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MATHEMATICS

Standard 9

(Semester II)

PLEDGE

India is my country.
All Indians are my brothers and sisters.
I love my country and I am proud of its rich and varied heritage.
I shall always strive to be worthy of it.
I shall respect my parents, teachers and all my elders and treat everyone with courtesy.
I pledge my devotion to my country and its people.
My happiness lies in their well-being and prosperity.

રાજ્ય સરકારની વિનામૂલ્યે યોજના હેઠળનું પુસ્તક

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PREFACE

The Gujarat State Secondary and Higher Secondary Education Board has prepared new syllabi in accordance with the new national syllabi prepared by the N.C.E.R.T. These syllabi are sanctioned by the Government of Gujarat.

It is pleasure for the Gujarat State Board of School Textbooks, to place before the students this textbook of **Mathematics** for **Standard 9 (Semester II)** prepared according to the new syllabus.

Before publishing the textbook, its manuscript has been fully reviewed by experts and teachers teaching at this level. Following suggestions given by teachers and experts, we have made necessary changes in the manuscript before publishing the textbook.

The Board has taken special care to ensure that this textbook is interesting, useful and free from errors. However, we welcome any suggestions, from people interested in education, to improve the quality of the textbook.

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FUNDAMENTAL DUTIES

It shall be the duty of every citizen of India

(A) to abide by the Constitution and respect its ideals and institutions, the National Flag and the National Anthem;

(B) to cherish and follow the noble ideals which inspired our national struggle for freedom;

(C) to uphold and protect the sovereignty, unity and integrity of India;

(D) to defend the country and render national service when called upon to do so;

(E) to promote harmony and the spirit of common brotherhood amongst all the people of India transcending religious, linguistic and regional or sectional diversities; to renounce practices derogatory to the dignity of women;

(F) to value and preserve the rich heritage of our composite culture;

(G) to protect and improve the natural environment including forests, lakes, rivers and wild life, and to have compassion for living creatures;

(H) to develop the scientific temper, humanism and the spirit of inquiry and reform;

(I) to safeguard public property and to abjure violence;

(J) to strive towards excellence in all spheres of individual and collective activity so that the nation constantly rises to higher levels of endeavour and achievement;

(K) to provide opportunities for education by the parent or the guardian, to his child or a ward between the age of 6-14 years as the case may be.

CONTENTS

CHAPTER 10

QUADRILATERALS

10.1 Introduction

We have learnt about triangles in the previous chapter using the terminology of the set theory. Now we shall study about quadrilaterals using the same terminology.

10.2 Plane Quadrilateral

We know that a triangle is the union of three line-segments determined by three non-collinear points.

Quadrilateral : A quadrilateral is the union of four line-segments determined by four distinct coplanar points of which no three are collinear and the line-segments intersect only at end points.

It is clear from the definition of a quadrilateral that for distinct coplanar points P, Q, R, S the following three conditions are essential to construct a quadrilateral :

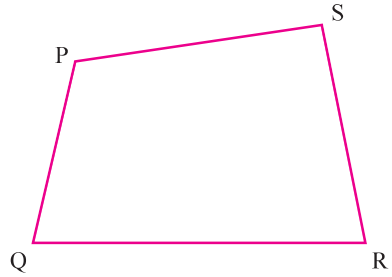

Figure 10.1

(i) P, Q, R and S are distinct and coplanar points.

(ii) No three of points P, Q, R and S are collinear.

(iii) Line-segments $\overline{PQ}$, $\overline{QR}$, $\overline{RS}$ and $\overline{SP}$ intersect at their end points only. Then the union of $\overline{PQ}$, $\overline{QR}$, $\overline{RS}$ and $\overline{SP}$ is the quadrilateral PQRS. We denote quadrilateral PQRS by $\square$ PQRS.

$\therefore \square PQRS = \overline{PQ} \cup \overline{QR} \cup \overline{RS} \cup \overline{SP}$

Now we see why above three conditions are essential :

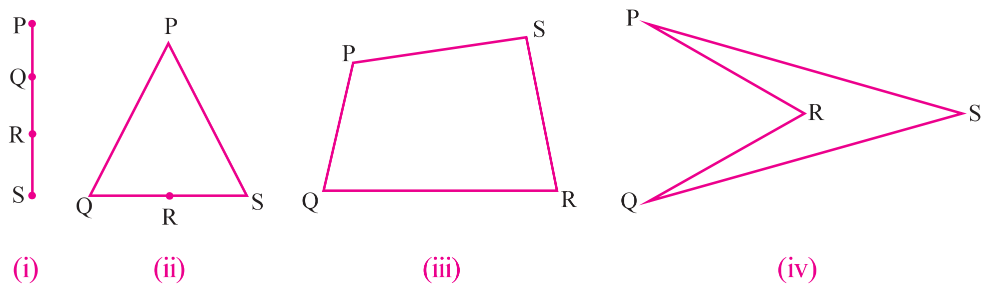

Figure 10.2

If all the four points are collinear, we obtain line-segments as given in figure 10.2 (i).

If three out of four points are collinear, we may get a triangle as given in figure 10.2 (ii).

If no three points out of four points are collinear, we obtain a closed figure with four sides given in figure 10.2 (iii) and 10.2 (iv).

In our study, we will consider only quadrilaterals of type as in figure 10.2 (iii).

Convex quadrilateral : If in a quadrilateral, no side intersects the line containing its opposite side, then the quadrilateral is called a convex quadrilateral. The diagonals of a convex quadrilateral intersect each other.

We will refer to convex quadrilaterals as quadrilaterals in the rest of the chapter.

Quadrilaterals of type given in figure 10.2 (iv) are called **concave** quadrilaterals.

10.3 Parts of a Quadrilateral

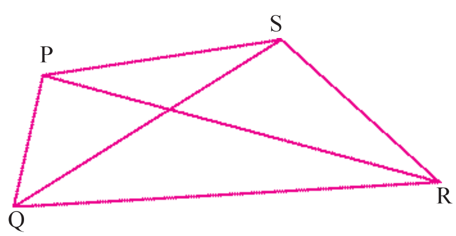

Figure 10.3

In the □ PQRS,

(i) Points P, Q, R, S are called the vertices of □ PQRS.

(ii) $\overline{PQ}$, $\overline{QR}$, $\overline{RS}$, $\overline{SP}$ are called sides of □ PQRS.

(iii) ∠SPQ, ∠PQR, ∠QRS, ∠RSP are called the angles of □ PQRS.

If there is no confusion, we denote these angles as ∠P, ∠Q, ∠R and ∠S respectively.

(iv) $\overline{PR}$ and $\overline{QS}$ are diagonals of □ PQRS.

It is clear that **the diagonals of a convex quadrilateral intersect each other.**

A quadralateral has 10 parts namely four sides, four angles and two diagonals.

Now we will learn about special pair of sides and angles of a quadrilateral.

(1) Two sides of a quadrilateral intersecting in a vertex are called adjacent sides.

As shown in figure 10.4, $\overline{PS}$ and $\overline{SR}$ have a common end point S. So, $\overline{PS}$ and $\overline{SR}$ are adjacent sides.

$\overline{PQ}$, $\overline{QR}$; $\overline{QR}$, $\overline{RS}$ and $\overline{PQ}$, $\overline{PS}$ are other pairs of adjacent sides of □ PQRS.

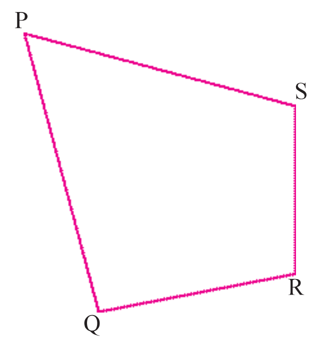

Figure 10.4

(2) The sides of a quadrilateral with no common end point are called opposite sides. The intersection of opposite sides is ∅.

Sides $\overline{PQ}$ and $\overline{SR}$ of □ PQRS have no common end point, so $\overline{PQ}$ and $\overline{SR}$ are opposite sides of □ PQRS. $\overline{PS}$ and $\overline{QR}$ is also another pair of **opposite sides.**

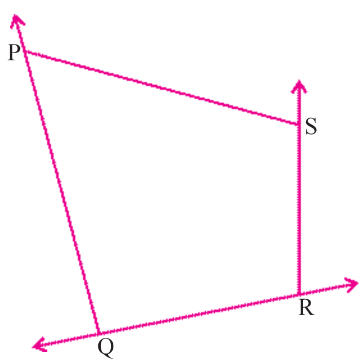

Figure 10.5

(3) If two angles of a quadrilateral intersect in a side of the quadrilateral, then these angles are called adjacent angles.

In figure 10.5, $\overline{QR}$ is the intersection of ∠Q and ∠R. Hence ∠Q and ∠R are adjacent angles of the quadrilateral. In this way, ∠Q and ∠R, ∠R and ∠S, ∠S and ∠P, ∠P and ∠Q are four pairs of the adjacent angles of □ PQRS.

(4) If the intersection of two angles of a quadrilateral is not a side of the quadrilateral, then the two angles are called opposite angles. Two angles are opposite if and only if they are not adjacent. Intersection of two opposite angles consists of two vertices only.

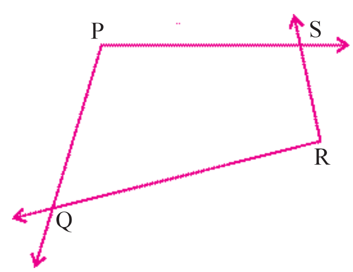

Figure 10.6

The intersection of two angles ∠P and ∠R does not contain any common side of the quadrilateral but consists of only two vertices Q and S. Hence ∠P and ∠R are opposite angles of the □ PQRS. Thus (i) ∠P and ∠R (ii) ∠Q and ∠S are two pairs of opposite angles in □ PQRS.

Now, with reference to □ PQRS it is clear from the above information that

(1) Every vertex of a quadrilateral is the common end point of two adjacent sides of the quadrilateral.

As in the figure 10.6, $\overline{PQ} \cap \overline{QR} = \{Q\}$, $\overline{QR} \cap \overline{RS} = \{R\}$, $\overline{SR} \cap \overline{SP} = \{S\}$, $\overline{SP} \cap \overline{PQ} = \{P\}$

(2) The union of the sides (line-segments) is a quadrilateral but the region enclosed by those line-segments is not a quadrilateral. (figure 10.6)

□ PQRS = $\overline{PQ} \cup \overline{QR} \cup \overline{RS} \cup \overline{SP}$

(3) All the vertices and sides of a quadrilateral are in the same plane. Thus a quadrilateral is a plane figure lying in a plane.

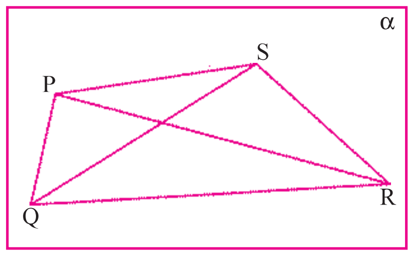

Figure 10.7

As shown in the figure 10.7, vertices P, Q, R, S are in the plane α and therefore $\overline{PQ}$, $\overline{QR}$, $\overline{RS}$ and $\overline{SP}$ are also in plane α. Thus □ PQRS is a plane figure lying in the plane α.

(4) The sides and set of vertices of a quadrilateral are subsets of the quadrilateral.

In the figure 10.7, $\overline{PQ} \subset$ □ PQRS, $\overline{QR} \subset$ □ PQRS, $\overline{RS} \subset$ □ PQRS, $\overline{SP} \subset$ □ PQRS and {P, Q, R, S} ⊂ □ PQRS.

(5) Angles and diagonals of a quadrilateral are not subsets of the quadrilateral.

In figure 10.7, ∠P ⊄ □ PQRS, ∠Q ⊄ □ PQRS, ∠R ⊄ □ PQRS, ∠S ⊄ □ PQRS, $\overline{PR}$ ⊄ □ PQRS, $\overline{QS}$ ⊄ □ PQRS.

(6) The plane containing a quadrilateral is partitioned into three mutually disjoint sets by the quadrilateral : (1) the quadrilateral (2) the interior of the quadrilateral (3) the exterior of the quadrilateral.

We get more clarity about naming of a quadrilateral from following examples :

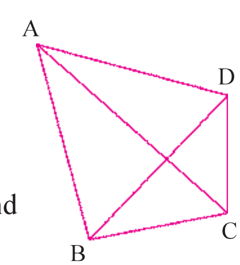

Figure 10.8

(1) Name the quadrilateral with diagonals $\overline{AC}$ and $\overline{BD}$:

In the figure 10.8, the quadrilateral with diagonals $\overline{AC}$ and $\overline{BD}$ is □ ABCD. It can also be called □ ADCB.

(2) Of which quadrilateral will $\overline{DF}$ and $\overline{GE}$ be the opposite sides and $\overline{DE}$ a diagonal ?

If $\overline{DF}$ and $\overline{GE}$ are the opposite sides of a quadrilateral and $\overline{DE}$ is the diagonal, then the quadrilateral is □ DGEF or □ DFEG.

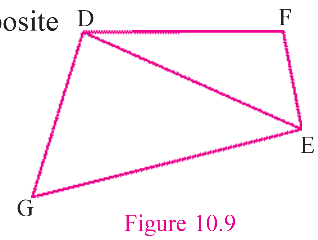

Figure 10.9

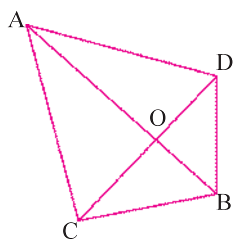

Figure 10.10

(3) If A–O–B and C–O–D and $\overline{AB} \cap \overline{CD} = \{O\}$, then which quadrilateral will be formed by A, B, C and D ?

If A–O–B and C–O–D and $\overline{AB} \cap \overline{CD} = \{O\}$, then □ ADBC or □ ACBD is formed.

(4) Is □ EFGH = □ HGFE ? Give reasons.

Yes, □ EFGH = □ HGFE,

because

□ EFGH $= \overline{EF} \cup \overline{FG} \cup \overline{GH} \cup \overline{HE}$

$= \overline{HG} \cup \overline{GF} \cup \overline{FE} \cup \overline{EH}$

= □ HGFE as $\overline{HG} = \overline{GH}$, $\overline{EF} = \overline{FE}$ etc.

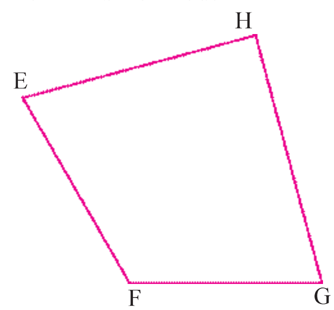

Figure 10.11

Thus, □ EFGH, □ HGFE, □ FGHE, □ GFEH, □ GHEF, □ FEHG and □ EHGF represent the same quadrilateral.

10.4 The Sum of the Measures of the Angles of a Quadrilateral

We know that the sum of the measures of all the angles of a triangle is 180. What should be sum of measures of all the angles of a quadrilateral ?

Drawing the diagonal $\overline{AC}$ of □ ABCD, we get Δ ABC and Δ ACD. Vertex C is in the interior of ∠DAB.

$m\angle DAC + m\angle CAB = m\angle DAB.$ **(i)**

Similarly vertex A is in the interior of ∠BCD.

$\therefore\ m\angle BCA + m\angle ACD = m\angle BCD$ **(ii)**

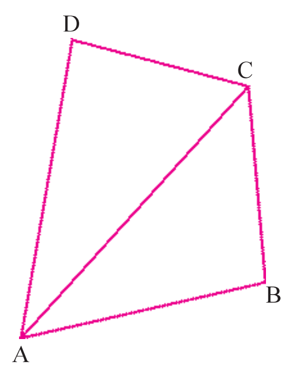

Figure 10.12

In ΔABC, $m\angle CAB + m\angle ABC + m\angle BCA = 180$ **(iii)**

In ΔACD, $m\angle ACD + m\angle CDA + m\angle DAC = 180$ **(iv)**

From (iii) and (iv),

$m\angle CAB + m\angle ABC + m\angle BCA + m\angle ACD + m\angle CDA + m\angle DAC = 360$

From (i) and (ii),

$\therefore\ m\angle DAB + m\angle ABC + m\angle BCD + m\angle ADC = 360$

Thus, **the sum of the measures of the angles of a quadrilateral is 360.**

Example 1 : In □ ABCD, the measures of ∠A, ∠B, ∠C and ∠D are in proportion 2 : 4 : 5 : 4. Find the measure of each angle.

Solution : The measures of ∠A, ∠B, ∠C and ∠D of □ ABCD are in proportion 2 : 4 : 5 : 4.

Let $m\angle A = 2x$, $m\angle B = 4x$, $m\angle C = 5x$ and $m\angle D = 4x$.

But in □ ABCD, $m\angle A + m\angle B + m\angle C + m\angle D = 360$

$\therefore\ 2x + 4x + 5x + 4x = 360$

$\therefore\ 15x = 360$

$\therefore\ x = \frac{360}{15} = 24$

$\therefore\ m\angle A = 2x = 48,\quad m\angle B = 4x = 96$

$m\angle C = 5x = 120,\quad m\angle D = 4x = 96$

EXERCISE 10.1

1. Describe the following for □ XYZW shown in the figure 10.13 :

(1) the sides (2) the angles (3) the diagonals

(4) pairs of adjacent sides

(5) pairs of opposite sides

(6) pairs of adjacent angles

(7) pairs of opposite angles

(8) $\overline{XW} \cap \overline{YZ}$ (9) $\overline{YX} \cap \overleftrightarrow{XW}$

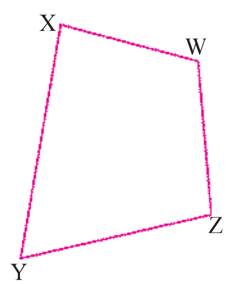

Figure 10.13

2. Is □ PQRS = □ PSQR ? Give reasons for your answer.

3. Solve the following :

(1) If in □ PQRS, $m\angle P = 2x$, $m\angle Q = 3x$, $m\angle R = 4x$ and $m\angle S = 6x$, then find the measure of each angle of □ PQRS.

(2) In □ ABCD, if $m\angle A = m\angle B = 70$, $m\angle C = 100$, find the measure of ∠D.

(3) In □ ABCD, the measures of ∠A, ∠B, ∠C and ∠D are in the proportion 2 : 5 : 6 : 7. Find the measure of each angle of □ ABCD.

(4) In □ ABCD, the measure of ∠A, ∠B, ∠C and ∠D are in proportion of 10 : 7 : 12 : 7. Find measure of each angle of □ ABCD.

4. For each of the following statements, state whether it is true or false :

(1) The angle of a quadrilateral is a subset of the quadrilateral.

(2) $\angle A$ and $\angle B$ are adjacent angles of □ ABCD.

(3) $\overline{GD}$ is a subset of □ DEFG.

(4) $\overline{AB}$ and $\overline{CD}$ are opposite sides of □ ABCD.

(5) $\overline{AC}$ is a diagonal of □ ABCD.

(6) If no three of E, F, G, H are collinear, then $\overline{EF} \cup \overline{FG} \cup \overline{GH} \cup \overline{HE}$ = □ EFGH.

(7) $\overline{ML}$ and $\overline{LN}$ are adjacent sides and $\overline{LO}$ is a diagonal, then MLON is a quadrilateral.

*

10.5 Types of Quadrilateral

We study different quadrilaterals given below :

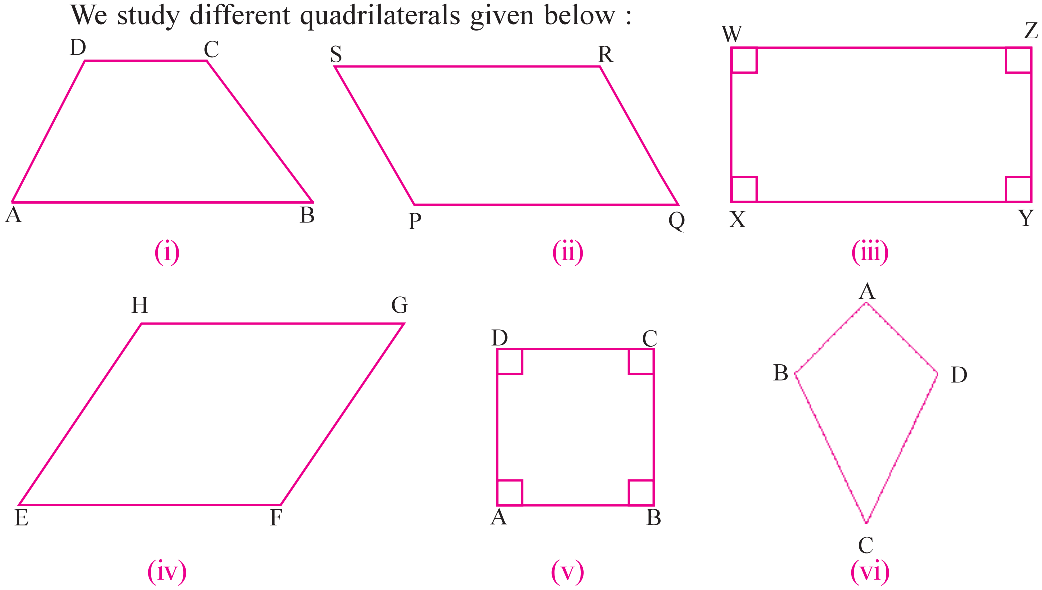

Figure 10.14

In figure 10.14 (i), in □ ABCD sides in only one pair of opposite sides $\overline{AB}$ and $\overline{CD}$ are parallel.

If in a quadrilateral, sides in only one pair of opposite sides are parallel to each other, then the quadrilateral is called a trapezium.

∴ □ ABCD is trapezium.

Sides in both the pairs of opposite sides are parallel in figure 10.14 (ii), (iii), (iv) and (v). Such quadrilaterals are called **parallelograms.**

Now let us get more information about each figure 10.14 (ii) to (v).

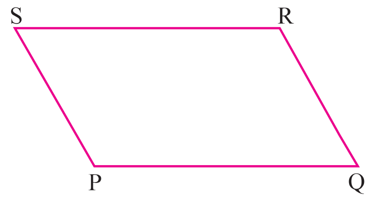

Figure 10.15

In a quadrilateral, if opposite sides are parallel to each other, then the quadrilateral is called a parallelogram.

In □ PQRS, $\overline{SP} \parallel \overline{RQ}$ and $\overline{SR} \parallel \overline{PQ}$. Hence it is a parallelogram and it is denoted by □^{m} PQRS.

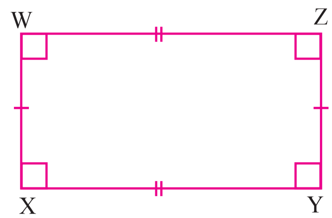

Figure 10.16

In □ XYZW, $\overline{XW} \parallel \overline{ZY}$ and $\overline{XY} \parallel \overline{WZ}$.

So □ XYZW is parallelogram, but also

$m\angle X = m\angle Y = m\angle Z = m\angle W = 90$.

□^{m} XYZW is known as a **rectangle**.

If all the angles of a parallelogram are right angles, then the parallelogram is called a rectangle.

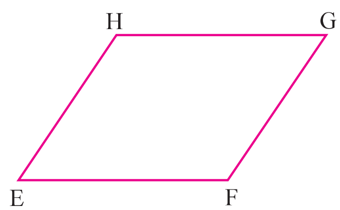

Figure 10.17

Here, we need to observe following facts :

(1) Each rectangle is parallelogram.

(2) All the four angles of a rectangle are congruent.

In □ EFGH, $\overline{HE} \parallel \overline{GF}$, $\overline{HG} \parallel \overline{EF}$. □ EFGH is a parallelogram. But in □^{m} EFGH, all sides are congruent. □ EFGH is known as a **rhombus.**

If all the sides of a parallelogram are congruent, then it is called a rhombus.

Here we note the following facts :

(1) Each rhombus is a parallelogram.

(2) All the four sides of a rhombus are congruent.

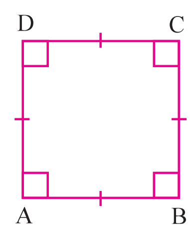

Figure 10.18

In □ ABCD, since $\overline{AD} \parallel \overline{BC}$ and $\overline{AB} \parallel \overline{CD}$, □ ABCD is parallelogram. But here, $m\angle A = m\angle B = m\angle C = m\angle D = 90$ and also all the sides of □ ABCD are congruent. So, □^{m}ABCD is known as a **square.**

This □^{m}ABCD is also a rectangle and □^{m}ABCD is a rhombus also.

If all the side of a rectangle are congruent, then it is called a square.

We observe,

(1) A square is a parallelogram.

(2) Since all the four sides of a square are congruent, it is a rhombus too.

(3) Since each angle of a square is a right angle, a square is also a rectangle.

In figure 10.19, □ ABCD, AB = AD and BC = CD. So adjacent sides are congruent, but □ ABCD is not parallelogram. □ ABCD is known as a **kite**.

Note : Diagonals of a kite are not congruent but intersect each other at right angles.

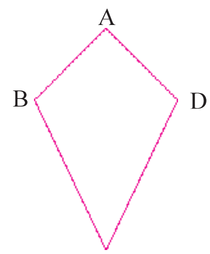

Figure 10.19

Example 2 : In a trapezium PQRS, if $\overline{PS} \parallel \overline{QR}$, $m\angle P : m\angle Q = 7 : 3$ and $m\angle R = 99$, then find the measures of all the remaining angles.

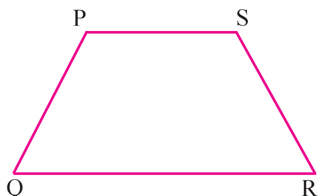

Figure 10.20

Solution : In □ PQRS, $\overline{QR} \parallel \overline{PS}$ and $\angle P$ and $\angle Q$ are the interior angles on one side of the transversal $\overleftrightarrow{PQ}$. Let $m\angle P = 7x$ and $m\angle Q = 3x$.

$\therefore\ m\angle P + m\angle Q = 180$

$\therefore\ 7x + 3x = 180$

$\therefore\ 10x = 180$

$\therefore\ x = 18$

$\therefore\ m\angle P = 7x = 7(18) = 126$

$\therefore\ m\angle Q = 3x = 3(18) = 54$

Now, in □ PQRS, $m\angle R + m\angle S = 180$ **($\overleftrightarrow{PS} \parallel \overleftrightarrow{RQ}$)**

$99 + m\angle S = 180$ **($m\angle R = 99$)**

$m\angle S = 180 - 99 = 81$

$\therefore\ m\angle S = 81$

EXERCISE 10.2

1. In a trapezium ABCD, $\overline{AB} \parallel \overline{CD}$. If $m\angle B = 60$ and $m\angle D = 100$, then find the measures of $\angle A$ and $\angle C$.

2. In a trapezium ABCD, $\overline{AB} \parallel \overline{DC}$. If $m\angle A = m\angle B = 60$, then find $m\angle C$ and $m\angle D$.

3. In a trapezium PQRS, $\overline{PQ} \parallel \overline{SR}$. If $m\angle P = 50$ and $m\angle R = 110$, then find $m\angle Q$ and $m\angle S$.

4. In a trapezium PQRS, if $\overline{PQ} \parallel \overline{RS}$, $m\angle S : m\angle P = 5 : 4$ and $m\angle Q = 72$, then find $m\angle R$, $m\angle S$, $m\angle P$.

5. In $\square$ ABCD, the measures of the angles are in proportion 6 : 7 : 11 : 12. Find the measure of each angle of $\square$ ABCD.

6. For each of the following statements, state whether it is true or false :

(1) Every square is a rectangle.

(2) Every rectangle is a parallelogram.

(3) Every rhombus is a square.

(4) Every trapezium is a parallelogram.

(5) Every rectangle is a trapezium.

(6) Every square is a rhombus.

(7) Every rhombus is a parallelogram.

(8) Every parallelogram is a rectangle.

(9) Every rectangle is a square.

*

10.6 Properties of Parallelograms

We have learnt about types of quadrilaterals. We have seen that a rectangle, a square, a rhombus are special types of parallelograms. A parallelogram is an important quadrilateral. Now we study some properties of parallelograms. We begin with proving following theorem asserting the congruence of triangles formed by each of its diagonals.

Theorem 10.1 : Two triangles formed by any diagonal of a parallelogram are congruent.

Data : Δ SPR and Δ QRP are formed by diagonal $\overline{PR}$ of $\square^m$ PQRS.

To Prove : $\Delta SPR \cong \Delta QRP$

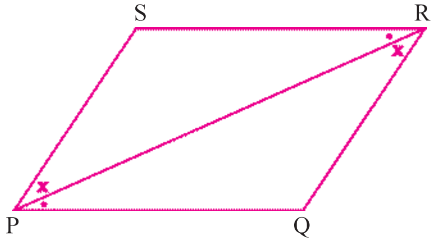

Figure 10.21

Proof : $\square$ PQRS is parallelogram.

$\therefore$ $\overline{PS} \parallel \overline{QR}$ and $\overline{SR} \parallel \overline{PQ}$

$\overleftrightarrow{PS} \parallel \overleftrightarrow{QR}$ and $\overleftrightarrow{PR}$ is their transversal.

$\therefore$ $\angle SPR \cong \angle QRP$ **(alternate angles) (i)**

$\overleftrightarrow{SR} \parallel \overleftrightarrow{PQ}$ and $\overleftrightarrow{PR}$ is their transversal.

$\angle SRP \cong \angle QPR$ **(alternate angles) (ii)**

For correspondence SPR $\leftrightarrow$ QRP

$\angle SPR \cong \angle QRP$ **(by (i))**

$\angle SRP \cong \angle QPR$ **(by (ii))**

$\overline{PR} \cong \overline{PR}$

∴ The correspondence SPR ↔ QRP is a congruence by ASA.

∴ Δ SPR ≅ Δ QRP

We know that if a correspondence between two triangles is a congruence, then corresponding sides and angles are congruent. Since two triangles formed by any one diagonal of a parallelogram are congruent; then it is obvious that opposite sides of the parallelogram are congruent. We accept this theorem without proof.

Theorem 10.2 : Opposite sides in a parallelogram are congruent.

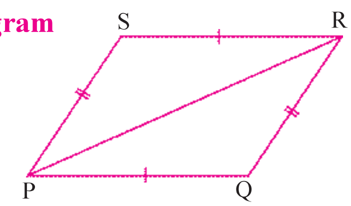

Figure 10.22

In □^{m} PQRS in figure 10.22, $\overline{PR}$ is diagonal.

∴ Δ SPR ≅ Δ QRP

∴ $\overline{SR} \cong \overline{QP}$ and $\overline{SP} \cong \overline{QR}$

Now if we construct a quadrilateral such that its opposite sides are congruent, then we get a parallelogram. This is the converse of the above theorem. We accept this theorem without proof.

Theorem 10.3 : If the sides in each pair of opposite sides in a quadrilateral are congruent, the quadrilateral is a parallelogram.

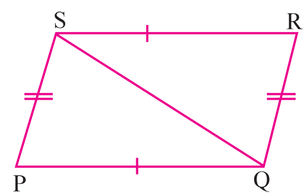

Figure 10.23

In figure 10.23, $\overline{SP} \cong \overline{QR}$ and $\overline{PQ} \cong \overline{SR}$.

