Chapter 13

Limits and Derivatives

Miscellaneous Exercise

Question 1: Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r and s are fixed non-zero constants and m and n are integers): (x + a)

Answer 1:

Let f(x) = x + a. Accordingly, $f(x+h) = x + h + a$

By first principle,

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{x+h+a-x-a}{h}$$

$$= \lim_{h \to 0} \left(\frac{h}{h}\right)$$

$$= \lim_{h \to 0} (1)$$

= 1

Question 2: Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r and s are fixed non-zero constants and m and n are integers): $(px + q)\left(\frac{r}{x} + s\right)$

Answer 2:

Let, $f(x) = (px + q)\left(\frac{r}{x} + s\right)$

$$f'(x) = (px + q)\left(\frac{r}{x} + s\right) + \left(\frac{r}{x} + s\right)(px + q)$$

$= (px + q)(rx^{-1} + s) + \left(\frac{r}{x} + s\right)(p)$

$= (px + q)(-rx^{-2}) + \left(\frac{r}{x} + s\right)p$

$= (px + q)\left(\frac{-r}{x^2}\right) + \left(\frac{r}{x} + s\right)p$

$= \frac{-pr}{x} - \frac{qr}{x^2} + \frac{pr}{x} + ps$

$= ps - \frac{qr}{x^2}$

Question 3: Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r and s are fixed non-zero constants and m and n are integers)

Answer 3:

Let, $f(x) = (ax + b)(cx + d)^2$

By product rule,

$f'(x) = (ax + b)\frac{dd}{x}(cx + d)^2 + (cx + d)^2(ax + b)$

$= (ax + b)\frac{d}{dx}(c^2x^2 + 2cdx + d^2) + (cx + d^2)\frac{d}{dx}(ax + b)$

$= (ax + b)\left[\frac{d}{dx}(c^2x^2) + \frac{d}{dx}(2cdx) + \frac{d}{dx}d^2\right] + (cx + d)^2\left[\frac{d}{dx}ax\frac{d}{dx}b\right]$

$= (ax + b)(2x^2x + 2cd) + (cx + d)^2a$

$= 2c(ax + b)(cx + d) + a(cx + d)^2$

Question 4: Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r and s are fixed non-zero constants and m and n are integers): $\frac{ax+b}{cx+d}$

Answer 4:

Let $f(x) = \frac{ax+b}{cx+d}$

$$f'(x) = \frac{(cx+d)\frac{d}{dx}(ax+b)-(ax+b)\frac{d}{dx}(cx+d)}{(cx+d)^2}$$

$$= \frac{(cx+d)(a)-(ax+b)(c)}{(cx+d)^2}$$

$$= \frac{acx+ad-acx-bc}{(cx+d)^2}$$

$$= \frac{ad-bc}{(cx+d)^2}$$

Question 5: Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r and s are fixed non-zero constants and m and n are integers): $\frac{1+\frac{1}{x}}{1-\frac{1}{x}}$

Answer 5:

Let, $f(x) = \frac{1+\frac{1}{x}}{1-\frac{1}{x}} = \frac{\frac{x+1}{x}}{\frac{x-1}{x}} = \frac{x+1}{x-1}$, where $x \neq 0$

By quotient rule,

$$f'(x) = \frac{(x-1)\frac{d}{dx}(x+1)-(x+1)\frac{d}{dx}(x-1)}{(x-1)^2}, x \neq 0, 1$$

$$= \frac{(x-1)(1)-(x+1)(1)}{(x-1)^2}, x \neq 0,1$$

$$= \frac{x-1-x-1}{(x-1)^2}, x \neq 0,1$$

$$= \frac{-2}{(x-1)^2}, x \neq 0,1$$

Question 6: Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r and s are fixed non-zero constants and m and n are integers): $\frac{1}{ax^2+bx+c}$

Answer 6:

Let $f(x) = \frac{1}{ax^2+bx+c}$

By quotient rule,

$$f'(x) = \frac{(ax^2+bx+c)\frac{d}{dx}(1)-(1)\frac{d}{dx}(ax^2+bx+c)}{(ax^2+bx+c)^2}$$

$$= \frac{(ax^2+bx+c)(0)-(1)(2ax+b)}{(ax^2+bx+c)^2}$$

$$= \frac{-(2ax+b)}{(ax^2+bx+c)^2}$$

Question 7: Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r and s are fixed non-zero constants and m and n are integers): $\frac{ax+b}{px^2+qx+r}$

Answer 7:

Let, $f(x) = \frac{ax+b}{px^2+qx+r}$

By quotient rule,

$$f'(x) = \frac{(px^2+qx+r)\frac{d}{dx}(ax+b)-(ax+b)\frac{d}{dx}(px^2+qx+r)}{(px^2+qx+r)^2}$$

$$= \frac{(px^2+qx+r)(a)-(ax+b)(2px+q)}{(px^2+qx+r)^2}$$

$$= \frac{apx^2+aqx+ar-2apx^2-aqx-2bpx-bq}{(px^2+qx+r)^2}$$

$$= \frac{-apx^2-2bpx+ar-bq}{(px^2+qx+r)^2}$$

Question 8: Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r and s are fixed non-zero constants and m and n are integers): $\frac{px^2+qx+r}{ax+b}$

