EXERCISE 9A (Pg.No.: 345)

1. Show that $f(x)=x^2$ is continuous at $x=2$.

Sol. $f(x)=x^2$

Left hand limit at $x=2$,

$$=\lim_{x\to 2^-} f(x)=\lim_{h\to 0^+} f(2-h)=\lim_{h\to 0^+}(2-h)^2=\lim_{h\to 0^+} h^2-4h+4=(0)^2-4(0)+4=4$$

Right hand limit at $x=2$,

$$=\lim_{x\to 2^+} f(x)=\lim_{h\to 0^+} f(2+h)=\lim_{h\to 0^+}(2+h)^2=\lim_{h\to 0^+}\left(4+h^2+4h\right)=\left(4+(0)^2+4(0)\right)=4$$

Value of function at $x=2$, $f(2)=(2)^2=4$

Since $\lim_{x\to 2^-} f(x)=\lim_{x\to 2^+} f(x)=f(2)=4$. Hence, $f(x)$ is continuous at $x=2$.

2. Show that $f(x)=\left(x^2+3x+4\right)$ is continuous at $x=1$.

Sol. $f(x)=\left(x^2+3x+4\right)$

Left hand limit at $x=1$,

$$=\lim_{x\to 1^-} f(x)=\lim_{h\to 0^+} f(1-h)=\lim_{h\to 0^+}\left\{(1-h)^2+3(1-h)+4\right\}=\lim_{h\to 0^+}\left(1+h^2-2h+3-3h+4\right)$$

$$=\lim_{h\to 0^+}\left(h^2-5h+8\right)=(0)^2-5(0)+8=8$$

Right hand limit at $x=1$,

$$=\lim_{x\to 1^+} f(x)=\lim_{h\to 0^+} f(1+h)=\lim_{h\to 0^+}\left\{(1+h)^2+3(1+h)+4\right\}$$

$$=\lim_{h\to 0^+}\left(1+h^2+2h+3+3h+4\right)=\lim_{h\to 0^+}\left(h^2+5h+8\right)=(0)^2+5(0)+8=8$$

Value of function at $x=1$,

$$f(1)=\left\{1^2+3(1)+4\right\}=(1+3+4)=8$$

Since $\lim_{x\to 1^-} f(x)=\lim_{x\to 1^+} f(x)=f(1)=8$, Hence, $f(x)$ is continuous at $x=1$

Prove that :

3. $f(x)=\begin{cases}\dfrac{x^2-x-6}{x-3}, & \text{when } x\neq 3\\ 5, & \text{when } x=3\end{cases}$ is continuous at $x=3$.

Sol. $f(x)=\begin{cases}\dfrac{x^2-x-6}{x-3}, & \text{when } x\neq 3\\ 5, & \text{when } x=3\end{cases}$

$$f(x)=\begin{cases}\dfrac{x(x-3)+2(x-3)}{(x-3)}, & \text{when } x\neq 3\\ 5, & \text{when } x=3\end{cases} \quad\Rightarrow f(x)=\begin{cases}\dfrac{(x-3)(x+2)}{(x-3)}, & \text{when } x\neq 3\\ 5, & \text{when } x=3\end{cases}$$

$$f(x)=\begin{cases}(x+2), & \text{when } x\neq 3\\ 5, & \text{when } x=3\end{cases}$$

Now, left hand limit at $x=3$,

$$\lim_{x\to 3^-} f(x)=\lim_{h\to 0^+}(3-h+2)=\lim_{h\to 0^+}(3-h+2)=\lim_{h\to 0^+}(5-h)=(5-0)=5$$

And, Right hand limit at $x=3$,

$$\lim_{x\to 3^+} f(x)=\lim_{h\to 0^+} f(3+h)=\lim_{h\to 0^+}\{(3+h)+2\}=\lim_{h\to 0^+}(3+h+2)=\lim_{h\to 0^+}(5+h)=(5+0)=5$$

Value of function at $x=3$,

$$f(x)=5 \Rightarrow f(3)=5$$

Since $\lim_{x\to 3^-} f(x)=\lim_{x\to 3^+} f(x)=f(3)=5$. Hence, $f(x)$ is continuous at $x=3$.

4. $f(x)=\begin{cases}\dfrac{x^2-25}{(x-5)}, & \text{when } x\neq 5\\ 10, & \text{when } x=5\end{cases}$, is continuous at $x=5$.