So □ PQRS is a parallelogram.

Example 3 : In □^{m} ABCD, AB = 10 *cm* and AD = 6 *cm*. Find the perimeter of □ ABCD.

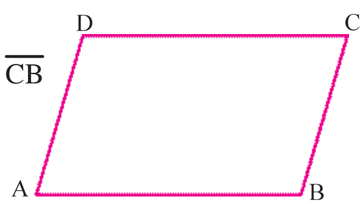

Figure 10.24

Soultion : In □^{m} ABCD, $\overline{AB} \cong \overline{DC}$ and $\overline{AD} \cong \overline{CB}$

AB = DC = 10 *cm*, AD = BC = 6 *cm*

∴ The perimeter of □^{m} ABCD

= AB + BC + CD + AD

= 10 + 6 + 10 + 6 = 32 *cm*

We construct a parallelogram and measure the opposite angles. We will find that they are congruent. We accept this theorem without proof.

Theorem 10.4 : Opposite angles in a parallelogram are congruent.

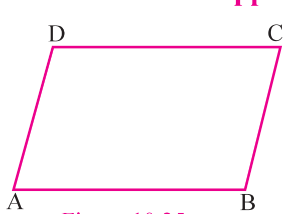

Figure 10.25

In figure 10.25, □ ABCD is a parallelogram.

∴ ∠B ≅ ∠D, ∠A ≅ ∠C

If the opposite angles of a quadrilateral are congruent, then the quadrilateral is a parallelogram. We accept this theorem without proof.

Theorm 10.5 : If in a quadrilateral, both the angles in each pair of opposite angles are congruent, then the quadrilateral is a parallelogram.

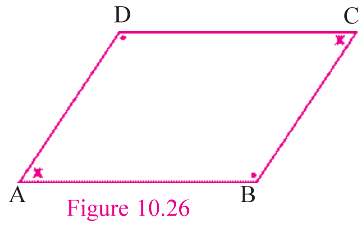

Figure 10.26

As shown in figure 10.26, for □ ABCD, ∠A ≅ ∠C and ∠B ≅ ∠D. So □ ABCD is a parallelogram.

In a □^{m} PQRS, diagonals $\overline{SQ}$ and $\overline{PR}$ intersect each other at O. If we measure $\overline{SO}$, $\overline{OQ}$ and $\overline{OR}$, $\overline{PO}$ then we see that SO = OQ and PO = OR. So O is the midpoint of both $\overline{SQ}$ and $\overline{PR}$. So diagonals bisect each other at O. We accept this theorem without proof.

Theorem 10.6 : Diagonals of a parallelogram bisect each other.

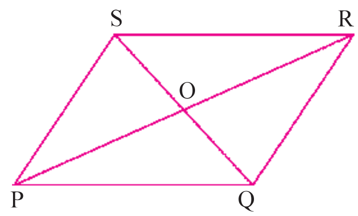

Figure 10.27

In figure 10.27, □ PQRS is parallelogram. The diagonals $\overline{PR}$ and $\overline{SQ}$ bisect each other at O.

PO = OR and SO = OQ

Converse of this theorem is also true. We accept this theorem without proof.

Theorem 10.7 If the diagonals of a quadrilateral bisect each other, then the quadrilateral is a parallelogram.

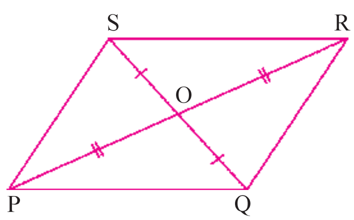

Figure 10.28

In the figure 10.28, the diagonals $\overline{PR}$ and $\overline{SQ}$ bisect each other at O. So $\overline{PO} \cong \overline{OR}$ and $\overline{SO} \cong \overline{OQ}$. □ PQRS is a parallelogram.

Example 4 : In □^{m} ABCD, $m\angle A$ = 75 and $m\angle DBC$ = 60. Find $m\angle CDB$ and $m\angle ADC$.

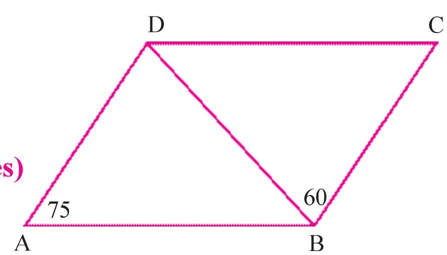

Figure 10.29

Solution : □ ABCD is a parallelogram.

$\overline{AD} \parallel \overline{BC}$ and $\overleftrightarrow{BD}$ is their transversal.

∴ ∠ADB ≅ ∠DBC **(alternate angles)**

But $m\angle DBC = 60$

∴ $m\angle ADB = 60$

In ΔABD, $m\angle A + m\angle ADB + m\angle DBA = 180$

$75 + 60 + m\angle DBA = 180$

$\therefore m\angle DBA = 180 - 135 = 45$

$\overline{CD} \parallel \overline{AB}$ and $\overleftrightarrow{BD}$ is their transversal.

$\angle DBA \cong \angle CDB$ **(alternate angles)**

$\therefore m\angle DBA = m\angle CDB$

$\therefore m\angle CDB = 45$

$\therefore m\angle ADC = m\angle ADB + m\angle CDB = 60 + 45 = 105$

Example 5 : If an angle of a parallelogram is a right angle, then prove that the parallelogram is a rectangle.

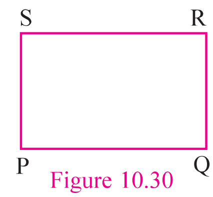

Figure 10.30

Solution : In $\square^m$ PQRS, $m\angle P = 90$

The opposite angles of a parallelogram are congruent.

$\therefore m\angle R = m\angle P = 90$

$\overleftrightarrow{PQ} \parallel \overleftrightarrow{SR}$ and $\overleftrightarrow{SP}$ is their transversal.

$\therefore$ $\angle P$ and $\angle S$ are the interior angles on the same side of the transversal $\overleftrightarrow{SP}$.

$\therefore m\angle P + m\angle S = 180$

But $m\angle P = 90$. So $m\angle S = 90$

Hence $m\angle Q = 90$ **(opposite angles in a parallelogram)**

$m\angle P = m\angle Q = m\angle R = m\angle S = 90$

$\square^m$ PQRS is a rectangle.

An Important result (1) : Show that the diagonals of a rhombus are perpendicular to each other. Diagonals bisect the angles at the vertices.

Solution : $\square$ ABCD is a rhombus.

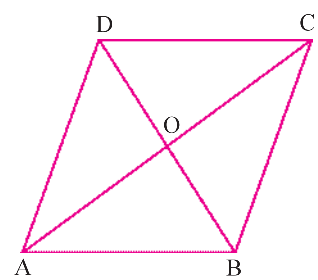

Figure 10.31

So, AB = BC = CD = DA.

$\square$ ABCD is also a parallelogram.

$\therefore$ Diagonals $\overline{AC}$ and $\overline{BD}$ bisect each other at O.

$\overline{AO} \cong \overline{OC}$, $\overline{DO} \cong \overline{OB}$ **(i)**

Now for the correspondence AOD $\leftrightarrow$ COD of ΔAOD and ΔCOD.

$\overline{AO} \cong \overline{CO}$

$\overline{OD} \cong \overline{OD}$ **(by (i))**

$\overline{AD} \cong \overline{CD}$ **(given)**

Thus, the correspondence AOD $\leftrightarrow$ COD is a congruence. **(SSS)**

$\therefore \Delta AOD \cong \Delta COD$ **(ii)**

$\angle AOD \cong \angle COD$ **(iii)**

But $m\angle AOD + m\angle COD = 180$ **(linear pair of angles)**

$\therefore$ $2m\angle AOD = 180$ **(by (iii))**

$\therefore$ $m\angle AOD = 90$

$\therefore$ $m\angle COD = 90$

$\therefore$ **Diagonals of a rhombus bisect each other at right angles.**

Also $\angle ODA \cong \angle ODC$ **(by (ii))**

but D – O – B.

$\therefore$ $\angle BDA \cong \angle BDC$

$\therefore$ **Diagonal $\overline{BD}$ bisects $\angle D$.**

Similarly we can prove that $\overline{BD}$ bisects $\angle B$, diagonal $\overline{AC}$ bisects $\angle A$ and $\angle C$

An Important result (2) : Prove that the diagonals of a square are congruent and perpendicular to each other.

Solution : For the correspondence ADB $\leftrightarrow$ BCA of ΔADB and ΔBCA.

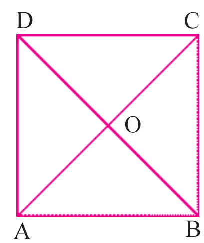

Figure 10.32

$\overline{AD} \cong \overline{BC}$ **(given)**

$\angle BAD \cong \angle ABC$ **(right angles)**

and $\overline{AB} \cong \overline{BA}$

$\therefore$ The correspondence ADB $\leftrightarrow$ BCA is a congruence. **(SAS)**

$\therefore$ Δ ADB $\cong$ Δ BCA

$\therefore$ $\overline{DB} \cong \overline{CA}$

$\therefore$ Diagonals are congruent.

Note : For the rest of the proof refer to previous result (1).

EXERCISE 10.3

1. In $\square^m$ PQRS, $m\angle P : m\angle Q = 5 : 4$. Find the measure of each angle.

2. In $\square^m$ DEFG, if $m\angle DFG = 60$, then find $m\angle FDE$.

3. In $\square^m$ ABCD, $m\angle A - m\angle B = 30$. Find $m\angle C$ and $m\angle D$.

4. In $\square^m$ PQRS, $m\angle P = 3x$ and $m\angle Q = 6x$. Find the measures of all the angles.

5. Prove that in $\square^m$ ABCD, the bisectors of $\angle C$ and $\angle D$ intersect each other at right angles.

6. The diagonals of a rectangle PQRS intersect at O. If $m\angle POS = 54$, find the measure of $\angle OPS$.

7. $\square$ ABCD is a square. Find the measure of $\angle DCA$.

8. $\square$ ABCD is a rectangle. If $m\angle BAC = 30$, find the measure of $\angle DBC$.

9. $\square$ DEFG is a rhombus. $m\angle DFE = 50$. Find the measures of $\angle DFG$ and $\angle DGE$.

10. $\square$ ABCD is square. $\overline{AC}$ and $\overline{BD}$ intersect at O. Find the measure of $\angle AOB$.

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10.7 Another Condition for a Quadrilateral to be a Parallelogram

If we construct a quadrilateral in such a way that the sides in only one pair of opposite sides are congruent and parallel, then the quadrilateral is also a parallelogram. We accept this theorem stated below without proof :

Theorem 10.8 : If in a quadrilateral, one pair of opposite sides consists of congruent and parallel line-segments, then the quadrilateral is a parallelogram.

In □ ABCD, $\overline{AB} \cong \overline{CD}$ and $\overline{AB} \parallel \overline{CD}$.

∴ □ ABCD is a parallelogram.

Now, we shall apply above theorem to an illustration.

Example 6 : $\overline{AB}$ and $\overline{CD}$ are the sides of $□^m$ABCD and their midpoint are P and R respectively. $\overline{AR}$ intersect $\overline{DP}$ in the point S and $\overline{BR}$ intersects $\overline{CP}$ in the point Q. Prove that □ PQRS is a parallelogram.

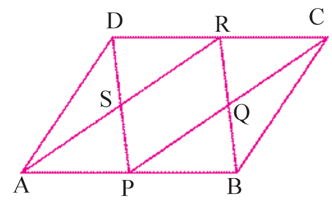

Figure 10.33

Solution : □ ABCD is a parallelogram.

∴ AB = CD

P and R are midpoints of $\overline{AB}$ and $\overline{CD}$ respectively.

AP = $\frac{1}{2}$ AB and CR = $\frac{1}{2}$ CD

∴ $\overline{AP} \cong \overline{CR}$ **(AB = CD) (i)**

Also, $\overline{AB} \parallel \overline{CD}$, A – P – B and C – R – D

∴ $\overline{AP} \parallel \overline{CR}$ **(ii)**

From (i) and (ii), $\overline{AP} \cong \overline{CR}$ and $\overline{AP} \parallel \overline{CR}$

□ APCR is a parallelogram.

∴ $\overline{AR} \parallel \overline{PC}$

∴ $\overline{SR} \parallel \overline{PQ}$ **(S ∈ $\overleftrightarrow{AR}$ and Q ∈ $\overleftrightarrow{PC}$) (iii)**

Similarly it can be proved that □ DRBP is a parallelogram.

∴ $\overline{BR} \parallel \overline{DP}$

∴ $\overline{RQ} \parallel \overline{SP}$ **(Q ∈ $\overleftrightarrow{BR}$ and S ∈ $\overleftrightarrow{PD}$) (iv)**

From (iii) and (iv), in □ PQRS, $\overline{SR} \parallel \overline{PQ}$ and $\overline{RQ} \parallel \overline{SP}$

∴ □ PQRS is a parallelogram.

An Important result : If the diagonals of a parallelogram are perpendicular to each other, then it is a rhombus.

Solution : In $□^m$ ABCD diagonals bisect each other at O.

∴ OA = OC

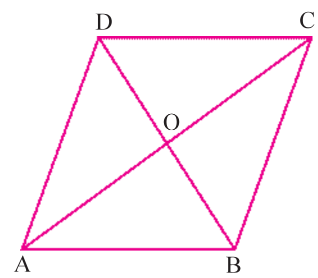

Figure 10.34

Now for the correspondence AOD $\leftrightarrow$ COD of ΔAOD and ΔCOD.

$\overline{OA} \cong \overline{OC}$

$\angle AOD \cong \angle COD$ **(right angles)**

$\overline{OD} \cong \overline{OD}$

$\therefore$ By SAS, the correspondence AOD $\leftrightarrow$ COD is a congruence.

$\therefore$ Δ AOD $\cong$ Δ COD

$\therefore$ AD = CD

but □ ABCD is parallelogram.

AD = BC and CD = AB

AD = BC = CD = AB

$\square^m$ ABCD is a rhombus.

An Important result : If the diagonals of a parallelogram are congruent and intersect at right angles, then the parallelogram is a square.

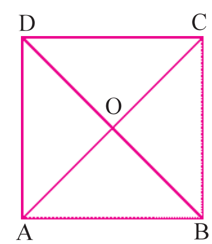

Figure 10.35

Solution : For correspondence AOB $\leftrightarrow$ AOD of ΔAOB and ΔAOD,

$\overline{AO} \cong \overline{AO}$

$\angle AOB \cong \angle AOD$ **(right angles)**

$\overline{OB} \cong \overline{OD}$

$\therefore$ By SAS, the correspondence AOB $\leftrightarrow$ AOD is congruence.

$\therefore$ AB = AD

But AB = CD and AD = BC

$\therefore$ AB = AD = CD = BC **(i)**

For the correspondence ABD $\leftrightarrow$ BAC of ΔABD and ΔBAC,

$\overline{AB} \cong \overline{BA}$

$\overline{AD} \cong \overline{BC}$

and $\overline{BD} \cong \overline{AC}$ **(given)**

By SSS, the correspondence ABD $\leftrightarrow$ BAC is a congruence.

$\therefore$ $\angle DAB \cong \angle CBA$

$\therefore$ $m\angle DAB = m\angle CBA$

But $\overleftrightarrow{AD} \parallel \overleftrightarrow{BC}$ and $\overleftrightarrow{AB}$ is a transeversal.

$m\angle DAB + m\angle CBA = 180$ **(interior angles on one side)**

$m\angle DAB = m\angle CBA = 90$ **(ii)**

From (i) and (ii) in $\square^m$ABCD, all the sides are congruent and all the angles are right angles.

$\therefore$ $\square^m$ ABCD is a square.

EXERCISE 10.4

1. Two sides of a rectangle have lengths 6 *cm* and 8 *cm*. Verify that the measures of the diagonals of the rectangle are same.

2. The perimeter of rectangle PQRS is 70 *cm*. If PQ : QR = 3 : 4, then find QR.

3. In rhombus ABCD, if AC = 10 *cm* and BD = 24 *cm*, then find the perimeter of rhombus ABCD.

4. $\square^m$ ABCD is neither a square nor a rhombus. Then prove that bisectors of its angles form a rectangle.

5. In $\square^m$ ABCD, $\overline{AP}$ and $\overline{CQ}$ are perpendicular from vertices A and C respectively to diagonal $\overline{BD}$. Prove that $\overline{AP} \cong \overline{CQ}$.

6. If the diagonals of a parallelogram are congruent, then prove that it is a rectangle.

7. $\square$ XYZW is a rectangle. If XY + YZ = 7 and XZ + YW = 10, then find XY.

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10.8 The Mid-point Theorem

We studied the properties of a parallelogram. Using them we shall study some properties of triangles and parallel lines.

In Δ ABC, E and F are the midpoints of the sides $\overline{AB}$ and $\overline{AC}$ respectively. If we measure $\overline{EF}$ and $\overline{BC}$, then we see that EF = $\frac{1}{2}$BC. We accept the theorem stated below without proof.

Theorm 10.9 : The line-segment joining the midpoints of two sides of a triangle is parallel to the third side and its measure is half the measure of the third side.

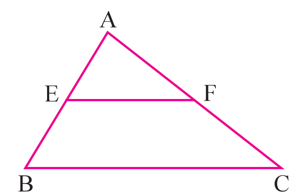


Figure 10.36

In Δ ABC, E and F are the midpoints of the sides $\overline{AB}$ and $\overline{AC}$ respectively. (i) $\overline{EF} \parallel \overline{BC}$ (ii) EF = $\frac{1}{2}$ BC.

We accept the following theorem without proof.

Theorem 10.10 A line passing through the midpoint of the one side and parallel to another side of a triangle bisects the third side of the triangle.

In Δ ABC, E is the midpoint of $\overline{AB}$. l is the line passing through E and parallel to $\overline{BC}$. l bisects $\overline{AC}$.

The following examples will help us in understanding the concept.

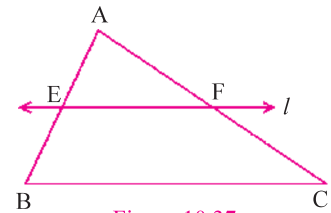


Figure 10.37

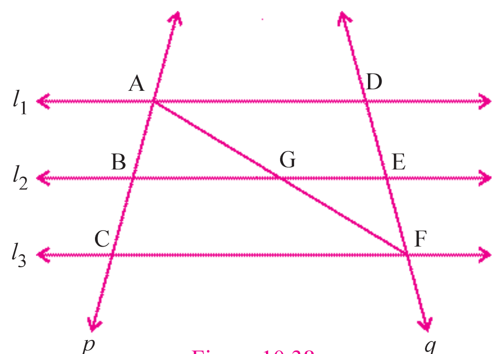


Figure 10.38

Example 7 : l_1, l_2 and l_3 are three parallel lines intersected by transversal p and q such that l_1, l_2 and l_3 cut off congruent intercepts $\overline{AB}$ and $\overline{BC}$ on p. Show that l_1, l_2 and l_3 cut off congruent intercepts $\overline{DE}$ and $\overline{EF}$ on q also.

Solution : We have AB = BC. **(given)**

Let $\overline{AF}$ intersect l_2 at G.

In Δ ACF, it is given that B is the midpoint of $\overline{AC}$ and $\overline{BG} \parallel \overline{CF}$ **($l_2 \parallel l_3$)**

$\therefore$ G is the midpoint of $\overline{AF}$

We apply the same theorem to Δ AFD. G is the midpoint of $\overline{AF}$. $\overline{GE} \parallel \overline{AD}$ and so by the theorem, E is the midpoint of $\overline{DF}$.

$\therefore$ $\overline{DE} \cong \overline{EF}$

In other words l_1, l_2 and l_3 cut off congruent intercepts on q also.

Example 8 : Δ ABC is an isosceles triangle with AB = AC and Let D, E and F be the midpoints of $\overline{BC}$, $\overline{CA}$ and $\overline{AB}$ respectively. Show that $\overline{AD} \perp \overline{EF}$ and $\overline{AD}$ bisects $\overline{EF}$.

Solution : In Δ ABC, D is the midpoint of $\overline{BC}$ and E is the midpoint of $\overline{AC}$.

$\therefore$ $\overleftrightarrow{DE} \parallel \overleftrightarrow{AB}$ and DE = $\frac{1}{2}$ AB **(i)**

Also AF = $\frac{1}{2}$ AB. **(ii)**

From (i) and (ii), DE = AF and $\overline{DE} \parallel \overline{AF}$. **(A–F–B)**

$\therefore$ $\square$ AFDE is a parallelogram.

$\therefore$ $\overline{AD}$ bisects $\overline{EF}$. **(iii)**

F and E are midpoints of $\overline{AB}$ and $\overline{AC}$ respectively.

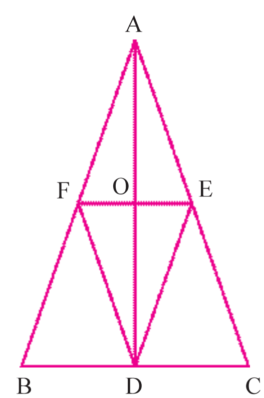


Figure 10.39

$\therefore$ AF = $\frac{1}{2}$ AB and AE = $\frac{1}{2}$ AC

But AB = AC **(given)**

$\therefore$ AE = AF **(iv)**

From (iii) and (iv), □ AFDE is a rhombus.

$\therefore$ $\overline{AD} \perp \overline{EF}$

Example 9 : Δ ABC is a triangle right angled at B and P is the midpoint of $\overline{AC}$ $\overline{PQ} \parallel \overline{BC}$ and Q $\in$ $\overline{AB}$. Prove that (i) $\overline{PQ} \perp \overline{AB}$ (ii) Q is the midpoint of $\overline{AB}$ (iii) PB = PA = $\frac{1}{2}$AC

Solution : P is the midpoint of $\overline{AC}$ **(given)**

Also $\overleftrightarrow{PQ} \parallel \overleftrightarrow{BC}$

$\overline{PQ}$ intersects $\overline{AB}$ at Q.

$\angle AQP \cong \angle ABC$.

But $m\angle ABC = 90$ **(given)**

$\therefore$ $m\angle AQP = 90$

$\therefore$ $\overline{PQ} \perp \overline{AB}$

In Δ ABC, P is the midpoint of $\overline{AC}$ and $\overline{PQ} \parallel \overline{BC}$. So Q is the midpoint of $\overline{AB}$.

$\therefore$ AQ = BQ

Now in ΔAPQ and ΔBPQ, consider the correspondence APQ $\leftrightarrow$ BPQ,

$\overline{AQ} \cong \overline{BQ}$

$\angle AQP \cong \angle BQP$ **(right angles)**

$\overline{PQ} \cong \overline{PQ}$

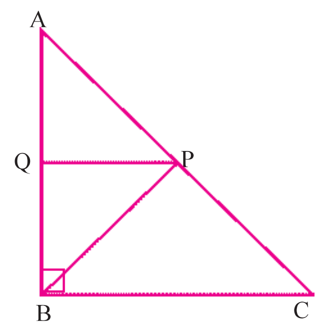


Figure 10.40

$\therefore$ The correspondence APQ $\leftrightarrow$ BPQ is a congruence by SAS.

$\therefore$ $\Delta APQ \cong \Delta BPQ$

$\therefore$ $\overline{PA} \cong \overline{PB}$

But P is the midpoint of $\overline{AC}$.

$\therefore$ PA = PB = $\frac{1}{2}$AC

Example 10 : In Δ ABC, $\overline{AD}$ is the median. E is the midpoint of $\overline{AD}$. $\overrightarrow{BE}$ intersects $\overline{AC}$ in F. Prove that AF = $\frac{1}{3}$AC.

Solution : Let $\overline{DK} \parallel \overline{BF}$ and K $\in$ $\overline{AC}$. In Δ ADK, E is the midpoint of $\overline{AD}$ and $\overline{EF} \parallel \overline{DK}$.

$\therefore$ F is the midpoint of $\overline{AK}$.

$\therefore$ AF = FK **(i)**

In Δ BCF, D is the midpoint of $\overline{BC}$ and $\overline{DK} \parallel \overline{BF}$

$\therefore$ K is the midpoint of $\overline{FC}$.

$\therefore$ FK = KC **(ii)**

From (i) and (ii), we have

AF = FK = KC

$\therefore$ AC = AF + FK + KC

$\therefore$ AC = AF + AF + AF

$\therefore$ AF = $\frac{1}{3}$ AC

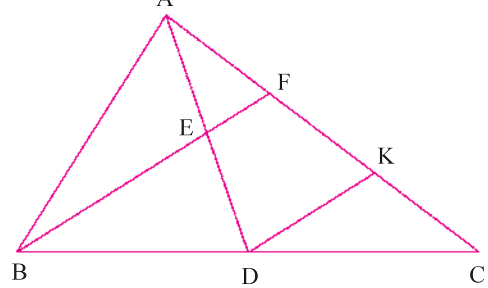


Figure 10.41

10.9 An Important Result

In a trapezium ABCD, $\overline{AB} \parallel \overline{CD}$. E and F are the midpoints of $\overline{AD}$ and $\overline{BC}$ respectively. Prove that $\overline{EF} \parallel \overline{AB}$ and EF = $\frac{1}{2}$ (AB + CD).

Solution : $\overrightarrow{DF}$ and $\overrightarrow{AB}$ intersect at P, so that A–B–P and D–F–P.

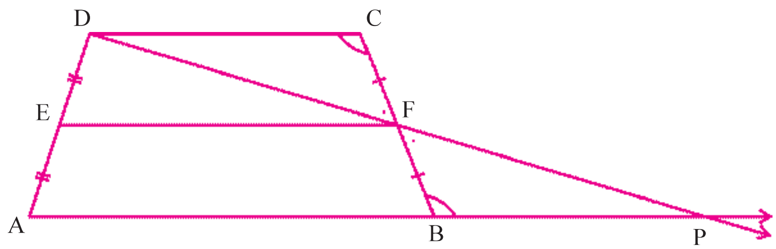


Figure 10.42

In the correspondence BPF $\leftrightarrow$ CDF of ΔBPF and ΔCDF.

$\angle$BFP $\cong$ $\angle$CFD **(Vertically opposite angles)**

$\overline{FB} \cong \overline{FC}$

and $\angle$FBP $\cong$ $\angle$FCD **(alternate angles made by transversal $\overleftrightarrow{BC}$ with $\overleftrightarrow{DC} \parallel \overrightarrow{BC}$)**

Thus, the correspondence BPF $\leftrightarrow$ CDF is a congruence **(ASA Theorem)**

So, $\overline{BP} \cong \overline{CD}$ and $\overline{PF} \cong \overline{DF}$. So, BP = CD and PF = DF.

So F is the midpoint of $\overline{DP}$. Now in Δ DAP, E is the midpoint of $\overline{DA}$ and F is the midpoint of $\overline{DP}$.

$\therefore$ $\overline{EF} \parallel \overline{AP}$ and EF = $\frac{1}{2}$AP

$\therefore$ $\overline{EF} \parallel \overline{AB}$ **(A – B – P)**

$\therefore$ EF = $\frac{1}{2}$AP = $\frac{1}{2}$(AB + BP)

$\therefore$ EF = $\frac{1}{2}$(AB + CD) **(BP = CD)**

Example 11 : In a trapezium PQRS, $\overline{PQ} \parallel \overline{SR}$ and PQ > SR. X and Y are midpoints of $\overline{SP}$ and $\overline{RQ}$ respectively. If SR = 12 and XY = 14.5, find PQ.

Solution : $XY = \frac{1}{2}(SR + PQ)$

$\therefore\ 14.5 = \frac{1}{2}(12 + PQ)$

$\therefore\ 29 = 12 + PQ$

$\therefore\ PQ = 17$

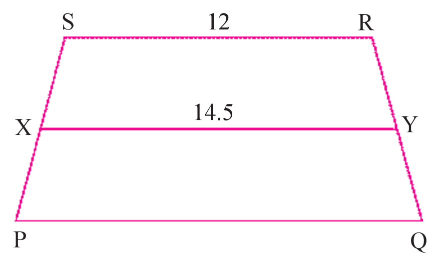


Figure 10.43

EXERCISE 10.5

1. In Δ ABC, the points E and F are the midpoints of $\overline{AB}$ and $\overline{AC}$. If EF = 6.5, then find BC.

2. In Δ DEF, the points X and Y are the midpoints of $\overline{DE}$ and $\overline{DF}$ respectively. If EF = 20, then find XY.

3. The perimeter of Δ XYZ is 25. P, Q and R are the midpoints of $\overline{XY}$, $\overline{YZ}$ and $\overline{ZX}$ respectively. Find perimeter of Δ PQR.

4. In Δ ABC, D, E and F are the mid points of $\overline{AB}$, $\overline{BC}$ and $\overline{CA}$ respectively. If AB = 9, BC = 12, CA = 18, find the perimeters of $\square$ DBCF and Δ CFE.

5. In a trapezium ABCD, $\overline{AB} \parallel \overline{CD}$, AB > DC. P and Q are the midpoints of $\overline{AD}$ and $\overline{CB}$ respectively. If AB = 15 and DC = 7, find PQ.

6. In a trapezium PQRS, $\overline{PQ} \parallel \overline{SR}$, PQ > SR. X and Y are the midpoints of $\overline{SP}$ and $\overline{QR}$ respectively. If XY = 7.5 and PQ = 12, then find RS.

7. In Δ ABC, the points P and Q are on $\overline{AB}$ and $\overline{AC}$ such that $AP = \frac{1}{4}AB$ and $AQ = \frac{1}{4}AC$. Prove that $PQ = \frac{1}{4}\ BC$.

8. In an equilateral Δ ABC, M and N are the midpoints of $\overline{AB}$ and $\overline{AC}$ respectively. If MN = 4.5, find the perimeter of Δ ABC.

9. In Δ ABC, E, F and G are the midpoints of $\overline{AB}$, $\overline{BC}$ and $\overline{AC}$ respectively. If EF + EG = 14 and AB = 7, find the perimeter of Δ ABC.

10. In Δ PQR, A, B and C are the midpoints of $\overline{PQ}$, $\overline{QR}$, $\overline{RP}$ respectively. If AB : BC : CA = 3 : 4 : 5 and QR = 20, find perimeter of Δ PQR.

11. In Δ ABC, D, E and F are the midpoints of $\overline{AB}$, $\overline{BC}$ and $\overline{CA}$ respectively. Prove that Δ ADF and Δ DBE, and Δ EFD and Δ FEC are congruent.

12. In Δ ABC, D, E and F are the midpoints of $\overline{BC}$, $\overline{CA}$ and $\overline{AB}$ respectively. Prove that $\overline{AD}$ and $\overline{EF}$ bisect each other.

13. In $\square$ ABCD, the midpoints of the sides $\overline{AB}$, $\overline{BC}$, $\overline{CD}$ and $\overline{DA}$ are P, Q, R and S respectively. Prove that $\square$ PQRS is a parallelogram.

