$ find the value of (*i*) $\sin(x + y)$ (*ii*) $\cos(x - y)$ (*iii*) $\tan(x + y)$.

Solution:



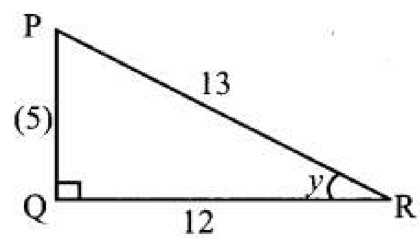
$\sin x = \frac{15}{17}$; x is in I quadrant

$\cos y = \frac{12}{13}$

y is in I quadrant

From ΔABC, $BC = \sqrt{17^2 - 15^2} = \sqrt{289 - 225}$

$= \sqrt{64} = 8$



From ΔPQR, $PQ = \sqrt{13^2 - 12^2} = \sqrt{169 - 144}$

$= \sqrt{25} = 5$

Now, $\sin x = \frac{15}{17}$, $\cos x = \frac{8}{17}$: $\tan x = \frac{15}{8}$

$\sin y = \frac{5}{13}$, $\cos y = \frac{12}{13}$: $\tan y = \frac{5}{12}$

(*i*) $\sin(x + y) = \sin x \cos y + \cos x \sin y$

$$= \frac{15}{17} \times \frac{12}{13} + \frac{8}{17} \times \frac{5}{13}$$

$$= \frac{180}{221} + \frac{40}{221} = \frac{220}{221}$$

(*ii*) $\cos(x - y) = \cos x \cos y + \sin x \sin y$

$$= \frac{8}{17} \times \frac{12}{13} + \frac{15}{17} \times \frac{5}{13}$$

$$= \frac{96}{221} + \frac{75}{221} = \frac{171}{221}$$

$$(iii)\ \tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x \tan y} = \frac{\frac{15}{8} + \frac{5}{12}}{1 - \frac{15}{8} \times \frac{5}{12}}$$

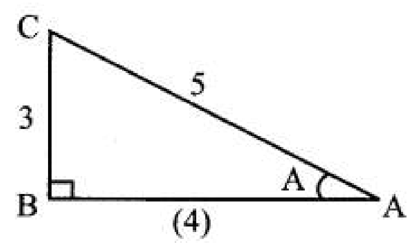
$$= \frac{45+10}{24} \Big/ 1 - \frac{75}{96} = \frac{55}{24} \Big/ \frac{96-75}{96}$$

$$= \frac{55}{24} \Big/ \frac{21}{96} = \frac{55}{24} \times \frac{96}{21} = \frac{220}{21}$$

Question 2.

If $\sin A = \frac{3}{5}$ and $\cos B = \frac{9}{41}$, $0 < A < \frac{\pi}{2}, 0 < B < \frac{\pi}{2}$, find the value of (*i*) $\sin(A+B)$ (*ii*) $\cos(A-B)$.

Solution:



$\sin A = \frac{3}{5}$

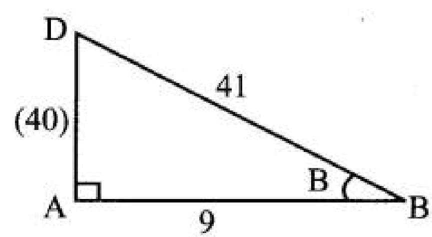
$0 < A < \pi/2$

From Δ ABC, AB = $\sqrt{5^2 - 3^2} = \sqrt{25-9} = \sqrt{16} = 4$

$\cos B = \frac{9}{41}$

$0 < B < \pi/2$

From Δ BAD



$$AD = \sqrt{41^2 - 9^2} = \sqrt{(41+9)(41-9)}$$

$$= \sqrt{50 \times 32} = \sqrt{100 \times 16}$$

$$= \sqrt{10^2 \times 4^2} = 10 \times 4 = 40$$

Now,

From ΔABC, $\sin A = \frac{3}{5}$; $\cos A = \frac{4}{5}$

From ΔABD, $\sin B = \frac{40}{41}$; $\cos B = \frac{9}{41}$

(*i*) $\sin(A+B) = \sin A\cos B + \cos A\sin B$

$$= \left(\frac{3}{5}\times\frac{9}{41}\right)+\left(\frac{4}{5}\times\frac{40}{41}\right)$$

$$= \frac{27}{205}+\frac{160}{205}=\frac{187}{205}$$

(*ii*) $\cos(A-B) = \cos A\cos B + \sin A\sin B$

$$= \left(\frac{4}{5}\times\frac{9}{41}\right)+\left(\frac{3}{5}\times\frac{40}{41}\right)$$

$$= \frac{36}{205}+\frac{120}{205}=\frac{156}{205}$$

Question 3.

Find $\cos(x-y)$, given that $\cos x = -\frac{4}{5}$ with $\pi < x < \frac{3\pi}{2}$ and $\sin y = -\frac{24}{25}$ with $\pi < y < \frac{3\pi}{2}$

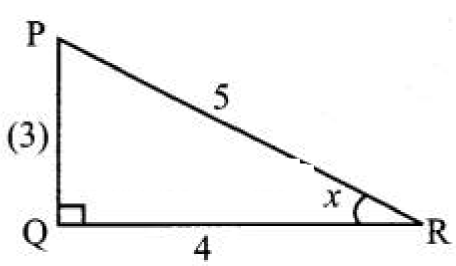
Solution:

$\cos x = -4/5$

$\pi < x < \frac{3\pi}{2}$

$\Rightarrow x$ is in III quadrant

From Δ PQR, $PQ = \sqrt{5^2-4^2} = \sqrt{25-16} = \sqrt{9} = 3$

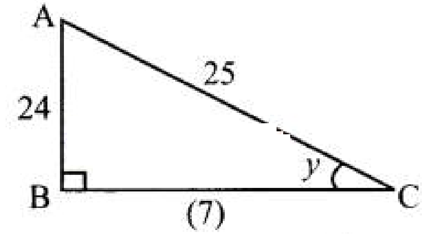


since x is in III quadrant

Both sin x and cos x are negative

$\therefore \sin x = -\frac{3}{5}$ and $\cos x = -\frac{4}{5}$

$\sin y = -\frac{24}{25}$ and y is in III quadrant



Both sin y and cos y are negative

From Δ ABC, BC $= \sqrt{25^2 - 24^2} = \sqrt{625-576}$

$= \sqrt{49} = 7$

So, $\sin y = -\frac{24}{25}$ and $\cos y = -\frac{7}{25}$

Now $\cos(x-y) = \cos x \cos y + \sin x \sin y$

$= \left(-\frac{4}{5}\right)\left(-\frac{7}{25}\right) + \left(-\frac{3}{5}\right)\left(-\frac{24}{25}\right)$

$= \frac{28}{125} + \frac{72}{125} = \frac{100}{125} = \frac{4}{5}$

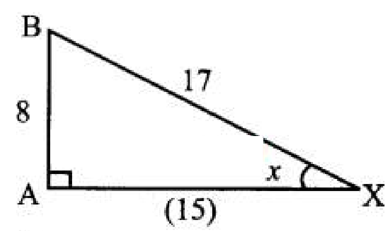
Question 4.

Find sin($x - y$), given that $\sin x = \frac{8}{17}$ with $0 < x < \frac{\pi}{2}$ and $\cos y = -\frac{24}{25}$ with $\pi < y < \frac{3\pi}{2}$

Solution:

$\sin x = \frac{8}{17}, 0 < x < \pi/2 \Rightarrow$ Where x is in I quadrant

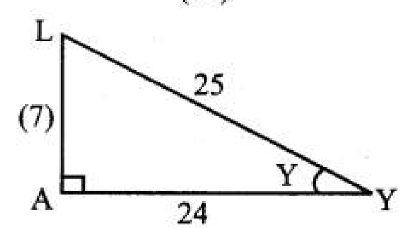
$\therefore$ sin x, cos x are +ve



From Δ ABX, AX $= \sqrt{17^2 - 8^2} = \sqrt{(17+8)(17-8)}$

$= \sqrt{(25)(9)} = 5 \times 3 = 15$

$\therefore \sin x = \frac{8}{17}$ and $\cos x = \frac{15}{17}$



$\cos y = -\frac{24}{25}, \pi < y < 3\pi/2 \Rightarrow$ Where y is in III quadrant

So, sin y and cos y are –ve

From Δ ALY, AL $= \sqrt{25^2 - 24^2}$

$= \sqrt{49} = 7$

$\therefore \cos y = -\frac{24}{25}$ and $\sin y = -\frac{7}{25}$

$$\sin(x - y) = \sin x \cos y - \cos x \sin y$$

$$= \left(\frac{8}{17}\right)\left(-\frac{24}{25}\right) - \left(\frac{15}{17}\right)\left(-\frac{7}{25}\right)$$

$$= -\frac{192}{425} + \frac{105}{425} = -\frac{87}{425}$$

Question 5.

Find the value of (*i*) cos 105° (*ii*) sin 105° (*iii*) tan $\frac{7\pi}{12}$

Solution:

$105° = 60° + 45°$

(*i*) $\cos 105° = \cos(60° + 45°)$

$$= \cos 60° \cos 45° - \sin 60° \sin 45°$$

$$= \frac{1}{2}\,\frac{1}{\sqrt{2}} - \frac{\sqrt{3}}{2}\,\frac{1}{\sqrt{2}} = \frac{1}{2\sqrt{2}} - \frac{\sqrt{3}}{2\sqrt{2}}$$

$$= \frac{1-\sqrt{3}}{2\sqrt{2}}$$

(*ii*) $\sin 105° = \sin(60° + 45°)$

$$= \sin 60° \cos 45° + \cos 60° \sin 45°$$

$$= \frac{\sqrt{3}}{2}\,\frac{1}{\sqrt{2}} + \frac{1}{2}\,\frac{1}{\sqrt{2}}$$

$$= \frac{\sqrt{3}}{2\sqrt{2}} + \frac{1}{2\sqrt{2}} = \frac{\sqrt{3}+1}{2\sqrt{2}}$$

(*iii*) Now $\tan\frac{7\pi}{12} = \tan\frac{7}{12}\times 180° = \tan 7\times 15° = \tan 105°$

$$= \frac{1+\sqrt{3}}{2\sqrt{2}} \Big/ \frac{1-\sqrt{3}}{2\sqrt{2}}$$

$$= \frac{1+\sqrt{3}}{2\sqrt{2}}\times\frac{2\sqrt{2}}{1-\sqrt{3}} = \frac{1+\sqrt{3}}{1-\sqrt{3}}$$

$$= \frac{1+\sqrt{3}}{1-\sqrt{3}}\times\frac{1+\sqrt{3}}{1+\sqrt{3}}$$

$$= \frac{1+\sqrt{3}+\sqrt{3}+3}{1-3} = \frac{4+2\sqrt{3}}{-2}$$

$$= \frac{2(2+\sqrt{3})}{-2} = -(2+\sqrt{3})$$

Question 6.

Prove that (*i*) $\cos(30° + x) = \frac{\sqrt{3}\cos x - \sin x}{2}$ (*ii*) $\cos(\pi + \theta) = -\cos\theta$
(*iii*) $\sin(\pi + \theta) = -\sin\theta$.

Solution:

(*i*) LHS $= \cos(30° + x) = \cos 30° \cos x - \sin 30° \sin x$

$$= \frac{\sqrt{3}}{2}\cos x - \frac{1}{2}\sin x$$

$$= \frac{\sqrt{3}\cos x - \sin x}{2} = \text{RHS}$$

(ii) $\cos(\pi + \theta) = -\cos\theta$
$\cos(\pi + \theta) = \cos\pi\cos\theta - \sin\pi\sin\theta$
$= (-1)\cos\theta - (0)\sin\theta$
$\cos(\pi + \theta) = -\cos\theta$

(iii) $\sin(\pi + \theta) = -\sin\theta$

$\sin(\pi + \theta) = \sin\pi\cos\theta + \cos\pi\sin\theta$

$= (0)\cos\theta + (-1)\sin\theta$

$\sin(\pi + \theta) = 0 - \sin\theta = -\sin\theta$

Question 7.

Find a quadratic equation whose roots are sin 15° and cos 15°

Solution:

$$\sin 15° = \sin(45° - 30°) = \sin 45° \cos 30° - \cos 45° \sin 30°$$

$$= \frac{1}{\sqrt{2}}\frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}}\frac{1}{2} = \frac{\sqrt{3}}{2\sqrt{2}} - \frac{1}{2\sqrt{2}}$$

$$= \frac{\sqrt{3}-1}{2\sqrt{2}} = \frac{\sqrt{3}-1}{2\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{6}-\sqrt{2}}{4}$$

$$\cos 15° = \cos(45° - 30°) = \cos 45° \cos 30° + \sin 45° \sin 30°$$

$$= \frac{1}{\sqrt{2}}.\frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}}\frac{1}{2} = \frac{\sqrt{3}}{2\sqrt{2}} + \frac{1}{2\sqrt{2}}$$

$$= \frac{\sqrt{3}+1}{2\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{6}+\sqrt{2}}{4}$$

Now the quadratic equation with roots cos 15° and sin 15° is

$x^2 - (\cos 15° + \sin 15°)\, x + (\cos 15° \sin 15°) = 0$

Now, $\cos 15° + \sin 15° = \frac{\sqrt{6}+\sqrt{2}}{4} + \frac{\sqrt{6}-\sqrt{2}}{4}$

$$= \frac{2\sqrt{6}}{4} = \frac{\sqrt{6}}{2}$$

$$\cos 15° \times \sin 15° = \left(\frac{\sqrt{6}+\sqrt{2}}{4}\right)\left(\frac{\sqrt{6}-\sqrt{2}}{4}\right)$$

$$= \frac{6-2}{16} = \frac{4}{16} = \frac{1}{4}$$

Now the required quadratic equations

$$x^2 - \left(\frac{\sqrt{6}}{2}\right)x + \left(\frac{1}{4}\right) = 0$$

(i.e) $4x^2 - 2\sqrt{6}x + 1 = 0$

Question 8.
Expand cos(A + B + C). Hence prove that cos A cos B cos C = sin A sin B cos C + sin B sin C cos A + sin C sin A cos B, if A + B + C = π/2

Solution:
cos(A + B + C) = cos (A + (B + C))
= cos A cos (B + C) – sin A sin (B + C)
= cos A [cos B cos C – sin B sin C] – sin A [sin B cos C + cos B sin C]
= cos A cos B cos C – cos A sin B sin C – sin A sin B cos C – sin A cos B sin C
Given A + B + C = π/2
∴ cos(π/2) = cos A cos B cos C – cos A sin B sin C – sin A sin B cos C – sin A cos B sin C
0 = cos A cos B cos C – cos A sin B sin C – sin A sin B cos C – sin A cos B sin C
cos A cos B cos C = cos A sin B sin C + sin A sin B cos C + sin A cos B sin C

Question 9.
Prove that
(i) sin(45° + θ) – sin(45° – θ) = √2sin θ.
(ii) sin(30° + θ) + cos(60° + θ) = cos θ.

Solution:
(*i*)

$$\sin(45^\circ + \theta) = \sin 45^\circ \cos\theta + \cos 45^\circ \sin\theta$$

$$= \frac{1}{\sqrt{2}}\cos\theta + \frac{1}{\sqrt{2}}\sin\theta \quad \text{...(1)}$$

$$\sin(45^\circ - \theta) = \sin 45^\circ \cos\theta - \cos 45^\circ \sin\theta$$

$$= \frac{1}{\sqrt{2}}\cos\theta - \frac{1}{\sqrt{2}}\sin\theta \quad \text{...(2)}$$

From (1), (2) ⇒ LHS = $\sin(45^\circ + \theta) - \sin(45^\circ - \theta)$

$$= \frac{1}{\sqrt{2}}\cos\theta + \frac{1}{\sqrt{2}}\sin\theta - \frac{1}{\sqrt{2}}\cos\theta + \frac{1}{\sqrt{2}}\sin\theta$$

$$= \frac{1}{\sqrt{2}}\cos\theta + \frac{1}{\sqrt{2}}\sin\theta - \frac{1}{\sqrt{2}}\cos\theta + \frac{1}{\sqrt{2}}\sin\theta$$

$$= \frac{2}{\sqrt{2}}\sin\theta = \frac{2\sqrt{2}}{\sqrt{2}\sqrt{2}}\sin\theta = \sqrt{2}\sin\theta = \text{RHS}$$

(*ii*) $\sin(30° + \theta) = \sin 30° \cos\theta + \cos 30° \sin\theta$

$$= \frac{1}{2}\cos\theta + \frac{\sqrt{3}}{2}\sin\theta \quad \text{...(3)}$$

$\cos(60° + \theta) = \cos 60° \cos\theta - \sin 60° \sin\theta$

$$= \frac{1}{2}\cos\theta - \frac{\sqrt{3}}{2}\sin\theta \quad \text{...(4)}$$

$(3) + (4) \Rightarrow \text{LHS} = \sin(30° + \theta) + \cos(60° + \theta) = \frac{1}{2}\cos\theta + \frac{\sqrt{3}}{2}\sin\theta + \frac{1}{2}\cos\theta - \frac{\sqrt{3}}{2}\sin\theta$

$$= \frac{1}{2}\cos\theta + \frac{1}{2}\cos\theta = \cos\theta = \text{RHS}$$

Question 10.
If a cos(x + y) = b cos(x – y), show that (a + b) tan x = (a – b) cot y.

Solution:
a cos (x + y) = b cos (x – y)
a[cos x cos y – sin x sin y] = 6[cos x cos y + sin x sin y]
(i.e) a cos x cos y – a sin x sin y = b cos x cos y + b sin x sin y
a cos x cos y – b sin x sin y = a sin x sin y + b cos x cos y

(i.e) $$\frac{a\frac{\cos y}{\sin y} - b\frac{\sin x}{\cos x}}{\sin y \cos x} = \frac{a\frac{\sin x}{\cos x} + b\frac{\cos y}{\sin y}}{\sin y \cos x}$$

⇒ a cot y – b tan x = a tan x + b cot y
a cot y – b cot y = a tan x + b tan x
⇒ (a + b) tan x = (a – b) cot y.

Question 11.
Prove that sin 105° + cos 105° = cos 45°.

Solution:
sin 105° = sin (60°+ 45°)
= sin 60° cos 45° + cos 60° cos 45°

$$= \frac{\sqrt{3}}{2}\frac{1}{\sqrt{2}} + \frac{1}{2}\frac{1}{\sqrt{2}} = \frac{\sqrt{3}+1}{2\sqrt{2}}$$

$$\cos 105° = \cos(60° + 45°) = \cos 60° \cos 45° - \sin 60° \sin 45°$$

$$= \frac{1}{2}\frac{1}{\sqrt{2}} - \frac{\sqrt{3}}{2}\frac{1}{\sqrt{2}} = \frac{1}{2\sqrt{2}} - \frac{\sqrt{3}}{2\sqrt{2}} = \frac{1-\sqrt{3}}{2\sqrt{2}}$$

$$\text{So, LHS} = \sin 105° + \cos 105° = \frac{\sqrt{3}+1}{2\sqrt{2}} + \frac{1-\sqrt{3}}{2\sqrt{2}}$$

$$= \frac{\sqrt{3}+1+1-\sqrt{3}}{2\sqrt{2}} = \frac{2}{2\sqrt{2}} = \frac{1}{\sqrt{2}}$$

= cos 45° = RHS

Question 12.
Prove that sin 75° – sin 15° = cos 105° + cos 15°.