Sol. $f(x)=\begin{cases}\dfrac{(x)^2-(5)^2}{(x-5)}, & \text{when } x\neq 5\\ 10, & \text{when } x=5\end{cases} \Rightarrow f(x)=\begin{cases}\dfrac{(x-5)(x+5)}{(x-5)}, & \text{when } x\neq 5\\ 10, & \text{when } x=5\end{cases}$

$$f(x)=\begin{cases}(x+5), & \text{when } x\neq 5\\ 10, & \text{when } x=5\end{cases}$$

Now, left hand limit at $x=5$,

$$\lim_{x\to 5^-} f(x)=\lim_{h\to 0^+} f(5-h)=\lim_{h\to 0^+}(5-h+5)=\lim_{h\to 0^+}(10-h)=(10-0)=10.$$

And, Right hand limit at $x=5$,

$$\lim_{x\to 5^+} f(x)=\lim_{h\to 0^+} f(5+h)=\lim_{h\to 0^+}(5+h+5)=\lim_{h\to 0^+}(10+h)=(10+0)=10$$

Value of function at $x=5$, $f(x)=10 \Rightarrow f(5)=10$

$\lim_{x\to 5^-} f(x)=\lim_{x\to 5^+} f(x)=f(5)=10$. Hence, $f(x)$ is continuous at $x=5$.

5. $f(x)=\begin{cases}\dfrac{\sin 3x}{x}, & \text{when } x\neq 0\\ 1, & \text{when } x=0\end{cases}$, is discontinuous at $x=0$.

Sol. $f(x)=\begin{cases}\dfrac{\sin 3x}{x}, & \text{when } x\neq 0\\ 1, & \text{when } x=0\end{cases}$

Left hand limit at $x=0$,

$$\lim_{x\to 0^-} f(x)=\lim_{h\to 0^+} f(0-h)=\lim_{h\to 0^+} f(-h)=\lim_{h\to 0^+}\frac{\sin 3(-h)}{-h}=\lim_{h\to 0^+}\frac{\sin 3h}{h}=\lim_{h\to 0^+}\frac{\sin 3h}{3h}.3=3$$

And, Right hand limit at $x=0$,

$$\lim_{x\to 0^+} f(x)=\lim_{h\to 0^+} f(0+h)=\lim_{h\to 0^+} f(h)=\lim_{h\to 0^+}\frac{\sin 3h}{h}=\lim_{h\to 0^+}\frac{\sin 3h}{3h}.3=3$$

Value of function at $x=0$, $f(x)=1 \Rightarrow f(0)=1$

Since, $\lim_{x\to 0^-} f(x)=\lim_{x\to 0^+} f(x)\neq f(0)$. Hence, $f(x)$ is discontinuous at $x=0$.

6. $f(x)=\begin{cases}\dfrac{1-\cos x}{x^2}, & \text{when } x\neq 0\\ 1, & \text{when } x=0\end{cases}$, is discontinuous at $x=0$.

Sol. $f(x)=\begin{cases}\dfrac{1-\cos x}{x^2}, & \text{when } x\neq 0\\ 1, & \text{when } x=0\end{cases}$

Left hand limit at $x=0$,

$$\lim_{x\to 0^-} f(x)=\lim_{h\to 0^+} f(0-h)=\lim_{h\to 0^+} f(-h)=\lim_{h\to 0^+}\frac{1-\cos(-h)}{(-h)^2}$$

$$=\lim_{h\to 0^+}\frac{1-\cos h}{h^2}=\lim_{h\to 0^+}\frac{2\sin^2 h/2}{h^2}=\lim_{h\to 0^+}\frac{2.\sin h/2.\sin h/2}{h^2}$$

$$=\lim_{h\to 0^+} 2\frac{(\sin h/2)}{h/2}\times\frac{(\sin h/2)}{h/2}\times\frac{1}{4}=\frac{1}{2}$$

And, Right hand limit at $x=0$,

$$\lim_{h\to 0^+} f(x)=\lim_{h\to 0^+}(0+h)=\lim_{h\to 0^+}\frac{1-\cos h}{h^2}=\lim_{h\to 0^+}\frac{2\sin^2 h/2}{h^2}$$

$$=\lim_{h\to 0^+} 2\frac{(\sin h/2)}{h/2}\times\frac{(\sin h/2)}{h/2}\times\frac{1}{4}=\frac{1}{2}$$

Value of function at $x=0$, $f(x)=1 \Rightarrow f(0)=1$

Since, $\lim_{x\to 0^-} f(x)=\lim_{x\to 0^+} f(x)\neq f(0)$. Hence, $f(x)$ is discontinuous at $x=0$

7. $f(x)=\begin{cases}2-x, & \text{when } x<2\\ 2+x, & \text{when } x\geq 2\end{cases}$ is discontinuous at $x=2$.