14. If A, B, C, D are the midpoints of the sides $\overline{PQ}$, $\overline{QR}$, $\overline{RS}$ and $\overline{SP}$ of a rectangle PQRS, then prove that $\square$ ABCD is a rhombus.

15. In an equilateral Δ ABC, P, Q and R are the midpoints of $\overline{AB}$, $\overline{BC}$ and $\overline{CA}$. Prove that Δ PQR is equilateral.

EXERCISE 10

1. Solve the following :

(1) □ PQRS is a rhombus. If $m\angle QRS = 60$ and QS = 15, find the perimeter of the rhombus.

(2) □ DEFG is a rhombus. If DF = 30 and EG = 16, find the perimeter of □ DEFG.

(3) □ PQRS is a rectangle. If its diagonals intersect each other at O and $m\angle POS = 120$, find the $m\angle QPO$.

(4) In a trapezium PQRS, $\overline{PS} \parallel \overline{QR}$, QR > PS and X and Y are the midpoints of $\overline{PQ}$ and $\overline{SR}$. If PS = 18, XY = 20, find QR.

(5) In a triangle PQR, $m\angle P = 75$, $m\angle Q = 60$, $m\angle R = 45$. Find the measures of the angles of the triangle formed by joining the midpoints of the sides of this triangle.

2. In $\square^m$ PQRS, A is a point on $\overline{PS}$ such that AP = $\frac{1}{3}$PS and B is a point on $\overline{QR}$ such that RB = $\frac{1}{3}$QR, prove that □ APBR is a parallelogram.

3. Show that the quadrilateral, formed by joining the midpoints of the sides of a square in order is also a square.

4. The diagonals of a □ PQRS are perpendicular to each other. Show that the quadrilateral formed by joining the midpoints of its sides is a rectangle.

5. □ PQRS is a rhombus and A, B, C and D are the midpoints of $\overline{PQ}$, $\overline{QR}$, $\overline{RS}$ and $\overline{SP}$ respectively. Prove that □ ABCD is a rectangle.

6. In figure 10.44, in Δ PQR, $\overline{PA}$ is the median of Δ PQR and $\overline{AB} \parallel \overline{PQ}$. Prove that $\overline{QB}$ is a median Δ PQR.

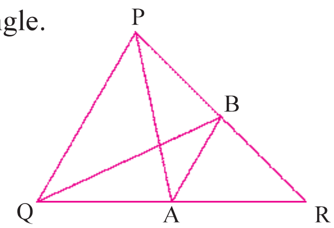


Figure 10.44

7. Select proper option (a), (b), (c) or (d) and write in the box given on the right so that the statement becomes correct :

(1) In $\square^m$ABCD, if $m\angle A : m\angle B = 2 : 3$, then $m\angle D$ is .... □

(a) 72 (b) 108 (c) 60 (d) 90

(2) In $\square^m$ ABCD, if $m\angle B - m\angle C = 40$, then $m\angle A$ is ...... □

(a) 70 (b) 110 (c) 55 (d) 35

(3) In $\square^m$ ABCD, $m\angle A : m\angle B = 1 : 3$, then $m\angle C$ is ...... □

(a) 90 (b) 120 (c) 45 (d) 135

(4) If the diagonals of quadrilateral are not congruent and bisect each other at right angles, then the quadrilateral is a ...... ☐

(a) square (b) rectangle (c) trapezium (d) rhombus

(5) The diagonals of a quadrilateral are congruent and bisect each other but not at right angles. Then the quadrilateral is a ...... ☐

(a) rectangle (b) rhombus (c) square (d) parallelogram

(6) All the four sides of a quadrilateral are congruent but all the four angles are not congruent. Then the quadrilateral is a .... ☐

(a) rhombus (b) square (c) rectangle (d) parallelogram

(7) All the four angles of a quadrilateral are congruent but all the four sides are not congruent. Then the quadrilateral is a ...... ☐

(a) rhombus (b) square (c) rectangle (d) trapezium

(8) A figure is formed by joining the midpoints of the sides of a quadrilateral. It is a ...... ☐

(a) square (b) rhombus (c) rectangle (d) parallelogram

(9) In rhombus PQRS if the diagonal PR = 8 and diagonal QS = 6, then perimeter of rhombus is ...... ☐

(a) 10 (b) 40 (c) 5 (d) 20

(10) The perimeter of rectangle ABCD is 36. If AB : BC = 4 : 5, then the length of $\overline{BC}$ is ...... ☐

(a) 8 (b) 16 (c) 10 (d) 9

(11) In ΔABC, D, E and F are the midpoints of $\overline{AB}$, $\overline{BC}$ and $\overline{CA}$ respectively. If the perimeter of Δ DEF is 12, then the perimeter of ΔABC is .... ☐

(a) 24 (b) 6 (c) 36 (d) 48

(12) Δ ABC is an equilateral triangle. AB = 6. The points P, Q and R are midpoints of $\overline{AB}$, $\overline{BC}$ and $\overline{CA}$ respectively. The perimeter of $\square$ PBCR is ...... ☐

(a) 18 (b) 15 (c) 9 (d) 12

(13) In trapezium ABCD, $\overline{AD} \parallel \overline{BC}$, BC > AD. Points P and Q are midpoints of $\overline{AB}$ and $\overline{CD}$. If AD = 6 and BC = 8, then the measure of $\overline{PQ}$ is ...... ☐

(a) 14 (b) 7 (c) 4 (d) 3

(14) In trapezium PQRS, $\overline{PS} \parallel \overline{QR}$, QR > PS and points M and N are the midpoints of $\overline{PQ}$ and $\overline{SR}$. If QR = 16 and MN = 14, then the measure of $\overline{PS}$ is ...... ☐

(a) 44 (b) 9 (c) 12 (d) 4

(15) In $\square^m$ PQRS the bisectors of $\angle P$ and $\angle Q$ intersect at X. If $m\angle P = 70$, then $m\angle PXQ$ is ...... ☐

(a) 90 (b) 35 (c) 55 (d) 110

(16) P and Q are the midpoints of $\overline{AB}$ and $\overline{AC}$ of Δ ABC. $\square$ PBCQ is a ...... ☐

(a) square (b) rhombus (c) trapezium (d) rectangle

(17) $\square$ ABCD is a rhombus. If the diagonals $\overline{AC}$ and $\overline{BD}$ intersect at M, then $m\angle AMB$ is ...... ☐

(a) 60 (b) 45 (c) 30 (d) 90

(18) $\square$ PQRS is a square. If PQ = 5, then QS is ...... ☐

(a) 10 (b) 50 (c) $5\sqrt{2}$ (d) 15

(19) Perimeter of rhombus PQRS is 96, then PQ is ...... ☐

(a) 24 (b) 48 (c) 12 (d) 6

*

Summary

In this chapter, we have learnt following points :

1. Plane quadrilateral and its parts

2. The sum of the measures of the angles of a quadrilateral

3. Types of quadrilateral

4. Properties of parallelograms and its theorems

5. Rhombus and its important result

(i) Diagonals of a rhombus are perpendicular to each other and vice-versa

(ii) Diagonals bisect the angle at vertices and vice-versa

6. Square and its properties

7. Diagonals of a square are congruent and perpendicular to each other and vice-versa.

8. The midpoint theorems for a triangle and vice-versa

9. For trapezium ABCD, $\overline{AB} \parallel \overline{CD}$ and E and F are midpoints of $\overline{AD}$ and $\overline{BC}$ then $EF = \frac{1}{2}(AB + CD)$.

●

CHAPTER 11

AREAS OF PARALLELOGRAMS AND TRIANGLES

11.1 Introduction

We have learnt earlier about areas of closed figures like triangles, quadrilaterals and circles. We know that **area is the 'measure' of the region enclosed by a closed figure in a plane. We know about units of area also.**

11.2 Interior of Triangle

We have learnt about interior of a triangle. The intersection of the interiors of all the three angles of a triangle is called the interior of the triangle. We also know that if we take the intersection of the interiors of any two angles of a triangle, then also we get the interior of the triangle.

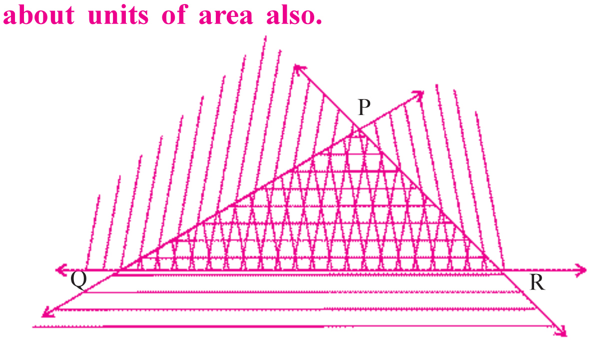


Figure 11.1

11.3 Triangular Region

For any ΔPQR, ΔPQR and interior of ΔPQR are two mutually disjoint sets. The union of these two sets is called the triangular region associated with ΔPQR.

Triangular region : The union of a triangle and its interior is called the triangular region associated with the given triangle. We denote the triangular region associated with the ΔPQR by Δ*PQR.

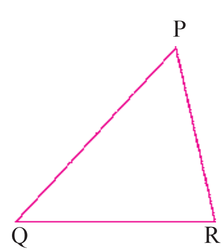


Figure 11.2 (i)

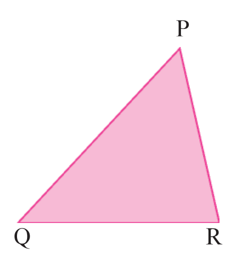


Figure 11.2 (ii)

ΔPQR is shown in figure 11.2(i) and triangular region Δ^*PQR as coloured region in figure 11.2(ii). $\Delta^*PQR = (\Delta PQR) \cup$ (interior of ΔPQR).

11.4 Interior of a Quadrilateral

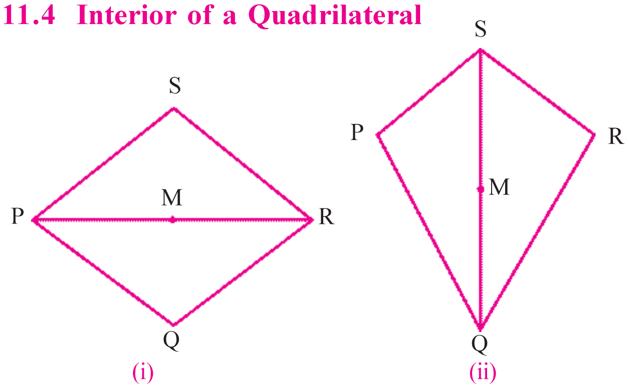


Figure 11.3

We have the concept of the interior of a triangle.

In figure 11.3 (i), we have a □ PQRS and $\overline{PR}$ is its diagonal. Then the interior of □ PQRS is the union of (1) The interior of Δ PSR (2) The interior of Δ PQR (3) The set of all the points M, such that P–M–R.

In figure 11.3 (ii), we have □ PQRS and $\overline{SQ}$ is its diagonal.

Then, the interior of □ PQRS is the union of (1) The interior of Δ PQS (2) The interior of Δ QRS (3) The set of all point M such that S–M–Q.

The intersection of the interiors of all the four angles of a quadrilateral is the interior of the quadrilateral.

If we take the intersection of the interiors of two opposite angles, then also we will get the interior of the quadrilateral.

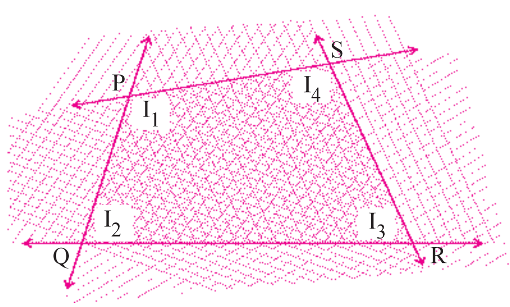


Figure 11.4

As in figure 11.4, let us denote interior of $\angle P$ by I_1, the interior of Q by I_2, the interior of R by I_3, the interior of S by I_4 and the interior of □ PQRS by I.

Then, $I = I_1 \cap I_2 \cap I_3 \cap I_4$

In □ PQRS, $\angle P$ and $\angle R$ are opposite angles. $\angle Q$ and $\angle S$ are opposite angles.

Then, $I = I_1 \cap I_3 = I_2 \cap I_4$

11.5 Quadrilateral Region

A quadrilateral and the interior of the quadrilateral are two mutually disjoint sets. The union of these two sets is called the quadrilateral region.

Figure 11.5 (i) shows □ PQRS and the coloured region in figure 11.5 (ii) shows the quadrilateral region of □ PQRS.

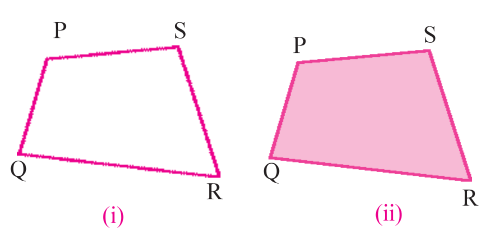


Figure 11.5

Quadrilateral region : The union of a quadrilateral and its interior is called the quadrilateral region associated with the given quadrilateral.

The quadrilateral region associated with □ PQRS contains all the points of □ PQRS as well as all the interior points of □ PQRS. The quadrilateral region associated with □ PQRS is denoted by □* PQRS.

Thus, □* PQRS = (□ PQRS) ∪ (interior of □ PQRS)

11.6 Postulates for Area

We know that area is a positive number and areas of congruent figures are equal. We shall take these natural ideas as postulates :

(1) The Postulate for Area : Corresponding to every triangular region, there is a unique positive number associated with it and it is called the area of the triangular region.

(2) Postulate for the Area of Congruent triangles : If two triangles are congruent, then the areas of their triangular regions are equal.

(3) Postulate for Addition of Areas : In Δ ABC, If B – D – C, then area of Δ*ABC = area of Δ*ABD + area of Δ*ADC

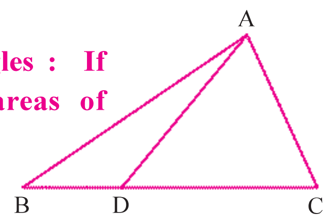


Figure 11.6

(Note that in the figure 11.6 interiors of Δ ABD and Δ ADC are mutually disjoint sets.)

If Δ* ABC is a union of several triangular regions, triangles having mutually disjoint interiors, then the area of Δ*ABC is the sum of the areas of these triangular regions. From now onwards, we shall denote the area of Δ*ABC by simply ABC and area of □ *PQRS by PQRS.

11.7 Area of a Rectangle

We know the formula to find the area of a rectangle.

Area of rectangle = length × breadth

We shall accept this idea in the form of a postulate.

Postulate for the area of a rectangle : The area of any rectangular region is the product of the lengths of any two adjacent sides of the rectangle.

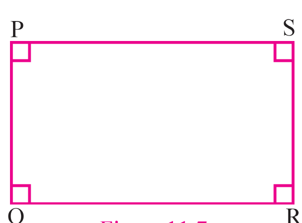


Figure 11.7

As shown in the figure 11.7, □ PQRS is a rectangle. Taking its adjacent sides $\overline{PQ}$ and $\overline{QR}$, we have, area of the rectangle PQRS, PQRS = PQ × QR.

Note : For the sake of simplicity, we shall use triangle for **'triangular region',** the words rectangle for **'rectangular region'** and side for the **'length of a side'** and similary quadrilateral for **'quadrilateral region'**.

Example 1 : The length of one side of a rectangle is thrice the length of its adjacent side. If the perimeter of the rectangle is 80 *cm*, find the area of the rectangle.

Solution : Let $\overline{DE}$ and $\overline{EF}$ be two adjacent sides of the rectangle DEFG. If the length of $\overline{DE}$ is x *cm*, then the length of $\overline{EF}$ is $3x$ *cm*. The perimeter of rectangle = 80 *cm*

$\therefore$ $2(x + 3x) = 80$

$\therefore$ $8x = 80$

$\therefore$ $x = 10$ *cm*

$\therefore$ $3x = 30$ *cm*

$\therefore$ DEFG $=$ DE $\times$ EF

$= 10 \times 30 = 300$ cm^2

$\therefore$ The area of the rectangle is 300 cm^2

Figure 11.8

11.8 The Area of a Right Triangle

The area of a right triangle is half the product of its sides forming the right angle.

In the figure 11.9, $\square$ PQRS is a rectangle and $\overline{PR}$ is diagonal.

Δ PQR is a right triangle with base $\overline{QR}$ and $\overline{PQ}$ is its altitude.

But since $\Delta PQR \cong \Delta RSP$, PQR = RSP

Also ΔPQR and ΔRSP have disjoint interiors.

$\therefore$ PQRS = PQR + RSP = PQR + PQR = 2 PQR

$\therefore$ PQR = $\frac{1}{2}$ PQRS

Now, PQRS = PQ $\times$ QR

Hence, PQR = $\frac{1}{2} \times$ QR $\times$ PQ

Hence, PQR = $\frac{1}{2}$ base $\times$ altitude

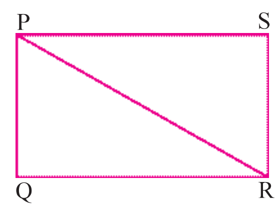


Figure 11.9

Example 2 : In a right triangle, the measure of one side is 12 *cm* and that of the hypotenuse is 13 *cm*. Find the area of the right triangle.

Solution : Let $\angle B$ be the right angle in Δ ABC.

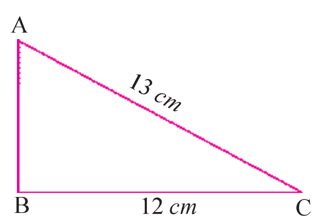


Figure 11.10

BC = 12 *cm* and AC = 13 *cm*.

In right triangle Δ ABC

$AC^2 = AB^2 + BC^2$

$\therefore$ $AB^2 = AC^2 - BC^2$

$= (13)^2 - (12)^2$

$= 169 - 144$

$= 25$

$\therefore$ AB $= 5$ *cm*

$\therefore$ Area of right triangle ABC $= \frac{1}{2} \times AB \times BC$

$= \frac{1}{2} \times 5 \times 12 = 30\ cm^2$

$\therefore$ The area of the right triangle is 30 cm^2.

11.9 Area of Triangle

The area of a triangle is one half the product of length of its altitude and the base corresponding to the altitude.

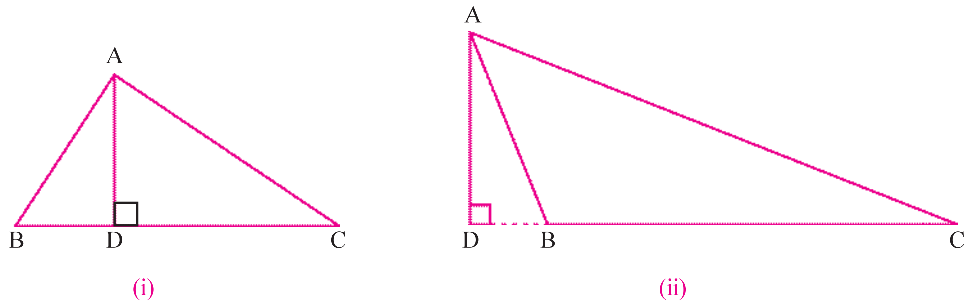


Figure 11.11

In figure 11.11 (i) $\overline{AD}$ is an altitude of Δ ABC, $\overline{BC}$ the corresponding base and B–D–C. Also ΔABC and ΔABD have disjoint interiors.

ABC = ABD + ADC **(postulate for addition of area)**

$= \frac{1}{2}$ AD $\times$ BD + $\frac{1}{2}$ AD $\times$ DC

$= \frac{1}{2}$ AD (BD + DC)

$\therefore$ ABC = $\frac{1}{2} \times$ AD $\times$ BC **(B – D – C)**

In figure 11.11 (ii), $\overline{AD}$ is the altitude to $\overleftrightarrow{BC}$ and it intersects $\overleftrightarrow{BC}$ in D such that D–B–C. $\overline{BC}$ is the base corresponding to the altitude $\overline{AD}$.

ΔABC and ΔADB have disjoint interiors.

$\therefore$ ADC = ADB + ABC

ABC = ADC – ADB **(postulate for addition of area)**

$= \frac{1}{2}$ AD $\times$ DC – $\frac{1}{2}$ AD $\times$ DB

$= \frac{1}{2}$ AD (DC – DB)

$= \frac{1}{2}$ AD $\times$ BC **(D – B – C)**

Every triangle has three altitudes and three corresponding bases so the **formula for area gives the area of the same triangle in three different ways.** However, for the same triangle, we get the same area by using any of these pairs of base and altitude.

11.10 Area of Parallelogram

A line-segment drawn from any vertex of a parallelogram and perpendicular to the line containing a side of the parallelogram which does not pass through that vertex, is called an altitude of the parallelogram and the side is called the base corresponding to the altitude.

In figure 11.12, sides $\overline{QR}$ and $\overline{SR}$ of $\square^m$ PQRS do not pass through vertex P. Line-segment $\overline{PM}$ passes through P and is perpendicular to $\overleftrightarrow{QR}$. So $\overline{QR}$ is the corresponding base and $\overline{PM}$ is the altitude.

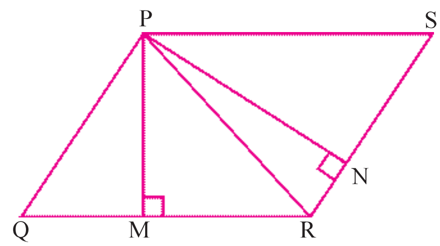


Figure 11.12

$\overline{PR}$ is a diagonal of $\square^m$ PQRS. Hence, Δ PQR $\cong$ Δ RSP. Also ΔPQR and ΔRSP have disjoint interiors. Thus area of $\square^m$ PQRS is twice the area of Δ PQR.

$$\text{PQRS} = 2\ (\text{PQR})$$
$$= 2\ (\tfrac{1}{2}\text{PM} \times \text{QR}) = \text{PM} \times \text{QR}$$

Hence, PQRS = altitude $\times$ corresponding base. Similarly, in figure 11.12, $\overline{SR}$ is also a side which does not pass through P. $\overline{PN}$ is the perpendicular line-segment from P to $\overleftrightarrow{SR}$. It is an altitude of $\square^m$ PQRS. Its corresponding base is $\overline{SR}$.

Since $\overline{PR}$ is the diagonal of $\square^m$ PQRS, Δ RSP $\cong$ Δ PQR. Hence the area of $\square^m$ PQRS is twice of Δ RSP.

As before PQRS = PN $\times$ SR

Thus, **the area of a parallelogram is the product of any of its altitude and its corresponding base.**

Note : Henceforth we will not mention about disjoint interiors, if it is obvious.

Example 3 : $\overline{EM}$ and $\overline{EN}$ are altitudes of $\square^m$ DEFG. Their corresponding bases are $\overline{DG}$ and $\overline{GF}$ respectively. If DG = 10 *cm*, EM = 8 *cm*, EN = 16 *cm*, find GF.

Solution : DEFG = EM $\times$ DG = EN $\times$ GF

$\therefore$ EM $\times$ DG = EN $\times$ GF

$\therefore$ $8 \times 10 = 16 \times$ GF

$\therefore$ GF = $\frac{8\times10}{16}$ = 5

$\therefore$ GF = 5 *cm*

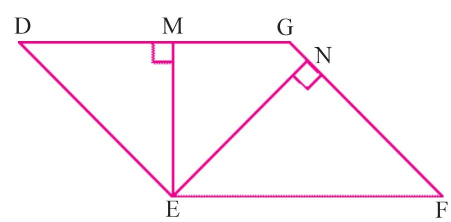


Figure 11.13

An Important Result : The area of a rhombus is half the product of its diagonals.

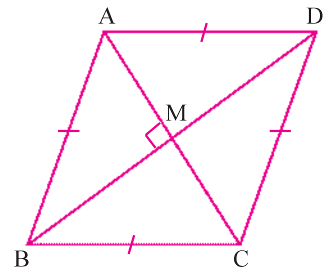


Figure 11.14

As shown in the figure 11.14, □ ABCD is a rhombus. Its diagonals $\overline{AC}$ and $\overline{BD}$ bisect each other at right angles at point M.

Hence $\overline{BM}$ and $\overline{MD}$ are altitudes to base $\overline{AC}$ in Δ ABC and Δ ACD respectively.

$$\begin{aligned} \text{Now } ABCD &= ABC + ACD \\ &= \frac{1}{2} AC \times BM + \frac{1}{2} AC \times MD \\ &= \frac{1}{2} AC (BM + MD) \\ &= \frac{1}{2} AC \times BD \end{aligned}$$

(B – M – D)

Example 4 : □ PQRS is a rhombus. The length of each side is 10 *cm*. If QS = 16 *cm*, find the area of □ PQRS.

Solution : □ PQRS is rhombus. Diagonals $\overline{SQ}$ and $\overline{PR}$ bisect each other at M at right angles.

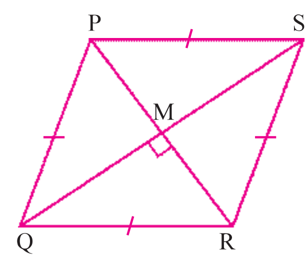


Figure 11.15

QS = 16 *cm* and M is the midpoint of $\overline{QS}$.

∴ QM = 8 *cm*

Now in right Δ PMQ,

$PM^2 = PQ^2 - QM^2 = (10)^2 - (8)^2 = 100 - 64 = 36$

∴ PM = 6 *cm*

∴ PR = 12 *cm*

$PQRS = \frac{1}{2} \times PR \times QS = \frac{1}{2} \times 12 \times 16 = 96\ cm^2$

∴ The area of the rhombus is 96 cm^2.

EXERCISE 11.1

1. State whether the following statements are true or false.

(1) A triangle and its triangular region are two disjoint sets.

(2) The intersection of a triangle and its interior is the empty set.

(3) If D, E and F are the midpoints of the sides of Δ PQR, then $\Delta^* DEF \cup \Delta^* PQR = \Delta^* PQR$.

(4) Every triangle is a subset of its triangular region.

(5) Interior of a triangle is a subset of its triangular region.

2. (1) In $\square^m$ ABCD, $\overline{CF} \perp \overline{AB}$ and $\overline{AE} \perp \overline{BC}$. If AB = 18 *cm*, AE = 10 *cm* and CF = 12 *cm*, find AD.

(2) If AD = 12 *cm*, CF = 20 *cm* and AE = 16 *cm*, find AB.

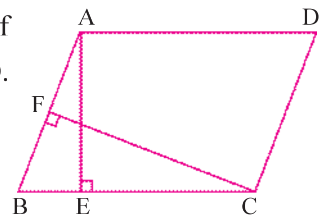


Figure 11.16

3. Let $\square^m$ ABCD be a parallelogram having area 250 cm^2. If E and F are the mid points of sides $\overline{AB}$ and $\overline{CD}$ respectively, then find the area of $\square$ AEFD.

4. In Δ ABC, $\overline{AD}$ is the altitude corresponding to base $\overline{BC}$. $\overline{BE}$ is the altitude corresponding to base $\overline{AC}$. If AD = 14, BC = 24 and AC = 35, find BE.

5. In Δ ABC, $\overline{BF}$ is the altitude to $\overline{AC}$ and $\overline{AE}$ is the altitude to $\overline{BC}$. If AC = 45 *cm*, BC = 15 *cm* and ABC = 225 cm^2, find BF and AE.

6. In $\square^m$ ABCD, $\overline{AM}$ and $\overline{BN}$ are altitudes and their corresponding bases are $\overline{BC}$ and $\overline{CD}$ respectively. If AM = 18, AB = 24, BC = 30, find BN.

7. Δ ABC is an equilateral triangle. If BC = 8, find ABC

8. In Δ ABC, P, Q and R are the midpoints of $\overline{AB}$, $\overline{BC}$ and $\overline{AC}$ respectively. If ABC = 64 cm^2. Find PQR, PQCR and PBCR.

9. In Δ ABC $m\angle B$ = 90, AB = 18 *cm*, BC = 24 *cm*, find ABC. Also find the measure of the altitude corresponding to $\overline{AC}$.

10. $\square$ ABCD is a rhombus. If AB = 25 and AC = 48, find ABCD.

*

11.11 Quadrilaterals on the Same Base and Between Two Parallel Lines

Let us observe the following figures 11.17 :

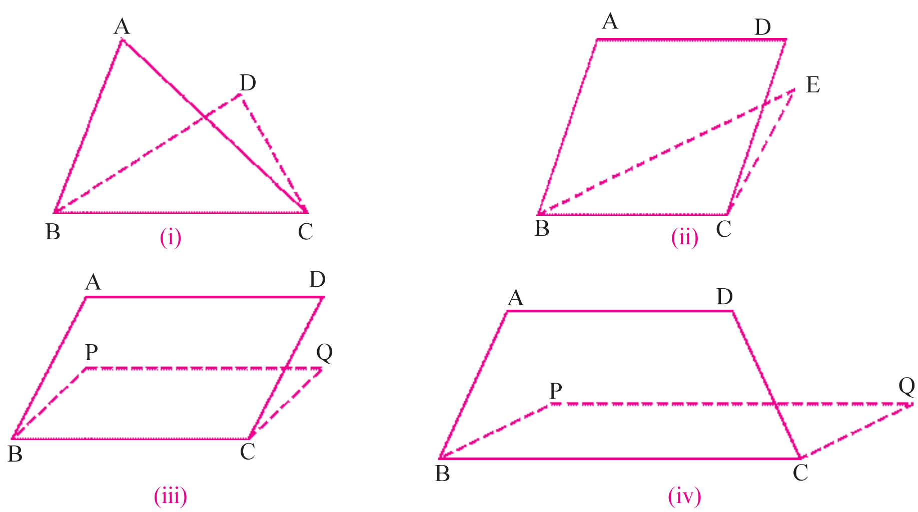
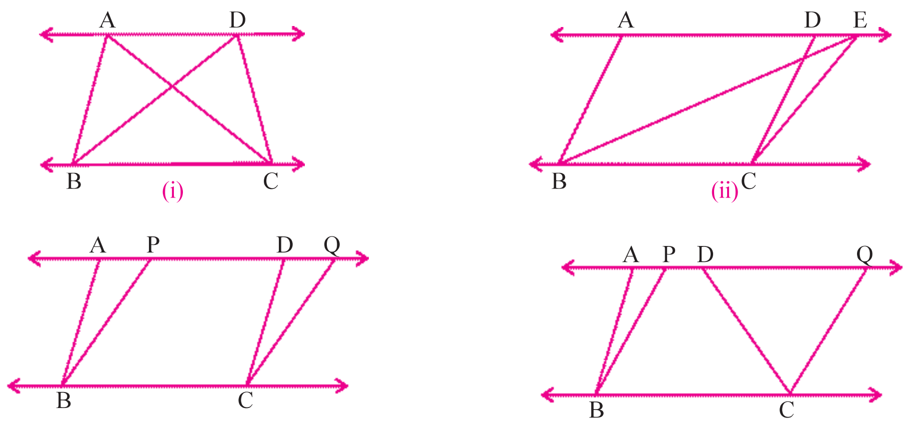


Figure 11.17

In figure 11.17 (i) ΔABC and ΔDBC have a common (same) base $\overline{BC}$. In figure 11.17 (ii) $\square^{m}$ ABCD and ΔEBC have the same base $\overline{BC}$. In figure 11.17 (iii) $\square^{m}$ ABCD and $\square^{m}$ PBCQ have the same base $\overline{BC}$. In figure 11.17 (iv) trapezium ABCD with $\overline{AD} \parallel \overline{BC}$ and $\square^{m}$ PBCQ have the same base $\overline{BC}$.