Answer 8:

Let, $f(x) = \frac{px^2+qx+r}{ax+b}$

By quotient rule,

$$f'(x) = \frac{(ax+b)\frac{d}{dx}(px^2+qx+r)-(px^2+qx+r)\frac{d}{dx}(ax+b)}{(ax+b)^2}$$

$$= \frac{(ax+b)(-px+q)-(px^2+qx+r)(a)}{(ax+b)^2}$$

$$= \frac{2apx^2+aqx+2bpx+bq-apx^2-aqx-ar}{(ax+b)^2}$$

$$= \frac{apx^2+2bpx+bq-ar}{(ax+b)^2}$$

Question 9:Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r and s are fixed non-zero constants and m and n are integers): $\frac{a}{x^4} - \frac{b}{x^2} + \cos x$

Answer 9:

Let, $f(x) = \frac{a}{x^4} - \frac{b}{x^2} + cosx$

$$f'(x) = \frac{d}{dx}\left(\frac{a}{x^4}\right) - \frac{d}{dx}\left(\frac{b}{x^2}\right) + \frac{d}{dx}(cosx)$$

$$= a\frac{d}{dx}(x^{-4}) - b\frac{d}{dx}(x^{-2}) + \frac{d}{dx}(cosx)$$

$$= a(-4x^{-5}) - b(-2x^{-3}) + (-sinx)$$

$$\left[\frac{d}{dx}(x^n) = nx^{n-1} and \frac{d}{dx}(cosx) = -sinx\right]$$

$$= \frac{-4a}{x^5} + \frac{2b}{x^3} - sinx$$

Question 10: Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r and s are fixed non-zero constants and m and n are integers): $4\sqrt{x} - 2$

Answer 10:

Let, $f(x) = 4\sqrt{x} - 2$

$$f'(x) = \frac{d}{dx}\left(4\sqrt{x} - 2\right) = \frac{d}{dx}\left(4\sqrt{x}\right) - \frac{d}{dx}(2)$$

$$= 4\,\frac{d}{dx}\left(x^{\frac{1}{2}}\right) - 0 = 4\left(\frac{1}{2}x^{\frac{1}{2}-1}\right)$$

$$= \left(2x^{\frac{1}{2}}\right) = \frac{2}{\sqrt{x}}$$

Question 11: Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r and s are fixed non-zero constants and m and n are integers): $\frac{a+bsinx}{c+dcosx}$

Answer 11:

Let, $f(x) = \frac{a+bsinx}{c+dcosx}$

By quotient rule,

$$f'(x) = \frac{(c+dcosx)\frac{d}{dx}(a+bsinx) - (a+bsinx)\frac{d}{dx}(c+dcosx)}{(c+dcosx)^2}$$

$$= \frac{(c+dcosx)(bcosx) - (a+bsinx)(-dsinx)}{(c+dcosx)^2}$$

$$= \frac{cbcosx + bdcos^2x + adsinx - bdsin^2x}{(c+dcosx)^2}$$

$$= \frac{bccosx + adsinx + bd(cos^2x + sin^2x)}{(c+dcosx)^2}$$

$$= \frac{bc\,cosx + ad\,sinx + bd}{(c+d\,cosx)^2}$$

Question 12: Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r and s are fixed non-zero constants and m and n are integers): x4 (5 sin x – 3 cos x)

Answer 12:

Let $f(x) =$ x4 (5 sin x – 3 cos x)

By product rule,

$$f'(x) = x^4 \frac{d}{dx}(5sinx - 3cosx) + (5sinx - 3cosx)\frac{d}{dx}(x^4)$$

$$= x^4\left[5\frac{d}{dx}(sinx) - 3\frac{d}{dx}(cosx)\right] + [5sinx - 3cosx]\frac{d}{dx}(x^4)$$

$$= x^4[5cosx - 3(-sinx)] + (5sinx - 3cosx)(4x^3)$$

$$= x^3[5xcosx + 3xsinx + 20sinx - 12cosx]$$

Question 13: Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r and s are fixed non-zero constants and m and n are integers): (x2 + 1) cos x

Answer 13:

Let $f(x) =$ (x2 + 1) cos x

By product rule,

$$f'(x) = (x^2 + 1)\frac{d}{dx}(cosx)cosx\frac{d}{dx}(x^2 + 1)$$

$$= (x^2 + 1)(-sinx) + cosx(2x)$$

$$= -x^2sinx - sinx + 2xcosx$$

Question 14: Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r and s are fixed non-zero constants and m and n are integers): (ax2 + sin x) (p + q cos x)

Answer 14:

Let $f(x) =$ (ax2 + sin x) (p + q cos x)