Sol. $f(x)=\begin{cases}2-x, & \text{when } x<2\\ 2+x, & \text{when } x\geq 2\end{cases}$

Left hand limit at $x=2$,

$$\lim_{x\to 2^-} f(x)=\lim_{h\to 0^+} f(2-h)=\lim_{h\to 0^+}\{2-(2-h)\}=\lim_{h\to 0^+}(2-2+h)=\lim_{h\to 0^+} h=0$$

And, Right hand limit at $x=2$,

$$\lim_{x\to 2^+} f(x)=\lim_{h\to 0^+}(2+h)=\lim_{h\to 0^+}(2+2+h)=\lim_{h\to 0^+}(4+h)=(4+0)=4$$

Value of function at $x=2$, $f(x)=2+x\Rightarrow f(2)=2+2\Rightarrow f(2)=4$

Since, $\lim_{x\to 2^-} f(x)\neq\lim_{x\to 2^+} f(x)=f(2)$. Hence, $f(x)$ is discontinuous at $x=2$.

8. $f(x)=\begin{cases}3-x, & \text{when } x\leq 0\\ x^2, & \text{when } x>0\end{cases}$ is discontinuous at $x=0$?

Sol. $f(x)=\begin{cases}3-x, & \text{when } x\leq 0\\ x^2, & \text{when } x>0\end{cases}$.

Left hand limit $x=0$,

$$\lim_{x\to 0^-} f(x)=\lim_{h\to 0^+} f(0-h)=\lim_{h\to 0^+}\{3-(-h)\}=\lim_{h\to 0^+}(3+h)=(3+0)=3$$

And, Right hand limit at $x=0$,

$$\lim_{x\to 0^+} f(x)=\lim_{h\to 0^+} f(0+h)=\lim_{h\to 0^+} f(h)=\lim_{h\to 0^+}(h)^2=\lim_{h\to 0^+} h^2=0$$

Value of function at $x=0$,

$$f(x)=3-x \Rightarrow f(0)=3-0=f(0)=3$$

Since, $\lim_{x\to 0^-} f(x) \neq \lim_{x\to 0^+} f(x)$. Hence, $f(x)$ is discontinuous at $x=0$.

9. $f(x)=\begin{cases} 5x-4, & \text{when } 0<x\le 1 \\ 4x^2-3x, & \text{when } 1<x<2 \end{cases}$ is continuous at $x=1$

Sol. $f(x)=\begin{cases} 5x-4, & \text{when } 0<x\le 1 \\ 4x^2-3x, & \text{when } 1<x<2 \end{cases}$

Left hand limit at $x=1$, $\lim_{x\to 1^-} f(x)=\lim_{h\to 0^+} f(1-h)=\lim_{h\to 0^+} 5(1-h)-4$

$$=\lim_{h\to 0^+} 5-5h-4=\lim_{h\to 0^+} 1-5h=1-5(0)=1$$

Right hand limit at $x=1$, $\lim_{x\to 1^+} f(x)=\lim_{h\to 0^+} f(1+h)=\lim_{h\to 0^+} 4(1+h)^2-3(1+h)$

$$=\lim_{h\to 0^+} 4(1+h^2+2h)-3-3h=4-3=1$$

Value of function at $x=1$, $f(x)=5x-4 \Rightarrow f(1)=5-4=1$

Since, $\lim_{x\to 1^-} f(x)=\lim_{x\to 1^+} f(x)=f(1)=1$. Hence, $f(x)$ is continuous at $x=1$.

10. $f(x)=\begin{cases} x-1, & \text{when } 1\le x<2 \\ 2x-3, & \text{when } 2\le x\le 3 \end{cases}$ is continuous at $x=2$.