Now look at the following figure 11.18 :



(iii) Figure 11.18 (iv)

In figure 11.18(i), we observe that ΔABC and ΔDBC are on same base $\overline{BC}$ and lie between two parallel lines $\overleftrightarrow{BC}$ and $\overleftrightarrow{AD}$. Vertices A and D of ΔABC and of ΔDBC are on the same side of the line containing the base $\overline{BC}$.

In figure 11.18(ii), $\square^{m}$ ABCD and ΔEBC are on same base $\overline{BC}$ and lie between two parallel lines $\overleftrightarrow{BC}$ and $\overleftrightarrow{AD}$. Vertices A and D of $\square^{m}$ ABCD and vertex E of ΔEBC are on same line $\overleftrightarrow{AE}$ and are on the same side of the line containing the base $\overline{BC}$.

In figure 11.18(iii), $\square^{m}$ ABCD and $\square^{m}$ PBCQ are on same base $\overline{BC}$ and lie between two parallel lines $\overleftrightarrow{BC}$ and $\overleftrightarrow{AQ}$. Vertices A and D of $\square^{m}$ ABCD and vertices P and Q of $\square^{m}$ PBCQ are on same line $\overleftrightarrow{AQ}$ and are on the same side of the line containing the base $\overline{BC}$.

In figure 11.18(iv), trapezium ABCD and $\square^{m}$ PBCQ are on same base $\overline{BC}$ and lie between two parallel lines $\overleftrightarrow{BC}$ and $\overleftrightarrow{AQ}$. Vertices A and D of trapezium ABCD and vertices A and Q of $\square^{m}$ ABCQ are on same line $\overleftrightarrow{AQ}$ and are on the same side of the line containing the base $\overline{BC}$.

We observed that a triangle and a quadrilateral, two figures have same base and are between two parallel lines and the vertices (or vertex) lie on a line parallel to the base. What can we say about the areas of such figures ?

We shall study some theorems regarding the areas of figures lying between a pair of parallel lines.

Theorem 11.1 : Parallelograms having the same base and lying between a pair of parallel lines, have the same area.

Data : $\square^m$ ABCD and $\square^m$ ABEF have the same base $\overline{AB}$ and lie between a pair of parallel lines l and m.

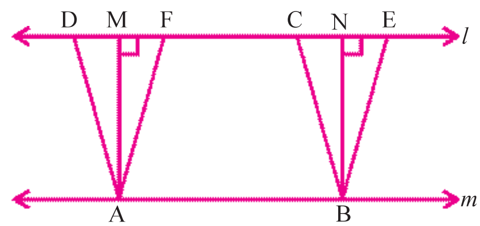


Figure 11.19

To prove : ABCD = ABEF

Proof : Let M and N be the feet of the perpendiculars from A and B respectively to l. We have $l \parallel m$.

AM and BN are perpendicular distances between l and m.

$\therefore$ AM = BN

Now ABCD = AM $\times$ CD

$\therefore$ ABCD = BN $\times$ CD **(AM = BN and AB = CD)**

Also ABEF = BN $\times$ EF = BN $\times$ CD **(EF = AB)**

$\therefore$ ABCD = ABEF

11.12 Triangles on the same Base and between a pair of Parallel Lines

Δ ABC and Δ PBC are on same base $\overline{BC}$ and lie between two parallel lines l and m.

Let us draw $\overline{CD} \parallel \overline{AB}$ and let D $\in$ l. Let $\overline{CQ} \parallel \overline{BP}$ and let Q $\in$ l.

$\therefore$ We get $\square^m$ ABCD and $\square^m$ PBCQ.

$\overline{AC}$ is diagonal of $\square^m$ ABCD. $\overline{PC}$ is diagonal of $\square^m$ PBCQ.

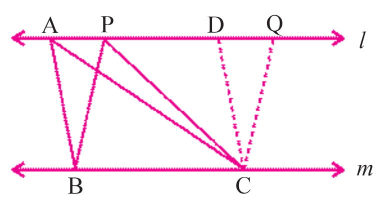


Figure 11.20

$\therefore$ ABC = $\frac{1}{2}$ ABCD and

PBC = $\frac{1}{2}$ PBCQ.

But ABCD = PBCQ

(on same base $\overline{BC}$ and between the pair of parallel lines l and m)

$\therefore$ $\frac{1}{2}$ ABCD = $\frac{1}{2}$ PBCQ

$\therefore$ ABC = PBC

We accept the theorem given below without proof.

Theorem 11.2 : Two triangles on the same base (or congruent bases) and lying between pair of parallel lines have same area.

The converse of theorem is also true and we accept the theorem without proof.

Theorem 11.3 : Two triangles having the same base (or congruent bases) and having their vertices (other than the base vertices) in the same half plane of the line containing the base (or congruent bases) and having equal areas lie between a pair of parallel lines.

Example 5 : Show that a median of a triangle divides a triangular region into two triangular regions with equal areas.

Solution : In Δ ABC, $\overline{AD}$ is the median.

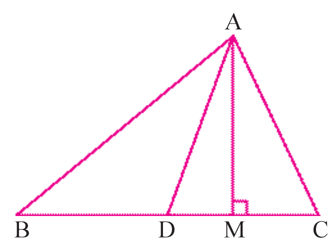


Figure 11.21

$\therefore$ BD = DC

Let $\overline{AM} \perp \overline{BC}$

ABC $= \frac{1}{2}$ AM $\times$ BD

ADC $= \frac{1}{2}$ AM $\times$ CD

but BD = DC

$\therefore$ ABD = ADC

Example 6 : D, E and F are the midpoints of the sides $\overline{AB}$, $\overline{BC}$ and $\overline{CA}$ respectively of ΔABC. Prove that $\square$BEFD, $\square$ECFD and $\square$EFAD have the same area.

Solution : In Δ ABC, D and F are the midpoints of the sides $\overline{AB}$ and $\overline{AC}$ respectively.

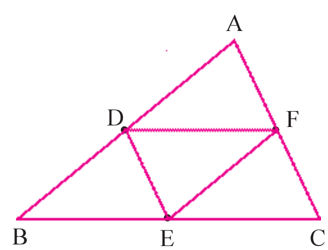


Figure 11.22

$\therefore$ DF $= \frac{1}{2}$ BC and $\overline{DF} \parallel \overline{BC}$

E is the midpoint of $\overline{BC}$.

$\therefore$ BE = EC $= \frac{1}{2}$BC = DF

$\therefore$ In $\square$BEFD, $\overline{BE} \cong \overline{DF}$ and $\overline{BE} \parallel \overline{DF}$ **(B – E – C)**

$\therefore$ $\square$BEFD is parallelogram.

Similarly, $\square$ ECFD is also parallelogram.

Now $\square^m$ BEFD and $\square^m$ ECFD have the same base $\overline{FD}$ and lie between the pair of parallel lines $\overline{DF}$ and $\overline{BC}$.

$\therefore$ BEFD = ECFD

Similarly, it can be proved that

EFAD = ECFD

$\therefore$ BEFD = ECFD = EFAD

An Important Result : In a trapezium ABCD, $\overline{AB} \parallel \overline{DC}$. M is the foot of perpendicular from D to $\overline{AB}$ and A – M – B.

Then ABCD = $\frac{1}{2}$(AB + CD) × DM

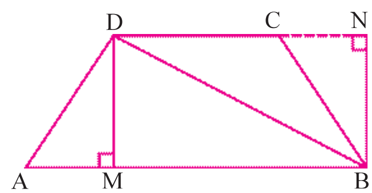


Figure 11.23

Solution : In trapezium ABCD, $\overline{AB} \parallel \overline{DC}$. M is the foot of the perpendicular from D to $\overline{AB}$ and A – M – B.

Let N be the foot of the perpendicular form B to $\overleftrightarrow{DC}$.

$\therefore$ DM and BN are perpendicular distances between parallel lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{DC}$.

$\therefore$ DM = BN

Now, $\overline{DM}$ is the altitude of Δ ABD and $\overline{AB}$ is the corresponding base.

$\therefore$ ABD = $\frac{1}{2}$ AB × DM

Similarly, $\overline{BN}$ is the altitude and $\overline{CD}$ the corresponding base in Δ BCD.

$\therefore$ BCD = $\frac{1}{2}$ CD × BN

$\therefore$ BCD = $\frac{1}{2}$ CD × DM **(DM = BN)**

Now ABCD = ABD + BCD

$= \frac{1}{2}$ AB × DM + $\frac{1}{2}$ CD × DM

$= \frac{1}{2}$ (AB + CD) × DM

$\therefore$ ABCD $= \frac{1}{2}$ (AB + CD) × DM

Example 7 : If a triangle and a parallelogram are on the same base and lie between a pair of two parallel lines, then prove that the area of the triangle is equal to half the area of the parallelogram.

Solution : Let Δ PAB and $\square^m$ DABC have same base $\overline{AB}$ and lie between parallel lines $\overleftrightarrow{PC}$ and $\overleftrightarrow{AB}$.

Draw $\overline{QB} \parallel \overline{PA}$ and let $\overleftrightarrow{BQ}$ intersect $\overleftrightarrow{PC}$ at Q.

$\overline{PA} \parallel \overline{QB}$ and $\overline{AB} \parallel \overline{PQ}$

$\therefore$ $\square$PABQ is parallelogram.

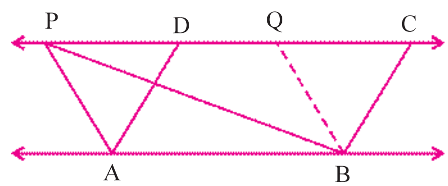


Figure 11.24

$\square^m$ ABCD and $\square^m$ ABQP are on the same base $\overline{AB}$ and lie between parallel lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{PC}$.

$\therefore$ ABCD = ABQP **(i)**

In $\square^m$ ABQP, $\overline{PB}$ is a diagonal.

$\therefore$ PAB = $\frac{1}{2}$ ABQP

PAB = $\frac{1}{2}$ ABCD. **(from (i))**

EXERCISE 11.2

1. In a trapezium ABCD, $\overline{AD} \parallel \overline{BC}$ and M and N are the midpoints of $\overline{AB}$ and $\overline{CD}$ respectively. $\overline{AE} \perp \overline{BC}$ such that B–E–C. If BC = 16 *cm* and MN = 10 *cm* and AE = 6 *cm*, find ABCD.

2. In figure 11.25, $l \parallel m$. A, B, C, D, E and F are distinct points such that A, B $\in m$ and C, D, E, F $\in l$. The perpendicular distance between the lines l and m is 5 *cm* and AB = 10 *cm*. Answer the following :

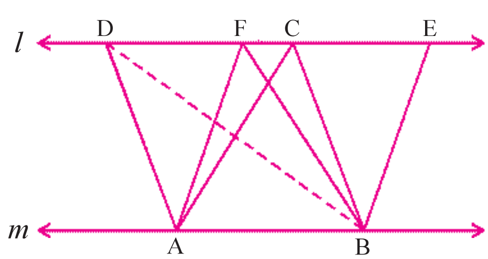


Figure 11.25

(1) Find the area of Δ ABD.

(2) Which other triangle has the same area as Δ ABD ? Why ?

(3) Find the area of $\square^m$ AFEB.

(4) Which other parallelogram has the same area as $\square^m$ AFEB ? Why ?

(5) Do Δ ADF and Δ BDF have the same area ? Why ?

(6) If DF = 3 *cm*, find the area of Δ ADF.

3. In Δ ABC, D is the midpoint of $\overline{BC}$ and E is the midpoint of $\overline{AD}$. Prove that BED = $\frac{1}{4}$ ABC.

4. Compute the area of the quadrilateral PQRS, where measures of sides are given in figure 11.26.

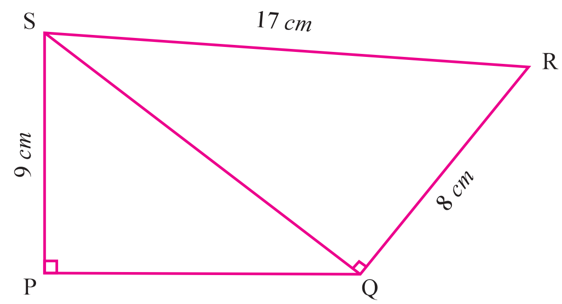


Figure 11.26

5. Compute the area of the trapezium ABCD using measures of sides given in figure 11.27.

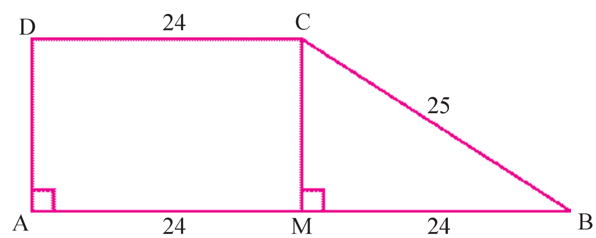


Figure 11.27

6. In the trapezium ABCD, AB = 14 *cm*, AD = BC = 10 *cm*, DC = x *cm* and distance between $\overline{AB}$ and $\overline{DC}$ is 8 *cm*. Find the value of x and area of the trapezium ABCD given in figure 11.28.

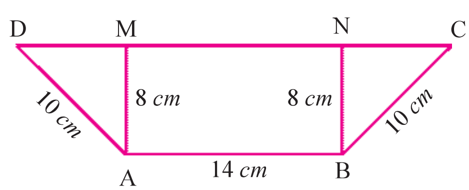


Figure 11.28

EXERCISE 11

1. If E, F, G and H are respectively the midpoints of the sides of a $\square^m$ PQRS, show that EFGH = $\frac{1}{2}$ (PQRS).

2. In figure 11.29, X is a point in the interior of a $\square^m$ PQRS. Show that,

(i) PXS + QXR = $\frac{1}{2}$ (PQRS)

(ii) PXQ + SXR = $\frac{1}{2}$ (PQRS)

(**Hint :** Draw a line through X parallel to $\overleftrightarrow{QR}$)

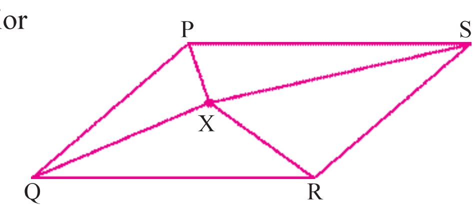


Figure 11.29

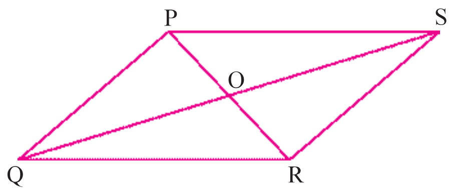


Figure 11.30

3. In figure 11.30, diagonals $\overline{PR}$ and $\overline{QS}$ intersect at O such that PO = OR. If SR = PQ, then show that (i) POQ = SOR (ii) PQR = SQR (iii) $\overleftrightarrow{PS} \parallel \overleftrightarrow{QR}$ and □ PQRS is parallelogram .

(**Hint :** Draw perpendicular to $\overline{QS}$ from P and R)

4. □^{m} PQRS and rectangle PQAB are on the same base $\overline{PQ}$ and have equal area. Show that the perimeter of the parallelogram is greater than that of the rectangle.

5. S is the midpoint of $\overline{QR}$ in Δ PQR and X is the midpoint of $\overline{QS}$. If Y is the midpoint of $\overline{PX}$, prove that QYX = $\frac{1}{8}$ (PQR)

6. Prove that the area of an equilateral triangle is equal to $\frac{\sqrt{3}}{4} l^2$, where l is the length of a side of the triangle.

7. A and B are any two points lying on the side $\overline{PS}$ and $\overline{PQ}$ respectively of a □^{m} PQRS. Show that AQR = BSR.

8. Δ ABC is equilateral triangle. If BC = 12 *cm*, find ABC

9. In ABC, P, Q, R are the midpoints of sides of $\overline{AB}$, $\overline{BC}$ and $\overline{AC}$ respectively. If ABC = 120 cm^2, find PQR, PQCR and PBCR.

10. Select proper option (a), (b), (c) or (d) and write in the box given on the right so that the statement becomes correct :

(1) In □^{m} ABCD, let $\overline{AM}$ be the altitude corresponding to the base $\overline{BC}$ and $\overline{CN}$ the altitude corresponding to the base $\overline{AB}$. If AB = 10 *cm*, AM = 6 *cm* and CN = 12 *cm*, then BC = ... *cm* □

(a) 20 (b) 10 (c) 12 (d) 5

(2) In □ ABCD, $\overline{AD} \parallel \overline{BC}$, $\overline{AM} \perp \overline{BC}$ such that B – M – C. If AD = 8 *cm*, BC = 12 *cm* and AM = 10 *cm*. ABCD = ... cm^2. □

(a) 100 (b) 50 (c) 200 (d) 400

(3) $\overline{AD}$ and $\overline{BE}$ are the altitudes of Δ ABC. If AD = 6 *cm*, BC = 16 *cm*, BE = 8 *cm*, then CA = ... *cm*. □

(a) 12 (b) 18 (c) 24 (d) 22

(4) $\overline{BE}$ and $\overline{CF}$ are the altitudes of Δ ABC. If BE = 10 *cm*, CA = 8 *cm*, AB = 16 *cm*, then CF = ... *cm*. □

(a) 2.5 (b) 5 (c) 10 (d) 6.4

(5) In $\square^m$ ABCD, $\overline{BC}$ is the base corresponding to the altitude $\overline{AM}$. If BC = 8 *cm* AM = 5 *cm*, then ABCD = ... cm^2.

(a) 40 (b) 20 (c) 80 (d) 10

(6) In a $\square$ ABCD, $\overline{AB} \parallel \overline{CD}$, $\overline{DM}$ is the altitude on $\overline{AB}$. If AB = 15 *cm*, CD = 25 *cm* and DM = 10 *cm*, then ABCD = ... cm^2.

(a) 400 (b) 250 (c) 100 (d) 200

(7) $\square$ ABCD is rhombus. If AC = 12 *cm* and BD = 15 *cm*, then the area of the rhombus ABCD = ... cm^2.

(a) 90 (b) 180 (c) 45 (d) 360

(8) $\square$ ABCD is a rhombus If ABCD = 80 cm^2 and AC = 8 *cm*, then BD = ... *cm*.

(a) 5 (b) 10 (c) 20 (d) 40

(9) If for $\square^m$ ABCD, ABCD = 48 cm^2, then ABC = ... cm^2.

(a) 12 (b) 24 (c) 96 (d) 6

(10) In Δ ABC, P, Q and R are the midpoints of $\overline{AB}$, $\overline{BC}$ and $\overline{CA}$ respectively. If ABC = 60 cm^2, then PBCR = ... cm^2.

(a) 15 (b) 30 (c) 45 (d) 75

*

Summary

In this chapter we have studied the following points :

1. Area of a figure is a number (in some units) associated with some part of the plane enclosed by that figure.
2. Two congruent figures have equal areas but the converse need not be true.
3. If a planer region formed by a figure T is made up of two non - overlaping planer regions formed by figures P and Q, then area of T = area of P + area of Q.
4. Area of a rectangle, area of a right triangle.
5. Area of a triangle is half the product of it base and the corresponding altitude.
6. Area of a parallelogram is product of its base and the corresponding altitude.
7. Parallelograms on a same base (or congruent bases) and lying between two parallel lines have equal area.
8. Parallelograms on the same base (or congruent bases) having equal areas lie between two parallel lines.
9. Triangles on the same base (or congruent bases) and lying between two parallel lines have equal area.
10. Triangles on the same base (or congruent bases) and having third vertex in the same semi plane of the line containing the base and having equal areas lie between the two parallel lines.
11. If a parallelogram and a triangle are on the same base and lie between a pair of parallel lines, then the area of the triangle is half the area of the parallelogram.
12. A median of a triangle divides it into two triangles of equal areas.

CHAPTER 12

CIRCLE

12.1 Introduction

Let us imagine about a routine scene of a village. A goat is tied up with a rope and the rope is fixed with a nail at some point on the ground. Now, think about the area that the goat can graze ! The boundary of that area and the fixed (nail) point gives us the idea of **a circle**. The length of the rope is **radius** and the nail where the rope is fixed is the **centre**.

We have already studied about a circle in earlier classes. Let us observe some circular objects in our neighbourhood. A circle is the edge of a wheel, edge of a button of a shirt, boundary of some coins, edge of full moon, etc.

Figure 12.1

12.2 Circle and its Related Terms

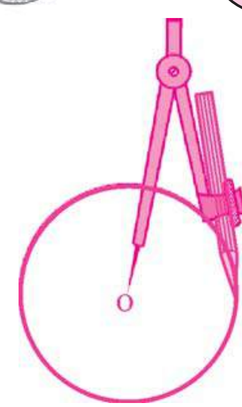


Figure 12.2

We can draw a circle by the use of a compass. Fix pointer at some fixed point O on a paper and fix the other end (where the pencil is inserted) at some distance and rotate this end through one revolution. The closed figure traced on the paper is a circle (figure 12.2). We have kept one point O fixed and that point is the **centre of the circle**. The circle is the arc traced by the pencil. The distance of any boundary point P from the fixed point O is called radius of the circle. Now, we define a circle.

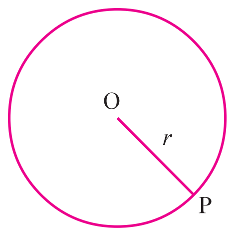


Figure 12.3

Circle : The set of points lying in a plane at a fixed positive distance from a fixed point in the plane is called a circle (figure 12.3).

If we denote the fixed point of the plane α, as O and fixed distance $r > 0$, then in the set form a circle can be defined as $\{P \mid OP = r, r > 0, P \in \alpha\}$.

Radius : The line-segment whose one end point is the centre and other end point is any of the points of the circle is called a radius of the circle. Its measure is also called radius and is denoted by *r*.

If O is the centre and r is the radius of a circle, then we denote the circle by $\odot(O, r)$.

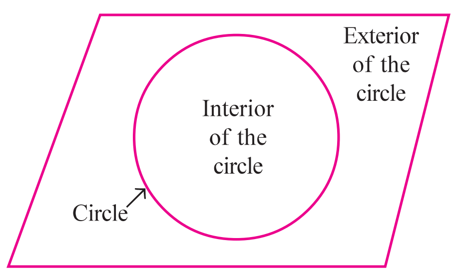


Figure 12.4

A circle divides plane into three parts,

(i) Interior : The set of points whose distance from the centre of the circle is less than its radius is called the interior of the circle.

(ii) Circle : points on the circle.

(iii) Exterior : The set of points whose distance from the centre of the circle is greater than its radius is called the exterior of the circle.

Circular region : Union of the set of the points of circle and its interior is called the circular region.

Chord : The line-segment both of whose end points are the elements of the circle is called a chord of the given circle. In figure 12.5, P, Q ∈ $\odot(O, r)$. So $\overline{PQ}$ is a chord of $\odot(O, r)$.

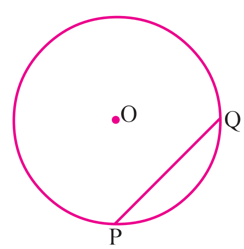


Figure 12.5

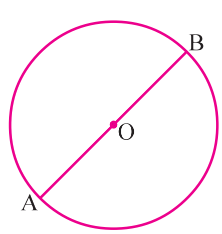


Figure 12.6

Diameter : If a chord of a circle passes through its centre, it is called a diameter of the circle (figure 12.6). $\overline{AB}$ is a diameter. **A diameter is the longest chord of the circle and has the length twice of its radius.** Length of the diameter is also called a diameter.

Arc : The set of points of a circle lying in each closed semi plane of a line passing through two distinct points of the circle is called an arc of the circle. The chord joining these two points is called the chord corresponding to the arc. The arc PQ, is denoted by $\overset{\frown}{PQ}$. (figure 12.7 and 12.8)

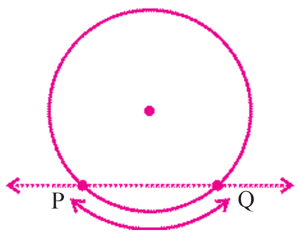


Figure 12.7

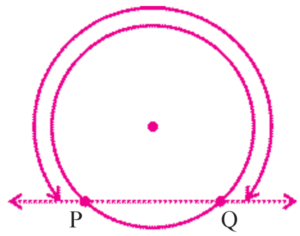


Figure 12.8

Minor arc : The set of points of a circle lying in the closed semi plane of the line containing a chord $\overline{PQ}$ and not containing the centre of the circle is called a minor arc of the circle (figure 12.7). We denote it by minor $\overset{\frown}{PQ}$.

Major arc : The set of points of a circle lying in the closed semi plane of the line containing a chord $\overline{PQ}$ and containing the centre of the circle is called a major arc (figure 12.8). $\overline{PQ}$ is not a diameter. We denote it by major $\overset{\frown}{PQ}$.

If a chord is a diameter of a circle, then arc corresponding to the chord is called a semi-circle arc.

We accept the following results about the length of an arc :

(i) If the measure of the angle subtended at the centre by minor $\overset{\frown}{AB}$ of $\odot(O, r)$ i.e. $m\angle AOB$ is α, then the length of minor $\overset{\frown}{AB}$ is $\frac{\pi r \alpha}{180}$.

(ii) The length of a semi circle arc of $\odot(O, r)$ is πr. we know 'length' of $\odot(O, r)$ i.e. its circumference is $2\pi r$.

(iii) If $\overline{AB}$ is the chord corresponding to major $\overset{\frown}{AB}$ of $\odot(O, r)$ and if $m\angle AOB = \alpha$, then the length of major $\overset{\frown}{AB}$ is $2\pi r - \frac{\pi r \alpha}{180}$.

Segment : The union of an arc and its corresponding chord of the circle is called a segment of the circle.

There are three types of segments : Minor segment, Major segment and Semi-circle segment.

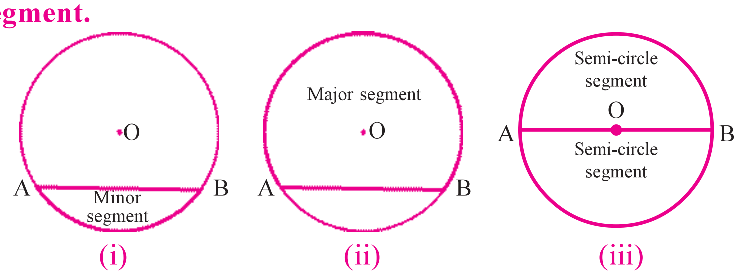


Figure 12.9

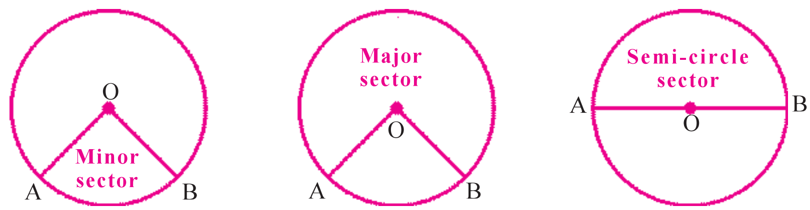
(i) **Minor segment : If an $\widehat{AB}$ is a minor arc, then $\widehat{AB} \cup \overline{AB}$ is called a minor segment** (figure 12.9 (i)).

(ii) **Major segment : If an $\widehat{AB}$ is a major arc, then $\widehat{AB} \cup \overline{AB}$ is called a major segment** (see figure 12.9 (ii)).

(iii) **Semi circle segment : If an $\widehat{AB}$ is a semi circle arc then $\widehat{AB} \cup \overline{AB}$ is called a semi-circle segment** (figure 12.9(iii)).

Sector : For the distinct points A and B of $\odot$(O, r), $\widehat{AB} \cup \overline{OA} \cup \overline{OB}$ is called a sector of the circle with centre O. As in case of a triangle, sector region OAB* is the corresponding region of sector OAB.

Minor sector, Major sector and Semi-circle sector.



(i) (ii) (iii)

Figure 12.10

Congruent circles : Two or more than two circles having congruent radii and different centres are called congruent circles. (figure 12.11)

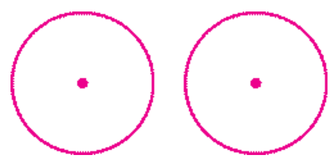


Figure 12.11

Concentric circles : If two or more than two circles in the same plane have the same centre and different radii, then they are called concentric circles. (figure 12.12)

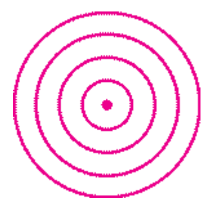


Figure 12.12

EXERCISE 12.1

1. Answer the following :

(1) If two circles having centres P and Q are concentric, then what can you say about P and Q ?

(2) If two circles having centres P and Q are congruent, then what can you say about their radii ?

(3) If P is in the interior and Q is in the exterior of the circle with centre O, which is larger between OP and OQ ?

2. State whether following statements are true or false. Give reasons for your answer.

(1) A line-segment joining the centre to any point of the circle is a diameter of the circle.

(2) An arc is a semi-circle arc, if its endpoints are the endpoints of a diameter.
(3) The set of points equidistant from a fixed point is called a circle.
(4) Union of two radii of a circle is a diameter of the circle.

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12.3 Angle Subtended by a Chord at a Point

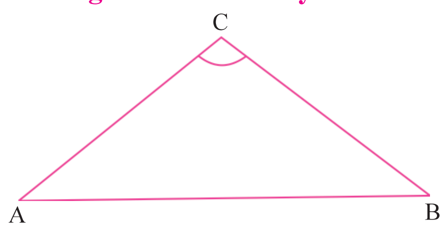

Figure 12.13

Angle subtended by a line segment : If the end points A and B of $\overline{AB}$ are joined to a third point C not lying on $\overleftrightarrow{AB}$, then $\angle ACB$ is called the angle subtended by, $\overline{AB}$ at C (figure 12.13).

The angle subtended by a chord (not a diameter) at the centre of the circle is called the angle subtended by the chord at the centre. If A and B lie on a circle (O, r) then $\angle AOB$ is called the angle subtended by chord $\overline{AB}$ at the centre O.

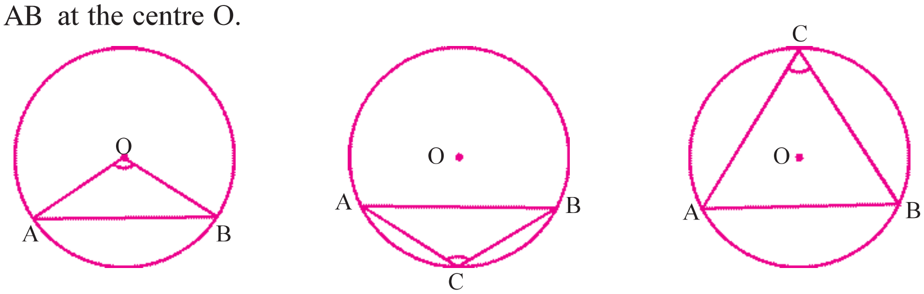

Figure 12.14

In figure 12.14 (i), $\angle AOB$ is the angle subtended by the chord $\overline{AB}$ at the centre O.