By product rule,

$$f'(x) = (ax^2 + sinx)\frac{d}{dx}(p + qcosx) + (p + qcosx)\frac{d}{dx}(ax^2 + sinx)$$

$$= (ax^2 + sinx)(-qsinx) + (p + qcosx)(2ax + cosx)$$

$= -qsinx(ax^2 + sinx) + (p + qcosx)(2ax + cosx)$

Question 15: Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r and s are fixed non-zero constants and m and n are integers): $\frac{4x+5sinx}{3x+7cosx}$

Answer 15:

Let $f(x) = \frac{4x+5sinx}{3x+7cosx}$

By quotient rule,

$$f'(x) = \frac{(3x+7cosx)\frac{d}{dx}(4x+5sinx) - (4x+5sinx)\frac{d}{dx}(3x+7cosx)}{(3x+7cosx)^2}$$

$$= \frac{(3x+7cosx)\left[4\frac{d}{dx}(x)+5\frac{d}{dx}(sinx)\right] - (4x+5sinx)\left[3\frac{d}{dx}x+7\frac{d}{dx}cosx\right]}{(3x+7cosx)^2}$$

$$= \frac{(3x+7cosx)(4+5cosx) - (4x+5sinx)(3-7sinx)}{(3x+7cosx)^2}$$

$$= \frac{12x+15xcosx+28cosx+35cos^2x-12x+28xsinx-15sinx+35sin^2x}{(3x+7\cos x)^2}$$

$$= \frac{15xcosx+28cosx+28xsinx-15sinx+35(cos^2x+sin^2x)}{(3x+7cosx)^2}$$

$$= \frac{35+15xcosx+28cosx+28xsinx-15sinx}{(3x+7cosx)^2}$$

Question 16: Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r and s are fixed non-zero constants and m and n are integers): $\frac{x^2cos\left(\frac{\pi}{4}\right)}{sinx}$

Answer 16:

Let $f(x) = \frac{x^2cos\left(\frac{\pi}{4}\right)}{sinx}$

By quotient rule,

$$f'(x) = cos\frac{\pi}{4}.\left[\frac{sinx\frac{d}{dx}(x^2)-x^2\frac{d}{dx}(sinx)}{sin^2x}\right]$$

$$= \cos\frac{\pi}{4}.\left[\frac{sinx.2x-x^2cosx}{sin^2x}\right]$$

$$= \frac{xcos\frac{\pi}{4}[2sinx-xcosx]}{sin^2x}$$

Question 17: Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r and s are fixed non-zero constants and m and n are integers): $\frac{x}{1+tanx}$

Answer 17:

Let $f(x) = \frac{x}{1+tanx}$

$$f'(x) = \frac{(1+tanx)\frac{d}{dx}(x)-x\frac{d}{dx}(1+tanx)}{(1+tanx)^2}$$

$$f'(x) = \frac{(1+tanx)-x.\frac{d}{dx}(1+tanx)}{(1+tanx)^2} \ldots (1)$$

Let, g (x) = 1 + tan x, accordingly, g ($x + h$) = 1 + tan ($x + h$)

By first principle,

$$g'(x) = \lim_{h\to 0}\frac{g\,(x+h)-g(x)}{h}$$

$$= \lim_{h\to 0}\left[\frac{1+\tan(x+h)-1-tanx}{h}\right]$$

$$= \lim_{h\to 0}\frac{1}{h}\left[\frac{\sin(x+h)}{\cos(x+h)}-\frac{sinx}{cosx}\right]$$

$$= \lim_{h\to 0}\frac{1}{h}\left[\frac{\sin(x+h)cosx-sinccos(x+h)}{\cos(x+h)cosx}\right]$$

$$= \lim_{h\to 0}\frac{1}{h}\left[\frac{\sin(x+h-x)}{\cos(x+h)cosx}\right]$$

$= \lim_{h\to 0} \frac{1}{h}\left[\frac{sinh}{cos(x+h)cosx}\right]$

$= \left\{\lim_{h\to 0} \frac{sinh}{h}\right\}.\left\{\lim_{h\to 0} \frac{1}{\cos(x+h)cosx}\right\}$

$= 1\times \frac{1}{cos^2 x} = sec^2 x$

$= \frac{d}{dx}(1 + tanx) = sec^2 x \ldots (2)$

From (i) and (ii), we obtain

$= f'(x) = \frac{1+tanx-xsec^2 x}{(1+tanx)^2}$

Question 18: Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r and s are fixed non-zero constants and m and n are integers): $\frac{x}{sin^n x}$

Answer 18:

Let $f(x) = \frac{x}{sin^n x}$

By quotient rule,

$f'(x) = \frac{sin^n x\frac{d}{dx}x - x\frac{d}{dx}sin^n x}{sin^{2n} x}$

It can be easily shown that $\frac{d}{dx} sin^n x = nsin^{n-1} xcosx$

Therefore,

$f'(x) = \frac{sin^n x\frac{d}{dx}x - x\frac{d}{dx}sin^n x}{sin^{2n} x}$

$= \frac{sin^n .1 - x\left(nsin^{n-1} xcosx\right)}{sin^{2x} x}$

$= \frac{sin^{n-1} x(sinx - nxcosx)}{sin^{2n} x}$

$$= \frac{sinx - nxcosx}{sin^{n+1}x}$$