Sol. $f(x)=\begin{cases} x-1, & \text{when } 1\le x<2 \\ 2x-3, & \text{when } 2\le x\le 3 \end{cases}$

Left hand limit at $x=2$, $\lim_{x\to 2^-} f(x)=\lim_{h\to 0^+} f(2-h)=\lim_{h\to 0^+} (2-h)-1=\lim_{h\to 0^+} 2-h-1=1$

Right hand limit at $x=2$, $\lim_{x\to 2^+} f(x)=\lim_{h\to 0^+} f(2+h)=\lim_{h\to 0^+} 2(2+h)-3=\lim_{h\to 0^+} 4+2h-3=1$

Value of function at $x=2$, $f(x)=2x-3 \Rightarrow f(2)=2(2)-3 \Rightarrow f(2)=4-3 \Rightarrow f(2)=1$

Since, $\lim_{x\to 2^-} f(x)=\lim_{x\to 2^+} f(x)=f(2)=1$. Hence, $f(x)$ is continuous at $x=2$.

11. $f(x)=\begin{cases} \cos x, & \text{when } x\ge 0 \\ -\cos x, & \text{when } x<0 \end{cases}$, is discontinuous at $x=0$.

Sol. $f(x)=\begin{cases} \cos x, & \text{when } x\ge 0 \\ -\cos x, & \text{when } x<0 \end{cases}$

Left hand limit at $x=0$,

$$\lim_{x\to 0^-} f(x)=\lim_{h\to 0^+} f(0-h)=\lim_{h\to 0^+} f(-h)=-\lim_{h\to 0^+} \cos(-h)=-\lim_{h\to 0^+} \cos h=-1$$

Right hand limit at $x=0$, $\lim_{x\to 0^+} f(x)=\lim_{h\to 0^+} f(0+h)=\lim_{h\to 0^+} f(h)=\lim_{h\to 0^+} \cos h=1$

Value of function at $x=0$, $f(x)=\cos x \Rightarrow f(0)=\cos 0 \Rightarrow f(0)=1$

Since, $\lim_{x\to 0^-} f(x)\neq \lim_{x\to 0^+} f(x)=f(0)$. Hence, $f(x)$ is discontinuous at $x=0$

12. $f(x)=\begin{cases} \dfrac{|x-a|}{x-a}, & \text{when } x\neq a \\ 1, & \text{when } x=a \end{cases}$, is discontinuous at $x=0$

Sol. $f(x)=\begin{cases}\dfrac{|x-a|}{x-a}, & \text{when } x\neq a\\ 1, & \text{when } x=a\end{cases}$, when $x\neq 0$

Left hand limit at $x=0$,

$$\lim_{x\to a^-} f(x)=\lim_{h\to 0^+} f(a-h)=\lim_{h\to 0^+}\frac{|a-h-a|}{a-h-a}=\lim_{h\to 0^+}\frac{|-h|}{-h}=\lim_{h\to 0}\frac{h}{-h}=-1$$

Right hand limit at $x=a$, $\lim_{x\to 0^+} f(x)=\lim_{h\to 0^+} f(0+h)=\lim_{h\to 0^+}\dfrac{|a+h-a|}{a+h-a}=\lim_{h\to 0^+}\dfrac{h}{h}=1$

Value of function at $x=a$, $f(x)=1 \Rightarrow f(a)=1$

Since, $\lim_{x\to a^-} f(x)\neq \lim_{x\to a^+} f(x)=f(a)$. Hence, $f(x)$ is discontinuous at $x=a$.

13. $f(x)=\begin{cases}\dfrac{1}{2}(x-|x|), & \text{when } x\neq 0\\ 2, & \text{when } x=0\end{cases}$ is discontinuous at $x=0$.

Sol. $f(x)=\begin{cases}\dfrac{1}{2}(x-|x|), & \text{when } x\neq 0\\ 2, & \text{when } x=0\end{cases}$

Left hand limit at $x=0$, $\lim_{x\to 0^-} f(x)=\lim_{x\to 0^-} f(0-h)=\lim_{h\to 0}(-h)=\lim_{h\to 0^+}\dfrac{1}{2}(-h-|-h|)$

$$=\lim_{h\to 0^+}\frac{1}{2}(-h-h)=\lim_{h\to 0}\frac{1}{2}\times(-2h)=\lim_{h\to 0^+}(-h)=-\lim_{h\to 0^+} h=0$$

Right hand limit at $x=0$, $\lim_{x\to 0^+} f(x)=\lim_{h\to 0^+} f(0+h)=\lim_{h\to 0^+}\dfrac{1}{2}(h-|h|)=\lim_{h\to 0^+}\dfrac{1}{2}\times 0=0$

Value of function at $x=0$, $f(x)=2 \Rightarrow f(0)=2$

Since, $\lim_{x\to 0^-} f(x)=\lim_{x\to 0^+} f(x)\neq f(0)$. Hence, $f(x)$ is discontinuous at $x=0$.