Solution:
sin 75° – sin 15° = sin (45° + 30°) – sin (45° – 30°)
= (sin 45° cos 30° + cos 45° sin 30°) – (sin 45° cos 30° – cos 45° sin 30°)
= sin 45° cos30° + cos 45° sin 30° – sin 45° cos 30° + cos 45° sin 30°
= 2 cos 45° sin 30°

$$\sin 75° - \sin 15° = 2 \times \frac{1}{\sqrt{2}} \times \frac{1}{2} = \frac{1}{\sqrt{2}} \text{ ——— } (1)$$

cos 105° + cos 15° = cos (90° + 15°) + cos 15°
= – sin 15° + cos 15°
= cos 15° – sin 15°
= cos(45° – 30°) – sin(45° – 30°)
= (cos 45° cos 30°+ sin 45° sin 30°) – (sin 45° cos 30° – cos 45° sin 30°)

$$= \left(\frac{1}{\sqrt{2}} \cdot \frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}} \cdot \frac{1}{2}\right) - \left(\frac{1}{\sqrt{2}} \cdot \frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}} \cdot \frac{1}{2}\right)$$

$$= \frac{\sqrt{3}}{2\sqrt{2}} + \frac{1}{2\sqrt{2}} - \frac{\sqrt{3}}{2\sqrt{2}} + \frac{1}{2\sqrt{2}}$$

cos 105° + cos 15° = $\frac{2}{2\sqrt{2}}$ = $\frac{1}{\sqrt{2}}$ ——— (2)

From equations (1) and (2)

sin 75° – sin 15° = cos 105° + cos 15°

Question 13.
Show that tan 75° + cot 75° = 4

Solution:

$$\tan 75^\circ = \tan(45^\circ + 30^\circ) = \frac{\tan 45^\circ + \tan 30^\circ}{1 - \tan 45^\circ \tan 30^\circ}$$

$$= \frac{1+\frac{1}{\sqrt{3}}}{1-\frac{1}{\sqrt{3}}} = \frac{\sqrt{3}+1}{\sqrt{3}} \Big/ \frac{\sqrt{3}-1}{\sqrt{3}} = \frac{\sqrt{3}+1}{\sqrt{3}-1}$$

$$\cot 75^\circ = \frac{1}{\tan 75^\circ} = \frac{\sqrt{3}-1}{\sqrt{3}+1}$$

So, LHS = tan 75° + cot 75°

$$= \frac{\sqrt{3}+1}{\sqrt{3}-1} + \frac{\sqrt{3}-1}{\sqrt{3}+1}$$

$$= \frac{(\sqrt{3}+1)^2 + (\sqrt{3}-1)^2}{(\sqrt{3}-1)(\sqrt{3}+1)} = \frac{3+1+2\sqrt{3}+3+1-2\sqrt{3}}{\sqrt{3}^2 - 1^2}$$

$$= \frac{8}{3-1} = \frac{8}{2} = 4 = \text{RHS}$$

Question 14.
Prove that cos(A + B) cos C – cos(B + C) cos A = sin B sin(C – A).

Solution:
cos (A + B) cos C = (cos A cos B – sin A sin B) cos C
cos (A + B) cos C = cos A cos B cos C – sin A sin B cos C ——— (1)

cos (B + C) cos A = (cos B cos C – sin B sin C) cos A
cos (B + C) cos A = cos A cos B cos C – cos A sin B sin C ——— (2)
Equation (1) – (2) ⇒
Cos (A + B) cos C – cos (B + C) cos A = cos A cos B cos C – sin A sin B cos C – cos A cos B cos C + cos A sin B sin C
= sin A sin B cos C + cos A sin B sin C
= sin B (cos C cos A + sin C sin A)
= sin B cos (C – A)

Question 15.
Prove that sin(n + 1) θ sin(n – 1) θ + cos(n + 1) θ cos(n – 1)θ = cos 2θ, n ∈ Z.

Solution:
cos (n + 1)θ cos(n – 1)θ + sin(n + 1)θ sin(n – 1)θ
= cos [(n + 1)θ – (n – 1)θ]
= cos [nθ + θ – nθ + θ]
= cos 2θ, n ∈ Z

Question 16.
If $x \cos\theta = y \cos\left(\theta + \frac{2\pi}{3}\right) = z \cos\left(\theta + \frac{4\pi}{3}\right)$, find the value of $xy + yz + zx$.

Solution:

$$x \cos\theta = y \cos\left(\theta + \frac{2\pi}{3}\right) = z \cos\left(\theta + \frac{4\pi}{3}\right) = k \text{ (say)}$$

$$\frac{k}{x} = \cos\theta$$

$$\frac{k}{y} = \cos\left(\theta + \frac{2\pi}{3}\right)$$

$$\frac{k}{z} = \cos\left(\theta + \frac{4\pi}{3}\right)$$

$$\frac{k}{x} + \frac{k}{y} + \frac{k}{z} = = \cos\theta + \cos\left(\theta + \frac{2\pi}{3}\right) + \cos\left(\theta + \frac{4\pi}{3}\right)$$

$$\frac{k}{x}+\frac{k}{y}+\frac{k}{z} = 0$$

$$k\left[\frac{yz+xz+xy}{xyz}\right] = 0 \Rightarrow xy+yz+zx=0$$

Question 17.
Prove that
(i) $\sin(A + B) \sin(A - B) = \sin^2 A - \sin^2 B$
(ii) $\cos(A + B) \cos(A - B) = \cos^2 A - \sin^2 B = \cos^2 B - \sin^2 A$
(iii) $\sin^2(A + B) - \sin^2(A - B) = \sin 2A \sin 2B$
(iv) $\cos 8\theta \cos 2\theta = \cos^2 5\theta - \sin^2 3\theta$

Solution:
(i) $\sin(A + B) \sin(A - B) = (\sin A \cos B + \cos A \sin B) (\sin A \cos B - \cos A \sin B)$
$= \sin^2 A \cos^2 B - \cos^2 A \sin^2 B = \sin^2 A (1 - \sin^2 B) - (1 - \sin^2 A) \sin^2 B$
$= \sin^2 A - \sin^2 A \sin^2 B - \sin^2 B + \sin^2 A \sin^2 B$
$= \sin^2 A - \sin^2 B$

(ii) LHS $= \cos (A + B) \cos (A - B) = (\cos A \cos B - \sin A \sin B) (\cos A \cos B + \sin (A \sin B)$
$= \cos^2 A \cos^2 B - \sin^2 A \sin^2 B$
$= \cos^2 A (1 - \sin^2 B) - (1 - \cos^2 A) \sin^2 B$
$= \cos^2 A - \cos^2 A \sin^2 B - \sin^2 B + \cos^2 A \sin^2 B$
$= \cos^2 A - \sin^2 B =$ RHS
Now $\cos^2 A - \sin^2 B = (1 - \sin^2 A) - (1 - \cos^2 B)$
$= 1 - \sin^2 A - 1 + \cos^2 B$
$= \cos^2 B - \sin^2 A$

(iii) $\sin^2(A + B) - \sin^2(A - B) = (\sin(A + B) + \sin(A - B)) (\sin(A + B) - \sin(A - B))$
$= [\sin A \cos B + \cos A \sin B + \sin A \cos B - \cos A \sin B] \times [(\sin A \cos B + \cos A \sin B) - (\sin A \cos B - \cos A \sin B)]$
$= (2 \sin A \cos B) \times [\sin A \cos B + \cos A \sin B - \sin A \cos B + \cos A \sin B]$
$= (2 \sin A \cos B) (2 \cos A \sin B)$
$= (2 \sin A \cos A)(2 \sin B \cos B)$
$= \sin^2 A . \sin^2 B$

(iv) LHS = $\cos 8\theta \cos 2\theta$
$= \cos (5\theta + 3\theta) \cos (5\theta - 3\theta)$.

We know $\cos (A + B) \cos (A - B) = \cos^2 A - \sin^2 B$
$\therefore \cos (5\theta + 3\theta) \cos (5\theta - 3\theta) = \cos^2 5\theta - \sin^2 3\theta$ = RHS

Question 18.
Show that $\cos^2 A + \cos^2 B - 2 \cos A \cos B \cos(A + B) = \sin^2(A + B)$.

Solution:
$\sin^2(A + B) = [\sin(A + B)]^2$
$= (\sin A \cos B + \cos A \sin B)^2$
$= \sin^2 A \cos^2 B + \cos^2 A \sin^2 B + 2 \sin A \cos B \cos A \sin B$
$= (1 - \cos^2 A) \cos^2 B + \cos^2 A (1 - \cos^2 B) + 2 \sin A \sin B \cos A \cos B$
$= \cos^2 B - \cos^2 A \cos^2 B + \cos^2 A - \cos^2 A \cos^2 B + 2 \sin A \sin B \cos A \cos B$
$= \cos^2 A + \cos^2 B - 2\cos^2 A \cos^2 B + 2 \sin A \sin B \cos A \cos B$
$= \cos^2 A + \cos^2 B - 2 \cos A \cos B (\cos A \cos B - \sin A \sin B)$
$\sin^2 (A + B) = \cos^2 A + \cos^2 B - 2 \cos A \cos B \cos(A + B)$

Question 19.
If $\cos(\alpha - \beta) + \cos(\beta - \gamma) + \cos(\gamma - \alpha) = -3/2$, then prove that $\cos \alpha + \cos \beta + \cos \gamma = \sin \alpha + \sin \beta + \sin \gamma$

Solution:
$\Rightarrow$ Given $\cos (\alpha - \beta) + \cos (\beta - \gamma) + \cos (\gamma - \alpha) = \frac{-3}{2}$

$2 \cos (\alpha - \beta) + 2\cos (\beta - \gamma) + 2\cos (\gamma - \alpha) = -3$
$2\cos(\alpha - \beta) + 2\cos(\beta - \gamma) + 2\cos (\gamma - \alpha) + 3 = 0$
$[2 \cos \alpha \cos \beta + 2 \sin \alpha \sin \beta] + [2 \cos \beta \cos \gamma + 2 \sin \beta \sin \gamma] + [2 \cos \gamma \cos \alpha + \sin \gamma \sin \alpha] + 3 = 0$
$= [2 \cos \alpha \cos \beta + 2 \cos \beta \cos \gamma + 2 \cos \gamma \cos \alpha] + [2 \sin \alpha \sin \beta + 2 \sin \beta \sin \gamma + 2 \sin \gamma \sin \alpha] + (\sin^2 \alpha + \cos^2 \alpha) + (\sin^2 \beta + \cos^2 \beta) + (\sin^2 \gamma + \cos^2 \gamma) = 0$
$\Rightarrow (\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma + 2 \cos \alpha \cos \beta + 2 \cos \beta \cos \gamma + 2 \cos \gamma \cos \alpha) + (\sin^2 \alpha + \sin^2 \beta) + (\sin^2 \gamma + 2 \sin \alpha \sin \beta + 2 \sin \beta \sin \gamma + 2 \sin \gamma \sin \alpha) = 0$
$(\cos \alpha + \cos \beta + \cos \gamma)^2 + (\sin \alpha + \sin \beta + \sin \gamma)^2 = 0$
$=(\cos \alpha + \cos \beta + \cos \gamma) = 0$ and $\sin \alpha + \sin \beta + \sin \gamma = 0$
Hence proved

Question 20.

Show that (*i*) $\tan(45° + A) = \frac{1+\tan A}{1-\tan A}$ (*ii*) $\tan(45° - A) = \frac{1-\tan A}{1+\tan A}$

Solution:

$$(i)\ \text{LHS} = \tan(45° + A) = \frac{\tan 45° + \tan A}{1 - \tan 45° \tan A}$$

$$= \frac{1+\tan A}{1-\tan A} = \text{RHS}$$

$$(ii)\ \text{LHS} = \tan(45° - A) = \frac{\tan 45° - \tan A}{1 + \tan 45° \tan A}$$

$$= \frac{1-\tan A}{1+\tan A} = \text{RHS}$$

Question 21.

Prove that $\cot(A + B) = \frac{\cot A \cot B - 1}{\cot A + \cot B}$.

Solution:

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

$$\text{LHS} = \cot(A+B) = \frac{1}{\tan(A+B)} = \frac{1}{\frac{\tan A + \tan B}{1 - \tan A \tan B}}$$

$$= \frac{1 - \tan A \tan B}{\tan A + \tan B}$$

÷ both Nr and Dr by tan A tan B

$$= \frac{1 - \tan A \tan B}{\tan A \tan B} \Big/ \frac{\tan A + \tan B}{\tan A \tan B}$$

$$= \frac{1}{\tan A \tan B} - 1 \Big/ \frac{1}{\tan B} + \frac{1}{\tan A} = \frac{\cot A \cot B - 1}{\cot B + \cot A} = \text{RHS}$$

Question 22.

If $\tan x = \dfrac{n}{n+1}$ and $\tan y = \dfrac{1}{2n+1}$, find $\tan(x+y)$

Solution:

$$\tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$$

$$= \frac{\dfrac{n}{n+1} + \dfrac{1}{2n+1}}{1 - \left(\dfrac{n}{n+1}\right)\left(\dfrac{1}{2n+1}\right)}$$

$$= \frac{n(2n+1) + 1(n+1)}{(n+1)(2n+1)} \Big/ \frac{(n+1)(2n+1) - n}{(n+1)(2n+1)}$$

$$= \frac{2n^2 + n + n + 1}{2n^2 + n + 2n + 1 - n} = \frac{2n^2 + 2n + 1}{2n^2 + 2n + 1} = 1$$

Question 23.

Prove that $\tan\left(\dfrac{\pi}{4} + \theta\right) \tan\left(\dfrac{3\pi}{4} + \theta\right) = -1$

Solution:

$$\tan\left(\frac{\pi}{4} + \theta\right) = \frac{\tan\dfrac{\pi}{4} + \tan\theta}{1 - \tan\dfrac{\pi}{4}\tan\theta} = \frac{1 + \tan\theta}{1 - \tan\theta}$$

$$\tan\left(\frac{3\pi}{4} + \theta\right) = \frac{\tan\dfrac{3\pi}{4} + \tan\theta}{1 - \tan\dfrac{3\pi}{4}\tan\theta} = \frac{(-1) + \tan\theta}{1 - (-1)\tan\theta}$$

$$= \frac{\tan\theta - 1}{1 + \tan\theta}$$

$$\therefore \tan\left(\frac{\pi}{4} + \theta\right)\tan\left(\frac{3\pi}{4} + \theta\right) = \frac{1 + \tan\theta}{1 - \tan\theta} \times \frac{-1 + \tan\theta}{1 + \tan\theta}$$

$$= \frac{\tan\theta - 1}{1 - \tan\theta} = -1 = \text{RHS}$$

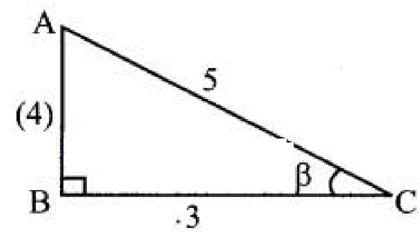
Question 24.

Find the values of $\tan(\alpha + \beta)$, given that $\cot\alpha = \frac{1}{2}$, $\alpha \in \left(\pi, \frac{3\pi}{2}\right)$ and $\sec\beta = -\frac{5}{3}$, $\beta \in \left(\frac{\pi}{2}, \pi\right)$

Solution:

$$\cot\alpha = \frac{1}{2}$$

$$\therefore \tan\alpha = \frac{1}{\cot\alpha} = 2, \ \alpha \text{ is in III quadrant}$$



$$\sec\beta = -\frac{5}{3}$$

$$\cos\beta = -\frac{3}{5}, \ \beta \text{ is in III quadrant}$$

$$AB = \sqrt{5^2 - 3^2} = \sqrt{25 - 9} = \sqrt{16} = 4$$

$$\therefore \tan\beta = \frac{\sin\beta}{\cos\beta} = \frac{4/5}{-3/5} = -\frac{4}{3}$$

$$\therefore \beta \text{ is in II quadrant}, \tan\beta = -\frac{4}{3}$$

$$\text{Now } \tan(\alpha + \beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta} = \frac{2 + \left(-\frac{4}{3}\right)}{1 - (2)\left(-\frac{4}{3}\right)}$$

$$= 2 - \frac{4}{3} \Big/ 1 + \frac{8}{3} = \frac{6 - 4}{3} \Big/ \frac{3 + 8}{3}$$

$$= \frac{2}{3} \Big/ \frac{11}{3} = \frac{2}{11}$$

Question 25.

If $\theta + \phi = \alpha$ and $\tan\theta = k\tan\phi$, then prove that $\sin(\theta - \phi) = \frac{k-1}{k+1}\sin\alpha$.

Solution:

$$\theta + \phi = \alpha,\ \tan\theta = k\tan\phi$$

$$k = \frac{\tan\theta}{\tan\phi}$$

$$\frac{k-1}{k+1} = \frac{\frac{\tan\theta}{\tan\phi} - 1}{\frac{\tan\theta}{\tan\phi} + 1} = \frac{\tan\theta - \tan\phi}{\tan\theta + \tan\phi}$$

$$= \frac{\frac{\sin\theta}{\cos\theta} - \frac{\sin\phi}{\cos\phi}}{\frac{\sin\theta}{\cos\theta} + \frac{\sin\phi}{\cos\phi}} = \frac{\sin\theta\cos\phi - \cos\theta\sin\phi}{\sin\theta\cos\phi + \cos\theta\sin\phi}$$

$$\frac{k-1}{k+1} = \frac{\sin(\theta - \phi)}{\sin(\theta + \phi)} = \frac{\sin(\theta - \phi)}{\sin\alpha}$$

$$\sin(\theta - \phi) = \frac{k-1}{k+1}\sin\alpha$$

Ex 3.5

Question 1.

Find the value of cos 2A, A lies in the first quadrant, when

(*i*) $\cos A = \frac{15}{17}$ (*ii*) $\sin A = \frac{4}{5}$ (*iii*) $\tan A = \frac{16}{63}$

Solution:

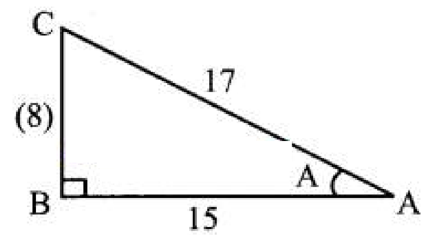
(*i*) $\cos A = \frac{15}{17}$

$$BC = \sqrt{17^2 - 15^2} = \sqrt{289 - 225} = \sqrt{64} = 8$$

Now $\cos A = \frac{15}{17}; \sin A \quad \frac{8}{17}$

$$\cos 2A = \cos^2 A - \sin^2 A = \left(\frac{15}{17}\right)^2 - \left(\frac{8}{17}\right)^2$$

$$= \frac{225}{289} - \frac{64}{289} = \frac{161}{289}$$



(*ii*) $\sin A = \frac{4}{5}$

$$\cos 2A = 1 - 2\sin^2 A = 1 - 2\left(\frac{4}{5}\right)^2$$

$$= 1 - 2\left(\frac{16}{25}\right) = 1 - \frac{32}{25}$$

$$= \frac{25 - 32}{25} = \frac{-7}{25}$$

(*iii*) $\tan A = 16/63$

$$\cos 2A = \frac{1 - \tan^2 A}{1 + \tan^2 A} = \frac{1 - (16/63)^2}{1 + (16/63)^2} = \frac{3713}{4225}$$

Question 2.

If θ is an acute angle, then find

(*i*) $\sin\left(\frac{\pi}{4} - \frac{\theta}{2}\right)$, when $\sin\theta = \frac{1}{25}$. (*ii*) $\cos\left(\frac{\pi}{4} + \frac{\theta}{2}\right)$, when $\sin\theta = \frac{8}{9}$

Solution:

(*i*) Given $\sin\theta = \frac{1}{25}$, To find $\sin\left(\frac{\pi}{4} - \frac{\theta}{2}\right)$

Now $\cos 2\left(\frac{\pi}{4}-\frac{\theta}{2}\right)=\cos\left(\frac{\pi}{2}-\theta\right)=\sin\theta=\frac{1}{25}$ (given)

We know $\cos 2\theta = 1 - 2\sin^2\theta$

$$\Rightarrow \quad 1-2\sin^2\left(\frac{\pi}{4}-\frac{\theta}{2}\right)= \frac{1}{25}$$

$$\Rightarrow \quad 2\sin^2\left(\frac{\pi}{4}-\frac{\theta}{2}\right) = 1-\frac{1}{25}=\frac{24}{25}$$

$$\text{So } \sin^2\left(\frac{\pi}{4}-\frac{\theta}{2}\right) = \frac{24}{25\times 2}=\frac{12}{25}$$

$$\Rightarrow \quad \sin\left(\frac{\pi}{4}-\frac{\theta}{2}\right) = \sqrt{\frac{12}{25}}=\frac{2\sqrt{3}}{5}$$

(*ii*) Given $\sin\theta = \frac{8}{9}$, To find $\cos\left(\frac{\pi}{4}+\frac{\theta}{2}\right)$

$$\text{Now } \cos 2\left(\frac{\pi}{4}+\frac{\theta}{2}\right) = \cos\left(\frac{\pi}{2}+\theta\right)=-\sin\theta=-\frac{8}{9}$$

[We know $\cos 2\theta = 2\cos^2\theta - 1$]

$$\Rightarrow \quad 2\cos^2\left(\frac{\pi}{4}+\frac{\theta}{2}\right)-1 = -\frac{8}{9}$$

$$\text{So, } \quad 2\cos^2\left(\frac{\pi}{4}+\frac{\theta}{2}\right) = 1-\frac{8}{9}=\frac{1}{9}$$

$$\Rightarrow \quad \cos^2\left(\frac{\pi}{4}+\frac{\theta}{2}\right) = \frac{1}{9\times 2}=\frac{1}{18}$$

$$\Rightarrow \quad \cos\left(\frac{\pi}{4}+\frac{\theta}{2}\right) = \sqrt{\frac{1}{18}}=\frac{1}{3\sqrt{2}}$$

Question 3.

