Chapter 4

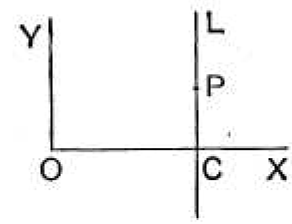
THE STRAIGHT LINE

: RECTANGULAR COORDINATES

46. *To find the equation to a straight line which is parallel to one of the coordinate axes.*

Let CL be any line parallel to the axis of y and passing through a point C on the axis of x such that $OC = c$.

Let P be any point on this line whose coordinates are x and y.



Then the abscissa of the point P is always c, so that

$$x = c \ldots\ldots\ldots\ldots\ldots (1).$$

This being true for every point on the line CL (produced indefinitely both ways), and for no other point, is, by Art. 42, the equation to the line.

It will be noted that the equation does not contain the coordinate y.

Similarly the equation to a straight line parallel to the axis of x is $y = d$.

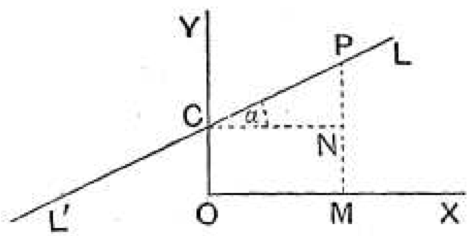
Cor. The equation to the axis of x is $y = 0$.

The equation to the axis of y is $x = 0$.

47. *To find the equation to a straight line which cuts off a given intercept on the axis of y and is inclined at a given angle to the axis of x.*

Let the given intercept be c and let the given angle be α.

Let C be a point on the axis of y such that OC is c. Through C draw a straight line LCL' inclined at an angle α $(=\tan^{-1} m)$ to the axis of x, so that $\tan \alpha = m$.



The straight line LCL' is therefore the straight line required, and we have to find the relation between the coordinates of any point P lying on it.

Draw PM perpendicular to OX to meet in N a line through C parallel to OX.

Let the coordinates of P be x and y, so that $OM = x$ and $MP = y$.

Then $MP = NP + MN = CN \tan \alpha + OC = m \,.\, x + c$,

i.e.
$$\mathbf{y = mx + c}.$$

This relation being true for *any* point on the given straight line is, by Art. 42, the equation to the straight line.

[In this, and other similar cases, it could be shewn, conversely, that the equation is only true for points lying on the given straight line.]

Cor. The equation to any straight line passing through the origin, *i.e.* which cuts off a zero intercept from the axis of y, is found by putting $c = 0$ and hence is $y = mx$.

48. The angle α which is used in the previous article is the angle through which a straight line, originally parallel to OX, would have to turn in order to coincide with the given direction, the rotation being always in the positive direction. Also m is always the tangent of this angle. In the case of such a straight line as AB, in the figure of Art. 50, m is equal to the tangent of the angle XAP (not of the angle PAO). In this case therefore m, being the tangent of an obtuse angle, is a negative quantity.

The student should verify the truth of the equation of the last article for *all* points on the straight line LCL', and also for straight lines in other positions, *e.g.* for such a straight line as A_2B_2 in the figure of Art. 59. In this latter case both m and c are negative quantities.

A careful consideration of all the possible cases of a few propositions will soon satisfy him that this verification is not always necessary, but that it is sufficient to consider the standard figure.

49. Ex. The equation to the straight line cutting off an intercept 3 from the negative direction of the axis of y, and inclined at $120°$ to the axis of x, is

$$y = x \tan 120° + (-3),$$

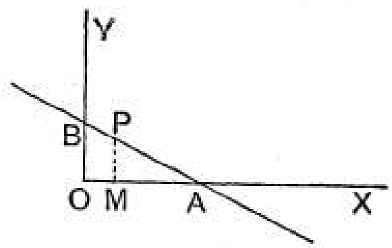
i.e. $$y = -x\sqrt{3} - 3,$$

i.e. $$y + x\sqrt{3} + 3 = 0.$$

50. *To find the equation to the straight line which cuts off given intercepts a and b from the axes.*

Let A and B be on OX and OY respectively, and be such that $OA = a$ and $OB = b$.

Join AB and produce it indefinitely both ways. Let P be any point (x, y) on this straight line, and draw PM perpendicular to OX.



We require the relation that always holds between x and y, so long as P lies on AB.

By Euc. VI. 4, we have

$$\frac{OM}{OA} = \frac{PB}{AB}, \text{ and } \frac{MP}{OB} = \frac{AP}{AB}.$$

$$\therefore \frac{OM}{OA} + \frac{MP}{OB} = \frac{PB + AP}{AB} = 1,$$

i.e. $$\mathbf{\frac{x}{a} + \frac{y}{b} = 1.}$$

This is therefore the required equation; for it is the relation that holds between the coordinates of *any* point lying on the given straight line.

51. The equation in the preceding article may be also obtained by expressing the fact that the sum of the areas of the triangles OPA and OPB is equal to OAB, so that

$$\tfrac{1}{2} a \times y + \tfrac{1}{2} b \times x = \tfrac{1}{2} a \times b,$$

and hence $$\frac{x}{a} + \frac{y}{b} = 1.$$

52. Ex. 1. *Find the equation to the straight line passing through the point* $(3, -4)$ *and cutting off intercepts, equal but of opposite signs, from the two axes.*

Let the intercepts cut off from the two axes be of lengths a and $-a$.

The equation to the straight line is then

$$\frac{x}{a}+\frac{y}{-a}=1,$$

i.e. $$x-y=a\text{..............................(1)}.$$

Since, in addition, the straight line is to go through the point $(3, -4)$, these coordinates must satisfy (1), so that

$$3-(-4)=a,$$

and therefore $$a=7.$$

The required equation is therefore

$$x-y=7.$$

Ex. 2. *Find the equation to the straight line which passes through the point* $(-5, 4)$ *and is such that the portion of it between the axes is divided by the point in the ratio of* $1 : 2$.

Let the required straight line be $\frac{x}{a}+\frac{y}{b}=1$. This meets the axes in the points whose coordinates are $(a, 0)$ and $(0, b)$.

The coordinates of the point dividing the line joining these points in the ratio $1 : 2$, are (Art. 22)

$$\frac{2.a+1.0}{2+1} \text{ and } \frac{2.0+1.b}{2+1}, \quad i.e. \ \frac{2a}{3} \text{ and } \frac{b}{3}.$$

If this be the point $(-5, 4)$ we have

$$-5=\frac{2a}{3} \text{ and } 4=\frac{b}{3},$$

so that $$a=-\tfrac{15}{2} \text{ and } b=12.$$

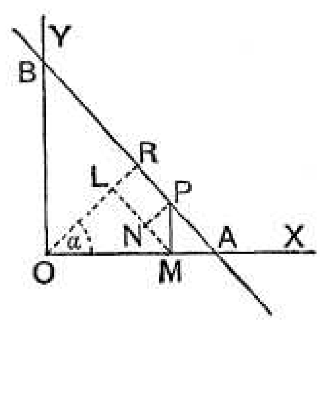
The required straight line is therefore

$$\frac{x}{-\frac{15}{2}}+\frac{y}{12}=1,$$

i.e. $$5y-8x=60.$$

53. *To find the equation to a straight line in terms of the perpendicular let fall upon it from the origin and the angle that this perpendicular makes with the axis of x.*

Let OR be the perpendicular from O and let its length be p.



Let α be the angle that OR makes with OX.