The angle subtended by a chord at any point of the arc is called the angle subtended by the chord on the arc.

In figure 12.14 (ii), $\angle ACB$ is the angle subtended by the chord $\overline{AB}$ on the minor $\overset{\frown}{AB}$.

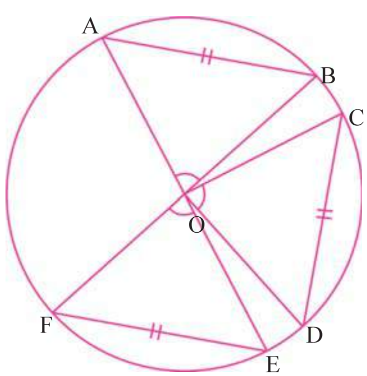

Figure 12.15

In figure 12.14 (iii), $\angle ACB$ is the angle subtended by the chord $\overline{AB}$ on major $\overset{\frown}{AB}$.

Activity : Draw a circle of desired radius on the plane paper.

Draw congruent chords in the circle. Measure angles subtended by them at the centre.

What can we say about the measures of such angles ? In fact, they are congruent angles. Let us prove this result as a theorem.

Theorem 12.1 : Congruent chords of a circle subtend congruent angles at the centre of the circle.

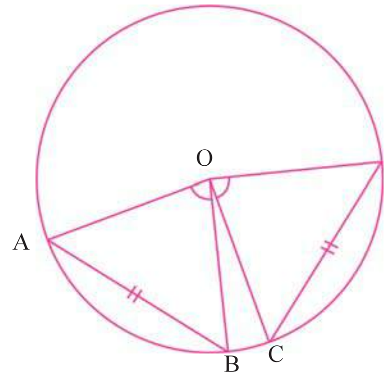

Figure 12.16

Data : Let O be the centre of the given circle and chords $\overline{AB} \cong \overline{CD}$.

To prove : $\angle AOB \cong \angle COD$

Proof : Consider the correspondence AOB $\leftrightarrow$ COD, for ΔAOB and ΔCOD,

$\overline{AB} \cong \overline{CD}$ **(given)**

$\overline{OA} \cong \overline{OC}$ **(radii of the same circle)**

$\overline{OB} \cong \overline{OD}$ **(radii of the same circle)**

$\therefore$ The correspondence AOB $\leftrightarrow$ COD is a congruence. **(SSS)**

$\therefore \angle AOB \cong \angle COD$.

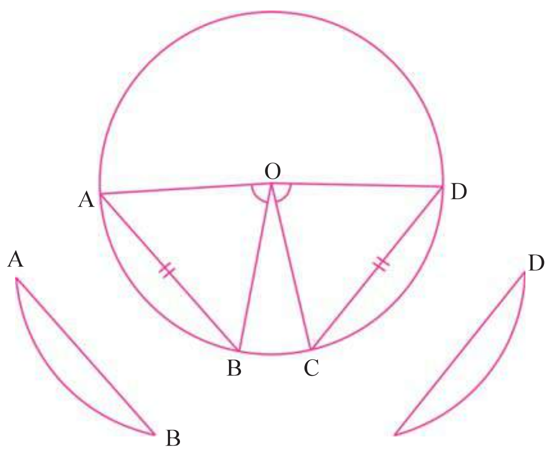

Figure 12.17

Activity : Draw a circle with centre O. Draw congruent angles $\angle AOB$ and $\angle COD$, where $\overline{AB}$ and $\overline{CD}$ are chords.

Now, cut regions enclosed by minor segments formed by $\overline{AB}$ and $\overline{CD}$. Place one segment on the other segment. Observe the result. They cover each other completely. So, the length of the chords have to be the same. This leads to the next theorem; the converse of theorem 12.1.

Theorem 12.2 : If the angles subtended by two chords at the centre of a circle are congruent, then the chords are congruent.

We accept this theorem without proof.

We note that theorms 12.1 and 12.2 are true for congruent circles also.

EXERCISE 12.2

1. Study figure 12.18 and answer the following questions :

(1) If $m\angle OCD=25$, then find $m\angle COD$.

(2) If the diameter of the circle is 10 *cm* and $m\angle COD=90$, then find CD.

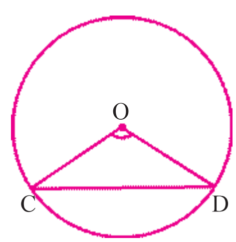

Figure 12.18

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12.4 Perpendicular drawn from the Centre to a Chord

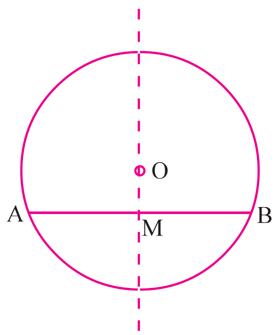

Figure 12.19

Activity : Draw a circle with centre O. Draw a chord $\overline{AB}$. Now fold the paper along the line through the centre O in such way that the portions of $\overline{AB}$ coincide with each other (i.e. point B falls on the point A). Let us cut $\overline{AB}$ at point M along the crease.

Observe that B coincides with A. What can you say about M ? Measure AM and BM. We can see that AM = MB. So M is the midpoint of $\overline{AB}$. This fact leads to the following theorem.

Theorem 12.3 : If a perpendicular is drawn to a chord from the centre of a circle, then it bisects the chord.

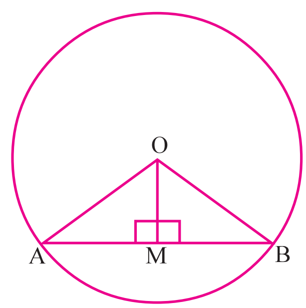

Figure 12.20

Data : Let O be the centre of the given circle. $\overline{AB}$ is a chord and $\overline{OM} \perp \overline{AB}$ and $M \in \overline{AB}$.

To prove : AM = BM.

Proof : Consider corrsepondence $AOM \leftrightarrow BOM$ for ΔAOM and ΔBOM.

$\overline{OA} \cong \overline{OB}$ **(radii)**

$\overline{OM} \cong \overline{OM}$ **(common segment)**

$\angle AMO \cong \angle BMO$ **(right angles)**

$\therefore$ The correspondence $AOM \leftrightarrow BOM$ is congruence. **(RHS theorem)**

$\therefore \overline{AM} \cong \overline{BM}$

$\therefore$ AM = BM

$\therefore$ M is the midpoint of $\overline{AB}$.

$\therefore \overline{OM}$ bisects chord $\overline{AB}$.

The converse of the theorem 12.3 is the theorem 12.4.

Theorem 12.4 : If a line from the centre of a circle bisects the chord, then it is perpendicular to the chord.

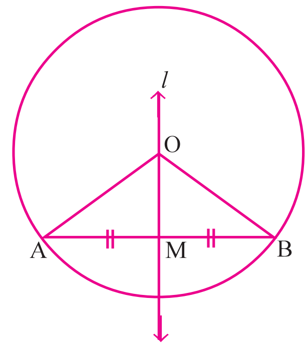

Figure 12.21

Data : Let O be the centre of the circle and l be the line through O bisecting the chord $\overline{AB}$ i.e. AM = BM.

To prove : $l \perp \overline{AB}$

Proof : In consider correspondence AOM $\leftrightarrow$ BOM for Δ AOM and Δ BOM.

$\overline{AO} \cong \overline{BO}$ **(radii)**

$\overline{AM} \cong \overline{BM}$ **(given)**

$\overline{OM} \cong \overline{OM}$ **(common)**

The correspondence AOM $\leftrightarrow$ BOM is a congruence. **(SSS rule)**

$\therefore \angle AMO \cong \angle BMO$

But $m\angle AMO + m\angle BMO = 180$ as $\angle AMO$ and $\angle BMO$ form a linear pair.

$\therefore m\angle AMO = m\angle BMO = 90$.

$\therefore \overline{OM} \perp \overline{AB}$

$\therefore l \perp \overline{AB}$

12.5 Circle Through Three Distinct Points

We know that two distinct points are sufficient to determine unique line. A question arises that, how many points are sufficient to determine a unique circle ?

If one point is given, then how many circles can be drawn through this point ? Obviously, infinitely many circles can be drawn through a given point A, (see figure 12.22).

Now if two distinct points are given, then how many circles can be drawn passing through both the points ? Here also infinitely many circles can be drawn through the given points A and B, (see figure 12.23). Take two distinct points A and B and draw the perpendicular bisector l of $\overline{AB}$. Now the points on l are equidistant from A and B. So taking distinct points on l as the centres and distances of them from A or B as radii we can draw infinitely many circles passing through A and B (see figure 12.24).

Considering above fact if one point A is given, then taking B anywhere in the same plane, we can draw infinitely many circles passing through A.

If we take three distinct points, then we have to think about two cases.

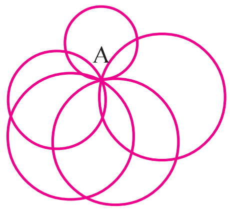

Figure 12.22

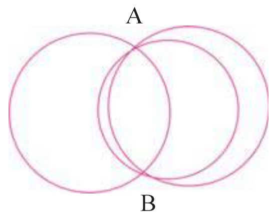

Figure 12.23

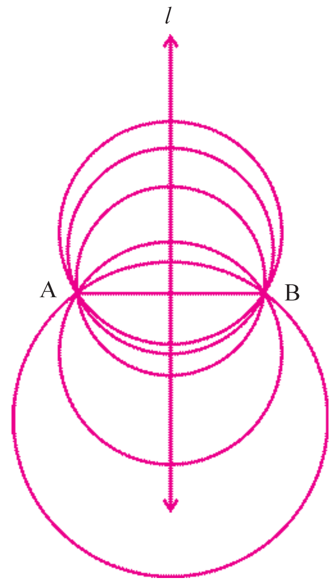

Figure 12.24

(i) collinear points and

(ii) non-collinear points.

If the points are collinear, the circle will not pass through all the three points. It will pass through two points and the remaining point lies in the interior or the exterior of the circle (figure 12.25 and 12.26).

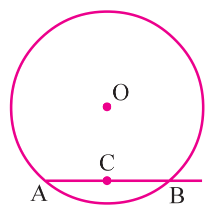

Figure 12.25

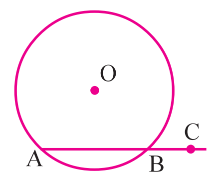

Figure 12.26

Now we take three distinct non-collinear points and we will try to draw a circle passing through them.

Let P, Q, R be three non-collinear points. To get a circle through P, Q, R, let us think in this way. Obviously, $\overline{PQ}$ and $\overline{QR}$ are going to be chords of the assumed circle. As we have learnt that the perpendicular bisector of a chord passes through the centre of the circle, perpendicular bisectors of $\overline{PQ}$ and $\overline{QR}$ both must pass through the centre of that assumed circle. Hence, the point of intersection of perpendicular bisectors of $\overline{PQ}$ and $\overline{QR}$ must be the centre of that assumed circle.

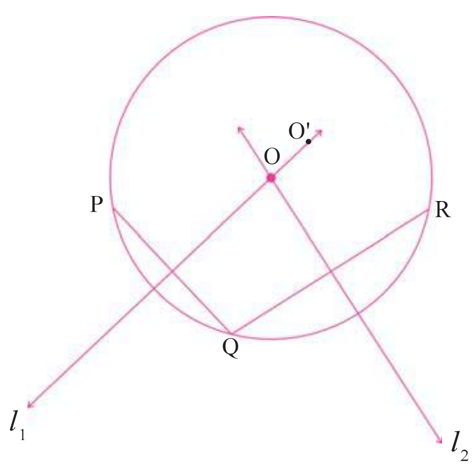

Figure 12.27

Draw perpendicular bisectors l_1 and l_2 of $\overline{PQ}$ and $\overline{QR}$ respectively. They intersect at a point say O. (figure 12.27). Here OP = OQ = OR.

i.e. O is equidistant from P, Q, R.

Now draw a circle with center O and radius OP. The circle passes through all the points P, Q and R.

Now take $O' \in l_1$, $O' \neq O$. Can we draw another circle passing through all the three points P, Q and R ? Obviously, our answer is no. Here O' is on the perpendicular bisector of $\overline{PQ}$ but not on the perpendicular bisector of $\overline{QR}$. So O' is equidistant from P and Q and so our circle, will pass through P and Q while $O'R \neq O'P$ (or $\neq O'Q$), so it will not pass through R. Thus, we observed that one and only one (unique) circle passes through three distinct non-collinear points.

The above discussion leads us to the following theorem. We accept it without proof.

Theorem 12.5 : There is a unique circle passing through three distinct non-collinear points.

A triangle has three vertices and they are non-collinear points, so from the above theorem we have a unique circle passing through the vertices of a triangle.

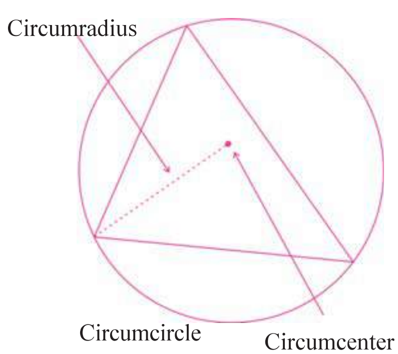

Figure 12.28

Circumcircle : A circle passing through the vertices of a triangle is called circumcircle of the triangle.

Circumcentre : The centre of the circumcircle of a triangle is called the circumcentre of the triangle.

Circumradius : The radius of the circumcircle of a triangle is called the circumradius of the triangle. It is usually denoted by R.

Example 1 : Draw the circle whose arc is given.

Solution : $\overset{\frown}{AB}$ is given. Let $C \in \overset{\frown}{AB}$. Join $\overline{AC}$ and $\overline{BC}$, Draw perpendicular bisectors of $\overline{AC}$ and $\overline{BC}$. They intersect at O.

Draw a circle with center O and radius OA. $\overset{\frown}{AB}$ is an arc of this Circle.

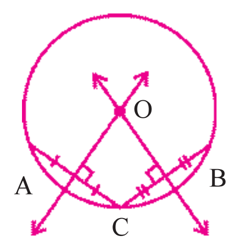

Figure 12.29

EXERCISE 12.3

1. Discuss the possible number of points of intersection of two distinct circles.
2. Explain how to find the centre of the circle of figure 12.30.

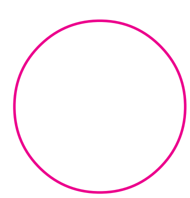
Figure 12.30

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12.6 Congruent Chords and their Distances from the Centre

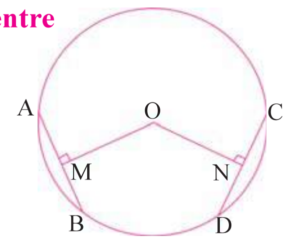


Figure 12.31

Now we will make an observation about the distance of congruent chords from the centre of a circle.

Activity : Draw a circle with centre O and having arbitrary radius. Draw two congruent chords $\overline{AB}$ and $\overline{CD}$. Also draw $\overline{OM}$, $\overline{ON}$ perpendiculars to $\overline{AB}$ and $\overline{CD}$ respectively (figure 12.31).

Now fold the figure in such a way that O will be on the crease, and C coincides with A, and D coincides with B. Now, where does N coincide ? Obviously, N coincides with M, i.e. OM = ON.

This activity leads us to the following theorem, which we accept without giving proof.

Theorem 12.6 : Congruent chords of a circle (or congruent chords of congruent circles) are equidistant from the centre of the of the circle (or centres).

Converse of this theorem is also true; we will do one activity to understand it.

Activity : Draw a circle with centre O. Draw two congruent segments $\overline{OM}$ and $\overline{ON}$ inside the circle.

Draw chords $\overline{AB}$ and $\overline{CD}$ perpendicular to $\overline{OM}$ and $\overline{ON}$ respectively (figure 12.31). Measure $\overline{AB}$ and $\overline{CD}$. We will observe that they are congruent.

Now we will state the converse of theorem 12.6, which we will accept without giving proof.

Theorem 12.7 : Chords equidistant from the centre of a circle (or centres of congruent circles) are congruent.

Example 2 : If two intersecting chords of a circle make congruent angles with the diameter passing through their point of intersection, then prove that chords are congruent.

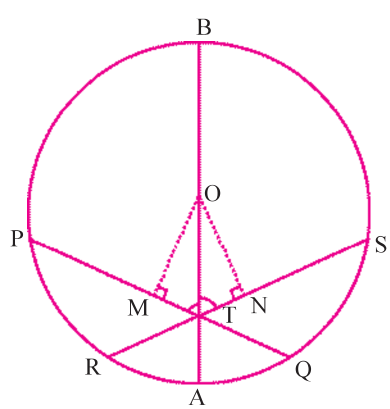


Figure 12.32

Solution : Take chords $\overline{PQ}$ and $\overline{RS}$ of a circle with centre O. Let $\overline{AB}$ be the diameter passing through T, the point of intersection of the given chords. Draw $\overline{OM}$ and $\overline{ON}$ perpendicular to $\overline{PQ}$ and $\overline{RS}$ respectively. We are given that

$\angle PTB \cong \angle STB$,

i.e. $\angle MTO \cong \angle NTO$

$(\overrightarrow{TP} = \overrightarrow{TM}$ and $\overrightarrow{TB} = \overrightarrow{TO})$ **(i)**

Now, consider the correspondance $MTO \leftrightarrow NTO$ for ΔMTO and ΔNTO.

$\angle OMT \cong \angle ONT$ **(right angles)**

$\angle MTO \cong \angle NTO$ **(given)**

$\overline{TO} \cong \overline{TO}$

$\therefore$ The correspondence MTO $\leftrightarrow$ NTO is a congruence. **(AAS)**

$\therefore \overline{OM} \cong \overline{ON}$

$\therefore$ OM = ON

$\therefore \overline{PQ} \cong \overline{RS}$

Example 3 : Find the length of the chord of $\odot$(O, 13) at distance 5 from the centre.

Solution : Let $\overline{OM}$ be perpendicular from centre O to chord $\overline{AB}$. M is the foot of perpendicular. Hence M is the midpoint of $\overline{AB}$.

OA = 13 and OM = 5 > 0. Hence O $\neq$ M,

for Δ OAM,

$\therefore OA^2 = OM^2 + AM^2$

$\therefore 169 = 25 + AM^2$

$\therefore AM^2 = 144$

$\therefore$ AM = 12

Also, AM = MB = $\frac{1}{2}$AB

$\therefore$ AB = 2AM = 24

$\therefore$ The length of the chord $\overline{AB}$ is 24.

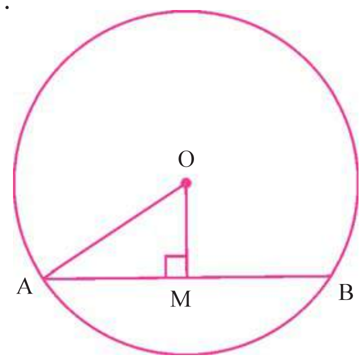


Figure 12.33

Example 4 : Lengths of two parallel chords of $\odot$(O,13) are 24 and 10. According as these chords are in different semi-planes or same semi-plane of the line containing the diameter parallel to these chords, find the distance between them.

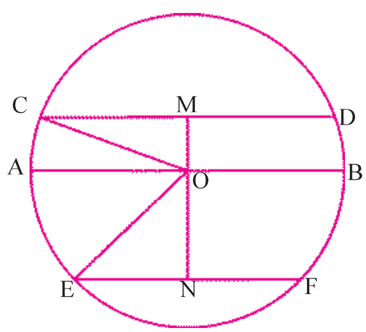


Figure 12.34

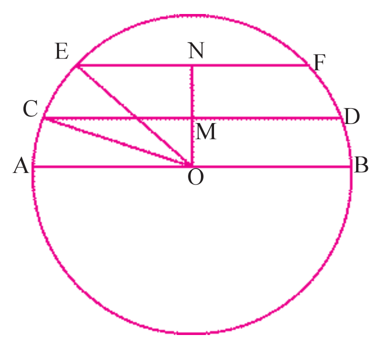


Figure 12.35

Solution : Let $\overline{CD}$ and $\overline{EF}$ be parallel chords. $\overline{AB}$ is the diameter parallel to them. CD = 24, EF = 10.

Perpendicular from O to $\overline{CD}$ is also perpendicular to $\overline{EF}$ as $\overline{CD} \parallel \overline{EF}$.

Let M and N be respectively the feet of perpendiculars from O to $\overline{CD}$ and $\overline{EF}$.

M is the midpoint of $\overline{CD}$ and N is the midpoint of $\overline{EF}$.

$\therefore$ CM = $\frac{1}{2}$ CD = 12, EN = $\frac{1}{2}$ EF = 5. Also radius r = 13.

For Δ OCM, $OC^2 = OM^2 + CM^2$

$\therefore$ $OM^2 = OC^2 - CM^2 = 169 - 144$

$\therefore$ $OM^2 = 25$

$\therefore$ OM = 5

Similarly, from Δ EON,

$\therefore$ $169 = 25 + ON^2$

$\therefore$ $ON^2 = 144$

$\therefore$ ON = 12

Now according to figure 12.34, $\overline{CD}$ and $\overline{EF}$ are on opposite sides of diameter $\overline{AB}$ and hence M–O–N.

$\therefore$ MN = OM + ON = 5 + 12 = 17

And according to figure 12.35, both the chords are on the same side of diameter $\overline{AB}$ and hence N - M - O. **(CD > EF)**

$\therefore$ OM + MN = ON

$\therefore$ 5 + MN = 12

$\therefore$ MN = 7

$\therefore$ If $\overline{CD}$ and $\overline{EF}$ are in different semi-planes of diameter $\overline{AB}$, then MN = 17 and if they are in the same semi-plane of diameter $\overline{AB}$, then MN = 7.

EXERCISE 12.4

1. Two congruent chords $\overline{AB}$ and $\overline{CD}$ which are not diameters, intersect at right angle in P. O is the centre of the circle. If M and N are the midpoints of $\overline{AB}$ and $\overline{CD}$ respectively, then prove that $\square$ OMPN is a square.

2. $\overline{AB}$ and $\overline{AC}$ are congruent chords of a circle with centre O. Feet of perpendiculars from O to $\overline{AB}$ and $\overline{AC}$ are D and E respectively. Prove Δ ADE is an isosceles triangle.

3. $\overline{AB}$ and $\overline{CD}$ are chords of a circle with radius r. AB = 2CD and the perpendicular distance of $\overline{CD}$ from the centre is twice perpendicular distance of $\overline{AB}$ from the centre. Prove that r = $\frac{\sqrt{5}}{2}$ CD.

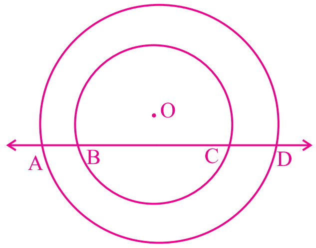


Figure 12.36

4. A line intersects two concentric circles at A, B, C and D. O is the centre, prove that $\overline{AB} \cong \overline{CD}$ (see figure 12.36).

5. If parallel chords $\overline{AB}$ and $\overline{CD}$ are in the same half-plane of a line containing a diameter parallel to them and AB = 8, CD = 6 and perpendicular distance between them is 1. Find the length of the diameter of the circle.

*

12.7 Angle Subtended by an Arc of a Circle

A chord other than diameter of a circle divides the circle into two subsets namely minor arc and major arc. **If chords of the same circle are congruent, then their coresponding arcs are also congruent.** (Here we will consider minor arc only).

Activity : Draw a circle with centre O on a piece of a paper. Draw two congruent chords $\overline{PQ}$ and $\overline{RS}$. Cut minor $\overset{\frown}{PQ}$ and place it on the minor $\overset{\frown}{RS}$. What do you observe ? $\overset{\frown}{PQ}$ will be exactly cover $\overset{\frown}{RS}$. This shows that $\overset{\frown}{PQ}$ and $\overset{\frown}{RS}$ are also congruent. This leads to the following result.

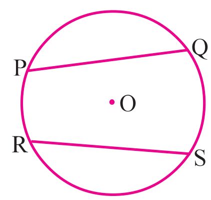


Figure 12.37

If two chords of a circle are congruent, then their corresponding arcs are also congruent and conversely, if two arcs of a circle are congruent then their corresponding chords are congruent.

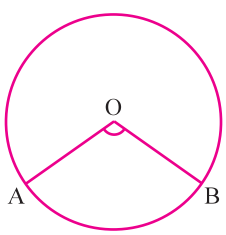


Figure 12.38

We define the angle subtended by an arc of a circle at the centre as the angle subtended by the corresponding chord of the arc at the centre. Here in figure 12.38, the angle subtended by the minor $\overset{\frown}{AB}$ is $\angle AOB$. In the same way, **we define the angle subtended by an arc at any point on the circle as the angle subtended by the corresponding chord of the arc at that point.**

From the property, **congruent chords of a circle subtend congruent angles at the centre, we can state that the congruent arcs also subtend congruent angles at the centre.**

Theorem 12.8 : The measure of the angle subtended by a minor arc of a circle at the centre is twice the measure of the angle subtended by the arc at any point on the remaining part of the circle.

Data : Minor $\overset{\frown}{AB}$ subtends $\angle AOB$ at the centre O of a circle and subtends $\angle APB$ at the remaining part of the circle.

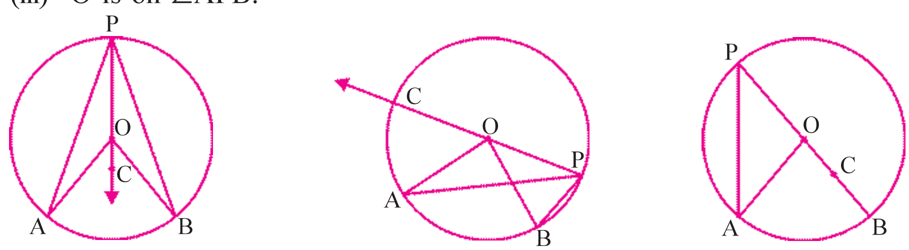
To prove : $m\angle AOB = 2\, m\angle APB$

Proof : Select a point C on $\overrightarrow{PO}$, which is not on $\overline{PO}$. We consider three alternatives :

(i) O is in the interior of $\angle APB$.

(ii) O is in the exterior of $\angle APB$

(iii) O is on $\angle APB$.



(i) (ii) (iii)

Figure 12.39

Let us consider alternatives (i) and (ii) to begin with.

For Δ AOP, $\angle AOC$ is an exterior angle.

$m\angle AOC = m\angle OPA + m\angle OAP$

But $OA = OP$,

$\therefore m\angle OPA = m\angle OAP$

$\therefore m\angle AOC = 2m\angle OPA$

Similarly, from consideration of Δ OPB, $m\angle BOC = 2m\angle OPB$.

According to alternative (i) (figure 12.39 (i)). O is in the interior of $\angle APB$ and C is also in the interior of $\angle AOB$.

$\therefore m\angle AOB = m\angle AOC + m\angle BOC$ **(C is in the interior of $\angle AOB$.)**

$= 2m\angle OPA + 2m\angle OPB$

$= 2\,(m\angle OPA + m\angle OPB)$

$= 2m\,\angle APB$ **(O is in the interior of $\angle APB$.)**

Similarly, if we consider alternative (ii) (see figure 12.39 (ii)), A is in the interior of $\angle BOC$ and $\angle OPB$.

$\therefore m\angle BOC = m\angle AOB + m\angle AOC$

$\therefore m\angle AOB = m\angle BOC - m\angle AOC$

$= 2m\angle OPB - 2m\,\angle OPA$

$= 2\,(m\angle OPB - m\angle OPA)$

Now A is an interior point of $\angle OPB$.

$m\angle OPA + m\angle APB = m\angle OPB$

$\therefore\ m\angle APB = m\angle OPB - m\angle OPA$

$\therefore\ m\angle AOB = 2m\angle APB$

As in alternative (iii) (see figure 12.39 (iii)). O is on an arm of $\angle APB$.

$\therefore\ m\angle AOB = m\angle OPA + m\angle OAP.$

$= 2m\angle APB.$

Hence in all the alternatives, $m\angle AOB = 2m\angle APB$.

If $\overline{AB}$ is a diameter and P is a point on semi circle $\widehat{AB}$, other than A or B, then $\angle APB$ is called an angle inscribed in semi-circle.

Corrollary : An angle inscribed in a semi-circle is a right angle.

Try to proove it !

Theorem 12.9 : Angles in the same segment of a circle are congruent. We will accept this theorem without proof.

Theorem 12.10 : If a line segment joining two distinct points A and B subtends congruent angles at two other points in the same semi plane of the line containing the line-segment, then all the four points lie on a circle whoes chord is $\overline{AB}$. (i.e. those four points are concyclic.)

Data : C and D are in the same semi plane of $\overleftrightarrow{AB}$ and $\angle ACB \cong \angle ADB$.

To prove : A, B, C, D lie on a circle or A, B, C, D are concyclic.

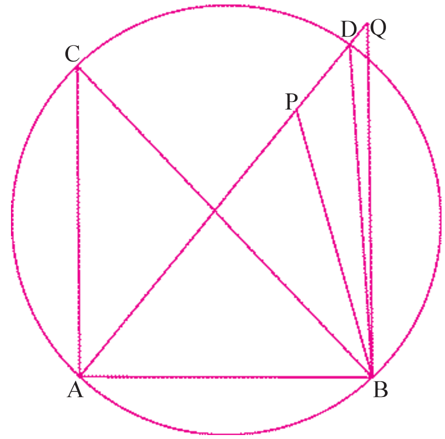


Figure 12.40

Proof : As A, B, C are non-collinear, there is a unique circle passing through A, B, C.

This circle may pass or may not pass through D.

If the circle passes through D, then nothing remains to prove.

If the circle does not pass through D, draw $\overrightarrow{AD}$ such that circle intersects $\overrightarrow{AD}$ at P or Q. ($Q \in \overrightarrow{AD}$, $Q \notin \overline{AD}$) (figure 12.40)

Also $\angle ACB \cong \angle ADB$. **(given)**

$\therefore\ \angle ACB \cong \angle APB$ **(angle in the same segment of a circle)**

So $\angle APB \cong \angle ADB$.

$\therefore$ P = D. **($P \in \overrightarrow{AD}$)**

Similarly we can prove that Q = D.

$\therefore$ D is on the circle.

$\therefore$ A, B, C, D are concyclic.

12.8 Cyclic Quadrilateral

If all the vertices of a quadrilateral lie on a circle, then that quadrilateral is called a cyclic quadrilateral.

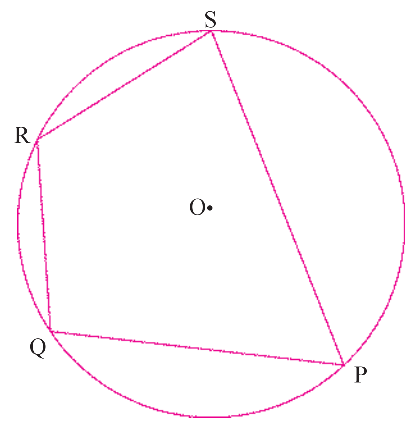


Figure 12.41

Draw several circles of different radii and inscribe quadrilateral PQRS in each circle. Measuring the angles of the quadrilateral, can we observe some relation in their measures ? We can see that sum of the measures of opposite angles is 180. i.e. opposite angles are supplementary. This result is reflected in the next theorem which we accept without proof.