If $\cos\theta = \frac{1}{2}\left(a + \frac{1}{a}\right)$, show that $\cos 3\theta = \frac{1}{2}\left(a^3 + \frac{1}{a^3}\right)$.

Solution:

Given, $\cos\theta = \frac{1}{2}\left(a+\frac{1}{a}\right)$

$$\cos^3\theta = \left[\frac{1}{2}\left(a+\frac{1}{a}\right)\right]^3 = \frac{1}{8}\left(a+\frac{1}{a}\right)^3$$

$$= \frac{1}{8}\left[a^3 + \frac{1}{a^3} + 3(a)\left(\frac{1}{a}\right)\left(a+\frac{1}{a}\right)\right]$$

$$= \frac{1}{8}\left[a^3 + \frac{1}{a^3} + 3\left(a+\frac{1}{a}\right)\right]$$

Now $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$

$$= 4\left[\frac{1}{8}\left\{\left(a^3+\frac{1}{a^3}\right)+3\left(a+\frac{1}{a}\right)\right\}\right] - 3\left[\frac{1}{2}\left(a+\frac{1}{a}\right)\right]$$

$$= \frac{4}{8}\left(a^3+\frac{1}{a^3}\right)+\frac{12}{8}\left(a+\frac{1}{a}\right)-\frac{3}{2}\left(a+\frac{1}{a}\right)$$

$$= \frac{1}{2}\left(a^3+\frac{1}{a^3}\right)+\frac{3}{2}\left(a+\frac{1}{a}\right)-\frac{3}{2}\left(a+\frac{1}{a}\right)$$

$$= \frac{1}{2}\left(a^3+\frac{1}{a^3}\right) = \text{RHS}$$

Question 4.

Prove that $\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$.

Solution:

$\cos 5\theta = \cos(2\theta + 3\theta) = \cos 2\theta \cos 3\theta - \sin 2\theta \sin 3\theta$

$= (2\cos^2\theta - 1)(4\cos^3\theta - 3\cos\theta) - 2\sin\theta\cos\theta(3\sin\theta - 4\sin^3\theta)$
$= 8\cos^5\theta - 6\cos^3\theta - 4\cos^3\theta + 3\cos\theta - 6\sin^2\theta\cos\theta + 8\cos\theta\sin^4\theta$
$= 8\cos^5\theta - 6\cos^3\theta - 4\cos^3\theta + 3\cos\theta - 6(1 - \cos^2\theta)\cos\theta + 8\cos\theta(1 - \cos^2\theta)^2$
$= 8\cos^5\theta - 6\cos^3\theta - 4\cos^3\theta + 3\cos\theta - 6\cos\theta + 6\cos^3\theta + 8\cos 0(1 + \cos^4\theta - 2\cos^2\theta)$
$= 8\cos^5\theta - 6\cos^3\theta - 4\cos^3\theta + 3\cos\theta - 6\cos\theta + 6\cos^3\theta + 8\cos\theta + 8\cos^5\theta - 16\cos^3\theta$
$= 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$ = RHS

Question 5.

Prove that $\sin 4\alpha = 4\tan\alpha \dfrac{1-\tan^2\alpha}{(1+\tan^2\alpha)^2}$.

Solution:

$$\text{RHS} = 4\tan\alpha \frac{1-\tan^2\alpha}{(1+\tan^2\alpha)^2}$$
$$= 2\left(\frac{2\tan\alpha}{1+\tan^2\alpha}\right)\times\left(\frac{1-\tan^2\alpha}{1+\tan^2\alpha}\right)$$
$$= 2\sin 2\alpha\cos 2\alpha = \sin 4\alpha = \text{LHS}$$

Question 6.
If A + B = 45°, show that (1 + tanA) (1 + tanB) = 2.

Solution:

$$A + B = 45^\circ$$
$$\Rightarrow \tan(A+B) = \tan 45^\circ = 1$$
$$\text{(i.c)} \quad \frac{\tan A + \tan B}{1 - \tan A\tan B} = 1 \Rightarrow \tan A + \tan B = 1 - \tan A\tan B \quad ...(1)$$

Now LHS = (1 + tan A) (1 + tan B)
= tan A + tan B + tan A tan B + 1
= (1 – tan A tan B) + (tan A tan B + 1) from (1)
= 2 = RHS

Question 7.
Prove that (1 + tan 1°)(1 + tan 2°)(1 + tan 3°)… (1 + tan 44°) is a multiple of 4.

Solution:
Let T = (1 + tan 1°) (1 + tan 2°) (1 + tan 3°) ………. (1 + tan44°)
T = (1 + tan 1°) (1 + tan 44°) (1 + tan 2°) (1 + tan 43°) (1 + tan 3°) (1 + tan 42°) ………. (1 + tan 22°) (1 + tan 23°)
[If A + B = 45°, then (1 + tan A) (1 + tan B) = 2]
= 2 × 2 × 2 × ………………….. 22 times
T = $2^{22} = (2^2)^{11} = 4^{11}$ which is a multiple of 4.
Therefore, (1 + tan 1°) (1 + tan 2°) (1 + tan 3°) ………….. (1 + tan 44°) is a multiple of 4

Question 8.

Prove that $\tan\left(\frac{\pi}{4}+\theta\right)-\tan\left(\frac{\pi}{4}-\theta\right)=2\tan 2\theta$.

Solution:

$$\tan\left(\frac{\pi}{4}+\theta\right) = \frac{\tan\frac{\pi}{4}+\tan\theta}{1-\tan\frac{\pi}{4}\tan\theta}=\frac{1+\tan\theta}{1-\tan\theta}$$

$$\tan\left(\frac{\pi}{4}-\theta\right) = \frac{\tan\frac{\pi}{4}-\tan\theta}{1+\tan\frac{\pi}{4}\tan\theta}=\frac{1-\tan\theta}{1+\tan\theta}$$

$$\text{Now LHS} = \tan\left(\frac{\pi}{4}+\theta\right)-\tan\left(\frac{\pi}{4}-\theta\right)$$

$$= \frac{1+\tan\theta}{1-\tan\theta}-\frac{1-\tan\theta}{1+\tan\theta}=\frac{(1+\tan\theta)^2-(1-\tan\theta)^2}{1-\tan^2\theta}$$

$$= \frac{1+\tan^2\theta+2\tan\theta-1-\tan^2\theta+2\tan\theta}{1-\tan^2\theta}$$

$$= \frac{4\tan\theta}{1-\tan^2\theta} = 2\left(\frac{2\tan\theta}{1-\tan^2\theta}\right)$$

$$= 2\tan 2\theta = \text{RHS}$$

Question 9.

Show that $\cot\left(7\frac{1}{2}^\circ\right) = \sqrt{2}+\sqrt{3}+\sqrt{4}+\sqrt{6}$.

Solution:

We have to prove that $\cot\left(7\frac{1}{2}^\circ\right) = \sqrt{2}+\sqrt{3}+\sqrt{4}+\sqrt{6}$

$$\text{LHS} = \cot\left(7\frac{1}{2}^\circ\right) = \frac{\cos\left(7\,{}^1\!/\!_2{}^\circ\right)}{\sin\left(7\,{}^1\!/\!_2{}^\circ\right)}$$

To find $\frac{\cos\theta}{\sin\theta}$, multiply numerator & denominator by 2 cos θ

Let $\theta = 7\ 1/2^\circ$
$2\theta = 15^\circ$

$$\frac{2\cos^2\theta}{2\sin\theta\cos\theta} = \frac{1+\cos 2\theta}{\sin 2\theta} = \frac{1+\cos 15^\circ}{\sin 15^\circ}$$

$$= \frac{1+\sqrt{3}+1\big/2\sqrt{2}}{\sqrt{3}-1\big/2\sqrt{2}} = \frac{2\sqrt{2}+\sqrt{3}+1}{\sqrt{3}-1}$$

Multiply numerator & denominator by $\sqrt{3}+1$

$$= \frac{(2\sqrt{2}+\sqrt{3}+1)(\sqrt{3}+1)}{(\sqrt{3}-1)(\sqrt{3}+1)}$$

$$= \frac{2\sqrt{2}+3+\sqrt{3}+\sqrt{3}+1}{3-1}$$

$$= \frac{2\sqrt{3}+2\sqrt{2}+4}{2}$$

$$= \frac{2\left(\sqrt{2}+\sqrt{3}+\sqrt{6}+2\right)}{2}$$

$$= \sqrt{2}+\sqrt{3}+\sqrt{4}+\sqrt{6}$$

$$= \text{RHS}$$

Question 10.
Prove that $(1 + \sec 2\theta)(1 + \sec 4\theta)\ldots\ldots (1 + \sec 2^n\theta) = \tan 2^n\theta$

Solution:

$$\text{LHS } (1 + \sec 2\theta) = 1 + \frac{1}{\cos 2\theta} + \frac{\cos 2\theta + 1}{\cos 2\theta} = \frac{2\cos^2\theta}{\cos 2\theta}$$

$$(1 + \sec 4\theta) = 1 + \frac{1}{\cos 4\theta} = \frac{\cos 4\theta + 1}{\cos 4\theta} = \frac{2\cos^2(2\theta)}{\cos 4\theta}$$

$$\vdots$$

$$(1 + \sec 2^n\theta) = 1 + \frac{1}{2^n\theta} = \frac{\cos 2^n\theta + 1}{2^n\theta} = \frac{2\cos^2 2^{n-1}\theta}{\cos 2^n\theta}$$

$(1 + \sec 2\theta)(1 + \sec 4\theta)\ldots. (1 + \sec 2^n\theta)$

$$= \frac{2^n\cos^2\theta}{\cos 2\theta}\cdot\frac{\cos^2 2\theta}{\cos 4\theta}\cdots\frac{\cos^2 2^{n-1}\theta}{\cos 2^n\theta}$$

$$= \frac{2^n\cos\theta}{\cos 2^n\theta}\left\{\cos\theta.\cos 2\theta\ldots\cos 2^{n-1}\theta\right\}$$

$$= \frac{2^n\cos\theta\left\{\sin 2^n\theta\right\}}{2^n\sin\theta\cos 2^n\theta} = \tan 2^n\theta.\cos\theta$$

Question 11.

Prove that $32\left(\sqrt{3}\right)\sin\frac{\pi}{48}\cos\frac{\pi}{48}\cos\frac{\pi}{24}\cos\frac{\pi}{12}\cos\frac{\pi}{6} = 3$

Solution:

$$32\sqrt{3}\left[\sin\frac{\pi}{48}\times\cos\frac{\pi}{48}\right] = 16\sqrt{3}\left[2\sin\frac{\pi}{48}\cos\frac{\pi}{48}\right] = 16\sqrt{3}\sin\frac{\pi}{24}\left(\frac{2\pi}{48}=\frac{\pi}{24}\right)$$

$$\text{Now } 16\sqrt{3}\left[\sin\frac{\pi}{24}\times\cos\frac{\pi}{24}\right] = 8\sqrt{3}\left[2\sin\frac{\pi}{24}\cos\frac{\pi}{24}\right] = 8\sqrt{3}\left[\sin\frac{2\pi}{24}\right] = 8\sqrt{3}\sin\frac{\pi}{12}$$

$$\text{Now } 8\sqrt{3}\left[\sin\frac{\pi}{12}\cos\frac{\pi}{12}\right] = 4\sqrt{3}\left[2\sin\frac{\pi}{12}\cos\frac{\pi}{12}\right]$$

$$= 4\sqrt{3}\left[\sin\frac{2\pi}{12}\right] = 4\sqrt{3}\left(\sin\frac{\pi}{6}\right)$$

$$\text{Now } 4\sqrt{3}\sin\frac{\pi}{6}\cos\frac{\pi}{6} = 2\sqrt{3}\left[2\sin\frac{\pi}{6}\cos\frac{\pi}{6}\right]$$

$$2\sqrt{3}\left[\sin\frac{2\pi}{6}\right] = 2\sqrt{3}\sin\frac{\pi}{3} = 2\sqrt{3}\times\frac{\sqrt{3}}{2} = 3 = \text{RHS}$$

Ex 3.6

Question 1.
Express each of the following as a sum or difference
(i) sin 35° cos 28°
(ii) sin 4x cos 2x
(iii) 2 sin 10θ cos 2θ
(iv) cos 5θ cos 2θ
(v) sin 5θ sin 4θ.

Solution:

(i)
$$\sin 35^\circ \cos 28^\circ = \frac{1}{2}[2\sin 35^\circ\cos 28^\circ]$$
$$= \frac{1}{2}[\sin(35^\circ+28^\circ)+\sin(35^\circ-28^\circ)]$$
$$= \frac{1}{2}[\sin 63^\circ+\sin 7^\circ]$$

(ii) $\sin 4x \cos 2x = \frac{1}{2}[2\sin 4x \cos 2x]$

$= \frac{1}{2}[\sin(4x+2x)+\sin(4x-2x)]$

$= \frac{1}{2}(\sin 6x+\sin 2x)$

(iii) $2\sin 10\theta \cos 2\theta = \frac{1}{2}[2\sin 10\theta \cos 2\theta]$

$= \frac{1}{2}[\sin(10\theta+2\theta)+\sin(10\theta-2\theta]$

$= \frac{1}{2}[\sin 12\theta+\sin 8\theta]$

(iv) $\cos 5\theta \cos 2\theta = \frac{1}{2}[2\cos 5\theta \cos 2\theta]$

$= \frac{1}{2}[\cos(5\theta+2\theta)+\cos(5\theta-2\theta)]$

$= \frac{1}{2}[\cos 7\theta+\cos 3\theta)$

(v) $\sin 5\theta \sin 4\theta = \frac{1}{2}[2\sin 5\theta \sin 4\theta]$

$= \frac{1}{2}[\cos(5\theta-4\theta)-\cos(5\theta+4\theta)]$

$= \frac{1}{2}[\cos\theta-\cos 9\theta)$

Question 2.

Express each of the following as a product

(i) sin 75° – sin 35°

(ii) cos 65° + cos 15°

(iii) sin 50° + sin 40°

(iv) cos 35° – cos 75°.

Solution:

(i) $\sin 75° - \sin 35° = 2\cos\left(\frac{75°+35°}{2}\right)\sin\left(\frac{75°-35°}{2}\right)$

$= 2\cos 55° \sin 20°$

(ii) $\cos 65° + \cos 15° = 2\cos\left(\frac{65°+15°}{2}\right)\cos\left(\frac{65°-15°}{2}\right)$

$= 2\cos 40° \cos 25°$

(iii) $\sin 50° + \sin 40° = 2\sin\left(\frac{50°+40°}{2}\right)\cos\left(\frac{50°-40°}{2}\right)$

$= 2\sin 45° \cos 5°$

(iv) $\cos 35° - \cos 75° = 2\sin\left(\frac{35°+75°}{2}\right)\sin\left(\frac{75°-35°}{2}\right)$

$= 2\sin 55° \sin 20°$

Question 3.

Show that $\sin 12° \sin 48° \sin 54° = \frac{1}{8}$.

Solution:

$$\text{LHS} = \sin 12° \sin 48° \sin 54°$$

$$= \frac{1}{2}[2\sin 12° \sin 48°]\sin 54°$$

$$= \frac{1}{2}[\cos(48° - 12°) - \cos(48° + 12°)] \sin(90° - 36°)]$$

$$= \frac{1}{2}[\cos 36° - \cos 60°]\sin 54°$$

$$= \frac{1}{2}[\cos 36° - \frac{1}{2}]\cos 36°$$

$$= \frac{1}{2}(\cos 36°)^2 - \frac{1}{4}(\cos 36°)$$

Since
* $2\sin x \sin y = \cos(x - y) - \cos(x + y)$
* $\sin 54° = \cos 36°$
* $\cos 36° = \frac{\sqrt{5}+1}{4}$

$$= \frac{1}{2}\left(\frac{\sqrt{5}+1}{4}\right)^2 - \frac{1}{4}\left(\frac{\sqrt{5}+1}{4}\right)$$

$$= \frac{1}{32}\left(5+1+2\sqrt{5}\right) - \frac{1}{16}\left(\sqrt{5}+1\right)$$

$$= \frac{1}{32}\left(6+2\sqrt{5}\right) - \frac{1}{16}\left(\sqrt{5}+1\right) = \frac{1}{16}\left[\left(3+\sqrt{5}\right) - \left(\sqrt{5}+1\right)\right]$$

$$= \frac{1}{16}[3+\sqrt{5}-\sqrt{5}-1] = \frac{1}{16}[2] = \frac{1}{8} = \text{RHS}$$

Question 4.

Show that $\cos\frac{\pi}{15}\cos\frac{2\pi}{15}\cos\frac{3\pi}{15}\cos\frac{4\pi}{15}\cos\frac{5\pi}{15}\cos\frac{6\pi}{15}\cos\frac{7\pi}{15} = \frac{1}{128}$

Solution:

$(\pi/15 = 12°)$

LHS = $\cos 12° \cos 24° \cos 36° \cos 48° \cos 60° \cos 72° \cos 84°$ ...(1)

consider (we know that)

$\cos A \cos (60° + A) \cos (60° - A)$

$$= \cos A\,[\cos^2 60° - \sin^2 A]$$

$$= \cos A\left[\frac{1}{4} - (1-\cos^2 A)\right]$$

$$\cos A \cos (60° + A) \cos (60° - A) = \frac{1}{4}\cos 3A$$

$$= \cos A\left[\cos^2 A - \frac{3}{4}\right]$$

$$= \frac{4\cos^3 A - 3\cos A}{4}$$

$$\cos 12° \cos 72° \cos 48° = \frac{1}{4} \cos 3(12°) = \frac{1}{4} \cos 36° = \frac{1}{4}\left[\frac{\sqrt{5}+1}{4}\right]$$

Similarly

$$\cos 24° \cos 84° \cos 36° = \frac{1}{4} \cos 3 (24°) = \frac{1}{4} \cos 72° = \frac{1}{4} \cos (90° - 18°)$$

$$= \frac{1}{4} \sin 18° = \frac{1}{4}\left[\frac{\sqrt{5}-1}{4}\right]$$

(1) $\Rightarrow$

$$\text{LHS} = \frac{1}{4}\left[\frac{\sqrt{5}+1}{4}\right] \cdot \frac{1}{4}\left[\frac{\sqrt{5}-1}{4}\right] \cdot \frac{1}{2} = \frac{1}{4}\left(\frac{\sqrt{5}+1}{4} \cdot \frac{\sqrt{5}-1}{4}\right) \cdot \frac{1}{2}$$

$$= \frac{5-1}{128 \times 4} = \frac{1}{128}$$

Question 5.