Let P be any point, whose coordinates are x and y, lying on AB; draw the ordinate PM, and also ML perpendicular to OR and PN perpendicular to ML.

Then $$OL = OM \cos\alpha \qquad\qquad (1),$$
and $$LR = NP = MP \sin NMP.$$

But $$\angle NMP = 90^\circ - \angle NMO = \angle MOL = \alpha.$$
$\therefore$ $$LR = MP\sin\alpha \qquad\qquad (2).$$

Hence, adding (1) and (2), we have
$$OM\cos\alpha + MP\sin\alpha = OL + LR = OR = p,$$
i.e. $$\mathbf{x\cos\alpha + y\sin\alpha = p.}$$

This is the required equation.

54. In Arts. 47—53 we have found that the corresponding equations are only of the first degree in x and y. We shall now prove that

Any equation of the first degree in x and y always represents a straight line.

For the most general form of such an equation is
$$Ax + By + C = 0 \qquad\qquad (1),$$
where A, B, and C are constants, *i.e.* quantities which do not contain x and y and which remain the same for all points on the locus.

Let (x_1, y_1), (x_2, y_2), and (x_3, y_3) be *any* three points on the locus of the equation (1).

Since the point (x_1, y_1) lies on the locus, its coordinates when substituted for x and y in (1) must satisfy it.

Hence $$Ax_1 + By_1 + C = 0 \qquad\qquad (2).$$
So $$Ax_2 + By_2 + C = 0 \qquad\qquad (3),$$
and $$Ax_3 + By_3 + C = 0 \qquad\qquad (4).$$
Since these three equations hold between the three quantities A, B, and C, we can, as in Art. 12, eliminate them.

The result is
$$\begin{vmatrix} x_1, & y_1, & 1 \\ x_2, & y_2, & 1 \\ x_3, & y_3, & 1 \end{vmatrix} = 0 \qquad\qquad (5).$$
But, by Art. 25, the relation (5) states that the area of the triangle whose vertices are (x_1, y_1), (x_2, y_2), and (x_3, y_3) is zero.

Also these are any three points on the locus.

The locus must therefore be a straight line; for a curved line could not be such that the triangle obtained by joining *any* three points on it should be zero.

55. The proposition of the preceding article may also be deduced from Art. 47. For the equation

$$Ax+By+C=0$$

may be written $$y=-\frac{A}{B}x-\frac{C}{B},$$

and this is the same as the straight line

$$y=mx+c,$$

if $$m=-\frac{A}{B} \text{ and } c=-\frac{C}{B}.$$

But in Art. 47 it was shewn that $y=mx+c$ was the equation to a straight line cutting off an intercept c from the axis of y and inclined at an angle $\tan^{-1}m$ to the axis of x.

The equation $$Ax+By+C=0$$

therefore represents a straight line cutting off an intercept $-\frac{C}{B}$ from the axis of y and inclined at an angle $\tan^{-1}\left(-\frac{A}{B}\right)$ to the axis of x.

56. We can reduce the general equation of the first degree $$Ax+By+C=0\text{.....................}(1)$$
to the form of Art. 53.

For, if p be the perpendicular from the origin on (1) and α the angle it makes with the axis, the equation to the straight line must be

$$x\cos\alpha+y\sin\alpha-p=0\text{ ....}(2).$$

This equation must therefore be the same as (1).

Hence $$\frac{\cos\alpha}{A}=\frac{\sin\alpha}{B}=\frac{-p}{C},$$

i.e. $$\frac{p}{C}=\frac{\cos\alpha}{-A}=\frac{\sin\alpha}{-B}=\frac{\sqrt{\cos^2\alpha+\sin^2\alpha}}{\sqrt{A^2+B^2}}=\frac{1}{\sqrt{A^2+B^2}}.$$

Hence

$$\cos\alpha=\frac{-A}{\sqrt{A^2+B^2}},\quad \sin\alpha=\frac{-B}{\sqrt{A^2+B^2}},\quad \text{and}\quad p=\frac{C}{\sqrt{A^2+B^2}}.$$

The equation (1) may therefore be reduced to the form (2) by dividing it by $\sqrt{A^2+B^2}$ and arranging it so that the constant term is negative.

57. Ex. *Reduce to the perpendicular form the equation*

$$x+y\sqrt{3}+7=0 \quad\ldots\ldots\ldots\ldots(1).$$

Here $\sqrt{A^2+B^2}=\sqrt{1+3}=\sqrt{4}=2.$

Dividing (1) by 2, we have

$$\tfrac{1}{2}x+y\frac{\sqrt{3}}{2}+\tfrac{7}{2}=0,$$

i.e. $$x(-\tfrac{1}{2})+y\left(-\frac{\sqrt{3}}{2}\right)-\tfrac{7}{2}=0,$$

i.e. $$x\cos 240^\circ+y\sin 240^\circ-\tfrac{7}{2}=0.$$

58. *To trace the straight line given by an equation of the first degree.*

Let the equation be

$$Ax+By+C=0 \quad\ldots\ldots\ldots\ldots(1).$$

(α) This can be written in the form

$$\frac{x}{-\frac{C}{A}}+\frac{y}{-\frac{C}{B}}=1.$$

Comparing this with the result of Art. 50, we see that it represents a straight line which cuts off intercepts $-\frac{C}{A}$ and $-\frac{C}{B}$ from the axes. Its position is therefore known.

If C be zero, the equation (1) reduces to the form

$$y=-\frac{A}{B}x,$$

and thus (by Art. 47, Cor.) represents a straight line passing through the origin inclined at an angle $\tan^{-1}\left(-\frac{A}{B}\right)$ to the axis of x. Its position is therefore known.

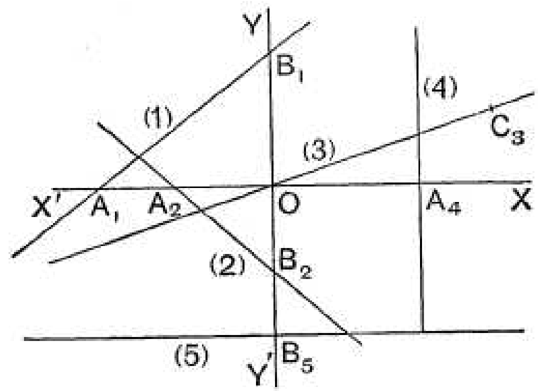
(β) The straight line may also be traced by finding the coordinates of any two points on it.

If we put $y=0$ in (1) we have $x=-\frac{C}{A}$. The point $\left(-\frac{C}{A}, 0\right)$ therefore lies on it.

If we put $x=0$, we have $y=-\frac{C}{B}$, so that the point $\left(0, -\frac{C}{B}\right)$ lies on it.

Hence, as before, we have the position of the straight line.

59. Ex. *Trace the straight lines*
(1) $3x-4y+7=0$; (2) $7x+8y+9=0$;
(3) $3y=x$; (4) $x=2$; (5) $y=-2$.



(1) Putting $y=0$, we have $x=-\frac{7}{3}$,
and putting $x=0$, we have $y=\frac{7}{4}$.

Measuring $OA_1\,(=-\frac{7}{3})$ along the axis of x we have one point on the line.

Measuring $OB_1\,(=\frac{7}{4})$ along the axis of y we have another point.

Hence A_1B_1, produced both ways, is the required line.

(2) Putting in succession y and x equal to zero, we have the intercepts on the axes equal to $-\frac{9}{7}$ and $-\frac{9}{8}$.