14. $f(x)=\begin{cases}\sin\dfrac{1}{x}, & \text{when } x\neq 0\\ 0, & \text{when } x=0\end{cases}$ is discontinuous at $x=0$.

Sol. Left hand limit at $x=0$, $\lim_{x\to 0^-} f(x)=\lim_{h\to 0} f(0-h)=\lim_{h\to 0^+} f(-h)=\lim_{h\to 0^+}\sin\left(\dfrac{1}{-h}\right)=-\infty$

Right hand limit at $x=0$, $\lim_{x\to 0^+} f(x)=\lim_{h\to 0} f(0+h)=\lim_{h\to 0^+} f(h)=\lim_{h\to 0^+}\sin\left(\dfrac{1}{h}\right)=\infty$

Value of function at $x=0$, $f(x)=0 \Rightarrow f(0)=0$

Since, $\lim_{x\to 0^-} f(x)\neq \lim_{x\to 0^+} f(x)\neq f(0)$. Hence, $f(x)$ is discontinuous at $x=0$.

15. $f(x)=\begin{cases}2x, & \text{when } x<2\\ 2, & \text{when } x=2\\ x^2, & \text{when } x>2\end{cases}$, is discontinuous at $x=2$.

Sol. $f(x)=\begin{cases}2x, & \text{when } x<2\\ 2, & \text{when } x=2\\ x^2, & \text{when } x>2\end{cases}$

Left hand limit at $x=2$,

$$\lim_{x\to 2^-} f(x)=\lim_{x\to 0^+} f(2-h)=\lim_{h\to 0^+} 2(2-h)=\lim_{h\to 0^+} 4-2h=4-2(0)=4$$

Right hand limit at $x=2$,

$$\lim_{x\to 2^+} f(x)=\lim_{h\to 0^+} f(2+h)=\lim_{h\to 0^+} (2+h)^2=\lim_{h\to 0^+} 4+h^2+4h=4+(0)^2+4(0)=4$$

Value of function at $x=2$, $f(x)=2 \Rightarrow f(2)=2$

Since, $\lim_{x\to 2^-} f(x)=\lim_{x\to 2^+} f(x)\neq f(2)$. Hence, $f(x)$ is discontinuous at $x=2$.

16. $f(x)=\begin{cases}-x, & \text{when } x<0\\ 1, & \text{when } x=0\\ x, & \text{when } x>0\end{cases}$ is discontinuous at $x=0$.

Sol. $f(x)=\begin{cases}-x, & \text{when } x<0\\ 1, & \text{when } x=0\\ x, & \text{when } x>0\end{cases}$

Left hand limit at $x=0$, $\lim_{x\to 0^-} f(x)=\lim_{h\to 0^+} f(0-h)=\lim_{h\to 0^+} f(-h)=\lim_{h\to 0^+} -(-h)=\lim_{h\to 0^+} h=0$

Right hand limit at $x=0$, $\lim_{x\to 0^+} f(x)=\lim_{h\to 0^+} f(0+h)=\lim_{h\to 0^+} h=0$

Value of function at $x=0$, $f(x)=1 \Rightarrow f(0)=1$

Since, $\lim_{x\to 0^-} f(x)=\lim_{x\to 0^+} f(x)\neq f(0)$. Hence, $f(x)$ is discontinuous at $x=0$

17. Find the value of k for which $f(x)=\begin{cases}\dfrac{\sin 2x}{5x}, & \text{when } x\neq 0\\ k, & \text{when } x=0\end{cases}$, is continuous at $x=0$.