Show that $\dfrac{\sin 8x \cos x - \sin 6x \cos 3x}{\cos 2x \cos x - \sin 3x \sin 4x} = \tan 2x$

Solution:

$$\sin 8x \cos x = \frac{1}{2}[2\sin 8x \ \cos x]$$

$$= \frac{1}{2}[\sin(8x + x) + (\sin(8x - x)]$$

$$= \frac{1}{2}[\sin 9x + \sin 7x]$$

$$\sin 6x \cos 3x = \frac{1}{2}[2\sin 6x \ \cos 3x]$$

$$= \frac{1}{2}[\sin(6x + 3x) + \sin(6x - 3x]$$

$$= \frac{1}{2}[\sin 9x + \sin 3x]$$

$$\cos 2x \cos x = \frac{1}{2}[2\cos 2x \ \cos x]$$

$$= \frac{1}{2}[\cos(2x+x)+\cos(2x-x]$$

$$= \frac{1}{2}[\cos 3x + \cos x]$$

$$\sin 3x \sin 4x = \frac{1}{2}[2\sin 3x \ \sin 4x]$$

$$= \frac{1}{2}[\cos(4x-3x)-\cos(4x+3x)]$$

$$= \frac{1}{2}[\cos x - \cos 7x]$$

$$\text{LHS} = \frac{\sin 8x \cos x - \sin 6x \cos 3x}{\cos 2x \cos x - \sin 3x \sin 4x}$$

$$= \frac{\frac{1}{2}(\sin 9x + \sin 7x) - \frac{1}{2}(\sin 9x + \sin 3x)}{\frac{1}{2}(\cos 3x + \cos x) - \frac{1}{2}(\cos x - \cos 7x)}$$

$$= \frac{\frac{1}{2}[(\sin 9x + \sin 7x) - (\sin 9x + \sin 3x)]}{\frac{1}{2}[(\cos 3x + \cos x) - (\cos x - \cos 7x)]}$$

$$= \frac{\sin 7x - \sin 3x}{\cos 3x + \cos 7x}$$

$$= \frac{2\cos\frac{7x+3x}{2}\sin\frac{7x-3x}{2}}{2\cos\frac{7x+3x}{2}\cos\frac{7x-3x}{2}} = \frac{\sin 2x}{\cos 2x} = \tan 2x = \text{RHS}$$

Question 6.

Show that $\dfrac{(\cos\theta-\cos 3\theta)(\sin 8\theta+\sin 2\theta)}{(\sin 5\theta-\sin\theta)(\cos 4\theta-\cos 6\theta)}=1$

Solution:

$$\cos\theta - \cos 3\theta = 2\sin\frac{\theta+3\theta}{2}\sin\frac{3\theta-\theta}{2}$$
$$= 2\sin 2\theta \sin\theta$$
$$\sin 8\theta + \sin 2\theta = 2\sin\frac{8\theta+2\theta}{2}\cos\frac{8\theta-2\theta}{2}$$
$$= 2\sin 5\theta\cos 3\theta$$
$$\sin 5\theta - \sin\theta = 2\cos\frac{(5\theta+\theta)}{2}\sin\frac{(5\theta-\theta)}{2}$$
$$= 2\cos 3\theta\sin 2\theta$$
$$\cos 4\theta - \cos 6\theta = 2\sin\frac{4\theta+6\theta}{2}\sin\frac{6\theta-4\theta}{2}$$
$$= 2\sin 5\theta\sin\theta$$
$$\therefore \text{LHS} = \frac{2\sin 2\theta\sin\theta\cdot 2\sin 5\theta\cos 3\theta}{2\cos 3\theta\sin 2\theta\cdot 2\sin 5\theta\sin\theta} = 1 = \text{RHS}$$

Question 7.

Prove that sin x + sin 2x + sin 3x = sin 2x (1 + 2 cos x).

Solution:

$$\sin x + \sin 3x = 2\sin\frac{x+3x}{2}\cos\frac{3x-x}{2}$$
$$= 2\sin 2x\cos x$$
$$\text{LHS} = \sin x + \sin 3x + \sin 2x = 2\sin 2x\cos x + \sin 2x$$
$$= \sin 2x\,(2\cos x + 1) = \text{RHS}$$

Question 8.

Prove that $\dfrac{\sin 4x+\sin 2x}{\cos 4x+\cos 2x}=\tan 3x$

Solution:

$$\sin 4x + \sin 2x = 2\sin\frac{4x+2x}{2}\cos\frac{4x-2x}{2}$$
$$= 2\sin 3x\cos x$$
$$\cos 4x + \cos 2x = 2\cos\frac{4x+2x}{2}\cos\frac{4x-2x}{2} = 2\cos 3x\cos x$$
$$\therefore \text{LHS} = \frac{\sin 4x+\sin 2x}{\cos 4x+\cos 2x} = \frac{2\sin 3x\cos x}{2\cos 3x\cos x} = \tan 3x = \text{RHS}$$

Question 9.
Prove that 1 + cos 2x + cos 4x + cos 6x = 4 cos x cos 2x cos 3x.

Solution:

$$1+\cos 6x = 2\cos^2 3x$$
$$\cos 2x + \cos 4x = 2\cos\frac{2x+4x}{2}\cos\frac{4x-2x}{2}$$
$$= 2\cos 3x\cos x$$

LHS = 1 + cos 2x + cos 4x + cos 6x
= (1 + cos 6x) + (cos 2x + cos 4x)
= $2\cos^2 3x + 2\cos 3x\cos x$
= 2 cos 3x (cos 3x + cos x)

$$= 2\cos 3x\left[2\cos\frac{3x+x}{2}\cos\frac{3x-x}{2}\right]$$
$$= 2\cos 3x\,(2\cos 2x\cos x)$$
$$= 4\cos x\cos 2x\cos 3x = \text{RHS}$$

Question 10.
Prove that $\sin\frac{\theta}{2}\sin\frac{7\theta}{2}+\sin\frac{3\theta}{2}\sin\frac{11\theta}{2}=\sin 2\theta\sin 5\theta$.

Solution:

$$\sin\frac{\theta}{2}\sin\frac{7\theta}{2} = \frac{1}{2}\left[2\sin\frac{\theta}{2}\sin\frac{7\theta}{2}\right]$$

$$= \frac{1}{2}\left[\cos\frac{7\theta-\theta}{2} - \cos\frac{7\theta+\theta}{2}\right]$$

$$= \frac{1}{2}(\cos 3\theta - \cos 4\theta)$$

$$\sin\frac{3\theta}{2}\sin\frac{11\theta}{2} = \frac{1}{2}\left[2\sin\frac{3\theta}{2}\sin\frac{11\theta}{2}\right]$$

$$= \frac{1}{2}\left[\cos\frac{11\theta-3\theta}{2} - \cos\frac{11\theta+3\theta}{2}\right]$$

$$= \frac{1}{2}[\cos 4\theta - \cos 7\theta]$$

$$\text{LHS} = \sin\frac{\theta}{2}\sin\frac{7\theta}{2} + \sin\frac{3\theta}{2}\sin\frac{11\theta}{2}$$

$$= \frac{1}{2}[\cos 3\theta - \cos 4\theta) + \frac{1}{2}[\cos 4\theta - \cos 7\theta]$$

$$= \frac{1}{2}[\cos 3\theta - \cos 4\theta + \cos 4\theta - \cos 7\theta]$$

$$= \frac{1}{2}[\cos 3\theta - \cos 7\theta]$$

$$= \frac{1}{2}\left[2\sin\frac{7\theta+3\theta}{2}\sin\frac{7\theta-3\theta}{2}\right]$$

$$= \sin 5\theta \sin 2\theta = \text{RHS}$$

Question 11.

Prove that $\cos(30° - A)\cos(30° + A) + \cos(45° - A)\cos(45° + A) = \cos 2A + \frac{1}{4}$.

Solution:

$$\cos(30° - A)\cos(30° + A) = (\cos 30° \cos A + \sin 30° \sin A)(\cos 30° \cos A - \sin 30° \sin A)$$

$$= \cos^2 30° \cos^2 A - \sin^2 30° \sin^2 A$$

$$= \left(\frac{\sqrt{3}}{2}\right)^2 \cos^2 A - \left(\frac{1}{2}\right)^2 \sin^2 A$$

$$= \frac{3}{4}\cos^2 A - \frac{1}{4}\sin^2 A \qquad ...(1)$$

$$\cos(45° - A)\cos(45° + A) = (\cos 45° \cos A + \sin 45° \sin A)(\cos 45° \cos A - \sin 45° \sin A)$$

$$= \cos^2 45° \cos^2 A - \sin^2 45° \sin^2 A$$

$$= \left(\frac{1}{\sqrt{2}}\right)^2 \cos^2 A - \left(\frac{1}{\sqrt{2}}\right)^2 \sin^2 A$$

$$= \frac{1}{2}\cos^2 A - \frac{1}{2}\sin^2 A \qquad ...(2)$$

$$\text{LHS} = (1) + (2) = \frac{3}{4}\cos^2 A - \frac{1}{4}\sin^2 A + \frac{1}{2}\cos^2 A - \frac{1}{2}\sin^2 A$$

$$= \frac{5}{4}\cos^2 A - \frac{3}{4}\sin^2 A$$

$$= \cos^2 A - \frac{3}{4}\sin^2 A - \frac{1}{4}\sin^2 A + \frac{1}{4}\sin^2 A + \frac{1}{4}\cos^2 A$$

$$= (\cos^2 A - \sin^2 A) + \frac{1}{4}(\sin^2 A + \cos^2 A)$$

$$= \cos 2A + \frac{1}{4}(1) = \cos 2A + \frac{1}{4} = \text{RHS}$$

Question 12.

Prove that $\dfrac{\sin x + \sin 3x + \sin 5x + \sin 7x}{\cos x + \cos 3x + \cos 5x + \cos 7x} = \tan 4x$

Solution:

Nr: $(\sin x + \sin 7x) + (\sin 3x + \sin 5x)$

$$= \left[2\sin\frac{7x+x}{2}\cos\frac{7x-x}{2}\right]+\left[2\sin\frac{5x+3x}{2}\cos\frac{5x-3x}{2}\right]$$

$$= 2\sin 4x \cos 3x + 2\sin 4x \cos x$$

$$= 2\sin 4x (\cos 3x + \cos x) \quad ...(1)$$

Dr. $(\cos x + \cos 7x) + (\cos 3x + \cos 5x)$

$$= \left[2\cos\frac{7x+x}{2}\cos\frac{7x-x}{2}\right]+\left[2\cos\frac{5x+3x}{2}\cos\frac{5x-3x}{2}\right]$$

$$= 2\cos 4x \cos 3x + 2\cos 4x \cos x$$

$$= 2\cos 4x (\cos 3x + \cos x) \quad ...(2)$$

$$\text{LHS} = \frac{(1)}{(2)} = \frac{2\sin 4x\,(\cos 3x+\cos x)}{2\cos 4x\,(\cos 3x+\cos x)}$$

$$= \tan 4x = \text{RHS}$$

Question 13.

Prove that $\mathbf{\dfrac{\sin(4A-2B)+\sin(4B-2A)}{\cos(4A-2B)+\cos(4B-2A)}=\tan(A+B)}$

Solution:

$$\text{LHS} = \frac{2\sin\left(\dfrac{4A-2B+4B-2A}{2}\right)\cos\left(\dfrac{4A-2B-4B+2A}{2}\right)}{2\cos\left(\dfrac{4A-2B+4B-2A}{2}\right)\cos\left(\dfrac{4A-2B-4B+2A}{2}\right)}$$

$$= \frac{2\sin(A+B)\cos(3A-3B)}{2\cos(A+B)\cos(3A-3B)} = \tan(A+B) = \text{RHS}$$

Question 14.

Show that $\mathbf{\cot(A+15^\circ)-\tan(A-15^\circ)=\dfrac{4\cos 2A}{1+2\sin 2A}}$

Solution:

$$\text{LHS} = \frac{\cos(A+15^\circ)}{\sin(A+15^\circ)} - \frac{\sin(A-15^\circ)}{\cos(A-15^\circ)}$$

$$= \frac{\cos(A+15^\circ)\cos(A-15^\circ) - \sin(A+15^\circ)\sin(A-15^\circ)}{\sin(A+15^\circ)\cos(A-15^\circ)}$$

$$= \frac{\cos(A+15^\circ+A-15^\circ)}{\frac{1}{2}[\sin(A+15^\circ+A-15^\circ)+\sin(A+15^\circ-A+15^\circ)]}$$

$$= \frac{2\cos 2A}{\sin 2A + \sin 30^\circ} = \frac{2\cos 2A}{\frac{1}{2}+\sin 2A}$$

$$= \frac{4\cos 2A}{1+2\sin 2A} = \text{RHS}$$

Ex 3.7

Question 1.

If A + B + C = 180°, prove that

(i) sin 2A + sin 2B + sin 2C = 4 sin A sin B sin C

(ii) $\cos A + \cos B - \cos C = -1 + 4\cos\frac{A}{2}\cos\frac{B}{2}\sin\frac{C}{2}$

(iii) $\sin^2 A + \sin^2 B + \sin^2 C = 2 + 2\cos A\cos B\cos C$

(iv) $\sin^2 A + \sin^2 B - \sin^2 C = 2\sin A\sin B\cos C$

(v) $\tan\frac{A}{2}\tan\frac{B}{2} + \tan\frac{B}{2}\tan\frac{C}{2} + \tan\frac{C}{2}\tan\frac{A}{2} = 1$

(vi) sin A + sin B + sin C = 4 cos A2 cos B2 cos c2

(vii) sin(B + C – A) + sin(C + A – B) + sin(A + B – C) = 4 sin A sin B sin C.

Solution:

(i) LHS = (sin 2A + sin 2B) + sin 2C

= 2 sin (A + B) cos (A – B) + 2 sin C cos C

[sin (A + B) = sin (180° – C) = sin C]

= 2 sin C cos (A – B) + 2 sin C cos C

= 2 sin C [cos (A – B) + cos C]

{cos C = cos [180° – (A + B)] = – cos (A + B)}
= 2 sin C [cos (A – B) – cos (A + B)]

$$= 2\sin C\left\{2\sin\frac{2A}{2}\sin\frac{2B}{2}\right\}$$

$$= 4 \sin A \sin B \sin C = \text{RHS}$$

(ii)

$$(\cos A + \cos B) - \cos C = 2\cos\frac{A+B}{2}\cos\frac{A-B}{2} - [1 - 2\sin^2 C/2]$$

Hint: $\left[\cos\frac{A+B}{2} = \sin\frac{C}{2}\right]$

$$= 2\sin\frac{C}{2}\cos\frac{A-B}{2} - 1 + 2\sin^2\frac{C}{2}$$

$$= -1 + 2\sin\frac{C}{2}\left[\cos\frac{A-B}{2} + \sin\frac{C}{2}\right]$$

$$= -1 + 2\sin\frac{C}{2}\left[\cos\frac{A-B}{2} + \cos\frac{A+B}{2}\right]$$

$$= -1 + 2\sin\frac{C}{2}\left[2\cos\frac{2A}{4}\cos\frac{2B}{4}\right]$$

$$= -1 + 2\sin\frac{C}{2}\left[2\cos\frac{A}{2}\cos\frac{B}{2}\right]$$

$$= -1 + 4\cos\frac{A}{2}\cos\frac{B}{2}\sin\frac{C}{2} = \text{RHS}$$

(iii)

$$\text{LHS} = \frac{1-\cos 2A}{2} + \frac{1-\cos 2B}{2} + \frac{1-\cos 2C}{2}$$

$$= \frac{3}{2} - \frac{1}{2}[\cos 2A + \cos 2B + \cos 2C]$$

Hint:

$$\left[\sin^2 A = \frac{1-\cos 2A}{2}\right] \qquad = \frac{3}{2} - \frac{1}{2}[2\cos(A+B)\cos(A-B) + 2\cos^2 C - 1]$$

$$= \frac{3}{2} - \cos(A+B)\cos(A-B) - \cos^2 C + \frac{1}{2}$$

$$= 2 + \cos C \cos (A-B) - \cos^2 C$$

$$= 2 + \cos C[\cos (A-B) (\cos (A+B)]$$

[cos (180° – C) – cos C – cos C]

= 2 + cos C [cos (A – B) + cos (A + B)]

= 2+ cos C[2 cos A cos B]

= 2 + 2 cos A cos B cos C = RHS

(iv)

$$\text{LHS} = \frac{1-\cos 2A}{2} + \frac{1-\cos 2B}{2} - \frac{1-\cos 2C}{2}$$

$$= \left[\frac{1}{2} + \frac{1}{2} - \frac{1}{2}\right] - \frac{1}{2}\,[\cos 2A + \cos 2B - \cos 2C]$$

Hint:

$$= \frac{1}{2} - \frac{1}{2}\,[2 \cos (A+B) \cos (A-B) - (2 \cos^2 C - 1)]$$

$$\left[\sin^2 A = \frac{1-\cos 2A}{2}\right] \qquad = \frac{1}{2} - \cos(A+B)\cos(A-B) + \cos^2 C - \frac{1}{2}$$

$$= \cos C \cos (A-B) + \cos^2 C$$

$$= \cos C\,[\cos (A-B) - \cos (A+B)]$$

$$= \cos C\,[2 \sin A \sin B] = 2 \sin A \sin B \cos C = \text{RHS}$$

(v)

Given $A + B + C = 180° \Rightarrow \dfrac{A+B+C}{2} = 90°$

$\Rightarrow \dfrac{A+B}{2} = 90° - \dfrac{C}{2}$

So $\tan\left(\dfrac{A+B}{2}\right) = \tan\left(90° - \dfrac{C}{2}\right) = \cot\dfrac{C}{2}$

(i.e) $$\frac{\tan\frac{A}{2} + \tan\frac{B}{2}}{1 - \tan\frac{A}{2}\tan\frac{B}{2}} = \cot\frac{C}{2} = \frac{1}{\tan\frac{C}{2}}$$

$$\Rightarrow \left(\tan\frac{A}{2} + \tan\frac{B}{2}\right)\tan\frac{C}{2} = 1 - \tan\frac{A}{2}\tan\frac{B}{2}$$

(i.e.) $$\tan\frac{A}{2}\tan\frac{C}{2} + \tan\frac{B}{2}\tan\frac{C}{2} = 1 - \tan\frac{A}{2}\tan\frac{B}{2}$$

(i.e.) $$\tan\frac{A}{2}\tan\frac{B}{2} + \tan\frac{B}{2}\tan\frac{C}{2} + \tan\frac{C}{2}\tan\frac{A}{2} = 1$$

(vi)

LHS = (sin A + sin B) + sin C

$$= 2\sin\frac{A+B}{2}\cos\left(\frac{A-B}{2}\right) + 2\sin\frac{C}{2}\cos\frac{C}{2}$$

$$= 2\cos\frac{C}{2}\left[\cos\left(\frac{A-B}{2}\right) + \sin\frac{C}{2}\right]$$

$$= 2\cos\frac{C}{2}\left[\cos\left(\frac{A-B}{2}\right) + \cos\frac{A+B}{2}\right]$$

$$= 2\cos\frac{C}{2}\left[2\cos\frac{A}{2}\cos\frac{B}{2}\right] = 4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2} = RHS$$

(vii) Now $A + B + C = 180°$

So $B + C = 180° - A$

sin (B + C – A) = sin (180° – A – A)
= sin(180° – 2A) = sin 2A
Now LHS = sin 2A + sin 2B + sin 2C
= 4 sin A sin B sin C (from (i) ans) = RHS

Question 2.
If A + B + C = 2s, then prove that sin(s – A) sin(s – B) + sin s sin(s – C) = sin A sin B.

Solution:

$$\text{Now } \sin(s-A)\sin(s-B) = \frac{1}{2}\{\cos[(s-A)-(s-B)]-\cos[(s-A)+(s-B)]\}$$

$$= \frac{1}{2}\cos(s-A-s+B)-\cos[2s-(A+B)]$$

$$= \frac{1}{2}\{\cos(A-B)-\cos C\}$$

$$[\therefore \cos(A-B)=\cos(B-A)]$$

$$\text{Again } \sin s \sin s - C = \frac{1}{2}[\cos C-\cos(A+B)]$$

$$\text{So, LHS} = \frac{1}{2}\{\cos(A-B)-\cos C+\cos C-\cos(A+B)\}$$

$$= \frac{1}{2}[\cos(A-B)-\cos(A+B)]$$

$$= \frac{1}{2}[2\sin A\sin B] = \sin A \sin B = \text{RHS}$$

Question 3.
If $x+y+z=xyz$, then prove that $\frac{2x}{1-x^2}+\frac{2y}{1-y^2}+\frac{2z}{1-z^2}=\frac{2x}{1-x^2}\frac{2y}{1-y^2}\frac{2z}{1-z^2}$.