If then $OA_2=-\frac{9}{7}$ and $OB_2=-\frac{9}{8}$, we have A_2B_2 the required line.

(3) The point (0, 0) satisfies the equation so that the origin is on the line.

Also the point (3, 1), *i.e.* C_3, lies on it. The required line is therefore OC_3.

(4) The line $x=2$ is, by Art. 46, parallel to the axis of y and passes through the point A_4 on the axis of x such that $OA_4=2$.

(5) The line $y=-2$ is parallel to the axis of x and passes through the point B_5 on the axis of y, such that $OB_5=-2$.

60. Straight Line at Infinity. We have seen that the equation $Ax+By+C=0$ represents a straight line which cuts off intercepts $-\frac{C}{A}$ and $-\frac{C}{B}$ from the axes of coordinates.

If A vanish, but not B or C, the intercept on the axis of x is infinitely great. The equation of the straight line then reduces to the form $y = \text{constant}$, and hence, as in Art. 46, represents a straight line parallel to Ox.

So if B vanish, but not A or C, the straight line meets the axis of y at an infinite distance and is therefore parallel to it.

If A and B both vanish, but not C, these two intercepts are both infinite and therefore the straight line $0 \cdot x + 0 \cdot y + C = 0$ is altogether at infinity.

61. The multiplication of an equation by a constant does not alter it. Thus the equations

$$2x - 3y + 5 = 0 \text{ and } 10x - 15y + 25 = 0$$

represent the same straight line.

Conversely, if two equations of the first degree represent the same straight line, one equation must be equal to the other multiplied by a constant quantity, so that the ratios of the corresponding coefficients must be the same.

For example, if the equations

$$a_1x + b_1y + c_1 = 0 \text{ and } A_1x + B_1y + C_1 = 0$$

represent the same straight line, we must have

$$\frac{a_1}{A_1} = \frac{b_1}{B_1} = \frac{c_1}{C_1}.$$

62. *To find the equation to the straight line which passes through the two given points (x', y') and (x'', y'').*

By Art. 47, the equation to **any** straight line is

$$y = mx + c \qquad \text{..........(1).}$$

By properly determining the quantities m and c we can make (1) represent any straight line we please.

If (1) pass through the point (x', y'), we have

$$y' = mx' + c \qquad \text{..........(2).}$$

Substituting for c from (2), the equation (1) becomes

$$\mathbf{y - y' = m\,(x - x')} \qquad \text{..........(3).}$$

This is the equation to the line going through (x', y') making an angle $\tan^{-1} m$ with OX. If in addition (3) passes through the point (x'', y''), then

$$y'' - y' = m(x'' - x'),$$

giving
$$m = \frac{y'' - y'}{x'' - x'}.$$

Substituting this value in (3), we get as the required equation

$$\mathbf{y - y' = \frac{y'' - y'}{x'' - x'}(x - x')}.$$

63. Ex. *Find the equation to the straight line which passes through the points* $(-1, 3)$ *and* $(4, -2)$.

Let the required equation be

$$y = mx + c \qquad \text{.................................(1).}$$

Since (1) goes through the first point, we have

$$3 = -m + c, \text{ so that } c = m + 3.$$

Hence (1) becomes

$$y = mx + m + 3 \qquad \text{.............................(2).}$$

If in addition the line goes through the second point, we have

$$-2 = 4m + m + 3, \text{ so that } m = -1.$$

Hence (2) becomes

$$y = -x + 2, \; i.e. \; x + y = 2.$$

Or, again, using the result of the last article the equation is

$$y - 3 = \frac{-2 - 3}{4 - (-1)}(x + 1) = -x - 1,$$

i.e.
$$y + x = 2.$$

64. To fix definitely the position of a straight line we must have always two quantities given. Thus one point on the straight line and the direction of the straight line will determine it; or again two points lying on the straight line will determine it.

Analytically, the general equation to a straight line will contain two arbitrary constants, which will have to be determined so that the general equation may represent any particular straight line.

Thus, in Art. 47, the quantities m and c which remain the same, *so long as we are considering the same straight line*, are the two constants for the straight line.

Similarly, in Art. 50, the quantities a and b are the constants for the straight line.

65. In any equation to a locus the quantities x and y, which are the coordinates of any point on the locus, are called Current Coordinates; the curve may be conceived as traced out by a point which "runs" along the locus.

EXAMPLES V

Find the equation to the straight line

1. cutting off an intercept unity from the positive direction of the axis of y and inclined at 45° to the axis of x.

2. cutting off an intercept -5 from the axis of y and being equally inclined to the axes.

3. cutting off an intercept 2 from the negative direction of the axis of y and inclined at 30° to OX.

4. cutting off an intercept -3 from the axis of y and inclined at an angle $\tan^{-1}\frac{3}{5}$ to the axis of x.

Find the equation to the straight line

5. cutting off intercepts 3 and 2 from the axes.

6. cutting off intercepts -5 and 6 from the axes.

7. Find the equation to the straight line which passes through the point (5, 6) and has intercepts on the axes

(1) equal in magnitude and both positive,

(2) equal in magnitude but opposite in sign.

8. Find the equations to the straight lines which pass through the point $(1, -2)$ and cut off equal distances from the two axes.

9. Find the equation to the straight line which passes through the given point (x', y') and is such that the given point bisects the part intercepted between the axes.

10. Find the equation to the straight line which passes through the point $(-4, 3)$ and is such that the portion of it between the axes is divided by the point in the ratio 5 : 3.

Trace the straight lines whose equations are

11. $x+2y+3=0$.

12. $5x-7y-9=0$.

13. $3x+7y=0$.

14. $2x-3y+4=0$.

Find the equations to the straight lines passing through the following pairs of points.

15. (0, 0) and $(2, -2)$.

16. (3, 4) and (5, 6).

17. $(-1, 3)$ and $(6, -7)$.

18. $(0, -a)$ and $(b, 0)$.

19. (a, b) and $(a+b,\ a-b)$.

20. $(at_1^2,\ 2at_1)$ and $(at_2^2,\ 2at_2)$. **21.** $\left(at_1, \frac{a}{t_1}\right)$ and $\left(at_2, \frac{a}{t_2}\right)$.

22. $(a\cos\phi_1,\ a\sin\phi_1)$ and $(a\cos\phi_2,\ a\sin\phi_2)$.

23. $(a\cos\phi_1,\ b\sin\phi_1)$ and $(a\cos\phi_2,\ b\sin\phi_2)$.

24. $(a\sec\phi_1,\ b\tan\phi_1)$ and $(a\sec\phi_2,\ b\tan\phi_2)$.

Find the equations to the sides of the triangles the coordinates of whose angular points are respectively

25. $(1, 4)$, $(2, -3)$, and $(-1, -2)$.

26. $(0, 1)$, $(2, 0)$, and $(-1, -2)$.

27. Find the equations to the diagonals of the rectangle the equations of whose sides are $x=a$, $x=a'$, $y=b$, and $y=b'$.

28. Find the equation to the straight line which bisects the distance between the points (a, b) and (a', b') and also bisects the distance between the points $(-a, b)$ and $(a', -b')$.

29. Find the equations to the straight lines which go through the origin and trisect the portion of the straight line $3x+y=12$ which is intercepted between the axes of coordinates.