Sol. $f(x)=\begin{cases}\dfrac{\sin 2x}{5x}, & \text{when } x\neq 0\\ k, & \text{when } x=0\end{cases}$

Left hand limit at $x=0$,

$$\lim_{x\to 0^-} f(x)=\lim_{h\to 0^+} f(0-h)=\lim_{h\to 0^+} f(-h)=\lim_{h\to 0^+}\frac{\sin 2(-h)}{5(-h)}=\lim_{h\to 0^+}\frac{-\sin 2h}{-5h}=\lim_{h\to 0^+}\frac{\frac{\sin 2h}{2h}\times 2h}{5h}=\frac{2}{5}$$

Value of function at $x=0$, $f(x)=k \Rightarrow f(0)=k$

Since, $\lim_{x\to 0^-} f(x)=\lim_{x\to 0^+} f(x)=f(k) \quad \Rightarrow \frac{2}{5}=k$. Hence, $k=\frac{2}{5}$

18. Find the value of λ for which $f(x)=\begin{cases}\dfrac{\quad}{x+1} & \text{when } x\neq -1\\ \lambda, & \text{when } x=-1\end{cases}$, is continuous at $x=-1$.

Sol. $f(x)=\begin{cases}\dfrac{x^2-2x-3}{x+1}, & \text{when } x\neq -1\\ \lambda, & \text{when } x=-1\end{cases}$

$\Rightarrow f(x)=\begin{cases}\dfrac{x^2-3x+x-3}{x+1}, & \text{when } x\neq -1\\ \lambda, & \text{when } x=-1\end{cases} \quad \Rightarrow f(x)=\begin{cases}\dfrac{x(x-3)+1(x-3)}{x+1}, & \text{when } x\neq -1\\ \lambda, & \text{when } x=-1\end{cases}$

$$\Rightarrow f(x)=\begin{cases}\dfrac{(x-3)(x+1)}{x+1}, & \text{when } x\neq -1\\ \lambda, & \text{when } x=-1\end{cases} \quad \Rightarrow f(x)=\begin{cases}x-3, & \text{when } x\neq -1\\ \lambda, & \text{when } x=-1\end{cases}$$

Left hand limit at $x=-1$, $\lim\limits_{x\to -1^-} f(x)=\lim\limits_{h\to 0^+} f(-1-h)=\lim\limits_{h\to 0^+}-1-h-3=\lim\limits_{h\to 0^+}-4-h=-4-0=-4$

And value of function at $x=-1$, $f(x)=\lambda \Rightarrow f(-1)=\lambda$

Since, $\lim\limits_{x\to -1^-} f(x)=\lim\limits_{x\to -1^+} f(x)=f(-1) \Rightarrow -4=\lambda$. Hence, $\lambda=-4$

19. For what value of k is the following function continuous at $x=2$?

$$f(x)=\begin{cases}2x+1, & x<2\\ k, & x=2\\ 3x-1, & x>2\end{cases}$$

Sol. $f(x)=\begin{cases}2x+1, & x<2\\ k, & x=2\\ 3x-1, & x>2\end{cases}$

Left hand limit at $x=2$,

$$\lim_{x\to 2^-} f(x)=\lim_{h\to 0^+} f(2-h)=\lim_{h\to 0^+} 2(2-h)+1=\lim_{h\to 0^+} 4-2h+1=\lim_{h\to 0^+} 5-2h=5-2(0)=5$$

Right hand limit at $x=2$,

$$\lim_{x\to 2^+} f(x)=\lim_{h\to 0^+} f(2+h)=\lim_{h\to 0^+} 3(2+h)-1=\lim_{h\to 0^+} 6+3h-1=\lim_{h\to 0^+} 5+3h=5+3(0)=5$$

Value of function at $x=2$, $f(x)=k=f(2)=k$

$\because$ Left hand limit $=$ right hand limit $=$ value of function at $x=2$

Hence, this function is continuous at $x=2$. $\therefore k=5$

20. For the what value of k is the function $f(x)=\begin{cases}\dfrac{x^2-9}{x-3}, & \text{when } x\neq 3\\ k, & \text{when } x=3\end{cases}$, is continuous at $x=3$?