Solution:

Taking $x = \tan A$, $y = \tan B$ and $z = \tan C$

$$\frac{2x}{1-x^2} = \frac{2\tan A}{1-\tan^2 A} = \tan 2A$$

Similarly, $\frac{2y}{1-y^2} = \tan 2B$ and $\frac{2z}{1-z^2} = \tan 2C$

Given $x + y + z = xyz$

(i.e) we are given tan A + tan B + tan C = tan A tan B tan C

⇒ A+B+C = 180°

⇒ A + B = 180° – C

multiply 2 on both sides

⇒ 2A + 2B = 360° – 2C

⇒ 2(A + B) = 360° – 2C

⇒ tan(2A + 2B) = tan(360° – 2C) = -tan 2C

(i.e) $$\frac{\tan 2A + \tan 2B}{1-\tan 2A \tan 2B} = -\tan 2C$$

⇒ tan 2A + tan 2B = -tan2C[1 – tan 2A tan 2B]

⇒ tan 2A + tan 2B = -tan 2C + tan 2A tan 2B tan 2C

⇒ tan 2A + tan 2B + tan 2C = tan 2A tan 2B tan 2C

(i.e.) $$\frac{2x}{1-x^2} + \frac{2y}{1-y^2} + \frac{2z}{1-z^2} = \frac{2x}{1-x^2} \times \frac{2y}{1-y^2} \times \frac{2z}{1-z^2}$$

Question 4.

If A + B + C = π/2, prove the following

(i) sin 2A + sin 2B + sin 2C = 4 cos A cos B cos C

(ii) cos 2A + cos 2B + cos 2C = 1 + 4 sin A sin B sin C.

Solution:

(i) LHS = (sin 2A + sin 2B) + sin 2C

= 2 sin (A + B) cos (A – B) + 2 sin C cos C = 2 sin (90° – C) cos (A – B) + 2 sin C cos C

= 2 cos C [cos (A – B) + sin C] + cos (A + B) (∴ A + B = π/2 – C)

= 2 cos C [cos (A – B) + cos (A + B)]

= 2 cos C [2 cos A cos B]

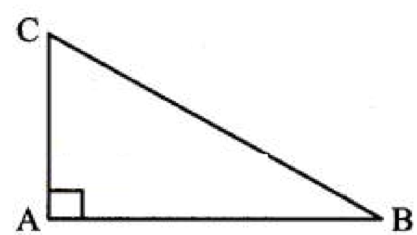
= 4 cos A cos B cos C = RHS

(ii) LHS = (cos 2A + cos 2B) + cos 2C
= 2 cos (A + B) cos (A – B) + 1 – 2 $\sin^2$ C
= 1 + 2 sin C (cos (A – B) – 2 $\sin^2$ C)
{∴ cos (A + B) = cos (90° – C) = sin C}
= 1 + 2 sin C [cos (A- B) – sin C]
= 1 + 2 sin C [cos (A – B) – cos (A + B)]
= 1 + 2 sin C [2 sin A sin B]
= 1 + 4 sin A sin B sin C
= RHS

Question 5.
If ΔABC is a right triangle and if ∠A = π/2, then prove that
(i) $\cos^2 B + \cos^2 C = 1$
(ii) $\sin^2 B + \sin^2 C = 1$
(iii) $\cos B - \cos C = -1 + 2\sqrt{2}\cos\frac{B}{2}\sin\frac{C}{2}$

Solution:

(i) $\angle A = 90°$; $\cos B = \frac{AB}{BC}$; $\cos C = \frac{AC}{BC}$



$\therefore$ LHS = $\cos^2 B + \cos^2 C$

$$= \frac{AB^2}{BC^2} + \frac{AC^2}{BC^2} = \frac{AB^2 + AC^2}{BC^2}$$

$$= \frac{BC^2}{BC^2} = 1 = \text{RHS}$$

(ii) From diagram, $\sin B = \frac{AC}{BC}$, $\sin C = \frac{AB}{BC}$

$$\text{L.H.S} = \sin^2 B + \sin^2 C$$

$$= \frac{AC^2}{BC^2} + \frac{AB^2}{BC^2} = \frac{AC^2 + AB^2}{BC^2}$$

$$= \frac{BC^2}{BC^2} = 1 = \text{RHS}$$

(*iii*) $A + B + C = 180°$, Given $A = 90°$

$$\therefore B + C = 90° \Rightarrow \frac{B+C}{2} = 45°$$

$$\frac{B}{2}+\frac{C}{2} = 45°$$

$$\text{RHS} = -1 + 2\sqrt{2}\cos\frac{B}{2}\sin\frac{C}{2}$$

$$= 1 + \sqrt{2}\,(2\cos\frac{B}{2}\sin\frac{C}{2})$$

We know that $2\cos A\sin B = \sin(A+B) - \sin(A-B)$

$$= -1 + \sqrt{2}\left(\sin\frac{(B+C)}{2} - \sin\frac{(B-C)}{2}\right)$$

$$= -1 + \sqrt{2}\left(\sin 45° - \sin\frac{(B-C)}{2}\right)$$

$$= -1 + \sqrt{2}\left(\frac{1}{\sqrt{2}} - \sin\frac{(B-C)^2}{2}\right)$$

$$= -1 + 1 - \sqrt{2}\sin\frac{(B-C)}{2}$$

$$= -\sqrt{2}\sin\frac{(B-C)}{2} \quad \dots (1)$$

$$\text{LHS} = \cos B - \cos C$$

$$= 2\sin\frac{(B+C)}{2}\sin\frac{(C-B)}{2}$$

$$= 2\sin 45°\sin\frac{(C-B)}{2}$$

$$= 2\left(\frac{1}{\sqrt{2}}\right)\sin\left(\frac{-(B-C)}{2}\right)$$

$$= -\sqrt{2}\sin\frac{(B-C)}{2} \quad \dots (2)$$

From (1) & (2) $\Rightarrow$ LHS = RHS

Ex 3.8

Question 1.
Find the principal solution and general solutions of the following:

(i) $\sin\theta = -\frac{1}{\sqrt{2}}$ **(ii)** $\cot\theta = \sqrt{3}$ **(iii)** $\tan\theta = -\frac{1}{\sqrt{3}}$

Solution:

(i)

$$\sin\theta = -\frac{1}{\sqrt{2}} < 0, \text{ the principal value lies in IV quadrant}$$

$$\sin\theta = \frac{1}{\sqrt{2}} \Rightarrow \theta = \frac{\pi}{4}$$

Hint: $\sin\theta = \sin\alpha$

$\theta = n\pi + (-1)^n\alpha$

$$\text{Here } \sin\theta = -\frac{1}{\sqrt{2}} = \sin\left(-\frac{\pi}{4}\right)$$

$$\Rightarrow \alpha = -\frac{\pi}{4}$$

The general soln is $\theta = n\pi + (-1)^n\alpha$

(i.e.) $$\theta = n\pi + (-1)^n -\frac{\pi}{4}, n \in \mathbb{Z}$$

(ii)

$$\cot\theta = \sqrt{3}$$

$$\Rightarrow \tan\theta = \frac{1}{\sqrt{3}} = \tan\pi/6$$

$$\Rightarrow \theta = \pi/6$$

The general soln is $\theta = n\pi + \pi/6, n \in \mathbb{Z}$

(iii)

$$\tan\theta = -\frac{1}{\sqrt{3}}$$

$$\text{Here } \tan\theta = -\frac{1}{\sqrt{3}} \Rightarrow \theta = -\pi/6$$

The general soln is $\theta = n\pi + (-\pi/6)$

$$= n\pi - \pi/6, \ n \in \mathbb{Z}$$

Question 2.
Solve the following equations for which solutions lies in the interval $0° < \theta < 360°$

(i) $\sin^4 = \sin^2 x$

(ii) $2\cos^2 x + 1 = -3\cos x$
(iii) $2\sin^2 x + 1 = 3\sin x$
(iv) $\cos 2x = 1 - 3\sin x - 3\sin x$

Solution:

(*i*)
$$\sin^2 x - \sin^4 x = 0$$
$$\sin^2 x(1-\sin^2 x) = 0$$
$$\sin^2 x(\cos^2 x) = 0$$
$$\left[\frac{1}{2}(2\sin x\cos x)\right]^2 = 0$$
$$\Rightarrow \quad (\sin 2x)^2 = 0$$
$$\Rightarrow \quad \sin 2x = 0 = 0, \pi, 2\pi, 3\pi, 4\pi$$
$$x = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi$$

(*ii*)
$$2\cos^2 x + 3\cos x + 1 = 0$$
$$2\cos^2 x + 2\cos x + \cos x + 1 = 0$$
$$2\cos x(\cos x + 1) + 1(\cos x + 1) = 0$$
$$(2\cos x + 1)(\cos x + 1) = 0$$

$2\cos x + 1 = 0$
$2\cos x = -1$
$\cos x = -\frac{1}{2}$

$\cos x + 1 = 0$
$\cos x = -1$
$\Rightarrow x = \pi$

$x = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$ $\quad$ $x = \pi + \frac{\pi}{3} = \frac{4\pi}{3}$ and $x = \pi$

(*iii*) $2\sin^2 x - 3\sin x + 1 = 0$
$2\sin^2 x - 2\sin x - \sin x + 1 = 0$
$2\sin x(\sin x - 1) - (\sin x - 1) = 0$
$(2\sin x - 1)(\sin x - 1) = 0$
$2\sin x - 1 = 0$
$2\sin x = 1$

$(\sin x - 1) = 0$
$\sin x = 1 = \sin \pi/2$
$x = \pi/2$

$$\sin x = \frac{1}{2} = \sin \pi/6$$

$$x = \pi/6$$

(iv)

$$\cos 2x = 1 - 3\sin x$$

$$1 - 2\sin^2 x = 1 - 3\sin x$$

$$3\sin x - 2\sin^2 x = 0$$

$$\sin x(3 - 2\sin x) = 0$$

$\sin x = 0$	$3 - 2\sin x = 0$
$\Rightarrow x = 0$	$\Rightarrow \sin x = 3/2$ is not possible

$\therefore \sin x = 0 \Rightarrow x = 0$ (or) π

Question 3.
Solve the following equations:
(i) sin 5x – sin x = cos 3x
(ii) 2 $\cos^2$ θ + 3 sin θ – 3 = 0
(iii) cos θ + cos 3θ = 2 cos 2θ
(iv) sin θ + sin 3θ + sin 5θ = 0
(v) sin 2θ – cos 2θ – sin θ + cos θ = 0
(vi) sin θ + cos θ = 3–√
(vii) sin θ + 3–√ cos θ = 1
(viii) cot θ + cosec θ = 3–√

(ix) $\tan\theta + \tan\left(\theta + \frac{\pi}{3}\right) + \tan\left(\theta + \frac{2\pi}{3}\right) = \sqrt{3}$

(x) $\cos 2\theta = \frac{\sqrt{5}+1}{4}$

(xi) $2\cos^2 x - 7\cos x + 3 = 0$

Solution:

(i) $\sin 5x - \sin x = \cos 3x$

$$2\cos\frac{5x+x}{2}\sin\frac{5x-x}{2} = \cos 3x$$

$$2\cos 3x \sin 2x - \cos 3x = 0$$
$$\cos 3x\,[2\sin 2x - 1] = 0$$

$$\cos 3x = 0 = \cos\frac{\pi}{2}$$

The general soln is

$$3x = (2n+1)\frac{\pi}{2} \text{ (or)}$$

$$x = (2n+1)\frac{\pi}{6}$$

$$2\sin 2x - 1 = 0$$

$$\sin 2x = \frac{1}{2} = \sin\frac{\pi}{6}$$

$$2x = n\pi + (-1)^n\frac{\pi}{6} \text{ (or)}$$

$$x = \frac{n\pi}{2} + (-1)^n\frac{\pi}{12}$$

The general soln is

$x = (2n+1)\,\pi/6$ (or) $x = n\,\pi/2 + (-1)^n\,\pi/12,\ n \in \mathbb{Z}$

(*ii*) $2\cos^2\theta + 3\sin\theta - 3 = 0$

$$2(1 - \sin^2\theta) + 3\sin\theta - 3 = 0$$
$$2 - 2\sin^2\theta + 3\sin\theta - 3 = 0$$

$$\sin\theta - 1 = 0$$
$$\sin\theta = 1$$
$$\Rightarrow \theta = \pi/2$$

$$2\sin\theta - 1 = 0$$
$$\sin\theta = 1/2$$
$$\theta = \pi/6$$

The general soln is

$$\theta = n\pi + (-1)^n\,\pi/2\ ;\ \theta = n\pi + (-1)^n\,\pi/6,\ n \in \mathbb{Z}$$

(*iii*) $\cos\theta + \cos 3\theta = 2\cos 2\theta$

$$2\cos\frac{\theta + 3\theta}{2}\cos\frac{3\theta - \theta}{2} = 2\cos 2\theta$$

$$2\cos 2\theta\cos\theta - 2\cos 2\theta = 0$$
$$2\cos 2\theta\,(\cos\theta - 1) = 0$$
$$2\cos 2\theta = 0$$
$$\cos 2\theta = 0$$
$$\Rightarrow 2\theta = (2n+1)\,\pi/2$$
$$\theta = (2n+1)\,\pi/4$$

$$\cos\theta = 1$$
$$\theta = 0$$
$$\theta = 2n\pi + 0$$
$$\theta = 2n\pi$$

The general soln is $\theta = (2n+1)\,\pi/4$ (or) $\theta = 2n\pi,\ n \in \mathbb{Z}$

(*iv*) $\sin\theta + \sin 3\theta + \sin 5\theta = 0$

$$(\sin\theta + \sin 5\theta) + \sin 3\theta = 0$$

$$2\sin\frac{5\theta+\theta}{2}\cos\frac{5\theta-\theta}{2} + \sin 3\theta = 0$$

$$2\sin 3\theta\cos 2\theta + \sin 3\theta = 0$$

$$\sin 3\theta\,[2\cos 2\theta + 1] = 0$$

$\sin 3\theta = 0$	$2\cos 2\theta = -1$
$\Rightarrow\ 3\theta = n\pi$	$\cos 2\theta = -1/2 = \cos(\pi - \pi/3) = \cos 2\pi/3$
$\theta = n\pi/3$	$\Rightarrow\ 2\theta = 2\pi/3$
	$2\theta = 2n\pi \pm 2\pi/3$

The general soln is $\theta = n\pi/3$ (or) $\theta = n\pi \pm \pi/3,\ n \in Z$

(*v*) $(\sin 2\theta - \sin\theta) - (\cos 2\theta - \cos\theta) = 0$

$$2\cos\frac{3\theta}{2}\sin\frac{\theta}{2} - \left(-2\sin\frac{3\theta}{2}\sin\frac{\theta}{2}\right) = 0$$

$$2\sin\frac{\theta}{2}\left[\cos\frac{3\theta}{2} + \sin\frac{3\theta}{2}\right] = 0$$

$2\sin\frac{\theta}{2} = 0 = \sin 0$	$\left(\cos\frac{3\theta}{2} + \sin\frac{3\theta}{2}\right) = 0$
$\frac{\theta}{2} = n\pi$	$\cos\frac{3\theta}{2} = -\sin\frac{3\theta}{2}$

$2\sin\frac{\theta}{2} = 0 = \sin 0$	$\left(\cos\frac{3\theta}{2} + \sin\frac{3\theta}{2}\right) = 0$
$\frac{\theta}{2} = n\pi$	$\cos\frac{3\theta}{2} = -\sin\frac{3\theta}{2}$
General soln is $\theta = 2n\pi$	$\Rightarrow \tan\frac{3\theta}{2} = -1 = \tan\left(\frac{-\pi}{4}\right)$

$$\Rightarrow \frac{3\theta}{2} = -\frac{\pi}{4}$$

$$\frac{3\theta}{2} = n\pi - \frac{\pi}{4}$$

$$\Rightarrow \theta = \frac{2}{3}n\pi - \frac{2}{3}\frac{\pi}{4}$$

$$\theta = \frac{2n\pi}{3} - \frac{\pi}{6}$$

The general soln is $\theta = 2n\pi$ (or) $\frac{2n\pi}{3} - \frac{\pi}{6}, n \in Z$

(*vi*) $\sin\theta + \cos\theta = \sqrt{2}$

Multiplying by $\frac{1}{\sqrt{2}}$ on both sides

$$\frac{1}{\sqrt{2}}\sin\theta + \frac{1}{\sqrt{2}}\cos\theta = \frac{1}{\sqrt{2}}\sqrt{2} = 1$$

$$\cos\theta\cos\frac{\pi}{4} + \sin\theta\sin\frac{\pi}{4} = 1$$

The general soln is $\cos(\theta - \pi/4) = 1$

$$\cos(\theta - \pi/4) = \cos 0$$

$$\therefore \theta - \pi/4 = 2n\pi \pm 0$$

The general soln is $\theta = 2n\pi + \pi/4, n \in Z$

$$= \frac{8n\pi + \pi}{4} = \frac{\pi}{4}(8n+1), n \in z$$

(*vii*) $\sin\theta + \sqrt{3}\cos\theta = 1$

Multiplying by $\frac{1}{\sqrt{1+3}} = \frac{1}{2}$ we get

$$\frac{1}{2}\sin\theta + \frac{\sqrt{3}}{2}\cos\theta = \frac{1}{2}$$

$$\cos\theta\cos\frac{\pi}{6} + \sin\theta\sin\frac{\pi}{6} = \frac{1}{2}$$

$$\cos(\theta - \pi/6) = \frac{1}{2} = \cos \pi/3$$

$$\Rightarrow \qquad \theta - \pi/6 = \pi/3$$

General soln is $\theta - \pi/6 = 2n\pi \pm \pi/3$

$$\theta = 2n\pi + \pi/6 \pm \pi/3, n \in Z$$

(*viii*) $\dfrac{\cos\theta}{\sin\theta} + \dfrac{1}{\sin\theta} = \sqrt{3}$

$\cos\theta + 1 = \sqrt{3}\ \sin\theta$

$\sqrt{3}\ \sin\theta - \cos\theta = 1$

($\div$ by 2) $\dfrac{\sqrt{3}}{2}\sin\theta - \dfrac{1}{2}\cos\theta = \dfrac{1}{2}$

(i.e) $\cos\theta\cos\dfrac{\pi}{3} - \sin\theta\sin\dfrac{\pi}{3} = -\dfrac{1}{2}$

$$\Rightarrow \qquad \cos\left(\theta + \frac{\pi}{3}\right) = \cos\left(\pi - \frac{\pi}{3}\right) = \cos\frac{2\pi}{3}$$

$$\Rightarrow \qquad \theta + \frac{\pi}{3} = \frac{2\pi}{3}$$

The general soln is $\theta = 2n\pi \pm \dfrac{2\pi}{3} - \dfrac{\pi}{3}, n \in Z$

(*ix*) Now $\tan\left(\theta + \dfrac{\pi}{3}\right) = \dfrac{\tan\theta + \sqrt{3}}{1 - \sqrt{3}\tan\theta}$ and $\tan\left(\theta + \dfrac{2\pi}{3}\right) = \dfrac{\tan\theta - \sqrt{3}}{1 + \sqrt{3}\tan\theta}$

So, $\tan\left(\theta + \dfrac{\pi}{3}\right) + \tan\left(\theta + \dfrac{2\pi}{3}\right)$

$$= \frac{\tan\theta + \sqrt{3}}{1 - \sqrt{3}\tan\theta} + \frac{\tan\theta - \sqrt{3}}{1 + \sqrt{3}\tan\theta} = \frac{(\tan\theta + \sqrt{3})(1 + \sqrt{3}\tan\theta) + (\tan\theta - \sqrt{3})(1 - \sqrt{3}\tan\theta)}{1 - 3\tan^2\theta}$$

$$= \frac{\tan\theta + \sqrt{3} + \sqrt{3}\tan^2\theta + 3\tan\theta + \tan\theta - \sqrt{3}\tan^2\theta - \sqrt{3} + 3\tan\theta}{1 - 3\tan^2\theta} = \frac{8\tan\theta}{1 - 3\tan^2\theta}$$

Given, $\tan\theta + \tan\left(\theta + \frac{\pi}{3}\right) + \tan\left(\theta + \frac{2\pi}{3}\right) = \sqrt{3}$

$$\Rightarrow \quad \tan\theta + \frac{8\tan\theta}{1 - 3\tan^2\theta} = \sqrt{3} \Rightarrow \frac{\tan\theta - 3\tan^3\theta + 8\tan\theta}{1 - 3\tan^2\theta} = \sqrt{3}$$

(*x*) $\cos 2\theta = \frac{\sqrt{5}+1}{4} = \cos 36° = \cos\frac{2\pi}{10}$

$\Rightarrow \quad 2\theta = \frac{2\pi}{10}$

The general soln is $2\theta = 2n\pi \pm \frac{2\pi}{10}$

$\theta = n\pi \pm \frac{\pi}{10}, n \in \mathbb{Z}$

(*xi*) $2\cos^2 x - 7\cos x + 3 = 0$

$\cos x = \frac{7 \pm \sqrt{49-24}}{2(2)} = \frac{7 \pm 5}{4}$

$\cos x = \frac{7+5}{4}, \frac{7-5}{4}$

$\cos x = 3 \text{ (or) } \frac{1}{2}$

$\cos x = 3$ is not possible

$\therefore \quad \cos x = \frac{1}{2} = \cos\frac{\pi}{3}$

$\Rightarrow \quad x = \pi/3$

The general soln is $x = 2n\pi \pm \pi/3, n \in Z$

Ex 3.9

Question 1.