ANSWERS

1. $y=x+1$. **2.** $x-y-5=0$. **3.** $x-y\sqrt{3}-2\sqrt{3}=0$.
4. $5y-3x+15=0$. **5.** $2x+3y=6$. **6.** $6x-5y+30=0$.
7. (1) $x+y=11$; (2) $y-x=1$. **8.** $x+y+1=0$; $x-y=3$.
9. $xy'+x'y=2x'y'$. **10.** $20y-9x=96$. **15.** $x+y=0$.
16. $y-x=1$. **17.** $7y+10x=11$.
18. $ax-by=ab$. **19.** $(a-2b)x-by+b^2+2ab-a^2=0$.
20. $y(t_1+t_2)-2x=2at_1t_2$. **21.** $t_1t_2y+x=a(t_1+t_2)$.
22. $x\cos\frac{1}{2}(\phi_1+\phi_2)+y\sin\frac{1}{2}(\phi_1+\phi_2)=a\cos\frac{1}{2}(\phi_1-\phi_2)$.
23. $\frac{x}{a}\cos\frac{\phi_1+\phi_2}{2}+\frac{y}{b}\sin\frac{\phi_1+\phi_2}{2}=\cos\frac{\phi_1-\phi_2}{2}$.
24. $bx\cos\frac{1}{2}(\phi_1-\phi_2)-ay\sin\frac{1}{2}(\phi_1+\phi_2)=ab\cos\frac{1}{2}(\phi_1+\phi_2)$.
25. $x+3y+7=0$; $y-3x=1$; $y+7x=11$.
26. $2x-3y=4$; $y-3x=1$; $x+2y=2$.
27. $y(a'-a)-x(b'-b)=a'b-ab'$; $y(a'-a)+x(b'-b)=a'b'-ab$.
28. $2ay-2b'x=ab-a'b'$. **29.** $y=6x$; $2y=3x$.

SOLUTIONS/HINTS

1. By Art. 47, we have $y = x + 1$.

2. By Art. 47, since the line is inclined to the axis of x at 45°, the equation is

$$y = +x - 5.$$

3. By Art. 47, we have $y = \frac{1}{\sqrt{3}}x - 2$.

4. By Art. 47, $y = \frac{3}{5}x - 3$. $\therefore$ $3x - 5y = 15$.

5. By Art. 50, $\frac{x}{3} + \frac{y}{2} = 1$; $\therefore 2x + 3y = 6$.

6. By Art. 50, $\frac{x}{-5} + \frac{y}{6} = 1$; $\therefore$ $5y - 6x = 30$.

7. (1) Let the intercepts each $= a$.

$\therefore$ equation is $\frac{x}{a} + \frac{y}{a} = 1$.

Since it passes through (5, 6), $\therefore \frac{5}{a} + \frac{6}{a} = 1$; $\therefore$ $a = 11$.

$\therefore$ equation becomes $x + y = 11$.

(2) Let the intercepts be a and $-a$.

$\therefore$ equation is $\frac{x}{a} - \frac{y}{a} = 1$.

Since it passes through (5, 6), $\therefore \frac{5}{a} - \frac{6}{a} = 1$. $\therefore a = -1$.

$\therefore$ equation becomes $y - x = 1$.

8. Let the equation of the lines be $\frac{x}{a} \pm \frac{y}{a} = 1$.

Since they pass through $(1, -2)$,

$$\therefore \frac{1}{a} \pm \frac{2}{a} = 1; \quad \therefore a = -1 \text{ or } 3.$$

Equations become $x + y + 1 = 0$, and $x - y = 3$.

9. It is easily seen from a figure that the intercepts on the axes are $2x'$, $2y'$.

$\therefore$ the equation is $\dfrac{x}{2x'} + \dfrac{y}{2y'} = 1$, or $xy' + yx' = 2x'y'$.

10. Let $\dfrac{x}{a} + \dfrac{y}{b} = 1$ be the equation of the line. The coordinates of the point dividing the line joining $(a, 0)$ and $(0, b)$ in the ratio of $5 : 3$ are (Art. 22),

$$\left(\frac{3a}{8}, \frac{5b}{8}\right); \quad \therefore \frac{3a}{8} = -4, \quad \frac{5b}{8} = 3.$$

$$\therefore a = -\tfrac{32}{3}, \quad b = \tfrac{24}{5}.$$

Hence the equation becomes $\dfrac{5y}{24} - \dfrac{3x}{32} = 1$,

or $$20y - 9x = 96.$$

11—14. See Art. 59. **15—19.** See Art. 63.

20. By Art. 62, the equation is

$$y - 2at_1 = \frac{2(t_2 - t_1)}{t_2^2 - t_1^2}(x - at_1^2),$$

i.e. $$(t_1 + t_2)y - 2at_1^2 - 2at_1t_2 = 2x - 2at_1^2,$$

i.e. $$(t_1 + t_2)y = 2x + 2at_1t_2.$$

21. By Art. 62, the equation is $y - \dfrac{a}{t_1} = \dfrac{\dfrac{1}{t_2} - \dfrac{1}{t_1}}{t_2 - t_1}(x - at_1)$,

i.e. $t_1t_2\left(y - \dfrac{a}{t_1}\right) + x - at_1 = 0$, *i.e.* $x + t_1t_2y = a(t_1 + t_2)$.

22. See Number 23, and put $b = a$.

23. Let $lx + my = 1$, be the equation.

Then we have $la\cos\phi_1 + mb\sin\phi_1 = 1$,

and $$la\cos\phi_2 + mb\sin\phi_2 = 1.$$

$$\therefore la\sin(\phi_1 - \phi_2) = \sin\phi_1 - \sin\phi_2.$$

$$\therefore\ 2la\sin\frac{\phi_1-\phi_2}{2}\cos\frac{\phi_1-\phi_2}{2}=2\cos\frac{\phi_1+\phi_2}{2}.\sin\frac{\phi_1-\phi_2}{2}.$$

$$\therefore\ l=\frac{1}{a}\cdot\frac{\cos\frac{\phi_1+\phi_2}{2}}{\cos\frac{\phi_1-\phi_2}{2}}.\quad \text{Similarly,}\ m=\frac{1}{b}\cdot\frac{\sin\frac{\phi_1+\phi_2}{2}}{\cos\frac{\phi_1-\phi_2}{2}}.$$

The equation becomes

$$\frac{x}{a}\cdot\cos\frac{\phi_1+\phi_2}{2}+\frac{y}{b}\sin\frac{\phi_1+\phi_2}{2}=\cos\frac{\phi_1-\phi_2}{2}.$$

24. By Art. 62, the equation is

$$\frac{y-b\tan\phi_1}{x-a\sec\phi_1}=\frac{b(\tan\phi_2-\tan\phi_1)}{a(\sec\phi_2-\sec\phi_1)}=\frac{b\sin(\phi_2-\phi_1)}{a\cos(\phi_1-\phi_2)}$$

$$=\frac{b\cos\frac{\phi_1-\phi_2}{2}}{a\sin\frac{\phi_1+\phi_2}{2}}.$$

$$\therefore\ bx\cos\frac{\phi_1-\phi_2}{2}-ay\sin\frac{\phi_1+\phi_2}{2}$$

$$=ab\sec\phi_1\left\{\cos\frac{\phi_1-\phi_2}{2}-\sin\phi_1\sin\frac{\phi_1+\phi_2}{2}\right\}$$

$$=ab\sec\phi_1\left\{\cos\frac{\phi_1-\phi_2}{2}-\frac{1}{2}\cos\frac{\phi_1-\phi_2}{2}+\frac{1}{2}\cos\frac{3\phi_1+\phi_2}{2}\right\}$$

$$=\frac{1}{2}ab\sec\phi_1\left\{\cos\frac{\phi_1-\phi_2}{2}+\cos\frac{3\phi_1+\phi_2}{2}\right\}=ab\cos\frac{\phi_1+\phi_2}{2}.$$

25, 26. See Art. 63.