Theorem 12.11 : Opposite angles of a cyclic quadrilateral are supplementary.

The converse of this theorem is also true.

Theorem 12.12 : If the opposite angles of a quadrilateral are supplementary, then the quadrilateral is cyclic.

We will accept above theorem also without proof.

Example 5 : If the non-parallel sides of a trapezium are congruent, then prove that the trapezium is cyclic.

Solution : In trapezium ABCD, $\overline{AB} \parallel \overline{CD}$ and $\overline{AD} \cong \overline{BC}$, AB > DC.

Draw $\overline{DM} \perp \overline{AB}$ and $\overline{CN} \perp \overline{AB}$ and $M \in \overline{AB}$, $N \in \overline{AB}$.

Consider the correspondence AMD $\leftrightarrow$ BNC for for Δ AMD and Δ BNC.

$\overline{AD} \cong \overline{BC}$ **(given)**

$\angle AMD \cong \angle BNC$ **(right angles)**

$\overline{DM} \cong \overline{CN}$ **($\overline{AB} \parallel \overline{CD}$)**

$\therefore$ The correspondence AMD $\leftrightarrow$ BNC is a congruence. **(RHS)**

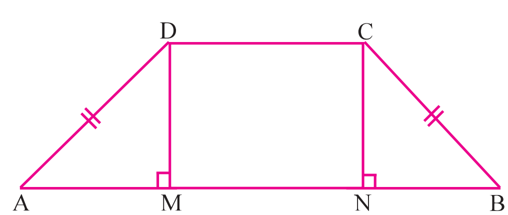


Figure 12.42

$\therefore \angle MAD \cong \angle NBC$

$\angle DCB$ and $\angle ABC$ are supplementary.

(interior angles on the same side of the transversal)

$\therefore \angle DCB$ and $\angle NBC$ are supplementary. **($N \in \overrightarrow{BA}$)**

$\therefore \angle DCB$ and $\angle BAD$ are supplementary. **($\angle BAD = \angle MAD$ as $\overrightarrow{AB} = \overrightarrow{AM}$)**

Similarly, $\angle ADC$ and $\angle ABC$ are supplementary.

$\therefore$ The trapezium ABCD is cyclic.

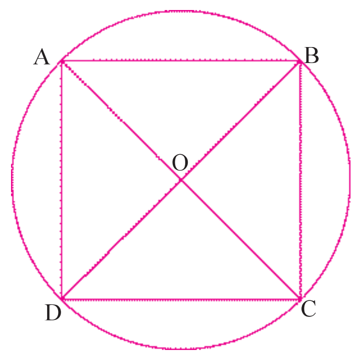


Figure 12.43

Example 6 : $\overline{AC}$ and $\overline{BD}$ are different diameters of a circle. Prove $\square$ ABCD is a rectangle.

Solution : Diagonals $\overline{AC}$ and $\overline{BD}$ are different diameters of a circle.

$\angle ABC$ and $\angle ADC$ are inscribed in a semi-circle arc whose diameter is $\overline{AC}$.

$\therefore m\angle ABC = m\angle ADC = 90$

Similarly $m \angle BAD = m\angle BCD = 90$

$\therefore \square$ ABCD is a rectangle.

(**Note :** Diagonals of $\square$ ABCD bisect each other and are congruent. Hence $\square$ ABCD is a rectangle.)

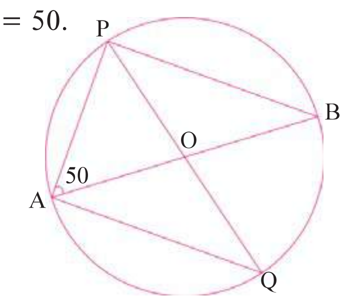


Figure 12.44

Example 7 : In figure 12.44, $\overline{AB}$ is a diameter. $m\angle PAB = 50$. Find $m\angle AQP$.

Solution : $m\angle APB = 90$, as $\overline{AB}$ is a diameter.

Also $m\angle PAB = 50$

$m\angle ABP = 90 - 50 = 40$

Being angles of same segment, $\overset{\frown}{AP} \cup \overline{AP}$

$\angle AQP \cong \angle ABP$.

$\therefore m\angle AQP = 40$

Example 8 : Prove that the quadrilateral formed (if possible) by the angle bisectors of any quadrilateral is cyclic.

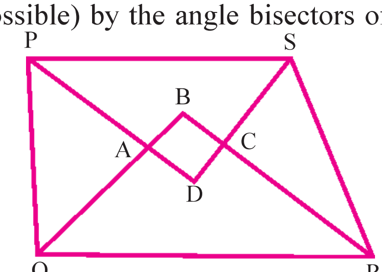


Figure 12.45

Solution : PQRS is a quadrilateral in which the angle bisectors $\overrightarrow{PD}$, $\overrightarrow{QB}$, $\overrightarrow{RB}$ and $\overrightarrow{SD}$ of angles $\angle P$, $\angle Q$, $\angle R$ and $\angle S$ respectively form a quadrilateral ABCD. (see figure 12.45)

Now, $m\angle BAD = m\angle PAQ = 180 - m\angle APQ - m\angle AQP$

$= 180 - \frac{1}{2}(m\angle SPQ + m\angle PQR)$

Similarly $m\angle BCD = m\angle RCS = 180 - \frac{1}{2}(m\angle QRS + m\angle RSP)$

Therefore, $m\angle BAD + m\angle BCD$

$= 180 - \frac{1}{2}(m\angle SPQ + m\angle PQR) + 180 - \frac{1}{2}(m\angle QRS + m\angle RSP)$

$= 360 - \frac{1}{2}(m\angle SPQ + m\angle PQR + m\angle QRS + m\angle RSP)$

$= 360 - \frac{1}{2}(360) = 360 - 180 = 180$

Hence, a pair of opposite angles of □ ABCD is supplementary.

∴ □ ABCD is cyclic.

EXERCISE 12.5

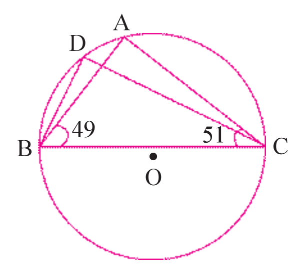


Figure 12.46

1. If D is on the major $\overset{\frown}{AB}$ of the circle with center O and $m\angle ADB = 45$, then find the measure of $\angle AOB$.
2. If $m\angle ABC = 49$, $m\angle ACB = 51$, find $m\angle BDC$. (Refer figure 12.46)
3. A chord of a circle is congruent to the radius of the circle. Find the measure of the angle subtended by the chord at a point on the minor arc and also at a point on the major arc.

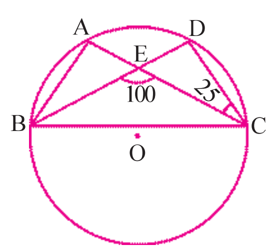


Figure 12.47

4. A, B, C and D are four points on a circle. $\overline{AC}$ and $\overline{BD}$ intersect at a point E such that $m\angle BEC = 100$ and $m\angle ECD = 25$. Find $m\angle BAC$. (see figure 12.47).
5. □ PQRS is a cyclic quadrilateral whose diagonals intersect at the point E. If $m\angle SQR = 70$, $m\angle QPR = 30$, find $m\angle QRS$. Further, if PQ = PR, find $m\angle ERS$.

6. Bisector of $\angle A$ intersects circumcircle of Δ ABC at D. If $m\angle BCD = 50$, then find $m\angle BAC$. (figure 12.48).

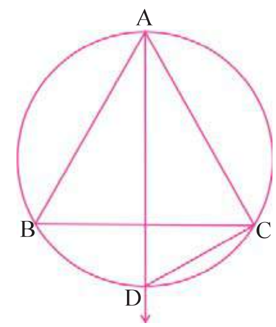


Figure 12.48

7. $\angle ABC$ is an angle inscribed in a semi-circle arc of $\odot(O, r)$. Δ ABC is isosceles and AB = $3\sqrt{2}$. Find area of the circle.
8. Prove that a cyclic parallelogram is a rectangle.
9. In a cyclic quadrilateral ABCD, $\overline{AB} \parallel \overline{CD}$. Prove that $\overline{AD} \cong \overline{BC}$.
10. If in a cyclic $\square$ ABCD, $\overline{AD} \cong \overline{BC}$, prove $\overline{AB} \parallel \overline{CD}$.

EXERCISE 12

1. Congruent parallel chords $\overline{AB}$ and $\overline{CD}$ have mid points M and N respectively and the centre is O. $\overleftrightarrow{MN}$ intersects the circle in P and Q. Prove that PM = QN.
2. In Δ ABC, bisector of $\angle A$ passes through its circumcentre. Prove that AB = AC.
3. $\overline{AB}$ and $\overline{CD}$ are two parallel chords of a circle and AB = 24 *cm* and CD = 10 *cm*. If the perpendicular distance between them is 7 *cm*, then find the radius of the circle. Chords are in the same semiplane of the line containing the diameter parallel to them.
4. Chords $\overline{AB}$ and $\overline{CD}$ are parallel and they lie in the same semi plane of the line containing the diameter parallel to them. AB = 8 *cm*, CD = 6 *cm* and radius of the circle is 5 *cm*. Find the perpendicular distance between them.
5. $\overline{AC}$ and $\overline{BD}$ are different diameters of a circle. Prove that $\square$ ABCD is a rectangle.
6. $\overline{AD}$ and $\overline{BE}$ are altitudes of Δ ABC. D $\in \overline{BC}$, E $\in \overline{AC}$. Prove that $\angle A$, $\angle B$, $\angle D$, $\angle E$ are angles of the same segment of a circle.
7. $\overline{AB}$ and $\overline{CD}$ are two parallel chords of a circle with centre O. If AB = 10, CD = 24 and distance between them is 17, then find its radius. (Chords are in different semi planes of the line containing the diameter parallel to them.)
8. Prove that the perpendicular bisector of a chord of a circle is the bisector of the corresponding arc of the circle.
9. If congruent chords of a circle with centre O are given, prove that $\overrightarrow{BO}$ is the bisector of $\angle ABC$, where $\overline{AB} \cong \overline{CB}$.

10. ΔABC is inscribed in a circle with centre O. If $m\angle BAC=30$, then prove that Δ OBC is an equilateral triangle.

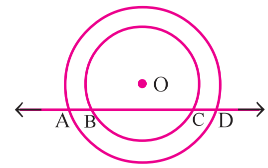


Figure 12.49

11. In the figure 12.49, AD = 12, BC = 8. Find AB, CD, AC and BD. (Here two circles are concentric.)

12. Select proper option (a), (b), (c) or (d) and write in the box given on the right so that the statement becomes correct :

(1) The centre of a circle lies .... ☐
(a) in the interior of the circle. (b) in the exterior of the circle.
(c) on the circle. (d) anywhere in the plane.

(2) A point whose distance from the centre of a circle is less than its radius lies... ☐
(a) in the interior of the circle. (b) in the exterior of the circle.
(c) on the circle (d) anywhere in the plane.

(3) The longest chord of a circle is...... ☐
(a) a line segment joining the centre and any point on the circle
(b) a chord joining the end points of a minor arc.
(c) a chord joining the end points of the major arc.
(d) a chord joining the end points of the semi circle arc.

(4) Line-segment joining the centre to any point on the circle is called .... ☐
(a) a diameter (b) a chord (c) a line (d) a radius

(5) If a chord $\overline{AB}$ subtends an angle with measure 60 at the centre O, then Δ OAB is .... ☐
(a) a right angled triangle (b) an obtuse angled triangle
(c) an equilateral triangle (d) an isosceles right angled triangle

(6) If a line-segment $\overline{AB}$ is a chord of a circle with centre O, then Δ OAB is always .... ☐
(a) acute angled triangle (b) equilateral triangle
(c) obtuse angled triangle (d) isosceles triangle

(7) If the circle is a union of four disjoint congruent arcs, then the angle subtended by one of these arcs at the centre of the circle has measure ...... ☐
(a) 30 (b) 45 (c) 60 (d) 90

(8) The measure of the angle subtended by a chord of length equal to radius has measure ...... ☐
(a) 30 (b) 45 (c) 60 (d) 90

(9) If the measure of the angle between two radii of a circle is 50, then the region formed by these radii and the arc corresponding to this angle is ...... ☐

(a) a semi circle (b) a minor sector

(c) a major sector (d) the interior of the circle

(10) The perpendicular bisector of the chord of a circle passes through ...... ☐

(a) an end-point of the diameter (b) the mid-point of the diameter

(c) an end-point of the given chord (d) an end-point of an arc

(11) If the chord is at distance 3 *cm* from the centre of a circle having radius 5 *cm*, then the length of the chord is ...... ☐

(a) 4 *cm* (b) 6 *cm* (c) 8 *cm* (d) 10 *cm*

(12) The chord of the length 12 *cm* is at a distance 3 *cm* from the centre of a circle whose radius is ...... *cm*. ☐

(a) $2\sqrt{5}$ (b) $3\sqrt{5}$ (c) $4\sqrt{5}$ (d) $6\sqrt{5}$

(13) Number of circle / circles passing through three distinct non-collinear points is / are ...... ☐

(a) zero (b) one (c) three (d) infinite

(14) Number of circles passing through a single given point are ...... ☐

(a) two (b) four (c) three (d) infinite

(15) A, B, C are three distinct non-collinear points. The point of intersection of the perpendicular bisectors of $\overline{AB}$ and $\overline{BC}$ is ...... ☐

(a) the centre of a circle passing through only B.

(b) the centre of a circle passing through only A.

(c) the centre of the circle passing through all A, B, C.

(d) the centre of a circle passing through none of A, B, C.

(16) A line passing through the centres of two circles intersecting in two distinct points is not ...... ☐

(a) a line bisecting the common chord.

(b) a line perpendicular to the common chord.

(c) a line which is the perpendicular bisector of the common chord.

(d) a line passing through one of the end points of the common chord.

(17) If 50 and 100 are the measures of the angles of a cyclic quadrilateral, then the remaining angles are of measure ...... and .... ☐

(a) 130, 80 (b) 100, 50 (c) 100, 130 (d) 80, 50

(18) ☐ PQRS is a cyclic quadrilateral in which $m\angle SQR = 65$ and $m\angle QPR = 30$, then $m\angle QRS =$ .... ☐

(a) 85 (b) 95 (c) 115 (d) 150

(19) In a cyclic quadrilateral ABCD, $m\angle CAB=45$ and $m\angle ABC=100$, then $m\angle ADB$ = ... ☐

(a) 55 (b) 105 (c) 80 (d) 35

(20) If $\overline{AB}$ is a diameter of the circle and P is on the semi-circle, and if $m\angle PAB = 40$, then $m\angle PBA$ is .... ☐

(a) 30 (b) 40 (c) 50 (d) 90

(21) A circle passes through the vertices of an equilateral ΔABC. The measure of the angle subtended by the side $\overline{AB}$ at the centre of the circle has measure .... ☐

(a) 30 (b) 60 (c) 90 (d) 120

*

Summary

In this chapter we have studied the following points :

1. We have defined a circle, its centre and radius, different terms related to the circle and congruent circles.

2. Congruent chords of a circle subtend congruent angles at the centre of the circle and its converse is true.

3. The perpendicular drawn from the centre of the circle to a chord bisects the chord and its converse is true.

4. A unique circle passes through three non-collinear distinct points.

5. Congruent chords of a circle are equidistant from the centre of circle and its converse is true.

6. If two arcs are congruent, then their corresponding chords are also congruent and conversely.

7. Congruent arcs of a circle subtend congruent angles at the centre of the circle.

8. The angle subtended by an arc at the centre has measure twice the measure of the angle subtended by it at any point on the remaining part of the circle.

9. Angles in the same segment of a circle are congruent.

10. Angle in a semicircle is a right angle.

11. If a line-segment joining two points subtends congruent angles at two other points lying on the same side of the line containing the line-segment, the four points lie on a circle.

12. The pair of opposite angles of a cyclic quadrilateral are supplementary and its converse is also true.

●

CHAPTER 13

CONSTRUCTIONS

13.1 Introduction

In earlier chapters, the necessary rough diagrams drawn were just sufficient to represent the given situation. There was no precision required in the drawing of different figures. But in different walks of life, precise drawing is essential. For example in furniture design, fashion design, machine drawing, constructions of buildings etc, the geometrical figures must be in the precise form and with accurate measure. So, we shall learn some constructions with the help of a straight edge and compass only. Here we shall also see the mathematical justification for the procedure adopted for the constructions, which will also use the ideas discussed in the earlier chapters. Also such constructions will help us to develop the skill of correctness in our mathematical understanding.

13.2 Basic Constructions

We have learnt how to construct a circle, the perpendicular bisector of a line-segment, the bisector of a given angle and also the angles of measure 30, 45, 60, 90 and 120 with the help of straight edge and compass only. The justification of these constructions was not discussed there. In this chapter, mathematical justifiction is also given at the end of each constructions. It will justify the validity and correctness of the steps taken for the constructions.

Construction 1 : To construct the bisector of a given angle.

Data : $\angle ABC$ is given.

To construct : To construct the bisector of $\angle ABC$.

Steps of Construction :

(1) Taking B as a centre and an arbitrary radius, draw an arc intersecting both the arms $\overrightarrow{BA}$ and $\overrightarrow{BC}$ of $\angle ABC$ at D and E respectively.

(2) Draw arcs having equal radius with length more than $\frac{1}{2}$ DE by taking D and E as a centres.

These arcs intersect each other at some point P.

(3) Draw $\overrightarrow{BP}$. [see figure 13.1 (ii)]

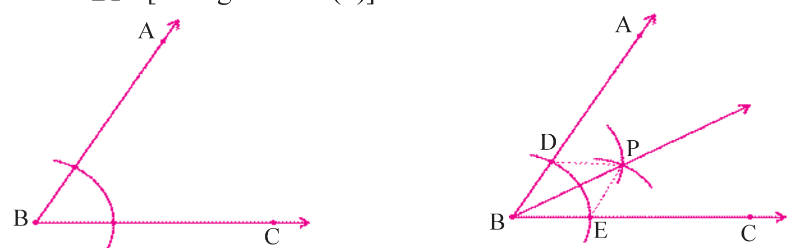


Figure 13.1

Thus $\overrightarrow{BP}$ is the required bisector of ∠ABC.

Now we justify our method of construction.

Draw $\overline{PD}$ and $\overline{PE}$.

For the correspondence BEP ↔ BDP of ΔBEP and ΔBDP.

$\overline{BE} \cong \overline{BD}$ **(radii of the same circle)**

$\overline{EP} \cong \overline{DP}$ **(congruent radii)**

$\overline{BP} \cong \overline{BP}$ **(common line-segment)**

∴ The correspondence BEP ↔ BDP is a congruence. **(SSS)**

∴ ∠EBP ≅ ∠DBP

∴ $\overrightarrow{BP}$ is the bisector of ∠ABC.

Construction 2 : To construct the perpendicular bisector of a given line-segment.

Data : $\overline{AB}$ is given.

To construct : The perpendicular bisector of $\overline{AB}$.

Steps of Construction :

(1) Draw arcs with equal radius having length more than $\frac{1}{2}$ AB taking as centres A and B in upper and lower half-planes of $\overleftrightarrow{AB}$.

(2) Let these arcs intersect, each other at points P and Q.

(3) Draw $\overleftrightarrow{PQ}$, which intersects $\overline{AB}$ at point say M.

Thus $\overleftrightarrow{PQ}$ is the perpendicular bisector of $\overline{AB}$.

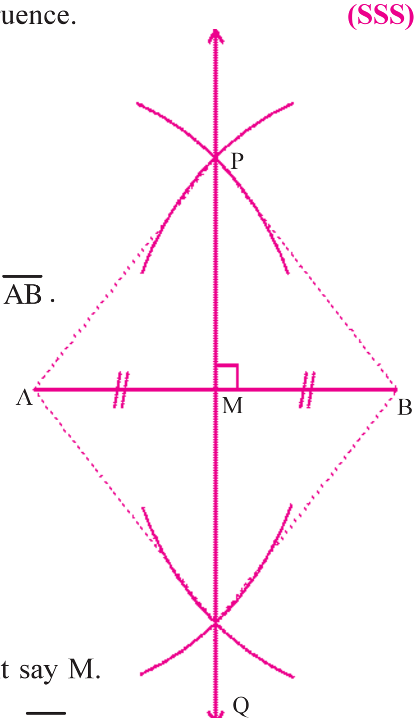


Figure 13.2

Now, we justify our method of constructions.

Join A and B with both P and Q to form $\overline{AP}$, $\overline{AQ}$, $\overline{BP}$ and $\overline{BQ}$.

For correspondence PAQ $\leftrightarrow$ PBQ of ΔPAQ and ΔPBQ.

$\overline{AP} \cong \overline{BP}$ **(radii of the congruent circles)**

$\overline{AQ} \cong \overline{BQ}$ **(radii of the cogruent circles)**

$\overline{PQ} \cong \overline{PQ}$ **(common line-segment)**

$\therefore$ The correspondence PAQ $\leftrightarrow$ PBQ is a congruence. **(SSS)**

$\therefore \angle APQ \cong \angle BPQ$

Hence $\angle APM \cong \angle BPM$ as P–M–Q

Now for correspondence PMA $\leftrightarrow$ PMB of ΔPMA and ΔPMB

$\overline{AP} \cong \overline{BP}$ **(radii of the congruent circles)**

$\angle APM \cong \angle BPM$ **(proved)**

$\overline{PM} \cong \overline{PM}$ **(common line-segment)**

$\therefore$ The correspondence PMA $\leftrightarrow$ PMB is a congruence. **(SAS)**

$\therefore \overline{AM} \cong \overline{BM}$ and $\angle AMP \cong \angle BMP$ **(i)**

As $\angle AMP$ and $\angle BMP$ form a linear pair of angles, they are supplementary angles and they are congruent also.

$\therefore m\angle AMP = m\angle BMP = 90$ **(ii)**

From (i) and (ii), we can say that $\overleftrightarrow{PQ}$ is the perpendicular bisector of $\overline{AB}$.

Construction 3 : To construct an angle having measure 60 at the initial point of a given ray.

Data : $\overrightarrow{BC}$ with initial point B is given.

(figure 13.3(i))

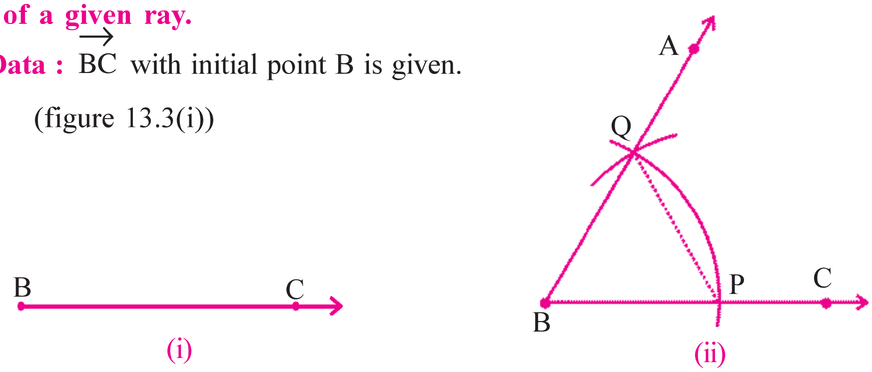


Figure 13.3

To construct : To construct $\overrightarrow{BA}$ such that $m\angle ABC = 60$.

Steps of Construction :

(1) Draw an arc with B as centre and arbitrary radius. Let this arc intersect $\overrightarrow{BC}$ at P.

(2) With centre at P and keeping the same radius as before, draw an arc to intersect the previous arc at a point, say Q.

(3) Draw $\overrightarrow{BA}$ passing through the point Q. (see figure 13.3 (ii))

Thus, we have $\angle ABC$ of measure 60.

Now, we justify our method of constructions.

Draw $\overline{PQ}$.

In ΔBPQ, $\overline{BP} \cong \overline{BQ} \cong \overline{PQ}$ **(radii of the same circle or congruent circles)**

ΔBPQ is an equilateral triangle and hence it is an equiangular triangle.

$m\angle QBP = 60$ and hence $m\angle ABC = 60$ **($Q \in \overrightarrow{BA}$ and $P \in \overrightarrow{BC}$)**

One can construct any angle having measure which is a multiple of 15 using constructions 1 and 3. Of course we remember that measure of an angle lies between 0 and 180 !

Example 1 : Draw ΔABC where BC = 4 *cm*, $m\angle B = 60$, $m\angle C = 90$

Data : In ΔABC, BC = 4 *cm*, $m\angle B = 60$, $m\angle C = 90$

To construct : To construct ΔABC having given measures for side and angles.

Steps of Construction :

(1) Draw $\overrightarrow{BX}$.

(2) Construct an angle of measure 60 at point B. (see construction 3) such that $m\angle YBX = 60$

(3) Mark points C and D on $\overrightarrow{BX}$ such that BC = 4 *cm* and CD = 4 *cm*.

(4) Draw $\angle BCZ$ such that $m\angle BCZ = 90$. $\overleftrightarrow{CZ}$ intersects $\overrightarrow{BY}$ at A.

Then ΔABC with given measure is constructed.

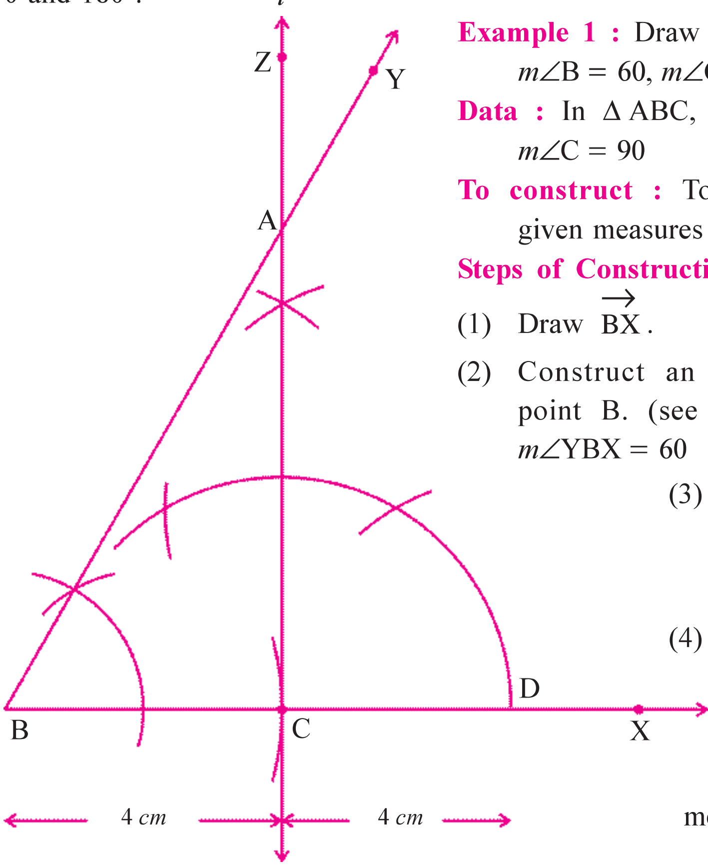


Figure 13.4

EXERCISE 13.1

1. Draw $\overline{AB}$ having length 10 *cm*. Construct its perpendiculer bisector $\overleftrightarrow{PQ}$, which intersects $\overline{AB}$ at M. Measure $\overline{AM}$ and $\overline{BM}$.

2. Construct an angle having measure 120 by using a pair of compass and a straight edge only.

3. Construct an angle having measure 30 by using a pair of compass and a straight edge only.

4. Construct an angle having measure (1) 15 (2) 90 (3) 150 by using a pair of compass and a straight edge only.

5. Construct an equilateral triangle having length of each side 6 *cm* by using a pair of compass and a straight edge only.

6. Construct Δ PQR, where $m\angle Q = 60$, $m\angle R = 90$ and QR = 5 *cm* by using a pair of compass and a straight edge only.

7. Construct Δ XYZ, where YZ = 4 *cm*, $m\angle X = 60$, $m\angle Z = 90$.

*

13.3 Some Constructions related to Triangles

Now we will construct triangles using the constructions learnt in our earlier classes and in this chapter.

We know that a triangle has six parts i.e. three sides and three angles. Because of the postulates and theorems of congruence of triangles, some definite three parts of a triangle determine the triangle completely. We shall now see how to construct a triangle when some definite relations among measures of angles and measures of sides are given. You may have noted that at least three parts of a triangle have to be given for the constructions of a triangle, but not all combinations of three parts are sufficient for our purpose. For example, if two sides and not included angle are given, then it is not possible to construct such a triangle. When we are given the measure of an angle for such constructions, we shall construct the angle with the help of a compass. We shall not use a protractor.

Construction 4 : To construct a triangle, given the base, one base angle and the sum of measures of two sides.

Data : Base QR, $m\angle PRQ$ and PQ + PR are given.

To construct : To construct ΔPQR with given measures.

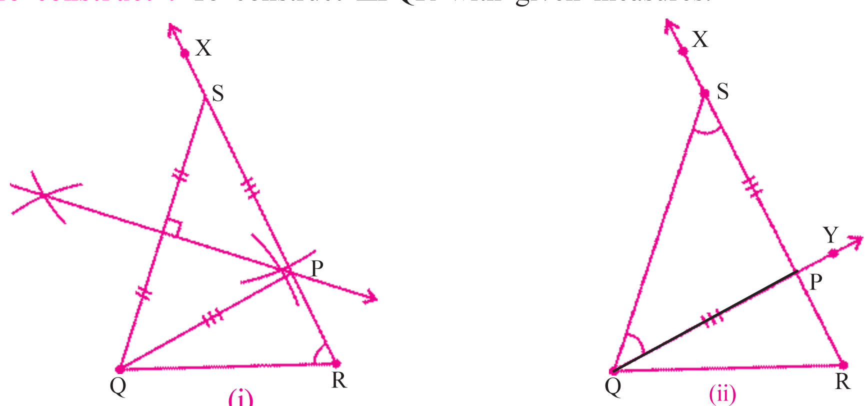


Figure 13.5

Steps of Construction :

(1) Draw $\overline{QR}$ having given measure.

(2) $\overrightarrow{RX}$ can be constructed such that $m\angle QRX$ is equal to the given $m\angle PRQ$.

(3) Select S on $\overrightarrow{RX}$ such that RS = PQ + PR.

(4) Draw $\overline{QS}$.

(5) Now to get P on $\overline{RS}$ such that PQ = PS, construct the perpendicular bisector of $\overline{QS}$, which intersects $\overline{RS}$ at P [see Figure 13.5 (i)] or Draw ∠SQY, whose measure is equal to $m\angle$RSQ. Let $\overrightarrow{QY}$ intersect $\overrightarrow{RX}$ at P (see figure 13.5 (ii)).