In a ΔABC, if $\frac{\sin A}{\sin C} = \frac{\sin(A-B)}{\sin(B-C)}$, prove that a^2, b^2, c^2 are in Arithmetic Progression.

Solution:

$$\text{LHS} = \frac{\sin A}{\sin C} = \frac{\frac{a}{2R}}{\frac{c}{2R}} = \frac{a}{c}$$

$$\text{RHS} = \frac{\sin(A-B)}{\sin(B-C)} = \frac{\sin A\cos B - \cos A\sin B}{\sin B\cos C - \cos B\sin C}$$

$$\text{Nr} = \frac{a}{2R}\left[\frac{a^2+c^2-b^2}{2ac}\right] - \frac{b}{2R}\left[\frac{b^2+c^2-a^2}{2bc}\right]$$

$$= \frac{1}{2R}\left\{\frac{a^2+c^2-b^2}{2c} - \frac{b^2+c^2-a^2}{2c}\right\}$$

$$= \frac{1}{2R}\left[\frac{a^2+c^2-b^2-b^2-c^2+a^2}{2c}\right]$$

$$= \frac{1}{2R}\left[\frac{2a^2-2b^2}{2ab}\right] = \frac{1}{2R}\left[\frac{a^2-b^2}{c}\right]$$

$$\text{Dr} = \frac{b}{2R}\left[\frac{a^2+b^2-c^2}{2ab}\right] - \frac{c}{2R}\left[\frac{a^2+c^2-b^2}{2ac}\right]$$

$$= \frac{1}{2R}\left[\frac{a^2+b^2-c^2}{2a} - \frac{a^2+c^2-b^2}{2a}\right]$$

$$= \frac{1}{2R}\left[\frac{a^2+b^2-c^2-a^2-c^2+b^2}{2a}\right] = \frac{1}{2R}\left(\frac{2b^2-2c^2}{2a}\right)$$

$$= \frac{1}{2R}\left(\frac{b^2 - c^2}{a}\right)$$

$$\therefore \qquad \text{RHS} = \frac{\text{Nr}}{\text{Dr}} = \frac{1}{2R}\left(\frac{a^2 - b^2}{c}\right) \Big/ \frac{1}{2R}\left(\frac{b^2 - c^2}{a}\right)$$

$$= \frac{a}{c}\left(\frac{a^2 - b^2}{b^2 - c^2}\right)$$

Given LHS = RHS $\Rightarrow \frac{a}{c} = \frac{a}{c}\left(\frac{a^2 - b^2}{b^2 - c^2}\right) \Rightarrow \frac{a^2 - b^2}{b^2 - c^2} = 1$

$$a^2 - b^2 = b^2 - c^2 \Rightarrow a^2 + c^2 = 2b^2$$

$\Rightarrow$ a^2, b^2, c^2 are in A.P.

Question 2.

The angles of a triangle ABC, are in Arithmetic Progression and if b : c = √3 : √2, find ∠A.

Solution:

Given $\frac{b}{c} = \frac{\sqrt{3}}{\sqrt{2}}$ and ∠A, ∠B and ∠C are in A.P.

$$\therefore \qquad 2\angle B = \angle A + \angle C$$

$$\Rightarrow \qquad 3\angle B = \angle A + \angle B + \angle C$$

Now from sine formula, we have

$$\frac{b}{\sin B} = \frac{c}{\sin C}$$

$$\Rightarrow \qquad \frac{b}{c} = \frac{\sin B}{\sin C} \Rightarrow \frac{\sqrt{3}}{\sqrt{2}} = \frac{\sin 60^\circ}{\sin C}$$

$$\Rightarrow \qquad \frac{\sqrt{3}}{\sqrt{2}} = \frac{\frac{\sqrt{3}}{2}}{\sin C}$$

$\Rightarrow \quad \sin C = \frac{1}{\sqrt{2}}$ i.e., $\angle C = 45°$

$\therefore \angle A = 180° - (\angle B + \angle C)$

$\angle A = 180° - (60° + 45°) = 75°$.

Question 3.

In a ΔABC, if $\cos C = \frac{\sin A}{2\sin B}$, show that the triangle is isosceles.

Solution:

Let $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} = k$ (say)

$\Rightarrow$ $\sin A = ak$, $\sin B = bk$ and $\sin C = ck$ we are given $\cos C = \frac{\sin A}{2\sin B}$

$$\Rightarrow \quad \frac{a^2+b^2-c^2}{2ab} = \frac{ak}{2bk} = \frac{a}{2b}$$

$$\Rightarrow \quad \frac{a^2+b^2-c^2}{2a} = \frac{a}{2}$$

$$\Rightarrow \quad \frac{a^2+b^2-c^2}{a} = \frac{a}{1}$$

$\Rightarrow a^2 + b^2 - c^2 = a^2 \Rightarrow b^2 - c^2 = a^2 - a^2$

$\Rightarrow b^2 - c^2 = 0 \Rightarrow b = c$

$\therefore$ Δ ABC is isosceles

Question 4.

In a ΔABC, prove that $\frac{\sin B}{\sin C} = \frac{c - a\cos B}{b - a\cos C}$.

Solution:

$$\text{RHS} = \frac{c - a\cos B}{b - a\cos C}$$

$$\text{Nr. } c - a\cos B = c - a\left(\frac{a^2 + c^2 - b^2}{2ac}\right)$$

$$= c - \left(\frac{a^2 + c^2 - b^2}{2c}\right) = \frac{2c^2 - a^2 - c^2 + b^2}{2c}$$

$$= \frac{b^2 + c^2 - a^2}{2c} = \frac{b^2 + c^2 - a^2}{2c} \times \frac{b}{b}$$

$$= b\left\{\frac{b^2 + c^2 - a^2}{2bc}\right\} = b\cos A$$

$$\text{Dr. } b - a\cos C = b - a\left[\frac{a^2 + b^2 - c^2}{2ab}\right]$$

$$= b - \left(\frac{a^2 + b^2 - c^2}{2b}\right) = \frac{2b^2 - a^2 - b^2 + c^2}{2b}$$

$$= \frac{b^2 + c^2 - a^2}{2b} = \frac{b^2 + c^2 - a^2}{2b} \times \frac{c}{c}$$

$$= c\left[\frac{b^2 + c^2 - a^2}{2bc}\right] = c\cos A$$

$$\therefore \text{RHS} = \frac{b\cos A}{c\cos A} = \frac{b}{c}$$

$$= \frac{2R\sin B}{2R\sin C} = \frac{\sin B}{\sin C} = \text{LHS}$$

Question 5.
In an ΔABC, prove that a cos A + b cos B + c cos C = 2a sin B sin C.

Solution:
We have A + B + C = 180°

$2A + 2B + 2C = 360°$

Also $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$

$\frac{a}{\sin A} = 2R \Rightarrow a = 2R \sin A$

$\frac{b}{\sin B} = 2R \Rightarrow b = 2R \sin B$

$\frac{c}{\sin C} = 2R \Rightarrow c = 2R \sin C$

a cos A + b cos B + c cos C = 2R sin A cos A + 2R sin B cos B + 2R sin C cos C
= R sin 2A + R sin 2B + R sin 2C
= R (sin 2A + sin 2B + sin 2C)
= R[sin (360° – (2B + 2C)) + sin 2B + sin 2C]
= R [- sin(2B + 2C) + sin 2B + sin 2C]
= R [- (sin 2B cos 2C + cos 2B sin 2C) + sin 2B + sin 2C]
= R [- sin 2B cos 2C – cos 2B sin 2C + sin 2B + sin 2C]
= R[sin 2B(1 – cos 2C) + sin 2C(1 – cos 2B)]
= R [sin 2B 2 $\sin^2$C + sin 2C . 2 $\sin^2$B]
= 2R [2 sin B cos B sin 2C + 2 sin C cos C sin ^{2}B]
= 2R. 2 sin B sin C [sin C cos B + cos C sin B]
= 4R sinB sinC [sin (B + C)]
= 4R sinB sinC [sin (180° – A)]
= 4R sin B sin C sin A
= 2 (2R sin A) sin B sin C
= 2a sin B sin C

Question 6.

In a ΔABC, $\angle A = 60°$. Prove that $b + c = 2a \cos\left(\frac{B-C}{2}\right)$.

Solution:

$$\text{LHS} = b + c = 2R\sin B + 2R\sin C$$

$$= 2R[\sin B + \sin C] = 2R\left[2\sin\frac{B+C}{2}\cos\frac{B-C}{2}\right]$$

$$= 4R\cos\frac{A}{2}\cos\frac{B-C}{2} = 4R\frac{\sqrt{3}}{2}\cos\frac{B-C}{2} \quad ...(1)$$

$$\text{RHS} = 2a\cos\left(\frac{B-C}{2}\right)$$

$$= 2\times 2R\sin A\cos\frac{B-C}{2}$$

$$= 4R\sin 60°\cos\frac{B-C}{2}$$

$$= 4R\left(\frac{\sqrt{3}}{2}\right)\cos\frac{B-C}{2} \quad ...(2)$$

$(1) = (2) \Rightarrow \text{LHS} = \text{RHS}$

Question 7.
In an ΔABC, prove the following.

(i) $a\sin\left(\frac{A}{2}+B\right) = (b+c)\sin\frac{A}{2}$ (ii) $a(\cos B + \cos C) = 2(b+c)\sin^2\frac{A}{2}$

(iii) $\frac{a^2-c^2}{b^2} = \frac{\sin(A-C)}{\sin(A+C)}$ (iv) $\frac{a\sin(B-C)}{b^2-c^2} = \frac{b\sin(C-A)}{c^2-a^2} = \frac{c\sin(A-B)}{a^2-b^2}$

(v) $\frac{a+b}{a-b} = \tan\left(\frac{A+B}{2}\right)\cot\left(\frac{A-B}{2}\right)$

Solution:

(i) $$\frac{a}{b+c} = \frac{2R\sin A}{2R\sin B + 2R\sin C}$$

$$= \frac{2R(\sin A)}{2R(\sin B + \sin C)} = \frac{\sin A}{\sin B + \sin C}$$

$$= \frac{2\sin A/2 \cos A/2}{2\sin\frac{C+B}{2}\cos\frac{C-B}{2}} = \frac{\sin A/2 \cos A/2}{\cos A/2 \cos\frac{C-B}{2}}$$

$$= \frac{\sin A/2}{\sin 90° - \left(\frac{C-B}{2}\right)} = \frac{\sin A/2}{\sin\frac{A+B+C}{2} - \frac{(C-B)}{2}}$$

$$= \frac{\sin A/2}{\sin\left(\frac{A}{2}+\frac{B}{2}+\frac{C}{2}-\frac{C}{2}+\frac{B}{2}\right)} = \frac{\sin A/2}{\sin\left(\frac{A}{2}+B\right)}$$

(i.e) $$\frac{a}{b+c} = \frac{\sin A/2}{\sin\left(\frac{A}{2}+B\right)}$$

$$\therefore a \sin\left(\frac{A}{2}+B\right) = (b+c)\sin\frac{A}{2}$$

(*ii*) $$\frac{a}{b+c} = \frac{2R\sin A}{2R\sin B + 2R\sin C} = \frac{\sin A/2}{\cos\frac{B-C}{2}} \text{ from } (i) \quad \dots (1)$$

Now $$\frac{2\sin^2 A/2}{\cos B + \cos C} = \frac{2\sin^2 A/2}{2\cos\frac{B+C}{2}\cos\frac{B-C}{2}}$$

$$= \frac{2\sin^2\frac{A}{2}}{2\sin\frac{A}{2}\cos\frac{B-C}{2}} = \frac{\sin\frac{A}{2}}{\cos\frac{B-C}{2}} \quad \dots(2)$$

$$(1) = (2) \Rightarrow \frac{a}{b+c} = \frac{2\sin^2 \frac{A}{2}}{\cos B + \cos C}$$

$$\Rightarrow a(\cos B + \cos C) = 2(b+c)\sin^2 \frac{A}{2}$$

(*iii*) Using sine formula

$$\text{LHS} = \frac{a^2 - c^2}{b^2} = \frac{k^2 \sin^2 A - k^2 \sin^2 C}{k^2 \sin^2 B}$$

$$= \frac{\sin^2 A - \sin^2 C}{\sin^2 B} = \frac{\sin(A+C).\sin(A-C)}{\sin^2 B}$$

$$= \frac{\sin B \sin(A-C)}{\sin^2 B} = \frac{\sin(A-C)}{\sin B}$$

$$= \sin(A-C)/\sin(180° - A + C)$$

$$= \frac{\sin(A-C)}{\sin(A+C)} = \text{RHS}$$

(*iv*)

$$a \sin(B - C) = a[\sin B \cos C - \cos B \sin C]$$

$$= a\left[\frac{b}{2R}\left(\frac{a^2+b^2-c^2}{2ab}\right) - \frac{a^2+c^2-b^2}{2ac}\left(\frac{c}{2R}\right)\right]$$

$$= \frac{1}{2R}\left(\frac{a^2+b^2-c^2}{2} - \frac{a^2+c^2-b^2}{2}\right)$$

$$= \frac{1}{4R}(a^2+b^2-c^2-a^2-c^2+b^2)$$

$$= \frac{2(b^2-c^2)}{4R}$$

$$\therefore \quad \frac{a\sin(B-C)}{b^2-c^2} = \frac{2(b^2-c^2)}{4R} \Big/ b^2 - c^2 = \frac{2}{4R} = \frac{1}{2R} \qquad ...(1)$$

$$b(\sin C - A) = b[\sin C \cos A - \cos C \sin A)$$

$$= b\left[\frac{c}{2R}\left(\frac{b^2+c^2-a^2}{2bc}\right) - \frac{a^2+b^2-c^2}{2ab} \times \frac{a}{2R}\right]$$

$$= \frac{1}{4R}[b^2 + c^2 - a^2 - a^2 - b^2 + c^2]$$

$$= \frac{1}{4R}\left(2(c^2 - a^2)\right) = \frac{c^2 - a^2}{2R}$$

$$\therefore \quad \frac{b\sin(C-A)}{c^2 - a^2} = \frac{c^2 - a^2}{2R} \Big/ c^2 - a^2 = \frac{1}{2R} \qquad \text{...(2)}$$

$$\text{Similarly } \frac{\sin(A \quad B)}{a^2 \quad b^2} = \frac{1}{2R}$$

$$\therefore \frac{a\sin(B-C)}{b^2 - c^2} = \frac{b\sin(C-A)}{c^2 - a^2} = \frac{c\sin(A-B)}{a^2 - b^2} = \frac{1}{2R}$$

(*v*)

$$\text{LHS} = \frac{a+b}{a-b} = \frac{2R\sin A + 2R\sin B}{2R\sin A - 2R\sin B}$$

$$= \frac{2R(\sin A + \sin B)}{2R(\sin A - \sin B)} = \frac{2\sin\frac{A+B}{2}\cos\frac{A-B}{2}}{2\cos\frac{A+B}{2}\sin\frac{A-B}{2}}$$

$$= \tan\left(\frac{A+B}{2}\right)\cot\left(\frac{A-B}{2}\right) = \text{RHS}$$

Question 8.

In a ΔABC, prove that $(a^2 - b^2 + c^2) \tan B = (a^2 + b^2 - c^2) \tan C$

Solution:

$$\left[\frac{a^2+c^2-b^2}{2ac}\right] = \cos B$$

$$\text{So, } a^2+c^2-b^2 = 2ac\cos B$$

$$\therefore \text{ LHS } (a^2+c^2-b^2)\tan B = 2ac\cos B \times \tan B$$

$$= 2ac\cos B \times \frac{\sin B}{\cos B}$$

$$= 2ac\sin B = 2ac\times\frac{b}{2R}$$

$$= \frac{2abc}{2R} = \frac{abc}{R} \qquad \ldots(1)$$

$$\frac{a^2+b^2-c^2}{2ab} = \cos C$$

$$\therefore\ a^2+b^2-c^2 = 2ab\cos C$$

$$\text{RHS} = (a^2-c^2+b^2)\tan C = 2ab\cos C\tan C$$

$$= 2ab\cos C\frac{\sin C}{\cos C} = 2ab\sin C$$

$$= 2ab\times\frac{c}{2R} = \frac{abc}{R} \qquad \ldots(2)$$

$(1) = (2) \Rightarrow \text{LHS} = \text{RHS}$

Question 9.

An Engineer has to develop a triangular-shaped park with a perimeter of 120 m in a village. The park to be developed must be of maximum area. Find out the dimensions of the park.

Solution:

Given, the perimeter of a triangular-shaped park = 120 m

Perimeter of triangle = $a + b + c = 120$ m and we know, $\frac{a+b+c}{2} = s$

$\therefore 2s = 120 \text{ m} \Rightarrow s = 60$ m

Maximum area of triangular shaped park $\leq \frac{s^3}{27}$

$\therefore (s-a)(s-b)(s-c) \leq \frac{s^3}{27}$

When area is maximum, then $a = b = c$

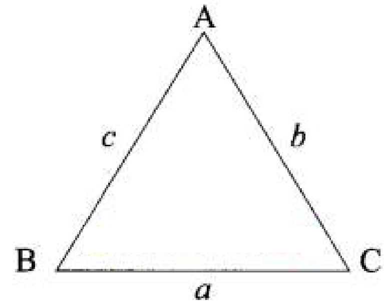
$\therefore (s-a)^3 \leq \frac{s^3}{27}$

$\therefore (s-a)^3 \leq \left(\frac{s}{3}\right)^3$

$\therefore s - a \leq \frac{s}{3}$

$60 - a \leq \frac{60}{3}$ (substitute $s = 600$)

$60 - a \leq 20 \quad \therefore 40 \leq a$



All sides of a triangular part would be 40 m.
i.e., a = 40 m,
b = 40 m,
c = 40 m.

Question 10.
A rope of length 12 m is given. Find the largest area of the triangle formed by this rope and find the dimensions of the triangle so formed.
Solution:
The largest triangle will be an equilateral triangle

$\therefore$ Side of the triangle $= \frac{12}{3} = 4 \text{ m} = a$

Area of the triangle $= \frac{a^2\sqrt{3}}{4} = \frac{4^2\sqrt{3}}{4} = 4\sqrt{3}$ sq.m

Question 11.
Derive Projection formula from
(i) Law of sines,
(ii) Law of cosines.

Solution:
(i) To Prove a = b cos c + c cos B

Using sine formula
RHS = b cos C + c cos B
= 2R sin B cos C + 2R sin C cos B
= 2R [sin B cos C + cos B sin C]
= 2R sin (B + C) = 2R [sin π – A)
= 2R sin A = a = LHS

(ii) To prove a = b cos c + c cos B
Using cosine formula

$$\text{RHS} = b\cos C + c\cos B$$
$$= b\left[\frac{a^2+b^2-c^2}{2ab}\right]+c\left[\frac{a^2+c^2-b^2}{2ac}\right]$$
$$= \frac{a^2+b^2-c^2}{2a}+\frac{a^2+c^2-b^2}{2a}$$
$$= \frac{a^2+b^2-c^2+a^2+c^2-b^2}{2a}=\frac{2a^2}{2a}$$
$$= a = \text{LHS}$$

Ex 3.10

Question 1.
Determine whether the following measurements produce one triangle, two triangles or no triangle:
∠B = 88°, a = 23, b = 2. Solve if solution exists.