27. The four corners are (a, b); (a', b); (a', b'); (a, b'). Let $lx+my=1$ be the equation of the line joining (a, b) and (a', b').

Then $\qquad la+mb=1,$ and $la'+mb'=1.$

Whence $\qquad l=\dfrac{b-b'}{a'b-ab'},$ and $m=\dfrac{a'-a}{a'b-ab'}$

and the equation becomes $(b-b')x+(a'-a)y=a'b-ab'$.

Similarly we may obtain the equation of the line joining (a', b) and (a, b').

28. The line passes through the point $\left(\frac{a+a'}{2}, \frac{b+b'}{2}\right)$, and also through the point $\left(\frac{a'-a}{2}, \frac{b-b'}{2}\right)$.

Let its equation be $lx+my=1$.

Then $(a+a')l+(b+b')m=2$, and $(a'-a)l+(b-b')m=2$.

Whence $l=\frac{2b'}{a'b'-ab}$, and $m=\frac{2a}{ab-a'b'}$.

$\therefore$ the equation becomes $2ay-2b'x=ab-a'b'$.

29. The coordinates of the points which trisect the line joining (4, 0) and (0, 12) are (Art. 22),

$$\left(\frac{2.0+1.4}{2+1}, \frac{2.12+1.0}{2+1}\right) \text{ and } \left(\frac{1.0+2.4}{1+2}, \frac{1.12+2.0}{1+2}\right),$$

that is $(\frac{4}{3}, 8)$ and $(\frac{8}{3}, 4)$.

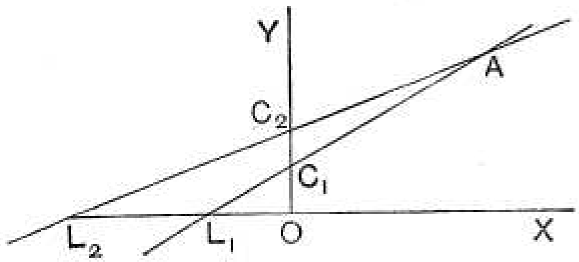
$\therefore$ the required equations are

$\frac{4}{3}y=8x$ and $\frac{8}{3}y=4x$, that is $y=6x$ and $2y=3x$.

Angles between straight lines.

66. *To find the angle between two given straight lines.*

Let the two straight lines be AL_1 and AL_2, meeting the axis of x in L_1 and L_2.



I. Let their equations be

$$y = m_1x + c_1 \text{ and } y = m_2x + c_2 \ldots\ldots\ldots\ldots(1).$$

By Art. 47 we therefore have

$$\tan AL_1X = m_1, \text{ and } \tan AL_2X = m_2.$$

Now $$\angle L_1AL_2 = \angle AL_1X - \angle AL_2X.$$

$\therefore$ $$\tan L_1AL_2 = \tan[AL_1X - AL_2X]$$

$$= \frac{\tan AL_1X - \tan AL_2X}{1 + \tan AL_1X \cdot \tan AL_2X} = \frac{m_1 - m_2}{1 + m_1m_2}.$$

Hence the required angle $= \angle L_1AL_2$

$$= \tan^{-1}\frac{\mathbf{m_1 - m_2}}{\mathbf{1 + m_1m_2}} \ldots\ldots\ldots\ldots\ldots\ldots(2).$$

[In any numerical example, if the quantity (2) be a positive quantity it is the tangent of the acute angle between the lines; if negative, it is the tangent of the obtuse angle.]

II. Let the equations of the straight lines be

$$A_1x + B_1y + C_1 = 0,$$

and $$A_2x + B_2y + C_2 = 0.$$

By dividing the equations by B_1 and B_2, they may be written

$$y = -\frac{A_1}{B_1}x - \frac{C_1}{B_1},$$

and
$$y = -\frac{A_2}{B_2}x - \frac{C_2}{B_2}.$$

Comparing these with the equations of (I.), we see that

$$m_1 = -\frac{A_1}{B_1}, \text{ and } m_2 = -\frac{A_2}{B_2}.$$

Hence the required angle

$$= \tan^{-1}\frac{m_1 - m_2}{1 + m_1 m_2} = \tan^{-1}\frac{-\frac{A_1}{B_1} - \left(-\frac{A_2}{B_2}\right)}{1 + \left(-\frac{A_1}{B_1}\right)\left(-\frac{A_2}{B_2}\right)}$$

$$= \tan^{-1}\frac{B_1 A_2 - A_1 B_2}{A_1 A_2 + B_1 B_2} \quad \text{.................. (3).}$$

III. If the equations be given in the form

$$x\cos\alpha + y\sin\alpha - p_1 = 0 \quad \text{and} \quad x\cos\beta + y\sin\beta - p_2 = 0,$$

the perpendiculars from the origin make angles α and β with the axis of x.

Now that angle between two straight lines, in which the origin lies, is the supplement of the angle between the perpendiculars, and the angle between these perpendiculars is $\beta - \alpha$.

[For, if OR_1 and OR_2 be the perpendiculars from the origin upon the two lines, then the points O, R_1, R_2, and A lie on a circle, and hence the angles R_1OR_2 and R_2AR_1 are either equal or supplementary.]

67. *To find the condition that two straight lines may be parallel.*

Two straight lines are parallel when the angle between them is zero and therefore the tangent of this angle is zero.

The equation (2) of the last article then gives

$$\mathbf{m_1 = m_2}.$$

Two straight lines whose equations are given in the "m" form are therefore parallel when their "m's" are the same, or, in other words, if their equations differ only in the constant term.

The straight line $Ax+By+C'=0$ is any straight line which is parallel to the straight line $Ax+By+C=0$. For the "m's" of the two equations are the same.

Again the equation $A(x-x')+B(y-y')=0$ clearly represents the straight line which passes through the point (x', y') and is parallel to $Ax+By+C=0$.

The result (3) of the last article gives, as the condition for parallel lines,

$$B_1A_2-A_1B_2=0,$$

i.e.

$$\frac{A_1}{B_1}=\frac{A_2}{B_2}.$$

68. Ex. *Find the equation to the straight line, which passes through the point* $(4, -5)$, *and which is parallel to the straight line*

$$3x+4y+5=0\qquad\qquad(1).$$

Any straight line which is parallel to (1) has its equation of the form

$$3x+4y+C=0\qquad\qquad(2).$$

[For the "m" of both (1) and (2) is the same.]

This straight line will pass through the point $(4, -5)$ if

$$3\times 4+4\times(-5)+C=0,$$

i.e. if

$$C=20-12=8.$$

The equation (2) then becomes

$$3x+4y+8=0.$$

69. *To find the condition that two straight lines, whose equations are given, may be perpendicular.*

Let the straight lines be

$$y=m_1x+c_1,$$

and

$$y=m_2x+c_2.$$

If θ be the angle between them we have, by Art. 66,

$$\tan\theta = \frac{m_1 - m_2}{1 + m_1 m_2} \quad \text{.....................(1).}$$

If the lines be perpendicular, then $\theta = 90°$, and therefore $\tan\theta = \infty$.