Then ΔPQR is the required triangle with given measures.

Now we justify our method of constructions.

In ΔPQS, PQ = PS. **(by construction)**

Then PR = RS – PS = RS –PQ

PR + PQ = RS

[if $m\angle$PSQ = $m\angle$PQS, then also PQ = PS]

Example 2 : Construct Δ ABC such that BC = 3 *cm*, $m\angle$BCA = 75 and AB + AC = 8 *cm*.

Data : In Δ ABC, BC = 3 *cm*, $m\angle$BCA = 75 and AB + AC = 8 *cm*.

To construct : To construct Δ ABC with given measures.

Steps of Construction :

(1) Draw $\overline{BC}$ such that BC = 3 *cm*.

(2) Draw $\overrightarrow{CX}$ such that $m\angle$BCX = 75 [using constructions 3 and 1].

(3) Take a point D on $\overrightarrow{CX}$ such that CD = 8 *cm*.

(4) Draw $\overline{BD}$.

(5) Draw the perpendicular bisector of $\overline{BD}$ which intersects $\overline{CD}$ at A.

(6) Draw $\overline{BA}$.

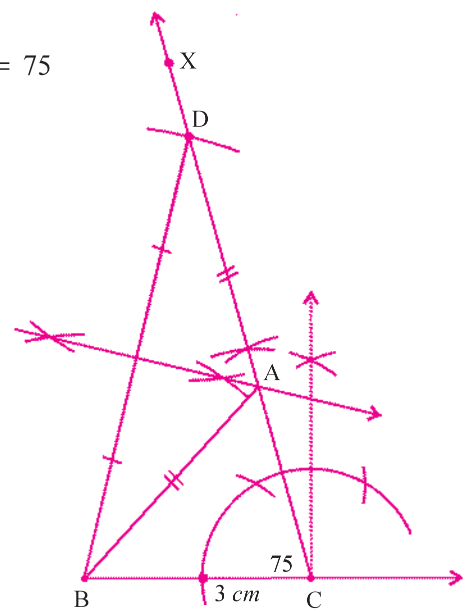


Figure 13.6

Then Δ ABC is the required triangle with given measures.

Construction 5 : To construct a triangle given its base, a base angle and the difference of the other two sides

Data : In Δ ABC, BC, $m\angle$ABC and AB–AC or AC–AB are given.

To construct : To construct Δ ABC with given measures.

Steps of Construction :

Case (1) Let AB > AC and AB–AC be given,

(1) Draw $\overline{BC}$ of given measure.

(2) Construct $\overrightarrow{BX}$ such that $m\angle CBX$ equal to given $m\angle ABC$.

(3) Select D on $\overrightarrow{BX}$ such that BD = AB – AC.

(4) Draw $\overline{CD}$.

(5) Draw the perpendicular bisector of $\overline{CD}$, which intersects $\overrightarrow{BX}$ at the point A.

(6) Draw $\overline{AC}$. (see Figure 13.7)

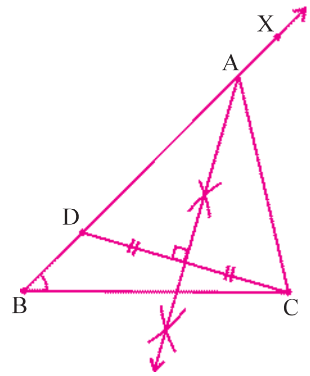


Figure 13.7

Then ΔABC is the required triangle with given measures.

Case (2) : Let AC > AB, AC – AB be given.

(1) Draw $\overline{BC}$ of given measure.

(2) Construct $\overrightarrow{BX}$ such that $m\angle CBX$ equal to given $m\angle ABC$.

(3) Draw $\overrightarrow{BY}$, opposite ray of $\overrightarrow{BX}$.

(4) Select $D \in \overrightarrow{BY}$ such that BD = AC – AB.

(5) Draw $\overline{CD}$.

(6) Draw the perpendicular bisector of $\overline{CD}$ which intersects $\overrightarrow{BX}$ at the point A.

(7) Draw $\overline{AC}$. (see figure 13.8)

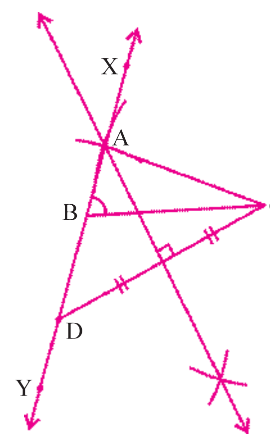


Figure 13.8

Select the point D in such a way that, if the base angle $\angle B$ is given and the side whose one of the end point is B is greater side (AB) then A–D–B, if that side (AB) is less, then A–B–D.

Then Δ ABC is the required triangle with given measures.

Now we justify our method of construction.

Case (1) $\overline{BC}$ and $\angle B$ of given measures are drawn

$\therefore$ AD = AC, as A is on the perpendicular bisector of $\overline{CD}$.

Now AD = AB – BD

$\therefore$ AC = AB – BD

$\therefore$ BD = AB – AC

Thus $\overline{BD}$ representes AB – AC.

Case (2) AC = AD as A is on the perpendicular bisector of $\overline{CD}$.

$\therefore$ AC = AB + BD

$\therefore$ BD = AC – AB

$\therefore$ $\overline{BD}$ represents AC – AB

Example 3 : Construct ΔPQR, where QR = 6 *cm*. *m*∠PRQ = 30, PQ – PR = 3 *cm*.

Data : In ΔPQR, QR = 6 *cm*. *m*∠PRQ = 30, PQ – PR = 3 *cm*.

To construct : To construct ΔPQR with given measures.

Steps of Construction :

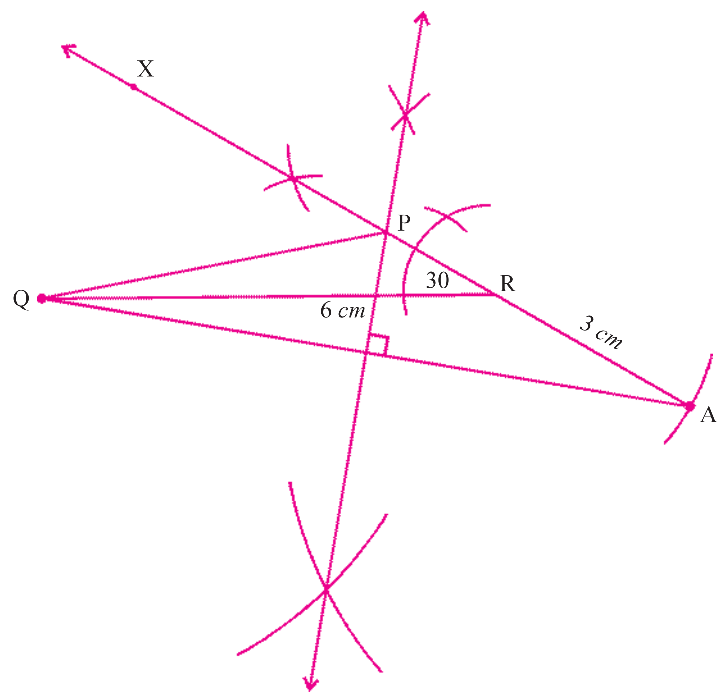


Figure 13.9

(1) Draw $\overline{QR}$ of length 6 *cm*.

(2) Draw $\overrightarrow{RX}$ such that *m*∠QRX = 30 (Construction of an angle of measure 30)

(3) Take a point A on the ray opposite to $\overrightarrow{RX}$ such that RA = 3 *cm*. (Why ?)

(4) Draw $\overline{QA}$.

(5) Draw the perpendicular bisector of $\overline{QA}$, which intersects $\overrightarrow{RX}$ at P

(6) Draw $\overline{PQ}$.

Thus ΔPQR with given conditions is constructed.

Example 4 : Construct ΔDEF such that EF = 5 *cm*, *m*∠DFE = 30, DF – DE = 2 *cm*

Data : In ΔDEF, EF = 5 *cm*, *m*∠DFE = 30, DF – DE = 2 *cm*.

Construction 5 : To construct ΔDEF with given measures.

Steps of Construction :

(1) Draw $\overline{EF}$ of length 5 *cm*.

(2) Draw $\overrightarrow{FX}$ such that $m\angle EFX = 30$.
(Construction of an angle of measure 30)

(3) Take a point C on $\overrightarrow{FX}$ such that FC = 2 *cm*.

(4) Draw $\overline{EC}$.

(5) Draw the perpendicular bisector of $\overline{EC}$ which intersects $\overrightarrow{FX}$ at D.

(6) Draw $\overline{DE}$.

Then ΔDEF is constructed in accordance with given conditions.

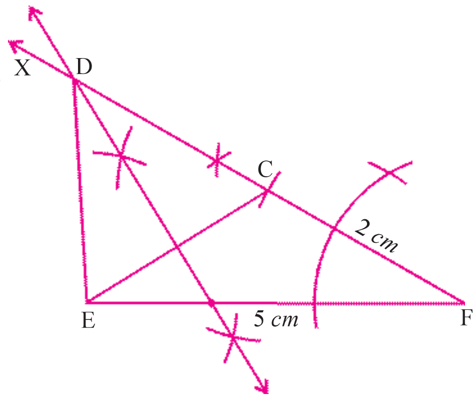


Figure 13.10

Construction 6 : To construct a triangle, given its perimeter and its two base angles.

Data : In ΔPQR, $m\angle Q$, $m\angle R$ and PQ + QR + RP are given.

To construct : To construct ΔPQR with given conditions.

Steps of Construction :

(1) Draw $\overline{XY}$ such that XY = PQ + QR + RP.

(2) Construct $\angle AXY$ and $\angle BYX$ such that $m\angle AXY = m\angle Q$ and $m\angle BYX = m\angle R$.

(3) Draw bisectors of $\angle AXY$ and $\angle BYX$, and they intersect at P.

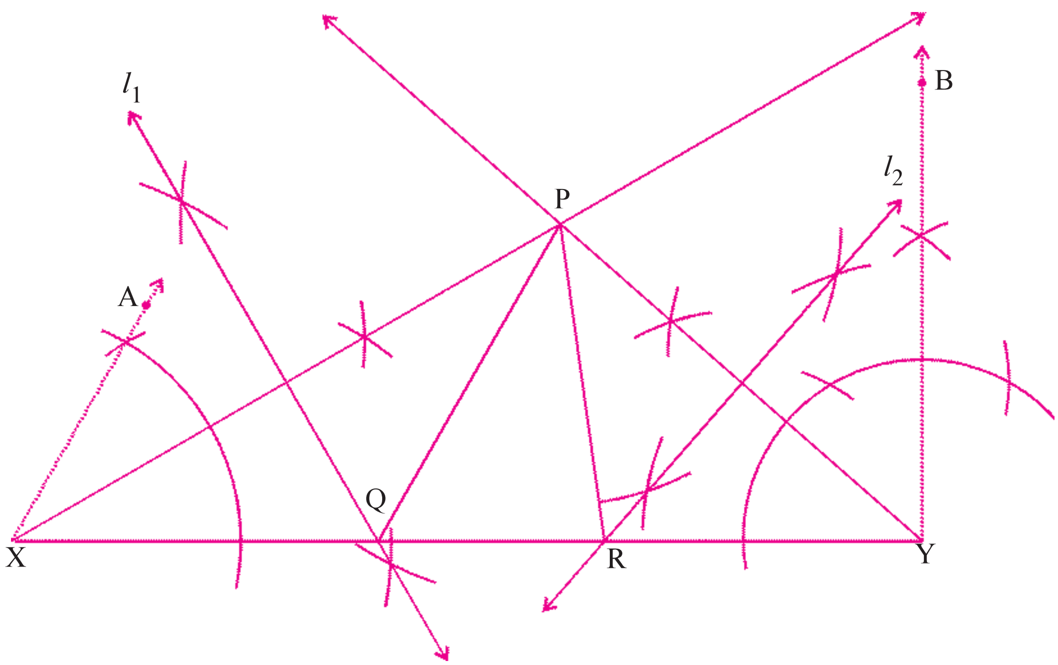


Figure 13.11

(4) Draw the perpendicular bisector, l_1 and l_2 of $\overline{PX}$ and $\overline{PY}$ respectively intersecting $\overline{XY}$ at Q and R respectively.

(5) Draw $\overline{PQ}$ and $\overline{PR}$.

Thus, ΔPQR with given conditions is constructed.

Now, we justify our method of construction.

$m\angle PYR = \frac{1}{2}\, m\angle R$ and $m\angle PXQ = \frac{1}{2}\, m\angle Q$

Line l_1 is the perpendicular bisector of $\overline{PX}$.

$\therefore\ \overline{PQ} \cong \overline{XQ}$ and similarly $\overline{PR} \cong \overline{RY}$

$\therefore$ PQ = QX and PR = RY

$\therefore\ m\angle PXQ = m\angle QPX = \frac{1}{2}\, m\angle PQR$

$\therefore\ m\angle PQR = 2\, m\angle PXQ = m\angle AXQ = m\angle AXY$

Similarly $m\angle PRQ = m\angle BYR = m\angle BYX$

Also XY = XQ + QR + RY = PQ + QR + PR

Example 5 : Construct ΔPQR Such that $m\angle Q = 60$, $m\angle R = 90$ and PQ + QR + RP = 12 *cm*.

Data : In ΔPQR, $m\angle Q = 60$, $m\angle R = 90$ and PQ + QR + RP = 12 *cm*

To construct : To construct ΔPQR with given conditions.

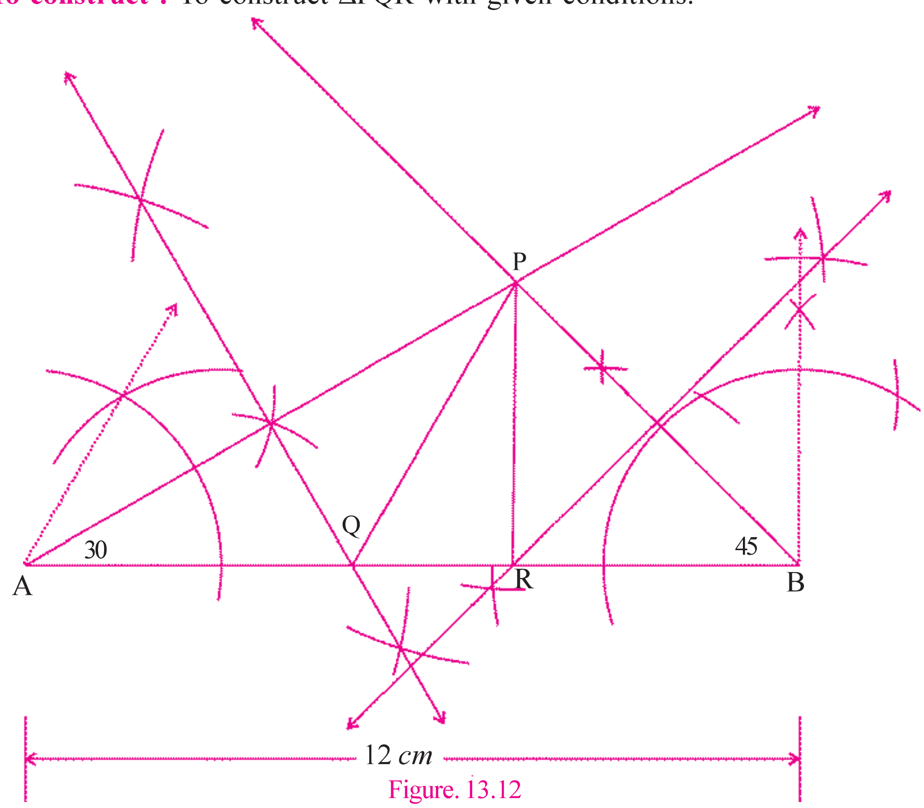


Figure. 13.12

Steps of Construction :

(1) Draw $\overline{AB}$ of length 12 *cm*.

(2) Construct ΔPAB with $m\angle A = 30$, $m\angle B = 45$ whose arms intersect at P.

(3) Construct the perpendicular bisectors of $\overline{AP}$ and $\overline{BP}$ which intersect $\overline{AB}$ at Q and R respectively.

(4) Draw $\overline{PQ}$ and $\overline{PR}$.

Thus, ΔPQR of given measures is constructed.

EXERCICE 13

1. Construct ΔABC such that BC = 6 *cm*. $m\angle B = 60$, AB + CA = 9 *cm*. Write the steps of the constuction.

2. Construct ΔPQR where PQ = 7 *cm*. $m\angle P = 30$, RP – QR = 3 *cm*. Write the steps of the construction.

3. Construct ΔABC in which $m\angle B = 30$ and $m\angle C = 30$, AB + BC + CA = 12 *cm*. Also write the steps of the construction.

4. Construct and write the steps of the construction for ΔPQR in which QR = 8 *cm* $m\angle Q = 45$ and PR – PQ = 2 *cm*.

*

Summary

In this chapter we have done the following constructions with the help of straight edge (ruler) and compass only :

1. To bisect a given angle.

2. To draw the perpendicular bisector of a line segment.

3. To draw an angle with measure 60.

4. To draw an angle having measure a multiple of 15.

5. To draw a triangle, whose base, a base angle and sum of other two sides are given.

6. To draw a triangle, whose base, a base angle and difference of other two sides are given.

7. To draw a triangle, given its two base angles and perimeter.

CHAPTER 14

HERON'S FORMULA

14.1 Introduction

In the previous classes, we have studied about the figures of different shapes such as a triangle, a square, a rectangle, a rhombus, a trapezium etc. Moreover, we had found out the areas of regions enclosed by the figures and also calculated the perimeters of them. For example, if we want to find out the perimeter of any floor of a room of our school or home, it is obvious that we walk around the boundary of that room. The total distance covered by us is considered as perimeter of that room and the floor of that room will have an area also.

So if the floor of our room is rectangular and its length is l and breath is b, then total distance covered will be $2(l+b)$ i.e. its perimeter and its area is lb.

How can we find the area of a triangle ? We know the following result about area.

Area $= \frac{1}{2} \times$ **base** $\times$ **altitude** **(i)**

For a right angled triangle we can use the above formula directly because an altitude from the vertex to the base of the triangle will be a side of the triangle. For example, in the right angled Δ ABC, $m\angle B = 90$, AB = 3 *cm*, BC = 4 *cm*, length of the hypotenuse AC = 5 *cm*. Then the area of the triangle is given by $\frac{1}{2} \times AB \times BC$ where AB is the altitude and BC is the base of the triangle.

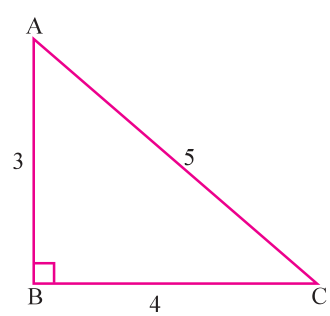


Figure 14.1

Area $= \frac{1}{2} \times 4 \times 3 = 6\ cm^2$

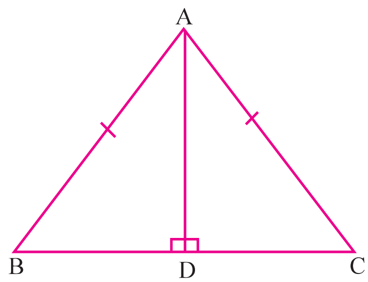


Figure 14.2

Let us find out the area of an isosceles triangle with the help of the above formula. In Δ ABC, let AB = AC. Now draw the perpendicular from the vertex A to the base $\overline{BC}$ which intersects $\overline{BC}$ at D. Thus, Δ*ABC is divided into two triangular regions, Δ*ABD and Δ*ACD.

$m\angle ADB = m\angle ADC = 90$

Now if AB = 5 *cm*, then AC is also 5 *cm* and let BC = 6 *cm*. Altitude from A divides $\overline{BC}$ in two congruent line-segments $\overline{BD}$ and $\overline{DC}$. Thus BD + DC = BC, so that BD = DC = 3 *cm* (figure 14.2)

Now, apply Pythagoras' theorem to the right angled Δ ADB

$AB^2 = BD^2 + AD^2$

$\therefore\ 5^2 = (3)^2 + AD^2$

$\therefore\ 25 - 9 = AD^2$

$\therefore\ AD^2 = 16$

$\therefore$ AD = 4 *cm* = length of the altitude

$\therefore$ By (i), area of the isosceles Δ ABC = $\frac{1}{2} \times 6 \times 4 = 12\ cm^2$

Similarly, we want to find the area of an equilateral Δ ABC, where the length of each side is 12 *cm*. For this triangle, if we draw a perpendicular from the vertex A to the base $\overline{BC}$ which intersects $\overline{BC}$ at D, then $\overline{AD}$ is an altitude of ΔABC. Here D is the midpoint of $\overline{BC}$.

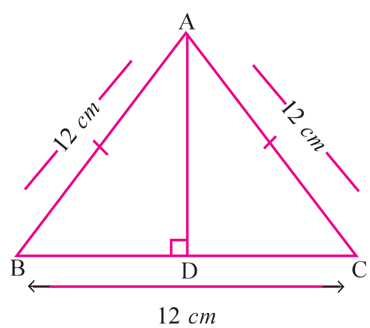


Figure 14.3

Thus, BD = DC = 6 *cm* (figure 14.3)

For right angled ΔADB, $AB^2 = BD^2 + AD^2$

$\therefore\ (12)^2 = AD^2 + (6)^2$

$\therefore\ AD^2 = 144 - 36$

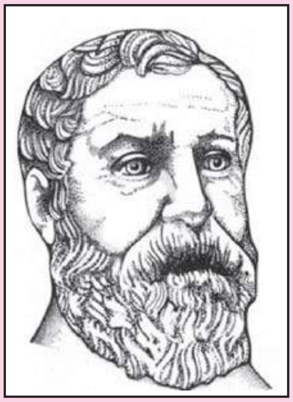
$\therefore\ AD^2 = 108$

$\therefore\ AD = 6\sqrt{3}\ cm$

$\therefore$ The area of equilateral Δ ABC is given by, $\frac{1}{2} \times AD \times BC = \frac{1}{2} \times 6\sqrt{3} \times 12$

$\therefore$ The area of $\Delta ABC = 36\sqrt{3}\ cm^2$

14.2 Heron's Formula



Heron (10AD - 75 AD)

Heron was born in about 10 A.D. possibly in Alexandria in Egypt. He worked in applied mathematics. His work on mathematical and physical subjects are so numerous and varied that he is considered to be an encyclopedic writer in these fields. His geometrical works deal largely with problems on mensuration written in three books. Book I deals with the area of squares, rectangles, triangles, trapezoids (trapezia), various other specialised quadrilaterals, the regular polygons, circles, surfaces of cylinders, cones, spheres etc. In this book, Heron has derived the famous formula for the area of a triangle in terms of its three sides.

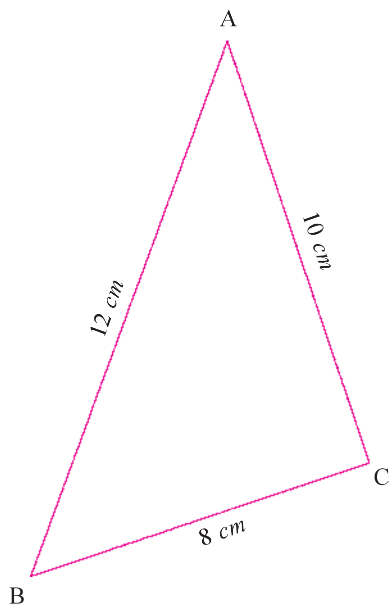


Figure 14.4

For an isosceles, equilateral and right angled triangle, we can draw the perpendiculars from the vertex to the base and we can find their lengths. Then we can find the area of the triangle by using the formula $\frac{1}{2} \times$ base $\times$ altitude. But if we have a scalene triangle, then we do not have any clue to find the length of an altitude (i.e. perpendicular from a vertex to the base of the triangle).

For an example, in Δ ABC, Let AB = 12 *cm*, BC = 8 *cm* and AC = 10 *cm*. Now there is a problem as to how can we calculate the area of this triangle ? For this, a formula is given by Heron, which is known as **Heron's formula**. It is as follows :

Area of a triangle $= \sqrt{s(s-a)(s-b)(s-c)}$ **(ii)**

Here a, b, c are the lengths of the sides of the triangle and s is semiperimeter of the triangle.

Thus, perimeter $= a + b + c = 2s$

$\therefore s = \frac{a+b+c}{2}$

So, if the length of the altitude is not given and it is not easy to find it, then this formula (ii) will be helpful to find the area of the triangle. So for the above example,

$$s = \frac{12+10+8}{2} = 15\ cm$$

$$\text{Area of } \Delta\ ABC = \sqrt{s(s-a)(s-b)(s-c)}$$

$$= \sqrt{15\,(15-12)\,(15-10)\,(15-8)}$$

$$= \sqrt{15(3)(5)(7)} = 15\sqrt{7}\ cm^2$$

Let us solve following examples to understand the application of Heron's formula.

Example 1 : Find the area of the triangle whose sides have lengths 15, 15, 12 *cm*.

Solution : Here, $s = \frac{a+b+c}{2} = \frac{15+15+12}{2} = \frac{42}{2} = 21\ cm$

$$\therefore \text{ The area of } \Delta\ ABC = \sqrt{s\,(s-a)\,(s-b)\,(s-c)}$$

$$= \sqrt{21\,(21-15)\,(21-15)\,(21-12)}$$

$$= \sqrt{21 \times 6 \times 6 \times 9}$$

$$= 18\sqrt{21}\ cm^2$$

(Do you have any other alternative method ?)

Example 2 : The lengths of the sides of a triangular park are in proportion 3 : 5 : 7 and its perimeter is 450 metre, then find out the area of this park. Also find the cost of fencing it with barbed wire at the rate of ₹ 25 per metre by leaving a space of 5 metre wide for a gate on all the sides.

Solution : The sides are in the proportion 3 : 5 : 7. Suppose the lengths of the sides of the triangular park are $3x$, $5x$ and $7x$. ($x > 0$).

Now, perimeter of triangular park = 450 metre

$\therefore\ 3x + 5x + 7x = 450$

$\therefore\ 15x = 450$

$\therefore\ x = 30$ metre

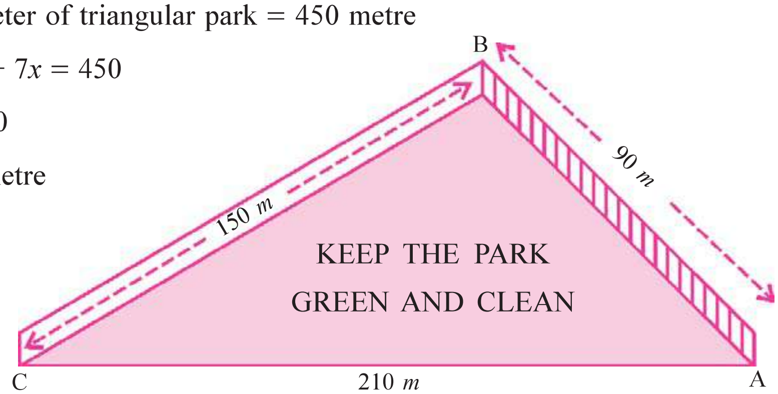


Figure 14.5

Thus, for Δ ABC, AB = c = $3x$ metre = 3(30) = 90 metre

BC = a = $5x$ metre = 5(30) = 150 metre

AC = b = $7x$ metre = 7(30) = 210 metre

Now, $s = \frac{a+b+c}{2} = \frac{90+150+210}{2} = \frac{450}{2}$ = 225 metre

$\therefore$ The area of Δ ABC $= \sqrt{225(225-90)(225-150)(225-210)}$

$= \sqrt{225(135)(75)(15)}$

$= \sqrt{15\times15\times15\times9\times25\times3\times15}$

$= \sqrt{(15)^4\times(5)^2\times(3)^2\times3}$

$= (15)^2 \times 5 \times 3 \times \sqrt{3}$

$= 3375\sqrt{3}\ m^2$

Now, for the fencing, 5 metre space is left on each side of the triangular park. Then total space left will be 5 × 3 = 15 m. Hence the total length for the fencing = length of the wire needed for fencing = Permeter of the triangular park – length of the gates

= 450 metre – 15 metre = 435 metre

$\therefore$ Total cost of fencing = 435 × 25

= ₹ 10875

Example 3 : Find the area of the triangle ΔABC where AB = 5 cm, BC = 8 cm and AC = 9 cm . Find the length of the perpendicular drawn from A to $\overline{BC}$

Solution : Here, $s = \frac{a+b+c}{2} = \frac{5+8+9}{2} = 11\ cm$

$\therefore$ The area of Δ ABC $= \sqrt{s(s-a)(s-b)(s-c)}$

$= \sqrt{11(11-8)(11-9)(11-5)}$

$= \sqrt{11\times3\times2\times6}$

$= \sqrt{11\times(6)^2}$

$= 6\sqrt{11}\ cm^2$

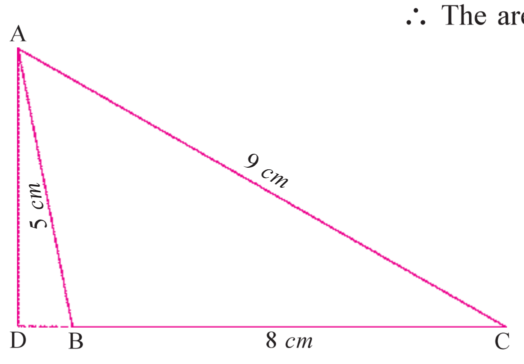


Figure 14.6

Here, $\overline{AD} \perp \overline{BC}$ (see figure 14.6)

Now we have, area of ΔABC

$= \frac{1}{2} \times$ base $\times$ altitude of Δ ABC

$= \frac{1}{2} \times 8 \times$ AD

$\therefore\ 6\sqrt{11} = 4\text{ AD}$

$\therefore\ \text{AD} = \frac{6\sqrt{11}}{4} = \frac{3}{2}\sqrt{11}\ cm$

$\therefore$ The length of the perpendicular from A to base $\overline{BC} = \frac{3}{2}\sqrt{11}\ cm$

EXERCISE 14.1

1. Find the area of the equilateral triangle having length of each side 6 units.

2. Find the area of the right angled triangle whose hypotenuse has the length 17 *cm* and has length of its base 15 *cm*.

3. Find the area of the triangle with the length of the sides 36 *cm*, 48 *cm* and 60 *cm*.

4. If the lengths of the sides of a triangle are in proportion 3 : 4 : 5 and the perimeter of the triangle is 120 metre, then find the area of the triangle.

5. An isosceles triangle has perimeter 30 *cm* and length of its congruent sides is 12 *cm*. Find the area of the triangle.

6. The triangular side walls of a flyover have been used for advertisements. The sides of the walls have lengths 100*m*, 35*m* and 105*m*. The rent per year for the advertisements is ₹ 4000 per m^2. A company hired one of its walls for 2 months. How much rent did it pay ? ($\sqrt{34} \cong 5.83$)

7. Find the area of the triangle with the lengths of the sides 5 *cm*, 7 *cm* and 10 *cm*. Also find the length of the altitude drawn from the vertex to the side whose length is 10 *cm*.