Solution:
We are given a = 23,
b = 2, and
∠B = 88°.
So we can

$$\text{(i.e.)} \quad \frac{a}{\sin A} = \frac{b}{\sin B}$$

$$\Rightarrow \quad \frac{23}{\sin A} = \frac{2}{\sin 88^\circ} = \frac{2}{(1)} \qquad [\because \sin 90^\circ = \sin 88^\circ \text{ is near } x]$$

$$\Rightarrow \quad \sin A = \frac{23}{2} \text{ which is not possible}$$

Question 2.
If the sides of a ΔABC are a = 4, b = 6 and c = 8, then show that 4 cos B + 3 cos C = 2.

Solution:
a = 4,
b = 6,
c = 8
To prove 4 cos B + 3 cos C = 2

$$\cos B = \frac{a^2+c^2-b^2}{2ac} = \frac{16+64-36}{2(4)(8)} = \frac{80-36}{64} = \frac{44}{64} = \frac{11}{16}$$

$$\cos C = \frac{a^2+b^2-c^2}{2ab} = \frac{16+36-64}{2(4)(6)}$$

$$= \frac{-12}{48} = -\frac{1}{4}$$

$$\text{So LHS} = 4 \cos B + 3 \cos C$$

$$= 4\left(\frac{11}{16}\right) + 3\left(-\frac{1}{4}\right)$$

$$= \frac{11}{4} - \frac{3}{4} = \frac{8}{4} = 2 = \text{RHS}$$

Question 3.
In a ΔABC, if a = √3 – 1, b = √3 + 1 and C = 60°, find the other side and other two angles.

Solution:

$$\cos C = \frac{a^2+b^2-c^2}{2ab} \Rightarrow \cos 60° = \frac{1}{2} = \frac{(3+1-2\sqrt{3})+(3+1+2\sqrt{3})-c^2}{2(\sqrt{3}-1)(\sqrt{3}+1)}$$

$$\Rightarrow \quad \frac{8-c^2}{2(3-1)} = \frac{1}{2} \Rightarrow \frac{8-c^2}{4} = \frac{1}{2}$$

$$4 = 16 - 2c^2$$

$$2c^2 = 16 - 4 = 12$$

$$c^2 = \frac{12}{2} = 6 ; \quad c = \sqrt{6}$$

Now $\dfrac{a}{\sin A} = \dfrac{c}{\sin C}$

(i.e.) $\dfrac{\sqrt{3}-1}{\sin A} = \dfrac{\sqrt{6}}{\sin 60°}$

$$\frac{\sqrt{3}-1}{\sin A} = \frac{\sqrt{3}\sqrt{2}}{\sqrt{3}/2} = 2\sqrt{2}$$

$$\Rightarrow \quad \sin A = \frac{\sqrt{3}-1}{2\sqrt{2}} \quad \Rightarrow A = 15°$$

Now $A = 15°$; $C = 60°$

$\therefore$ $B = 180° - (15 + 60°) = 105°$

Question 4.

In any ΔABC, prove that the area $\Delta = \dfrac{b^2+c^2-a^2}{4\cot A}$.

Solution:

$$\text{RHS} = \frac{b^2+c^2-a^2}{4\cot A}$$

$$\frac{b^2+c^2-a^2}{2bc} = \cos A$$

$$\therefore \quad b^2 + c^2 - a^2 = 2bc \cos A$$

$$\text{Now } \frac{b^2+c^2-a^2}{4\cot A} = \frac{2bc\cos A}{4\frac{\cos A}{\sin A}} = \frac{1}{2}bc\sin A$$

$$= \text{area of } \Delta = \text{LHS}$$

Question 5.
In a ΔABC, if a = 12 cm, b = 8 cm and C = 30°, then show that its area is 24 sq.cm.

Solution:
a = 12 cm,
b = 8 cm,
C = 30°

$$\text{Area of } \Delta ABC = \frac{1}{2}ab\sin C$$

$$= \frac{1}{2} \times 12 \times 8 \times \sin 30°$$

$$= \frac{1}{2} \times 12 \times 8 \times \frac{1}{2}$$

$$= \frac{96}{4} = 24 \text{ sq. cm}$$

Question 6.
In a ΔABC, if a = 18 cm, b = 24 cm and c = 30 cm, then show that its area is 216 sq.cm.

Solution:
a = 18 cm,
b = 24 cm,
c = 30 cm
The sides form a right-angled triangle

$\therefore$ Area $= \frac{1}{2} \times a \times b$

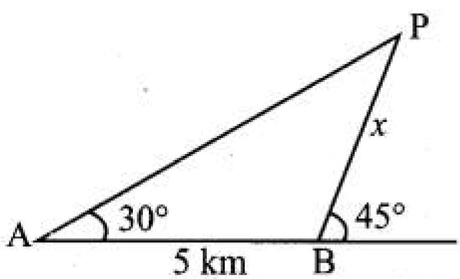
$= \frac{1}{2} \times 18 \times 24$

$= 216$ sq. cm

Question 7.
Two soldiers A and B in two different underground bunkers on a straight road, spot an intruder at the top of a hill. The angle of elevation of the intruder from A and B to the ground level in the eastern direction are 30° and 45° respectively. If A and B stand 5 km apart, find the distance of the intruder from B.

Solution:
By using sine formula

$$\frac{x}{\sin 30^\circ} = \frac{5}{\sin 15^\circ}$$



(i.e.) $\frac{x}{1/2} = \frac{5}{\sin 15^\circ}$

$x \sin 15^\circ = \frac{5}{2}$ ...(1)

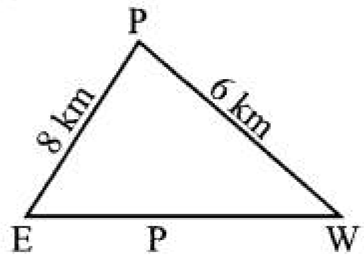
$\sin 15^\circ = \sin(45^\circ - 30^\circ) = \sin 45^\circ \cos 30^\circ - \cos 45^\circ \sin 30^\circ$

$= \frac{1}{\sqrt{2}} \frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}} \frac{1}{2} = \frac{\sqrt{3}}{2\sqrt{2}} - \frac{1}{2\sqrt{2}} = \frac{\sqrt{3}-1}{2\sqrt{2}}$

Substituting sin 15° value in (1) we get $x\left[\frac{\sqrt{3}-1}{2\sqrt{2}}\right] = \frac{5}{2} \Rightarrow x = \frac{\frac{5}{2} \times 2\sqrt{2}}{\sqrt{3}-1} = \frac{5\sqrt{2}}{\sqrt{3}-1}$ km

Question 8.
A researcher wants to determine the width of a pond from east to west, which cannot be done by actual measurement. From a point P, he finds the distance to the eastern-most point of the pond to be 8 km, while the distance to the western most point from P to be 6 km. If the angle between the two lines of sight is 60°, find the width of the pond.

Solution:
$p^2 = W^2 + E^2 - 2WE \cos P$
$P^2 = 64 + 36 - 2 \times 8 \times 6 \times \cos 60°$



$$= 100 - 96 \times \frac{1}{2} = 100 - 48 = 52$$

$$\Rightarrow P = \sqrt{52} = \sqrt{4 \times 13} = 2\sqrt{13} \text{ km}$$

Question 9.
Two Navy helicopters A and B are flying over the Bay of Bengal at same altitude from the sea level to search a missing boat. Pilots of both the helicopters sight the boat at the same time while they are apart 10 km from each other. If the distance of the boat from A is 6 km and if the line segment AB subtends 60° at the boat, find the distance of the boat from B.

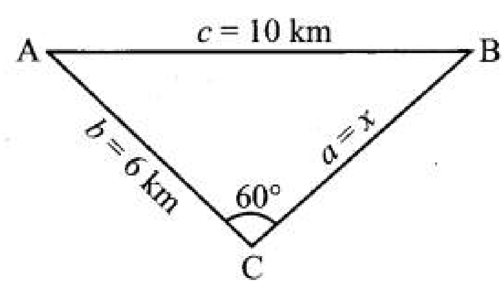
Solution:

Here $a = x$, $b = 6$, $c = 10$
By using cosine formula

$$c^2 = a^2 + b^2 - 2ab \cos 60°$$

(i.e.) $100 = x^2 + 36 - 2 \times x \times 6 \times \frac{1}{2}$

(i.e.) $x^2 - 6x - 64 = 0$



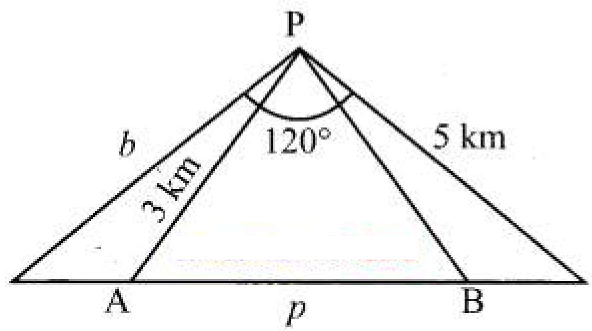
$$x = \frac{6 \pm \sqrt{36+256}}{2}$$

(i.e.) $$x = \frac{6 \pm \sqrt{292}}{2} = \frac{2(3 \pm \sqrt{73})}{2}$$

$\Rightarrow$ $x = 3 + \sqrt{73}$ km (as $3 - \sqrt{73}$ is not possible)

Question 10.
A straight tunnel is to be made through a mountain. A surveyor observes the two extremities A and B of the tunnel to be built from a point P in front of the mountain. If AP = 3 km, BP = 5 km and ∠APB = 120°, then find the length of the tunnel to be built.

Solution:



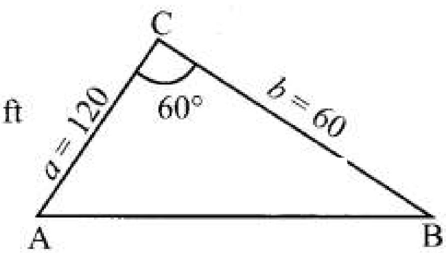
A, B are the two extremities of the tunnel.
P – Point of observation.
PA , PB are the directions of the points A, B as observed from the point P
AP = 3 km, BP = 5 km, ∠ APB = 120°
Using cosine formula in Δ APB
$AB^2 = AP^2 + BP^2 - 2AP.\ BP.\ \cos(120°)$
$AB^2 = 3^2 + 5^2 - 2 \times 3 \times 5 \cos(180° - 60°)$
$= 9 + 25 - 30(-\cos 60°)$
$= 34 + 30 \times 1/2$
$= 34 + 15 = 49$
$AB = \sqrt{49} = 7$
∴ The length of the tunnel AB = 7 k.m.

Question 11.
A farmer wants to purchase a triangular-shaped land with sides 120 feet and 60 feet and the angle included between these two sides is 60°. If the land costs ₹ 500 per sq. ft, find the amount he needed to purchase the land. Also, find the

perimeter of the land.

Solution:

$$\text{Area} = \frac{1}{2}ab\sin C$$
$$= \frac{1}{2}\times 120\times 60\times \sin 60^\circ$$
$$= 3600\times\frac{\sqrt{3}}{2} = 1800\times 1.732 = 3117.6 \text{ sq. ft}$$



Cost of the land $= 3117.6 \times 500 =$ ₹ 155880

Length of the third side c :

$$c^2 = a^2 + b^2 - 2ab\cos C$$
$$= 120^2 + 60^2 - 2(120^\circ)(60^\circ)\cos 60^\circ$$
$$= 14400 + 3600 - 2\times 120\times 60\times\frac{1}{2}$$
$$= 18000 - 7200 = 10800$$
$$c = \sqrt{10800} = 10\sqrt{108}$$
$$= 10\times 6\sqrt{3} = 60\sqrt{3}$$

Perimeter $= a + b + c = 120 + 60 + 60\sqrt{3} = 180 + 20\sqrt{27}$ feet

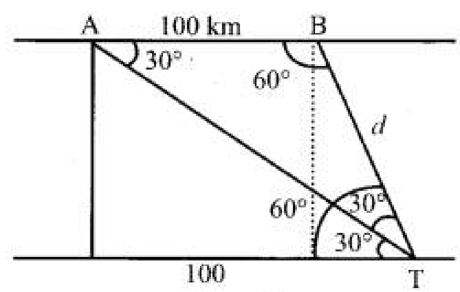
Question 12.

A fighter jet has to hit a small target by flying a horizontal distance. When the target is sighted, the pilot measures the angle of depression to be 30°. If after 100 km, the target has an angle of depression of 60°, how far is the target from the fighter jet at that instant?

Solution:

Using sin formula, $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$

$$\Rightarrow \frac{100}{\sin 30^\circ} = \frac{d}{\sin 30^\circ} \Rightarrow d = 100 \text{ km}$$



II method: In ΔABT

$\angle A = \angle ATB = 30^\circ$ (an isosceles triangle)

$\therefore AB = BT$

$\Rightarrow d = 100$ km

Question 13.
A plane is 1 km from one landmark and 2 km from another. From the plane's point of view, the land between them subtends an angle of 60°. How far apart are the landmarks?

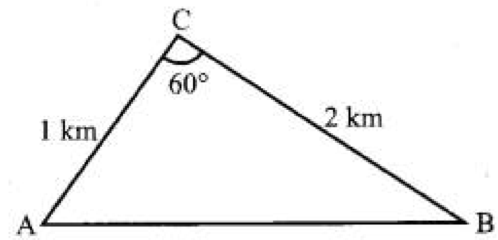
Solution:

$c^2 = a^2 + b^2 - 2ab \cos C$

$c^2 = 1 + 4 - 2(1)(2) \cos 45°$

$= 5 - 4 \times \frac{1}{\sqrt{2}} = 5 - \frac{4}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} = 5 - 2\sqrt{2}$

$\therefore c = \sqrt{5 - 2\sqrt{2}}$ km



Question 14.
A man starts his morning walk at a point A reaches two points B and C and finally back to A such that ∠A= 60° and ∠B = 45°, AC = 4 km in the ΔABC. Find the total distance he covered during his morning walk.

Solution:

$\frac{a}{\sin A} = \frac{b}{\sin B}$

(i.e.) $\frac{a}{\sin 60°} = \frac{4}{\sin 45°}$

$\frac{a}{\sqrt{3}/2} = \frac{4}{1/\sqrt{2}}$

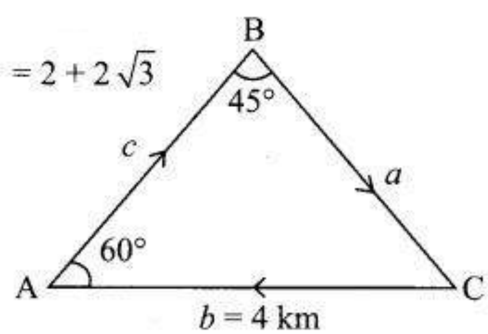
$a = 4\sqrt{2} \times \frac{\sqrt{3}}{2} = 2\sqrt{6}$

$c = b \cos A + a \cos B$

$c = 4 \cos 60° + 2\sqrt{6} \cos 45°$

$= 4 \times \frac{1}{2} + 2\sqrt{6} \frac{1}{\sqrt{2}}$

$c = 2 + 2\sqrt{3}$



Total distance covered = $a + b + c$

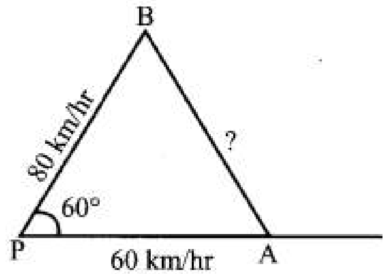
$= 2\sqrt{6} + 4 + 2 + 2\sqrt{3}$

$= 6 + 2\sqrt{6} + 2\sqrt{3}$ km

Question 15.
Two vehicles leave the same place P at the same time moving along two different roads. One vehicle moves at an average speed of 60 km/hr and the

other vehicle moves at an average speed of 80 km/hr. After half an hour the vehicle reaches destinations A and B. If AB subtends 60° at the initial point P, then find AB.

Solution:



Speed of vehicle A = 60 km / hr

Distance (AP) travelled in half an hour $= \frac{60}{2} = 30$ km

Speed of vehicle B = 80 km / hr

Distance (PB) travelled in half an hour

$$= \frac{80}{2} = 40 \text{ km}$$

$$AB^2 = AP^2 + PB^2 - 2\,(AP)\,(PB)\cos 60°$$

$$= 30^2 + 40^2 - 2\,(30)\,(40)\left(\frac{1}{2}\right)$$

= 900+ 1600 – 1200 = 1300

$$\therefore AB = \sqrt{1300} = 10\sqrt{13} \text{ km}$$

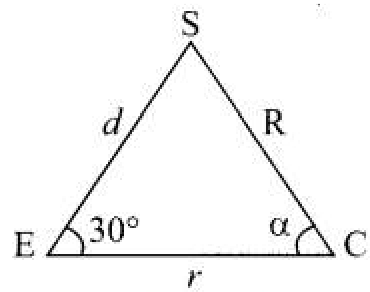
Distance between two destinations after one hour $= 10\sqrt{13} + 10\sqrt{13} = 20\sqrt{13}$ km

Question 16.

Suppose that a satellite in space, an earth station, and the centre of earth all lie in the same plane. Let r be the radius of earth and R be the distance from the centre of the earth to the satellite. Let d be the distance from the earth station to the satellite. Let 30° be the angle of elevation from the earth station to the satellite. If the line segment connecting earth station and satellite subtends angle α at the centre of the earth, then prove that

$$d = R\sqrt{1+\left(\frac{r}{R}\right)^2 - 2\frac{r}{R}\cos\alpha}\,.$$

Solution:



$$d^2 = r^2 + R^2 - 2Rr\cos\alpha$$

$$= R^2\left(\frac{r^2}{R^2} + 1 - \frac{2r}{R}\cos\alpha\right)$$

$$d = R\sqrt{\frac{r^2}{R^2} + 1 - \frac{2r}{R}\cos\alpha}$$

Ex 3.11

Question 1.

Find the principal value of (*i*) $\sin^{-1}\frac{1}{\sqrt{2}}$ (*ii*) $\cos^{-1}\frac{\sqrt{3}}{2}$ (*iii*) $\text{cosec}^{-1}(-1)$ (*iv*) $\sec^{-1}(-\sqrt{2})$ (*v*) $\tan^{-1}(\sqrt{3})$

Solution:

(*i*) $\sin^{-1}\frac{1}{\sqrt{2}} = \theta$

So $\sin\theta = \frac{1}{\sqrt{2}} \Rightarrow \theta = \frac{\pi}{4}$

(*ii*) $\cos^{-1}\frac{\sqrt{3}}{2} = \theta \Rightarrow \cos\theta = \frac{\sqrt{3}}{2} = \frac{\pi}{6}$

(*iii*) $\text{cosec}^{-1}(-1) = \theta \Rightarrow \text{cosec}\,\theta\ -1$

(i.e.) $\frac{1}{\sin\theta} = -1 \Rightarrow \sin\theta = -1 = -\frac{\pi}{2}$

(iv) $\sec^{-1}(-\sqrt{2})$ $= \theta \Rightarrow \sec\theta = -\sqrt{2}$

$$\Rightarrow \cos\theta = -\frac{1}{\sqrt{2}}$$

$$\theta = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$$

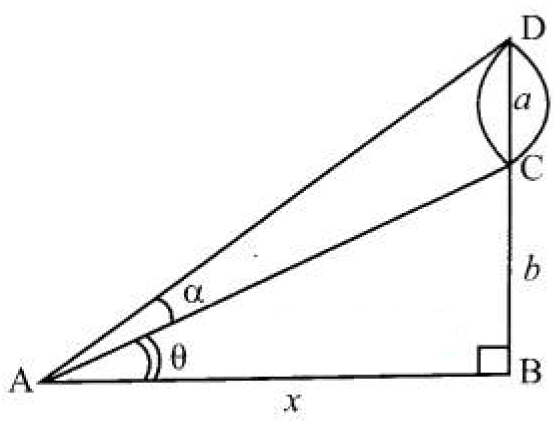
(v) $\tan^{-1}(\sqrt{3})$ $= \theta \Rightarrow \tan\theta = \sqrt{3} \Rightarrow \theta = \frac{\pi}{3}$

Question 2.