Sol. $f(x)=\begin{cases}\dfrac{x^2-9}{x-3}, & \text{when } x\neq 3\\ k, & \text{when } x=3\end{cases} \quad \Rightarrow f(x)=\begin{cases}\dfrac{(x)^2-(3)^2}{x-3}, & \text{when } x\neq 3\\ k, & \text{when } x=3\end{cases}$

$$\Rightarrow f(x)=\begin{cases}\dfrac{(x-3)(x+3)}{(x-3)}, & \text{when } x\neq 3\\ k, & \text{when } x=3\end{cases} \quad \Rightarrow f(x)=\begin{cases}x+3, & \text{when } x\neq 3\\ k, & \text{when } x=3\end{cases}$$

Left hand limit at $x=3$, $\lim\limits_{x\to 3^-} f(x)=\lim\limits_{h\to 0^+} f(3-h)=\lim\limits_{h\to 0^+}(3-h)+3=\lim\limits_{h\to 0^+} 6-h=6-0=6$

Value of function at $x=3$, $f(x)=k$, $f(3)=k$

Since, $\lim\limits_{x\to 3^-} f(x)=\lim\limits_{x\to 3^+} f(x)=f(k) \Rightarrow 6=k$. Hence, $k=6$

21. Find the value of k for which the function $f(x)=\begin{cases}\dfrac{k\cos x}{\pi-2x}, & \text{if } x\neq \dfrac{\pi}{2}\\ 3, & \text{if } x=\dfrac{\pi}{2}\end{cases}$ is continuous at $x=\pi/2$

Sol. $f(x)=\begin{cases}\dfrac{k\cos x}{\pi-2x}, & \text{if } x\neq\dfrac{\pi}{2}\\ 3, & \text{if } x=\dfrac{\pi}{2}\end{cases}$

Since, $f(x)$ is continuous at $x=\pi/2$

Left hand limit at $x=\pi/2$,

$$\lim_{x\to\pi/2^-} f(x)=\lim_{h\to0^+} f\left(\frac{\pi}{2}-h\right)=\lim_{h\to0^+}\frac{k\cos(\pi/2-h)}{\pi-2(\pi/2-h)}=\lim_{h\to0^+}\frac{k\sin h}{2h}=\lim_{h\to0^+}\frac{k}{2}.\frac{\sin h}{h}=\frac{k}{2}$$

Right hand limit at $x=\pi/2$,

Since, $\lim_{x\to\pi/2^+} f(x)=\lim_{h\to0^+} f\left(\frac{\pi}{2}+h\right)=\lim_{h\to0^+}\frac{k\cos(\pi/2+h)}{\pi-2(\pi/2+h)}=\lim_{h\to0^+}\frac{-k\sin h}{-2\sin h}=\frac{k}{2}.$

So, $\frac{k}{2}=3 \Rightarrow k=6$

22. Show that the function $f(x)=\begin{cases}x^2\sin\dfrac{1}{x}, & \text{if } x\neq0\\ 0, & \text{if } x=0\end{cases}$, is continuous at $x=0$

Sol. $f(x)=\begin{cases}x^2\sin\dfrac{1}{x}, & \text{if } x\neq0\\ 0, & \text{if } x=0\end{cases}$

Left hand limit at $x=0$,

$$\lim_{x\to0^-} f(x)=\lim_{h\to0^+} f(0-h)=\lim_{h\to0^+} f(-h)=\lim_{h\to0^+}(-h)^2\sin\left(\frac{1}{-h}\right)=\lim_{h\to0^+} h^2\sin\left(-\frac{1}{h}\right)=0$$

Right hand limit at $x=0$,

$$\lim_{x\to0^+} f(x)=\lim_{h\to0^+} f(0+h)=\lim_{h\to0^+} f(h)=\lim_{h\to0^+} h^2\sin\left(\frac{1}{h}\right)=0$$

Value of function at $x=0 \Rightarrow f(x)=0$

$\because$ left hand limit $=$ right hand limit $=$ value of function $=0$.

Hence this function is continuous at $x=0$.

23. Show that : $f(x)=\begin{cases}x+1, & \text{if } x\geq1\\ x^2+1, & \text{if } x<1\end{cases}$ is continuous at $x=1$

Sol. $f(x)=\begin{cases}x+1, & \text{if } x\geq1\\ x^2+1, & \text{if } x<1\end{cases}$

Left hand limit at $x=1$,

$$\lim_{x\to1^-} f(x)=\lim_{h\to0^+} f(1-h)=\lim_{h\to0^+}(1-h)^2+1=\lim_{h\to0^+} h^2-2h+2=(0)^2-2(0)+2=2$$

Right hand limit at $x=1$,

$$\lim_{x\to1^+} f(x)=\lim_{h\to0^+} f(1+h)=\lim_{h\to0^+} 1+h+1=\lim_{h\to0^+} 2+h=2+0=2$$

Value of function at $x=1$, $f(x)=x+1=f(1)=1+1=f(1)=2$

$\because$ left hand limit $=$ right hand limit $=$ value of function $=2$ at $x=1$

Hence, this function is continuous at $x=1$.