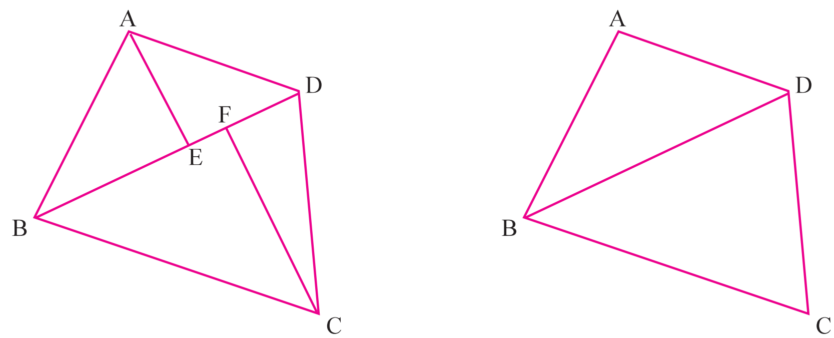
*

14.3 Application of Heron's Formula in Finding Area of Quadrilaterals

For a quadrilateral ABCD, if we join two opposite vertices, then we get a diagonal and if we draw the perpendiculars from remaining two vertices to the diagonals, then we have a formula to find the area of the quatrilateral as

Area of the quadrilateral = $\frac{1}{2}$ ***(length of a diagonal) (sum of the length of perpendiculars drawn to the diagonal from other two vertices)***

But it is a difficult and tedious process. So instead of it, if we draw a diagonal then quadrilateral region can be divided into two triangular regions and then we can use the fact that area of the quadrilateral = sum of the areas of both triangles. Both these cases are shown in the figure 14.7.



(i) Figure 14.7 (ii)

In figure 14.7 (i) we have the diagonal $\overline{BD}$ and the altitudes are $\overline{AE}$ and $\overline{CF}$. So by finding their lengths (i.e. AE and CF) we can use the result. In figure 14.7 (ii) by a single diagonal we get two triangles and by Heron's formula we can find the area of both the triangles and then take the sum of them. Thus we get the area of the quadrilateral. It will be easier to find the area of a quadrilateral in this manner.

Let us understand this discussion by the following examples.

Example 4 : In quadrilateral ABCD, AB = 3 *cm*, BC = 4 *cm*, CD = 6 *cm* and DA = 5 *cm* and the length of the diagonal $\overline{AC}$ is 5 *cm*. Find the area of □ABCD.

Solution : Here diagonal $\overline{AC}$ partitions □* ABCD in two triangular regions : Δ*ACD and Δ*ABC. For ΔACD,

$s = \frac{AD+DC+AC}{2} = \frac{5+6+5}{2} = 8\ cm$

Now the area of ΔACD $= \sqrt{8(8-5)(8-6)(8-5)}$

$= \sqrt{8(3)(2)(3)}$

$= 12\ cm^2$

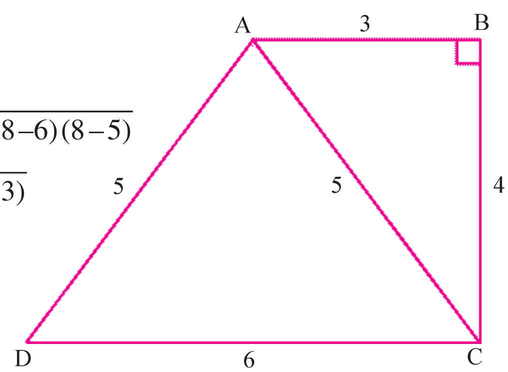


Figure 14.8

For ΔABC, $s = \frac{AB + BC + AC}{2}$

$= \frac{3+4+5}{2} = 6\ cm$

Now the area of ΔABC $= \sqrt{6(6-3)(6-4)(6-5)}$

$= \sqrt{6(3)(2)(1)} = 6\ cm^2$

$\therefore$ Area of $\square$ ABCD $=$ Area of Δ ACD + Area of Δ ABC
$= 12 + 6$
$= 18\ cm^2$

See that ΔABC is a right angled triangle. ΔADC is an isosceles triangle. So there is no need to use of Heron's formula. Do it by yourself.

Example 5 : A park is in the shape of a quadrilateral ABCD, where $m\angle C = 90$. Lengths of the sides are AB = 11 *m*; BC = 3 *m*, CD = 4 *m*, AD = 8 *m*. Then find the area of the park.

Solution : Here, for the quadrilateral ABCD, $m\angle C = 90$, and $\overline{BD}$ = diagonal. (figure 14.9). Thus for right angled Δ BCD, see that we $\overline{BD}$ is the hypotenuse.

$\therefore BD^2 = CD^2 + BC^2 = (4)^2 + (3)^2 = 25$

$\therefore$ BD = 5 = length of the diagonal

Now the area of quadrilateral ABCD

= The area of Δ BCD + The area of Δ ABD

$\therefore$ The area of Δ BCD

$= \frac{1}{2} \times \text{base} \times \text{altitude}$

$= \frac{1}{2} \times BC \times CD$

$= \frac{1}{2} \times 3 \times 4$

$= 6\ m^2$

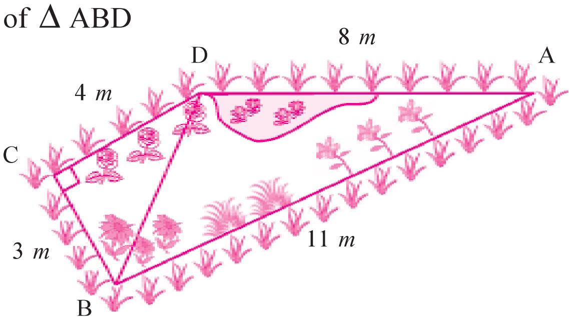


Figure 14.9

Now, for the area of Δ ABD,

$$s = \frac{AB+BD+AD}{2} = \frac{11+5+8}{2} = 12\ m$$

$\therefore$ Area of Δ ABD $= \sqrt{12(12-5)(12-8)(12-11)}$
$= \sqrt{12 \times 7 \times 4 \times 1}$
$= \sqrt{4 \times 3 \times 7 \times 4}$
$= 4\sqrt{21}\ m^2$

$\therefore$ Area of quadrilateral ABCD $= 6 + 4\sqrt{21}\ m^2$

EXERCISE 14.2

1. Find the area of the quadrilateral ABCD where AB = 7 *cm*, BC = 6 *cm*, CD = 12 *cm* and AD = 15 *cm* and the length of the diagonal $\overline{AC}$ is 11 *cm*.
2. Find the area of the quadrilateral ABCD where AB = 8 *m*, BC = 15 *m* and CD = 13 *m*, DA = 12 *m*, $m\angle B = 90$.

3. If the perimeter of a quadrilateral ABCD is 92 *m* and the perimeter of Δ ABD is 90 *m*, then find the length of the diagonal $\overline{BD}$. Also find the area of the quadrilateral ABCD where AB = 40 *m*, BC = 15 *m*, CD = 28 *m*, DA = 9 *m*.
4. If the lengths of the diagonals of a quadrilateral field are 40 *m* and 24 *m* and they bisect each other at right angles, then find its area.
5. If the lengths of the sides of a parallelogram are 13 *cm* and 10 *cm* and the length of one of its diagonal is 9 *cm,* then find its area.

*

EXERCISE 14

1. Find the area of regular hexagon ABCDEF (figure 14.10) where the length of each side is 4 *cm* and O is the midpoint of the diagonals $\overline{FC}$, $\overline{DA}$ and $\overline{BE}$ and their lengths are 8 *cm*.

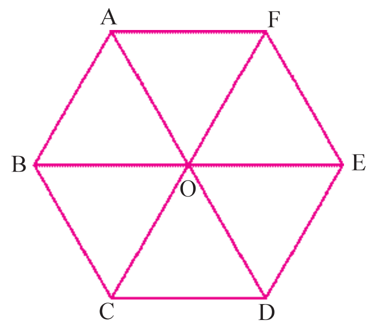


Figure 14.10

2. Find the area of the quadrilateral ABCD, where AB = 9 *cm*, BC = 10 *cm*, CD = 12 *cm*, DA = 11 *cm* and $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$.
3. A bulk of triangular tiles of the length 3 *cm*, 4 *cm* and 5 *cm* is to be used for the flooring of a room with area 216 cm^2. Find how many tiles should be used for the flooring. Find the total cost of polishing the tiles at the rate of ₹ 2.75 per cm^2.
4. An umbrella is to be made by stitching 8 triangular pieces of cloth with lengths 17 *cm*, 17 *cm* and 16 *cm*. Find how much cloth is required for the umbrella.
5. Find the area of the triangle whose length of the sides are 6 *cm*, 8 *cm* and 10 *cm*.
6. If the length of the sides of a triangle are in proportion 25 : 17 : 12 and its perimeter is 540 *m*, then find the lengths of the largest and smallest altitudes.
7. In figure 14.11, BC = 5 *cm*, CD = 3 *cm*, CF = 6 *cm*. Find the area occupied by the prism on the prism table.

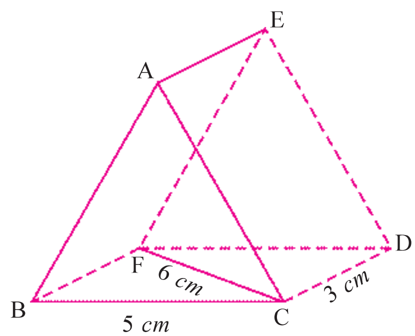


Figure 14.11

8. The base of a triangular field is twice to its altitude and the cost of cultivating the field is ₹ 30 per hectre and the total cost is ₹ 480. Find the length of the base and altitude of that trianguler field. (10000 m^2 = 1 Hector)
9. If the length of the side of a square is 5 *m* and it is converted into a rhombus whose major diagonal has length 8 *m,* then, find the length of the other diagonal and also find the area of the rhombus.

10. If the area of a rhombus is 100 cm^2 and the length of one of its digonal is 8 *cm*, then find the length of the other diagonal.

11. Both of the parallel sides of a trapezium are 8 *cm* and 16 *cm*. Non-parallel sides are congruent, each being 10 *cm*. Then find the area of the trapezium

12. Select proper option (a), (b), (c) or (d) and write in the box given on the right so that the statement becomes correct :

(1) For the Δ ABC, semiperimeter is ...... where AB = 8 *cm*, BC = 6 *cm*, AC = 10 *cm*. ☐

(a) 24 (b) 20 (c) 12 (d) 16

(2) For a $\square^m$ ABCD, $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$ and $\overleftrightarrow{BC} \parallel \overleftrightarrow{DA}$. If AB = 8 *cm* and BC = 10 *cm* the perimeter of the $\square^m$ ABCD is ...... *cm* ☐

(a) 18 (b) 20 (c) 36 (d) 56

(3) If the perimeter of a trapezium is 50 *cm* and the lengths of non-parallel sides are equal to 12 *cm,* then the sum of parallel sides is .... ☐

(a) 13 *cm* (b) 26 *cm* (c) 28 *cm* (d) 30 *cm*

(4) If the area of a rhombus is 54 cm^2 and the lengths of one of its diagonal is 9 *cm*, then the length of its other diagonal is ...... *cm*. ☐

(a) 9 (b) 12 (c) 27 (d) 90

(5) If the lengths of the sides of a triangle are in proportion 3 : 4 : 5 then the area of the triangle is ...... sq units where perimeter of the triangle is 144. ☐

(a) 64 (b) 364 (c) 564 (d) 864

(6) If the base of an isosceles triangle has length 10 *cm* and its perimeter is 28 *cm,* then the length of each congruent side is ...... *cm*. ☐

(a) 38 (b) 18 (c) 9 (d) 19

(7) If the lengths of the sides of a triangle are 8 *cm*, 11 *cm* and 13 *cm*, then area of the triangle is .... $(cm)^2$. ☐

(a) 44 (b) 43 (c) 42.82 (d) $8\sqrt{30}$

(8) If the length of the base of a triangle is 12 *cm* and the length of the altitude to that base is 8 *cm,* then the area of the triangle is ...... $(cm)^2$. ☐

(a) 12 (b) 24 (c) 36 (d) 48

(9) If the area of an equilateral triangle is $2\sqrt{3}$ cm^2, then the length of each side of the triangle is ...... *cm*. ☐

(a) $\sqrt{2}$ (b) $2\sqrt{3}$ (c) $2\sqrt{2}$ (d) $3\sqrt{2}$

(10) In a Δ ABC, $\overline{CD}$ is the altitude of Δ ABC where AD = 4 *cm*, CD = 5 *cm* and BD = 5 *cm*. Also the area of a square is the same as the area of Δ ABC. Then length of each side of the square is ...... *cm*. ☐

(a) $\frac{3\sqrt{2}}{5}$ (b) $\frac{3}{2}$ (c) $\frac{3\sqrt{10}}{2}$ (d) $\frac{3\sqrt{5}}{2}$

(11) In a square ABCD, length of each side is 7 *cm*. Then length of its diagonal is ...... *cm* ☐

(a) $\sqrt{2}$ (b) 7 (c) $7\sqrt{2}$ (d) $2\sqrt{7}$

(12) In quadrilateral ABCD, the lengths of each side is shown in the figure 14.12 then the length of the diagonal $\overline{AC}$ is ...... *m*. ☐

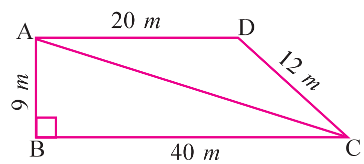


Figure 14.12

(a) 40 (b) 9 (c) 49 (d) 41

*

Summary

In this chapter we have studied the following points :

1. If the lengths of the sides of a triangle are *a*, *b* and *c*, then the perimeter of Δ ABC is $a + b + c = 2s$ and its semiperimeter is $s = \frac{a+b+c}{2}$.
2. The area of a triangle is given by Heron's formula and it is $\sqrt{s(s-a)(s-b)(s-c)}$.
3. To find the area of a quadrilateral whose sides and one diagonal are given. By a diagonal the quadrilateral region is partitioned into two triangular regions and then by Heron's formula we can find the area of each of the triangles. The sum of areas of both triangles gives us the area of quadrilateral.

CHAPTER 15

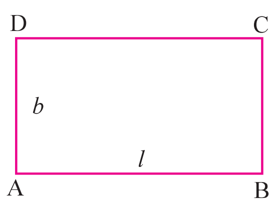
SURFACE AREA AND VOLUME

15.1 Introduction

We have learnt about plane figures like a rectangle, a square, a circle etc. We have also studied how to find out their perimeters and area in earlier classes. Now, we will learn about congruent figures made by cutting from cardboard sheet and stacking them up in a vertical pile. By this process we shall obtain a **'solid'**. We have already studied in earlier classes about cuboid, cube etc. We will now learn here about solids in detail.

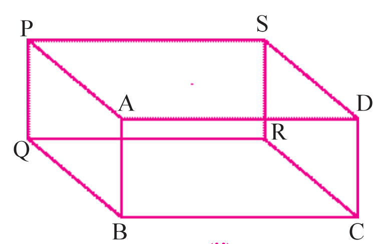
15.2 Introduction of a Cuboid and a Cube

We know about a rectangle and a square and formulae to find their areas and perimeters.



(i)

(i) Area = $l \times b$ Perimeter = $2(l + b)$



(ii)

Figure 15.1

Cuboid : A cuboid is a solid bounded by six rectangular plane regions.

Figure 15.1 (ii) represents a cuboid. We will study some solids.

In figure 15.1 (ii) □ ABCD, □ PQRS; □ SRCD, □ PQBA; □ PADS, □ QBCR are six faces of the cuboid. Each face is a rectangle. □ PADS and □ QBCR are **top and bottom faces** respectively. Also they are **opposite faces**. Similarly □ PQBA and □ SRCD; □ ABCD and □ PQRS are pairs of opposite faces. □ PQBA and □ ABCD are **adjacent faces**. Can you name another pair of adjacent faces from the figure ?

$\overline{AB}$, $\overline{BC}$, $\overline{CD}$, $\overline{DA}$; $\overline{PQ}$, $\overline{QR}$, $\overline{RS}$, $\overline{SP}$; $\overline{PA}$, $\overline{QB}$, $\overline{RC}$, $\overline{SD}$ are twelve **edges** of the cuboid. Adjacent faces intersect in an edge in one side of a rectangle only. Since opposite sides of a rectangle are congruent, BC = AD = QR = PS, AB = DC = SR = PQ, QB = PA = CR = SD.

A, B, C, D, P, Q, R and S are **vertices** of cuboid.

We can take any face of a cuboid as base of the cuboid. In this case, the four faces which meet the base are called **the lateral faces of cuboid.** In our cuboid type of classroom, four walls are faces of cuboid.

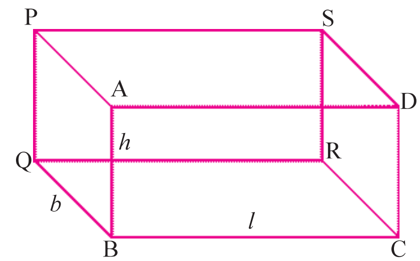


Figure 15.2

When we take, a rectangle, a face of a cuboid, as the base, then its length and breadth are known as the length and breadth of the cuboid. Any two lateral faces intersect in a line-segment called height of the cuboid. In figure 15.2 the rectangle QBCR is a base of cube. BC is the length l and QB is the breadth b. Intersection of faces □ ABCD and □ PQBA is $\overline{AB}$. Its length AB is the height of the cuboid.

The length, breadth and height of the cuboid are denoted by l, b and h respectively.

Cube : A cuboid whose length, breadth and height are equal is called a cube.

15.3 Surface Area of a Cuboid and Cube

We take a bundle of many congruent rectangular sheets of paper. The shape of this bundle is a cuboid. It is also called a **rectangular parallelopiped.**

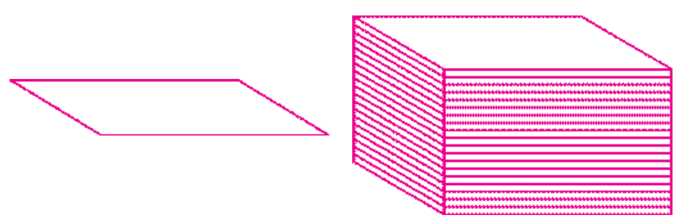


Figure 15.3

Activity (1) : First, we take an empty chalk-box. Open all the sides of the chalk-box carefully and arrange all the faces of the chalk-box on the table as given in the figure 15.4. Name all the faces.

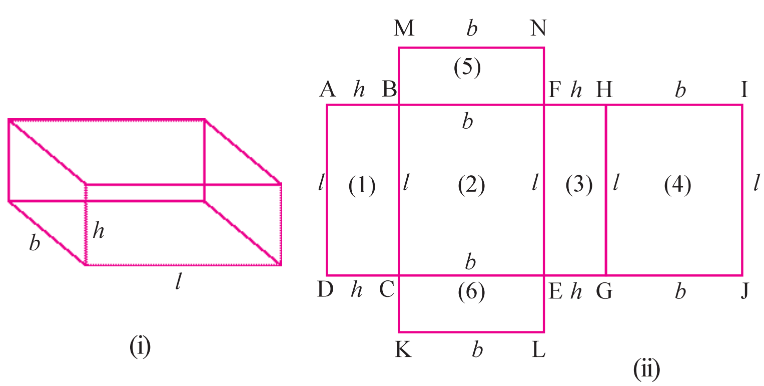


Figure 15.4

Area of the face ABCD = Area of the face FEGH = $l \times h$

Area of the face BCEF = Area of the face HGJI = $l \times b$

Area of the face CKLE = Area of the face BMNF = $b \times h$

Total surface area of a cuboid = Sum of the areas of all its six faces

$= 2\,(l \times h) + 2\,(l \times b) + 2\,(b \times h)$

$= \mathbf{2\,(lb + bh + hl)}$

Note : To find out the surface area of a cuboid, the length, breadth and height must be expressed in the same units.

Example 1 : If the dimensions of a cuboid are 20 cm × 15 cm × 10 cm, find its total surface area.

Solution : Total surface area $= 2\,(lb + bh + hl)$

$= 2\,(20 \times 15 + 15 \times 10 + 10 \times 20)$

$= 2\,(300 + 150 + 200)$

$= 2\,(650)$

$= 1300\ cm^2$

Surface Area of a Cube : For a cube, we have $l = b = h$. All the six faces of a cube are squares of the same size.

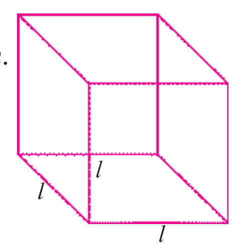


Figure 15.5

Total surface area of a cube $= 2\,(l \times l + l \times l + l \times l)$

$= 2\,(l^2 + l^2 + l^2)$

$= 6l^2$

$= 6\,(\text{length of cube})^2$

15.4 Lateral Surface Area of Cuboid and Cube :

Now we find the sum of the areas of the four faces of a cuboid excluding top and bottom faces. This sum is called the lateral surface area of the cuboid or the cube.

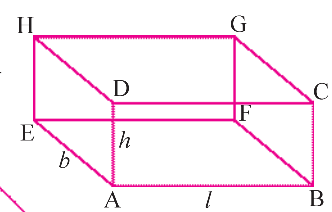


Figure 15.6

Lateral surface area of a cuboid

= Area of the face ABCD + Area of the face FBCG + Area of the face EFGH + Area of the face EADH.

$= l \times h + h \times b + l \times h + b \times h$

$= 2\,(l \times h) + 2\,(h \times b)$

$= \mathbf{2h\,(l + b) = h \cdot 2\,(l + b)}$

= **Height × Perimeter of base**

Cube : Lateral surface area of a cube

$= l^2 + l^2 + l^2 + l^2$

$= \mathbf{4l^2}$

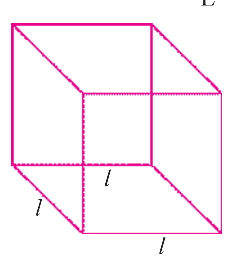


Figure 15.7

Example 2 : A cubical box has each edge having length 12 *cm* and another cuboidal box has edges 15 *cm* long, 12 *cm* wide and 8 *cm* high. (i) Which box has the smaller total surface area and by how much amount ? (ii) Which box has the greater lateral surface area and by how much amount ?

Solution : (i) Let the total surface areas of the cubical and cuboidal boxes be S_1 and S_2.

$S_1 = 6\,(l)^2 = 6\,(12)^2 = 6\,(144) = 864\ cm^2$

$S_2 = 2(lb + bh + hl)$

$= 2\,(15 \times 12 + 12 \times 8 + 8 \times 15)$

$= 2\,(180 + 96 + 120)$

$= 2\,(396)$

$= 792\ cm^2$

$\therefore\ S_1 - S_2 = 864 - 792 = 72\ cm^2$

$\therefore$ The cuboidal box has smaller surface area and is smaller by 72 cm^2

(ii) Let the lateral surface areas of the cubical and cuboid boxes be L_1 and L_2.

$L_1 = 4\,(l)^2$

$= 4\,(12)^2$

$= 4\,(144)$

$= 576\ cm^2$

$L_2 = 2h\,(l + b)$

$= 2 \times 8\,(15 + 12)$

$= 432\ cm^2$

$L_1 - L_2 = 576 - 432$

$= 144\ cm^2$

Thus, the cubical box has greater lateral surface area and is greater by 144 cm^2.

Example 3 : Kanjibhai had built closed cubical water tank with lid for his factory. The length, breadth and height of the tank are 2.5 *m*, 1.5 *m* and 1 *m* respectively. He wants to cover outer surface of the tank (excluding the base) with square tiles of side 25 *cm*. Find out the number of tiles and total cost, if the rate of the tiles is ₹ 480 per dozen.

(1 dozen = 12 units)

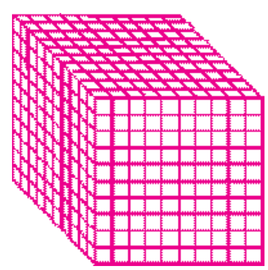


Figure 15.8

Solution : First we should find out total surface area of five outer faces of tank.

Length of the tank = 2.5 *m* = 250 *cm*

Breadth of the tank = 1.5 *m* = 150 *cm*

Height of the tank = 1 *m* = 100 *cm*

$\therefore$ Surface Area (excluding base) $= l \times b + 2(b \times h) + 2\,(h \times l)$

$= [250 \times 150 + 2\,(150 \times 100) + 2\,(100 \times 250)]$

$= (37500 + 30000 + 50000)$

$= 117500\ cm^2$

Area of each square tile = $(25 \times 25)\ cm^2$

$\therefore$ Number of tiles required = $\dfrac{\text{area of the tank}}{\text{area of one tile}} = \dfrac{117500}{25 \times 25}$ = 188 tiles

Since cost of 12 tiles is ₹ 480, cost of 188 tiles = $\dfrac{480 \times 188}{12}$ = ₹ 7520

$\therefore$ Number of tiles required is 188 and total cost is ₹ 7520.

Note : In fact $\frac{250}{25} \times \frac{150}{25}$ tiles are required for top.

$\therefore$ Total numbers of tiles required for top = $10 \times 6 = 60$

Similarly total numbers of tiles required for sides

$= 2\left(\frac{150}{25} \times \frac{100}{25} + \frac{250}{25} \times \frac{100}{25}\right)$

$= 2(6 \times 4 + 10 \times 4) = 128$

$\therefore$ Total number of tiles required is $128 + 60 = 188$.

If l or b or h is not a multiple of 25 then tiles would have to be broken ! Not a practical solution.

Example 4 : A hall for prayer in a school is 10 m long, 8 m wide and 5 m high. It has two doors each measuring $(3 \times 1.5)\ m^2$ and Four windows, each measuring $(2 \times 2)\ m^2$. Find the total expense for whitewashing the interior walls. The rate of whitewashing is ₹ 6 per m^2.

Solution : Area of four walls = (Lateral surface area of cuboidal hall)

$= 2h\ (l + b)$

$= 2 \times 5\ (10 + 8)$

$= 180\ m^2$

Area of two doors $= 2\ (3 \times 1.5) = 9\ m^2$

Area of four windows $= 4\ (2 \times 2) = 16\ m^2$

Area to be whitewashed = (Area of four walls with door and windows) – (Area of doors + Area of windows)

$= (180 - (9 + 16)) = 155\ m^2$

The rate of whitewashing is ₹ 6 per m^2.

$\therefore$ cost of whitewashing $= (155 \times 6)$

= ₹ 930

$\therefore$ The cost of whitewashing is ₹ 930.

EXERCISE 15.1

1. Fill in the blanks in each row in the following table from given information :

No.	length	breadth	height	lateral surface area	Total surface area
(1)	18 *cm*	10 *cm*	5 *cm*	.........cm^2	.........cm^2
(2)	3 *m*	3 *m*	3 *m*	.........m^2	.........m^2
(3)	1 *m*	75 *cm*	50 *cm*	.........cm^2	.........cm^2

2. A small indoor green house (herberium) is made entirely of glass panes (including base) held together with tape. It is 40 *cm* long, 30 *cm* wide and 25 *cm* high.

(1) What is the area of the glass panes used ?

(2) Find the cost of glass painting of four walls of the green-house. The rate of glass-painting is ₹ 500 per m^2.

3. Find the area of the four walls and ceiling of a room, whose length is 10 *m*, breadth is 8 *m* and height is 5 *m*. Also find the cost of whitewashing the walls and ceiling, at the rate of ₹ 15 per m^2.

4. The floor of a rectangular hall has a perimeter of 300 *m*. Its height is 10 *m*. There are two doors of 5 $m \times 3$ *m* and four windows of 3 $m \times 1.5$ *m*. Find the cost of painting of its four walls at the rate of ₹ 30 per m^2.

5. A cubical box is 15 *cm* long and another cuboidal box is 25 *cm* long, 20 *cm* wide and 10 *cm* high.

(1) Which box has the smaller lateral area and by how much ?

(2) Which box has the greater total surface area and by how much ?

*

15.5 Surface Area of a Right Circular Cylinder

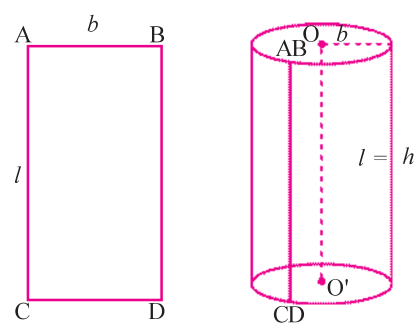


Figure 15.9

We know about a cylinder and formula to find its area.

Activity (1) : A cylinder is generated by the revolution of a rectangle about one of its sides. This cylinder is called a **right circular cylinder.**

Top and bottom of a right circular cylinder are parallel circular region.

In figure 15.9, breadth of the rectangle CD namely (b) becomes the circumference of the base. The radius of the base is the radius of the cylinder. The length of the rectangle (l) becomes the height (h) of the cylinder.

The line-segment joining the two centres of circular ends is perpendicular to base. This is the height (h) of cylinder. If the line-segment is not perpendicular to base, then what is the situation ? Let us see.

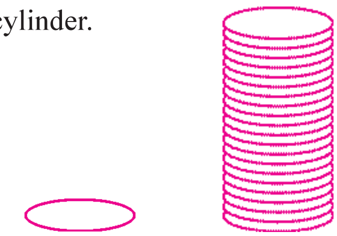


Figure 15.10

Activity (2) : If we take a number of coins of five rupees and stack them vertically up, then we get a right circular cylinder (figure 15.10).

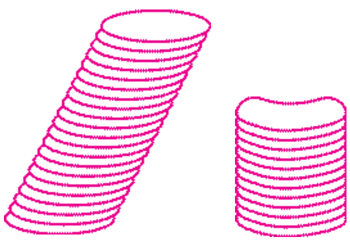


Figure 15.11

Keep in mind that stack of coins has been kept at right angle to the base and the base is circular.

Figure 15.11 does not represent right circular cylinder.

Note : In our study, a cylinder would mean a right circular cylinder.

Activity (3) : Now, we take a sufficiently large coloured rectangular paper, whose length is just enough to go round the cylinder and whose breadth is equal to the height of the cylinder (see figure 15.12).

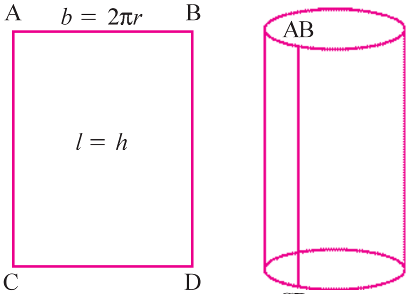


Figure 15.12

The rectangular region ABDC gives us curved surface of the cylinder. The breadth (b) of the rectangle is equal to the circumference of the circular base of the cylinder which is equal to $2\pi r$. The length (l) of the rectangle is the height (h) of the cylinder.

$\therefore$ Curved surface area of the cylinder = Area of the rectangle

= length $\times$ breadth

= perimeter of the base of the cylinder $\times$ height of the cylinder

= $2\pi r \times h = 2\pi rh$

$\therefore$ Curved surface area of the cylinder = $2\pi rh$