A man standing directly opposite to one side of a road of width x meter views a circular shaped traffic green signal of diameter a meter on the other side of the road. The bottom of the green signal is b meter height from the horizontal level of viewer's eye. If a denotes the angle subtended by the diameter of the green signal at the viewer's eye, then prove that

$$\alpha = \tan^{-1}\left(\frac{a+b}{x}\right) - \tan^{-1}\left(\frac{b}{x}\right)$$

Solution:



From the right angled triangle ABC

$$\tan\theta = \frac{b}{x} \Rightarrow \theta = \tan^{-1}\left(\frac{b}{x}\right) \quad ...(1)$$

From the right angled triangle ABD

$$\tan(\theta+\alpha) = \frac{a+b}{x} \Rightarrow \theta+\alpha = \tan^{-1}\frac{a+b}{x}$$

$$\text{So } \alpha = \tan^{-1}\left(\frac{a+b}{x}\right) - \theta$$

$$= \tan^{-1}\left(\frac{a+b}{x}\right) - \tan^{-1}\left(\frac{b}{x}\right)$$

$$\text{From (1) } \alpha = \tan^{-1}\left(\frac{a+b}{x}\right) - \tan^{-1}\left(\frac{b}{x}\right)$$

Ex 3.12

Question 1.

$$\frac{1}{\cos 80^\circ} - \frac{\sqrt{3}}{\sin 80^\circ} = \text{..................} .$$

(a) 2–√
(b) 3–√
(c) 2
(d) 4

Solution:
(d) 4
Hint:

$$\frac{1}{\cos 80^\circ} - \frac{\sqrt{3}}{\sin 80^\circ} = \frac{\frac{1}{2}}{\frac{1}{2}\times\cos 80^\circ} - \frac{\frac{\sqrt{3}}{2}}{\sin 80^\circ \times \frac{1}{2}}$$

$$= \frac{\frac{1}{2}\sin 80^\circ - \frac{\sqrt{3}}{2}\cos 80^\circ}{\frac{1}{2}\sin 80^\circ \cos 80^\circ} = \frac{\sin 80^\circ \cos 60^\circ - \cos 80^\circ \sin 60^\circ}{\frac{1}{2}\left(\frac{2\sin 80^\circ \cos 80^\circ}{2}\right)}$$

$$= \frac{\sin 20^\circ}{\frac{1}{4}[\sin 160^\circ]} = \frac{4\sin 20^\circ}{\sin(180^\circ - 20^\circ)} = \frac{4\sin 20^\circ}{\sin 20^\circ} = 4$$

Question 2.

If cos 28° + sin 28° = k^3, then cos 17° is equal to …….

(a) $\frac{k^3}{\sqrt{2}}$ (b) $-\frac{k^3}{\sqrt{2}}$ (c) $\pm\frac{k^3}{\sqrt{2}}$ (d) $-\frac{k^3}{\sqrt{3}}$

Solution:

(a) $\frac{k^3}{\sqrt{2}}$

Hint:

$$\cos 28° + \sin 28° = k^3$$

$$\text{(i.e.)} \quad \cos 28° + \sin (90° - 62°) = k^3$$

$$\Rightarrow \cos 28° + \cos 62° = k^3$$

$$2\cos\frac{90}{2}\cos\frac{34}{2} = k^3$$

$$2\cos 45° \cos 17° = k^3$$

$$2\times\frac{1}{\sqrt{2}}\cos 17° = k^3$$

$$\Rightarrow \cos 17° = k^3\times\frac{\sqrt{2}}{2} = \frac{k^3}{\sqrt{2}}$$

Question 3.

The maximum value of $4\sin^2 x + 3\cos^2 x + \sin\frac{x}{2} + \cos\frac{x}{2}$ is ………… .

(a) $4 + \sqrt{2}$
(b) $3 + \sqrt{2}$
(c) 9
(d) 4

Solution:

(a) $4 + \sqrt{2}$

Hint:

The max value of sin x is 1 at $x = \frac{\pi}{2}$ and the max value of cos x is 1 at $x = 0$

The value at $x = 0$ is 3 + 1 = 4

The value at $x = \frac{\pi}{2}$ is $4 + 0 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = 4 + \frac{2}{\sqrt{2}} = 4\sqrt{2}$

So the maximum value is $4 + \sqrt{2}$

Question 4.

$\left(1+\cos\frac{\pi}{8}\right)\left(1+\cos\frac{3\pi}{8}\right)\left(1+\cos\frac{5\pi}{8}\right)\left(1+\cos\frac{7\pi}{8}\right) =$ ....

Solution:

(a) 1/8

Hint:

$$\text{LHS} = \left(1+\cos\frac{\pi}{8}\right)\left(1+\cos\frac{3\pi}{8}\right)\left(1+\cos\pi-\frac{3\pi}{8}\right)\left(1+\cos\pi-\frac{\pi}{8}\right)$$

$$= \left[\left(1+\cos\frac{\pi}{8}\right)\left(1-\cos\frac{\pi}{8}\right)\left(1+\cos\frac{3\pi}{8}\right)\left(1-\cos\frac{3\pi}{8}\right)\right]$$

$$= \left(1-\cos^2\frac{\pi}{8}\right)\left(1-\cos^2\frac{3\pi}{8}\right) = \sin^2\frac{\pi}{8}\sin^2\frac{3\pi}{8}$$

$$= \left(\sin\frac{\pi}{8}\sin\frac{3\pi}{8}\right)^2 = \left\{\frac{1}{2}\left[2\sin\frac{\pi}{8}\sin\frac{3\pi}{8}\right]\right\}^2$$

$$= \frac{1}{4}[\cos(A-B) - \cos(A+B)]^2$$

$$= \frac{1}{4}\left[\cos\frac{\pi}{4} - \cos\frac{\pi}{2}\right]^2 = \frac{1}{4}\left(\frac{1}{\sqrt{2}}\right)^2 = \frac{1}{4}\times\frac{1}{2} = \frac{1}{8}$$

Question 5.

If $\pi < 2\theta < \frac{3\pi}{2}$, then $\sqrt{2+\sqrt{2+2\cos 4\theta}}$ equals to ....

Solution:
(c) 2 cos θ
Hint:

$$2 + 2\cos 4\theta = 2(1 + \cos 4\theta) = 2\,(2\cos^2 2\theta) = 4\cos^2 2\theta$$

$$\therefore \sqrt{4\cos^2 2\theta} = 2\cos 2\theta \qquad [1 + \cos\theta = 2\cos^2\theta]$$

$$\sqrt{2+2\cos 2\theta} = \sqrt{2(1+\cos 2\theta)} = \sqrt{2\times 2\cos^2\theta}$$

$$= \sqrt{4\cos^2\theta} = 2\cos\theta$$

Here θ is in II Quadrant

Question 6.

If $\tan 40° = \lambda$, then $\frac{\tan 140° - \tan 130°}{1+\tan 140° \tan 130°} = \ldots\ldots\ldots\ldots\ldots\ldots$.

(a) $\frac{1-\lambda^2}{\lambda}$ (b) $\frac{1+\lambda^2}{\lambda}$ (c) $\frac{1+\lambda^2}{2\lambda}$ (d) $\frac{1-\lambda^2}{2\lambda}$

Solution:
(d) $\frac{1-\lambda^2}{2\lambda}$
Hint:

Given $\tan 40° = \lambda$

$$\text{Now } \frac{\tan 140° - \tan 130°}{1+\tan 140° \tan 130°} = \tan(140° - 130°) = \tan 10°$$

$$= \frac{1}{\cot 10°} = \frac{1}{\cot(90° - 80°)} = \frac{1}{\tan 80°}$$

$$\text{Now } \tan 80° = \tan 2(40)$$

$$= \frac{2\tan 40°}{1-\tan^2 40°} = \frac{2\lambda}{1-\lambda^2}$$

$$\therefore \frac{1}{\tan 80°} = \frac{1-\lambda^2}{2\lambda}$$

Question 7.
cos 1° + cos 2° + cos 3° + …. + cos 179° = …….

(a) 0
(b) 1
(c) -1
(d) 89

Solution:
(a) 0
Hint:
LHS = (cos 10 + cos 179°)+(cos 2° ÷ cos 178°)+ ….. +cos(89° + cos 91°)+cos 90°
cos 179° = cos (180° – 1) = – cos 1°
cos 178° = cos(180° – 2)= – cos 2°
So (cos 1°- cos 1°)+(cos 2° – cos 2°) + (cos 89° – cos 89°) + cos 90°
= 0 + 0 …. + 0 + 0 = 0.

Question 8.

Let $f_k(x)=\frac{1}{k}[\sin^k x+\cos^k x]$ where $x \in$ R and $k \geq 1$. then $f_4(x)-f_6(x) =$ ....

(a) $\frac{1}{4}$ (b) $\frac{1}{12}$ (c) $\frac{1}{6}$ (d) $\frac{1}{3}$

Solution:
(b) 1/12
Hint:

$$f_k(x) = \frac{1}{k}[\sin^k x + \cos^k x] \Rightarrow f_4(x) = \frac{1}{4}(\sin^4 x + \cos^4 x)$$

$$f_4(x)-f_6(x) = \frac{1}{4}[\sin^4 x+\cos^4 x] - \frac{1}{6}[\sin^6 x+\cos^6 x]$$

$$= \frac{1}{4}[(\sin^2 x+\cos^2 x)-2\sin^2 x\cos^2 x]$$

$$-\frac{1}{6}[(\sin^2 x+\cos^2 x)^3-3\sin^2 x\cos^2 x]$$

$$= \frac{1}{4}(1-2\sin^2 x\cos^2 x)-\frac{1}{6}(1-3\sin^2 x\cos^2 x)$$

$$= \frac{1}{4}-\frac{1}{2}\sin^2 x\cos^2 x-\frac{1}{6}+\frac{1}{2}\sin^2 x\cos^2 x$$

$$= \frac{1}{4}-\frac{1}{6}=\frac{3-2}{12}=\frac{1}{12}$$

Question 9.
Which of the following is not true?
(a) $\sin\theta = -\frac{3}{4}$ (b) $\cos\theta = -1$ (c) $\tan\theta = 25$ (d) $\sec\theta = \frac{1}{4}$

Solution:
(d) sec θ = 1/4
Hint:
sec θ = 1/4 ⇒ cos θ = 4, which is not true.

Question 10.
cos 2θ cos 2ϕ+ $\sin^2$ (θ – ϕ) – $\sin^2$ (θ + ϕ) is equal to …….
(a) sin 2(θ + ϕ)
(b) cos 2(θ + ϕ)
(c) sin 2(θ – ϕ)
(d) cos 2(θ – ϕ)

Solution:
(b) cos 2(θ + ϕ)
Hint.
Given cos 2θ. cos 2ϕ + $\sin^2$ (θ – ϕ) – $\sin^2$ (θ + ϕ)
= cos 2θ cos 2ϕ + sin (θ – ϕ + θ + ϕ) sin (θ – ϕ – θ – ϕ)
= cos 2θ cos 2ϕ + sin 2θ sin(-2ϕ)
= cos 2θ cos 2ϕ – sin 2θ sin(2ϕ)
= cos (2θ + 2ϕ) = cos 2(θ + ϕ)

Question 11.
$\frac{\sin(A-B)}{\cos A\cos B}+\frac{\sin(B-C)}{\cos B\cos C}+\frac{\sin(C-A)}{\cos C\cos A}$ is ……………… .
(a) sin A + sin B + sin C
(b) 1
(c) 0

(d) cos A + cos B + cos C

Solution:
(c) 0
Hint:

$$\frac{\sin(A-B)}{\cos A\cos B} = \frac{\sin A\cos B - \cos A\sin B}{\cos A\cos B}$$

= tan A – tan B

Similarly $\frac{\sin(B-C)}{\cos B\cos C} = \tan B - \tan C$

and $\frac{\sin(C-A)}{\cos C\cos A} = \tan C - \tan A$ (adding we get 0)

Question 12.
cos pθ + cos qθ = 0 and if p ≠ q, then 0 is equal to (n is any integer) …….

(a) $\frac{\pi(3n+1)}{p-q}$ (b) $\frac{\pi(2n+1)}{p\pm q}$ (c) $\frac{\pi(n\pm 1)}{p\pm q}$ (d) $\frac{\pi(n+2)}{p+q}$

Solution:
(b) $\frac{\pi(2n+1)}{p\pm q}$

Hint:
cos pθ + cos qθ = 0

$$2\cos\left(\frac{p+q}{2}\right)\theta\cos\left(\frac{p-q}{2}\right)\theta = 0$$

as $2 \neq 0$, $\cos\left(\frac{p+q}{2}\right)\theta = 0$ (or) $\cos\left(\frac{p-q}{2}\right)\theta = 0$

$$\cos\left(\frac{p+q}{2}\right)\theta=0$$

$$\frac{p+q}{2}(\theta)=\left(2n+\frac{1}{2}\right)\pi/2$$

$$\theta = \frac{(2n+1)\pi}{p+q}$$

$$\cos\left(\frac{p-q}{2}\right)\theta=0$$

$$\frac{p-q}{2}\theta=(2n+1)\pi/2$$

$$\theta=\frac{(2n+1)\pi}{p-q}$$

$$\text{combining } \theta = \frac{(2n+1)\pi}{p\pm q}$$

Question 13.

If tan α and tan β are the roots $x^2 + ax + b = 0$, then $\frac{\sin(\alpha+\beta)}{\sin\alpha\sin\beta}$ is equal to …….

Solution:
(b) a/b

Hint:
$\tan^2 x + a \tan x + b = 0$
α and β are the roots of the equation
⇒ $\tan^2\alpha + a\tan\alpha + b = 0$ ….. (1)
$\tan^2\beta + a\tan\beta + b = 0$ …… (2)
(1) – (2) => $\tan^2\alpha - \tan^2\beta + a(\tan\alpha - \tan\beta) = 0$
$(\tan\alpha - \tan\beta)(\tan\alpha + \tan\beta) + a(\tan\alpha - \tan\beta) = 0$
⇒ $\tan\alpha + \tan\beta = -a$ …. (A)
(1) × tan β – (2) × tan α

$$\Rightarrow \tan^2\alpha\tan\beta - \tan^2\beta\tan\alpha + b(\tan\beta - \tan\alpha) = 0$$

$$\tan\alpha\tan\beta(\tan\alpha - \tan\beta) + b(\tan\beta - \tan\alpha) = 0$$

$$\Rightarrow \tan\alpha\tan\beta - b = 0$$

$$\Rightarrow \tan\alpha\tan\beta = b \quad \text{...(B)}$$

$$\text{Now } \frac{\sin(\alpha+\beta)}{\sin\alpha\sin\beta} = \frac{\sin\alpha\cos\beta+\cos\alpha\sin\beta}{\sin\alpha\sin\beta}$$

÷ by $\cos\alpha \cos\beta$

we get $$\frac{\tan\alpha + \tan\beta}{\tan\alpha \tan\beta} = -\frac{a}{b}$$

Question 14.
In a triangle ABC, $\sin^2 A + \sin^2 B + \sin^2 C = 2$, then the triangle is …….
(a) equilateral triangle
(b) isosceles triangle
(c) right triangle
(d) scalene triangle

Solution:
(c) right triangle
Hint.
Given $\sin^2 A + \sin^2 B + \sin^2 C = 2$
Suppose the given triangle is a right angle triangle with $\angle C = 90°$, then

$\sin^2 C = \sin^2 90° = 1$ …….. (1)
$\therefore \sin^2 A + \sin^2 B + 1 = 2$
$\sin^2 A + \sin^2 B = 1$
Also $A + B = 90° \Rightarrow A = 90° – B$

$\sin A = \sin (90° – B) = \cos B$ ——- (2)
$\sin^2 A + \sin^2 B + \sin^2 C = 2$
Using equations (1) and (2)
$\Rightarrow \cos^2 B + \sin^2 B + \sin^2 90° = 2$
$1 + 1 = 2$
$2 = 2$
$\therefore \sin^2 A + \sin^2 B + \sin^2 C = 2$ is true.
$\therefore \Delta$ ABC is a right-angle triangle.

Question 15.
If $f(\theta) = |\sin \theta| + |\cos \theta|, \theta \in R$, then $f(\theta)$ is in the interval …….
(a) [0, 2]
(b) [1, √2]
(c) [1, 2]
(d) [0, 1]

Solution:

(b) $[1, \sqrt{2}]$
Hint:

$$f(\theta) = |\sin \theta| + |\cos \theta|$$

at $\theta = 0, f(0) = 1$

$$\text{at } \theta = \pi, f(\pi) = |\sin \pi| + |\cos \pi|$$

$$= 0 + 1 = 1$$

$$\text{at } \theta = \frac{\pi}{6}, f(\frac{\pi}{6}) = \frac{1}{2} + \frac{\sqrt{3}}{2} = \frac{1+\sqrt{3}}{2} = 1.36$$

$$\text{at } \theta = \frac{\pi}{4}, f(\frac{\pi}{4}) = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \frac{2}{\sqrt{2}} = \sqrt{2} = 1.414$$

$$\text{at } \theta = \frac{\pi}{3}, f(\frac{\pi}{3}) = \frac{\sqrt{3}}{2} + \frac{1}{2} = 1.36$$

So the interval is $[1, \sqrt{2}]$

Question 16.

$$\frac{\cos 6x + 6\cos 4x + 15\cos 2x + 10}{\cos 5x + 5\cos 3x + 10\cos x} \text{ is equal to ....}$$

Solution:

Nr: cos 6x + cos 4x + 5 cos 4x + 5 cos 2x + 10 cos 2x + 10
= 2 cos 5x cos x+5(2 cos 3x cos x)+10(2 $\cos^2$x)
= 2 cos x [cos 5x + 5 cos 3x + 10 cos x]

$$\therefore \frac{\text{Nr}}{\text{Dr}} = \frac{2\cos x[\cos 5x + 5\cos 3x + 10\cos x]}{\cos 5x + 5\cos 3x + 10\cos x} = \cos 2x$$

Question 17.

The triangle of maximum area with constant perimeter 12m
(a) is an equilateral triangle with side of 4m
(b) is an isosceles triangle with sides 2m, 5m, 5m
(c) is a triangle with sides 3m, 4m, 5m
(d) does not exist

Solution:

(a) is an equilateral triangle with side of 4m

Hint.
Given the perimeter of the triangle is 12m
$2s = 12 \Rightarrow s = 6$
Maximum area is obtained when it is an equilateral triangle with a side of 4m each.

Question 18.
A wheel is spinning at 2 radians/second. How many seconds will it take to make 10 complete rotations?
(a) 10π seconds
(b) 20π seconds
(c) 5π seconds
(d) 15π seconds

Solution:
(a) 10π seconds
Hint.
In 1 second, it rotates = 2 radians
For 2 radians rotation time taken = 1 second
∴ For 1 complete rotation (2π radians) time taken
$= 1/2 \times 2\pi = \pi$ seconds.
∴ For 10 revolution time taken $= \pi \times 10$
$= 10\pi$ seconds.

Question 19.
If $\sin \alpha + \cos \alpha = b$, then $\sin 2\alpha$ is equal to ……..
(a) $b^2 - 1$, if $b \leq \sqrt{2}$
(b) $b^2 - 1$, if $b > \sqrt{2}$
(c) $b^2 - 1$, if $b \geq 1$
(d) $b^2 - 1$, if $b \geq \sqrt{2}$

Solution:
(b) $b^2 - 1$, if $b > \sqrt{2}$
Hint:
$\sin \alpha + \cos \beta = b$
$(\sin \alpha + \cos \beta)^2 = b^2$
$\sin^2 \alpha + \cos^2 \alpha + 2 \sin \alpha \cos \alpha = b^2$
$\sin^2 \alpha = b^2 - 1$

Question 20.

In a ΔABC, (*i*) $\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} > 0$ (*ii*) $\sin A \sin B \sin C > 0$ then ....

(a) Both (i) and (ii) are true
(b) Only (i) is true
(c) Only (ii) is true
(d) Neither (i) nor (ii) is true

Solution:
(a) Both (i) and (ii) are true
Hint.
We know in a Δ ABC

$\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} > 0$

Also $\sin A \sin B \sin C > 0$

∴ Statements (i) and (ii) are true.