The right-hand member of equation (1) must therefore be infinite, and this can only happen when its denominator is zero.

The condition of perpendicularity is therefore that

$$1 + m_1 m_2 = 0, \text{ i.e. } \mathbf{m_1 m_2 = -1.}$$

The straight line $y = m_2 x + c_2$ is therefore perpendicular to $y = m_1 x + c_1$, if $m_2 = -\dfrac{1}{m_1}$.

It follows that the straight lines

$$A_1 x + B_1 y + C_1 = 0 \text{ and } A_2 x + B_2 y + C_2 = 0,$$

for which $m_1 = -\dfrac{A_1}{B_1}$ and $m_2 = -\dfrac{A_2}{B_2}$, are at right angles if

$$\left(-\frac{A_1}{B_1}\right)\left(-\frac{A_2}{B_2}\right) = -1,$$

i.e. if

$$A_1 A_2 + B_1 B_2 = 0.$$

70. From the preceding article it follows that the two straight lines

$$A_1 x + B_1 y + C_1 = 0 \quad \text{.....................(1),}$$

and

$$B_1 x - A_1 y + C_2 = 0 \quad \text{.....................(2),}$$

are at right angles; for the product of their m's

$$= -\frac{A_1}{B_1} \times \frac{B_1}{A_1} = -1.$$

Also (2) is derived from (1) by interchanging the coefficients of x and y, changing the sign of one of them, and changing the constant into any other constant.

Ex. The straight line through (x', y') perpendicular to (1) is (2) where $B_1 x' - A_1 y' + C_2 = 0$, so that $C_2 = A_1 y' - B_1 x'$.

This straight line is therefore

$$B_1 (x - x') - A_1 (y - y') = 0.$$

71. Ex. 1. *Find the equation to the straight line which passes through the point* $(4, -5)$ *and is perpendicular to the straight line*

$$3x + 4y + 5 = 0 \quad \text{..............................(1)}.$$

First Method. Any straight line perpendicular to (1) is by the last article

$$4x - 3y + C = 0 \quad \text{..............................(2)}.$$

[We should expect an arbitrary constant in (2) because there are an infinite number of straight lines perpendicular to (1).]

The straight line (2) passes through the point $(4, -5)$ if

$$4 \times 4 - 3 \times (-5) + C = 0,$$

i.e. if $$C = -16 - 15 = -31.$$

The required equation is therefore

$$4x - 3y = 31.$$

Second Method. Any straight line passing through the given point is

$$y - (-5) = m(x - 4).$$

This straight line is perpendicular to (1) if the product of their m's is -1,

i.e. if $$m \times (-\tfrac{3}{4}) = -1,$$

i.e. if $$m = \tfrac{4}{3}.$$

The required equation is therefore

$$y + 5 = \tfrac{4}{3}(x - 4),$$

i.e. $$4x - 3y = 31.$$

Third Method. Any straight line is $y = mx + c$. It passes through the point $(4, -5)$, if

$$-5 = 4m + c \quad \text{..............................(3)}.$$

It is perpendicular to (1) if

$$m \times (-\tfrac{3}{4}) = -1 \quad \text{..............................(4)}.$$

Hence $m = \frac{4}{3}$ and then (3) gives $c = -\frac{31}{3}$.

The required equation is therefore $y = \frac{4}{3}x - \frac{31}{3}$,

i.e. $$4x - 3y = 31.$$

[In the first method, we start with any straight line which is perpendicular to the given straight line and pick out that particular straight line which goes through the given point.

In the second method, we start with any straight line passing through the given point and pick out that particular one which is perpendicular to the given straight line.

In the third method, we start with any straight line whatever and determine its constants, so that it may satisfy the two given conditions.

The student should illustrate by figures.]

Ex. 2. *Find the equation to the straight line which passes through the point* (x', y') *and is perpendicular to the given straight line*

$$yy' = 2a(x + x').$$

The given straight line is

$$yy' - 2ax - 2ax' = 0.$$

Any straight line perpendicular to it is (Art. 70)

$$2ay + xy' + C = 0 \qquad\qquad (1).$$

This will pass through the point (x', y') and therefore will be the straight line required if the coordinates x' and y' satisfy it,

i.e. if $\qquad 2ay' + x'y' + C = 0,$

i.e. if $\qquad C = -2ay' - x'y'.$

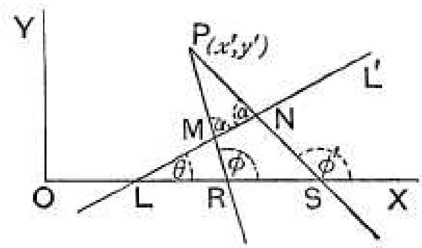
Substituting in (1) for C the required equation is therefore

$$2a(y - y') + y'(x - x') = 0.$$

72. *To find the equations to the straight lines which pass through a given point (x', y') and make a given angle α with the given straight line $y = mx + c$.*

Let P be the given point and let the given straight line be LMN, making an angle θ with the axis of x such that

$$\tan\theta = m.$$



In general (*i.e.* except when α is a right angle or zero) there are two straight lines PMR and PNS making an angle α with the given line.

Let these lines meet the axis of x in R and S and let them make angles ϕ and ϕ' with the positive direction of the axis of x.

The equations to the two required straight lines are therefore (by Art. 62)

$$y - y' = \tan\phi \times (x - x') \qquad\qquad (1),$$

and $\qquad y - y' = \tan\phi' \times (x - x') \qquad\qquad (2).$

Now $\qquad \phi = \angle LMR + \angle RLM = \alpha + \theta,$

and $\qquad \phi' = \angle LNS + \angle SLN = (180° - \alpha) + \theta.$

Hence

$$\tan\phi = \tan(\alpha + \theta) = \frac{\tan\alpha + \tan\theta}{1 - \tan\alpha\tan\theta} = \frac{\tan\alpha + m}{1 - m\tan\alpha},$$

and $\qquad \tan\phi' = \tan(180° + \theta - \alpha)$

$$= \tan(\theta - \alpha) = \frac{\tan\theta - \tan\alpha}{1 + \tan\theta\tan\alpha} = \frac{m - \tan\alpha}{1 + m\tan\alpha}.$$

On substituting these values in (1) and (2), we have as the required equations

$$y - y' = \frac{m + \tan\alpha}{1 - m\tan\alpha}(x - x'),$$

and

$$y - y' = \frac{m - \tan\alpha}{1 + m\tan\alpha}(x - x').$$

EXAMPLES VI

Find the angles between the pairs of straight lines

1. $x - y\sqrt{3} = 5$ and $\sqrt{3}x + y = 7$.

2. $x - 4y = 3$ and $6x - y = 11$.

3. $y = 3x + 7$ and $3y - x = 8$.

4. $y = (2 - \sqrt{3})x + 5$ and $y = (2 + \sqrt{3})x - 7$.

5. $(m^2 - mn)y = (mn + n^2)x + n^3$ and $(mn + m^2)y = (mn - n^2)x + m^3$.

6. Find the tangent of the angle between the lines whose intercepts on the axes are respectively $a, -b$ and $b, -a$.

7. Prove that the points $(2, -1)$, $(0, 2)$, $(2, 3)$, and $(4, 0)$ are the coordinates of the angular points of a parallelogram and find the angle between its diagonals.

Find the equation to the straight line

8. passing through the point $(2, 3)$ and perpendicular to the straight line $4x - 3y = 10$.

9. passing through the point $(-6, 10)$ and perpendicular to the straight line $7x + 8y = 5$.

10. passing through the point $(2, -3)$ and perpendicular to the straight line joining the points $(5, 7)$ and $(-6, 3)$.

11. passing through the point $(-4, -3)$ and perpendicular to the straight line joining $(1, 3)$ and $(2, 7)$.

12. Find the equation to the straight line drawn at right angles to the straight line $\frac{x}{a} - \frac{y}{b} = 1$ through the point where it meets the axis of x.

13. Find the equation to the straight line which bisects, and is perpendicular to, the straight line joining the points (a, b) and (a', b').

14. Prove that the equation to the straight line which passes through the point $(a\cos^3\theta, a\sin^3\theta)$ and is perpendicular to the straight line $x\sec\theta + y\,\mathrm{cosec}\,\theta = a$ is $x\cos\theta - y\sin\theta = a\cos 2\theta$.

15. Find the equations to the straight lines passing through (x', y') and respectively perpendicular to the straight lines

$$xx' + yy' = a^2,$$

$$\frac{xx'}{a^2} + \frac{yy'}{b^2} = 1,$$

and $$x'y + xy' = a^2.$$

16. Find the equations to the straight lines which divide, internally and externally, the line joining $(-3, 7)$ to $(5, -4)$ in the ratio of $4 : 7$ and which are perpendicular to this line.

17. Through the point $(3, 4)$ are drawn two straight lines each inclined at $45°$ to the straight line $x - y = 2$. Find their equations and find also the area included by the three lines.

18. Shew that the equations to the straight lines passing through the point $(3, -2)$ and inclined at $60°$ to the line

$\sqrt{3}x + y = 1$ are $y + 2 = 0$ and $y - \sqrt{3}x + 2 + 3\sqrt{3} = 0$.

19. Find the equations to the straight lines which pass through the origin and are inclined at $75°$ to the straight line

$$x + y + \sqrt{3}(y - x) = a.$$

20. Find the equations to the straight lines which pass through the point (h, k) and are inclined at an angle $\tan^{-1} m$ to the straight line $$y = mx + c.$$

21. Find the angle between the two straight lines $3x = 4y + 7$ and $5y = 12x + 6$ and also the equations to the two straight lines which pass through the point $(4, 5)$ and make equal angles with the two given lines.

ANSWERS

1. $90°$. **2.** $\tan^{-1}\frac{23}{11}$. **3.** $\tan^{-1}\frac{4}{3}$. **4.** $60°$.

5. $\tan^{-1}\dfrac{4m^2n^2}{m^4 - n^4}$. **6.** $\tan^{-1}\dfrac{a^2 - b^2}{2ab}$. **7.** $\tan^{-1}(2)$.

8. $4y + 3x = 18$. **9.** $7y - 8x = 118$. **10.** $4y + 11x = 10$.

11. $x + 4y + 16 = 0$. **12.** $ax + by = a^2$.

13. $2x(a - a') + 2y(b - b') = a^2 - a'^2 + b^2 - b'^2$.

15. $yx' - xy' = 0$; $a^2xy' - b^2x'y = (a^2 - b^2)x'y'$; $xx' - yy' = x'^2 - y'^2$.

16. $121y - 88x = 371$; $33y - 24x = 1043$.

17. $x = 3$; $y = 4$; $4\frac{1}{2}$. **19.** $x = 0$; $y + \sqrt{3}x = 0$.

20. $y = k$; $(1 - m^2)(y - k) = 2m(x - h)$.

21. $\tan^{-1}\frac{33}{56}$; $9x - 7y = 1$; $7x + 9y = 73$.

SOLUTIONS/HINTS

1. By Art. 66, the angle $= \tan^{-1} \dfrac{\dfrac{1}{\sqrt{3}} + \sqrt{3}}{1 - \dfrac{1}{\sqrt{3}} \cdot \sqrt{3}}$

$$= \tan^{-1} \infty = 90^\circ.$$

2. The angle $= \tan^{-1} \dfrac{6 - \frac{1}{4}}{1 + \frac{6}{4}} = \tan^{-1} \dfrac{23}{10}$.

3. The angle $= \tan^{-1} \dfrac{3 - \frac{1}{3}}{1 + 3 \cdot \frac{1}{3}} = \tan^{-1} \dfrac{4}{3}$.

4. The angle $= \tan^{-1} \dfrac{(2 + \sqrt{3}) - (2 - \sqrt{3})}{1 + (4 - 3)}$

$$= \tan^{-1} \sqrt{3} = 60^\circ.$$

5. The angle $= \tan^{-1} \cdot \dfrac{\dfrac{mn + n^2}{m^2 - mn} - \dfrac{mn - n^2}{mn + m^2}}{1 + \dfrac{n(m + n)}{m(m - n)} \cdot \dfrac{n(m - n)}{m(n + m)}}$

$$= \tan^{-1} \frac{4m^2n^2}{m^4 - n^4}.$$

6. The equations of the lines are (Art. 50)

$$\frac{x}{a} - \frac{y}{b} = 1 \text{ and } \frac{x}{b} - \frac{y}{a} = 1.$$

$\therefore$ The angle $= \tan^{-1} \dfrac{\dfrac{a}{b} - \dfrac{b}{a}}{1 + \dfrac{a}{b} \cdot \dfrac{b}{a}} = \tan^{-1} \dfrac{a^2 - b^2}{2ab}$.

7. The middle point of each diagonal is (2, 1). The equations of the diagonals are (Art. 62)

$$x - 2 = 0, \text{ and } y = 2 - \tfrac{1}{2}x.$$

$\therefore$ the angle $= \tan^{-1}(-\frac{1}{2}) - 90^\circ = \tan^{-1} 2.$

8. The equation of any line perpendicular to $4x - 3y = 10$ is $3x + 4y = c$. Since it passes through (2, 3),

$6 + 12 = c$; $\therefore$ $c = 18$. The equation becomes $3x + 4y = 18$.

9. The equation of any line perpendicular to $7x + 8y = 5$ is $8x - 7y = c$. Since it passes through $(-6, 10)$,

$$-48 - 70 = c; \quad \therefore \; c = -118.$$

$\therefore$ the equation becomes $7y - 8x = 118$.

10. The "m" of the line joining (5, 7) and $(-6, 3)$

$$= \frac{7-3}{5+6} = \frac{4}{11}. \quad \therefore \text{ required equation is}$$

$$y + 3 = -\tfrac{11}{4}(x - 2), \text{ or } 4y + 11x = 10.$$

11. The "m" of the line joining (1, 3) and (2, 7) is

$$\frac{7-3}{2-1} = 4. \quad \therefore \text{ required equation is}$$

$$y + 3 = -\tfrac{1}{4}(x + 4), \text{ or } x + 4y + 16 = 0.$$

12. Any line at right angles to $\dfrac{x}{a} - \dfrac{y}{b} = 1$ is $ax + by = c$.

If it passes through $(a, 0)$, then $c = a^2$.

Hence the equation becomes $ax + by = a^2$.

13. The "m" of the line joining (a, b) and (a', b') is

$$\frac{b-b'}{a-a'}. \quad \therefore \text{ required equation is}$$

$$y - \frac{b+b'}{2} = -\frac{a-a'}{b-b'}\left(x - \frac{a+a'}{2}\right),$$

or $$2(a-a')x + 2(b-b')y = a^2 - a'^2 + b^2 - b'^2.$$

14. Any line perpendicular to $x \sec\theta + y \operatorname{cosec}\theta = a$ is

$$x\cos\theta - y\sin\theta = c.$$

Since it passes through $(a\cos^3\theta, \; a\sin^3\theta)$,

$$\therefore \; a\cos^4\theta - a\sin^4\theta = c. \quad \therefore \; c = a\cos 2\theta.$$

$\therefore$ the equation becomes $x\cos\theta - y\sin\theta = a\cos 2\theta$.

15. Let the equations be

$$xy' - x'y = c_1, \quad a^2y'x - b^2x'y = c_2, \quad xx' - yy' = c_3.$$

Since each passes through (x', y')

$$c_1 = 0, \quad c_2 = (a^2 - b^2)\,x'y', \quad c_3 = x'^2 - y'^2.$$

Substituting, we have the required equations.

16. The lines pass through the points

$$\left(\frac{4\,.\,5 - 3\,.\,7}{4+7}, \frac{-4\,.\,4 + 7\,.\,7}{4+7}\right); \ \left(\frac{4\,.\,5 + 3\,.\,7}{4-7}, \frac{-4\,.\,4 - 7\,.\,7}{4-7}\right),$$

that is $(-\frac{1}{11}, 3)$ and $(-\frac{41}{3}, \frac{65}{3})$ and the "m" of the line joining $(-3, 7)$ and $(5, -4)$ is $\dfrac{7+4}{-3-5} = -\dfrac{11}{8}$.

Hence the required equations are

$$y - 3 = \tfrac{8}{11}\left(x + \tfrac{1}{11}\right), \text{ and } y - \tfrac{65}{3} = \tfrac{8}{11}\left(x + \tfrac{41}{3}\right),$$

or $\qquad 121y - 88x = 371$, and $33y - 24x = 1043$.

17. The required equations are (Art. 72)

$$y - 4 = \frac{1+1}{1-1}(x-3), \text{ and } y - 4 = \frac{1-1}{1+1}(x-3),$$

i.e. $\qquad x = 3$ and $y = 4$.

From a figure it is easily seen that the area is a right-angled isosceles triangle whose equal sides $= 3$;

$$\therefore \text{ area} = \tfrac{9}{2} = 4\tfrac{1}{2}.$$

18. The required equations are (Art. 72)

$$y + 2 = \frac{-\sqrt{3} + \sqrt{3}}{1+3}(x-3), \text{ and } y + 2 = \frac{-\sqrt{3} - \sqrt{3}}{1-3}(x-3),$$

or $\qquad y + 2 = 0$ and $y - \sqrt{3}x + 2 + 3\sqrt{3} = 0$.

19. The required equations are (Art. 72)

$$y = \frac{(2-\sqrt{3}) + (2+\sqrt{3})}{1 - (2-\sqrt{3})(2+\sqrt{3})} \cdot x,$$

and

$$y = \frac{(2-\sqrt{3}) - (2+\sqrt{3})}{1 + (2-\sqrt{3})(2+\sqrt{3})} \cdot x,$$

or $\qquad x = 0$ and $y + \sqrt{3}x = 0$.

20. The required equations are (Art. 72)

$$y-k=\frac{m+m}{1-m^2}(x-h), \text{ and } y-k=\frac{m-m}{1+m^2}(x-h),$$

or $$y=k \text{ and } (1-m^2)(y-k)=2m(x-h).$$

21. The angle $=\tan^{-1}\dfrac{\frac{12}{5}-\frac{3}{4}}{1+\frac{12}{5}\cdot\frac{3}{4}}=\tan^{-1}\dfrac{33}{56}$.

The angle θ which either line makes with the axis of x is given by

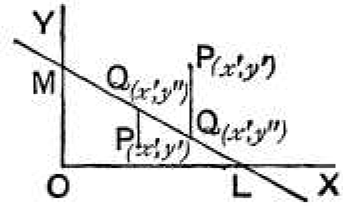
$$\tan 2\theta=\tan(\theta_1+\theta_2)=\frac{\frac{12}{5}+\frac{3}{4}}{1-\frac{12}{5}\cdot\frac{3}{4}}=-\frac{63}{16},$$

whence $$\tan\theta=\tfrac{9}{7} \text{ or } -\tfrac{7}{9}.$$

Hence the required equations are $y-5=\frac{9}{7}(x-4)$, and $y-5=-\frac{7}{9}(x-4)$, or $9x-7y=1$, and $7x+9y=73$.

73. *To shew that the point (x', y') is on one side or the other of the straight line $Ax + By + C = 0$ according as the quantity $Ax' + By' + C$ is positive or negative.*

Let LM be the given straight line and P any point (x', y').



Through P draw PQ, parallel to the axis of y, to meet the given straight line in Q, and let the coordinates of Q be (x', y'').

Since Q lies on the given line, we have

$$Ax' + By'' + C = 0,$$

so that

$$y'' = -\frac{Ax' + C}{B} \qquad \text{.....................(1).}$$

It is clear from the figure that PQ is drawn parallel to the positive or negative direction of the axis of y according as P is on one side, or the other, of the straight line LM,

i.e. according as y'' is $>$ or $< y'$,

i.e. according as $y'' - y'$ is positive or negative.

Now, by (1),

$$y'' - y' = -\frac{Ax' + C}{B} - y' = -\frac{1}{B}[Ax' + By' + C].$$

The point (x', y') is therefore on one side or the other of LM according as the quantity $Ax' + By' + C$ is negative or positive.

Cor. The point (x', y') and the origin are on the same side of the given line if $Ax' + By' + C$ and $A \times 0 + B \times 0 + C$ have the same signs, *i.e.* if $Ax' + By' + C$ has the same sign as C.

If these two quantities have opposite signs, then the origin and the point (x', y') are on opposite sides of the given line.

74. The condition that two points may lie on the same or opposite sides of a given line may also be obtained by considering the ratio in which the line joining the two points is cut by the given line.

For let the equation to the given line be

$$Ax + By + C = 0 \quad \dots\dots\dots\dots (1),$$

and let the coordinates of the two given points be (x_1, y_1) and (x_2, y_2).

The coordinates of the point which divides in the ratio $m_1 : m_2$ the line joining these points are, by Art. 22,

$$\frac{m_1x_2 + m_2x_1}{m_1 + m_2} \text{ and } \frac{m_1y_2 + m_2y_1}{m_1 + m_2} \quad \dots\dots\dots\dots (2).$$

If this point lie on the given line we have

$$A\frac{m_1x_2 + m_2x_1}{m_1 + m_2} + B\frac{m_1y_2 + m_2y_1}{m_1 + m_2} + C = 0,$$

so that

$$\frac{m_1}{m_2} = -\frac{Ax_1 + By_1 + C}{Ax_2 + By_2 + C} \quad \dots\dots\dots\dots (3).$$

If the point (2) be *between* the two given points (x_1, y_1) and (x_2, y_2), *i.e.* if these two points be on *opposite* sides of the given line, the ratio $m_1 : m_2$ is positive.

In this case, by (3) the two quantities $Ax_1 + By_1 + C$ and $Ax_2 + By_2 + C$ have opposite signs.

The two points (x_1, y_1) and (x_2, y_2) therefore lie on the opposite (or the same) sides of the straight line $Ax + By + C = 0$ according as the quantities $Ax_1 + By_1 + C$ and $Ax_2 + By_2 + C$ have opposite (or the same) signs.