24. Show that : $f(x)=\begin{cases} x^3-3, & \text{if } x\le 2 \\ x^2+1, & \text{if } x>2 \end{cases}$, is continuous at $x=2$.

Sol. $f(x)=\begin{cases} x^3-3, & \text{if } x\le 2 \\ x^2+1, & \text{if } x>2 \end{cases}$

Left hand limit at $x=2$,

$$\lim_{x\to 2^-} f(x)=\lim_{h\to 0^+} f(2-h)=\lim_{h\to 0^+}(2-h)^3-3=\lim_{x\to 0^+}2^3-h^3-3.2.h(2-h)-3=8-0-0-3=5$$

Right hand limit at $x=2$,

$$\lim_{x\to 2^+} f(x)=\lim_{h\to 0^+} f(2+h)=\lim_{h\to 0^+}(2+h)^2+1$$

$$=\lim_{h\to 0^+}4+h^2+4h+1=\lim_{h\to 0^+}h^2+4h+5=(0)^2+4(0)+5=5$$

Value of function at $x=2 \Rightarrow f(2)=(2)^3-3 \Rightarrow f(2)=8-3 \Rightarrow f(2)=5$

Since, left hand limit = right hand limit = value of function = 5 at $x=2$.

Hence, this function is continuous at $x=2$.

25. Find the values of a and b such that the following function is continuous

$$f(x)=\begin{cases} 5, \text{ when } x\le 2 \\ ax+b, \text{ when } 2<x<10 \\ 21, \text{ when } x\ge 10 \end{cases}$$

Sol. We note that domain of $f=R$.

As the given function is continuous, it is continuous for all $x\in R$.

In particular, the function is continuous at $x=2$ and at $x=10$.

Continuity at $x=2$ **:** Here, $f(2)=5$; $\underset{x\to 2^-}{Lt} f(x)=\underset{x\to 2^-}{Lt} 5=5$ and $\underset{x\to 2^+}{Lt} f(x)=\underset{x\to 2^+}{Lt}(ax+b)=2a+b$

As the function f is continuous at $x=2$,

$$\underset{x\to 2^+}{Lt} f(x)=\underset{x\to 2^-}{Lt} f(x)=f(2) \quad \Rightarrow 5=2a+b=5 \quad \Rightarrow 2a+b=5$$

Continuity at $x=10$ **:** Here, $f(10)=21$;

$$\underset{x\to 10^-}{Lt} f(x)=\underset{x\to 10^-}{Lt}(ax+b)=10a+b \text{ and } \underset{x\to 10^+}{Lt} f(x)=\underset{x\to 10^+}{Lt} 21=21$$

As the function f, is continuous at $x=10$,

$$\underset{x\to 10^-}{Lt} f(x)=\underset{x\to 10^+}{Lt} f(x)=f(10) \quad \Rightarrow 10a+b=21=21 \quad \Rightarrow 10a+b=21$$

Solving (i) and (ii) simultaneously, we get $a=2, b=1$.

Hence, the given function f is continuous if $a=2$ and $b=1$.

26. Find the value of a for which the function f, defined as

$$f(x)=\begin{cases} a\sin\frac{\pi}{2}(x+1), & x\le 0 \\ \frac{\tan x-\sin x}{x^3}, & x>0 \end{cases} \quad \text{is continuous at } x=0$$

Sol. Here $f(0)=a\sin\frac{\pi}{2}(0+1)=a\sin\frac{\pi}{2}=a\times 1=a=a\times 1=a$

and $\underset{x\to 0^+}{Lt} = \underset{x\to 0^+}{Lt}\dfrac{\tan x-\sin x}{x^3} = \underset{x\to 0^+}{Lt}\dfrac{\dfrac{\sin x}{\cos x}-\sin x}{x^3}$

$= \underset{x\to 0^+}{Lt}\dfrac{\sin x(1-\cos x)}{x^3\cos x} = \underset{x\to 0^+}{Lt}\dfrac{\sin x(1-\cos x)}{x^3\cos x}\times\dfrac{1+\cos x}{1+\cos x}$

$= \underset{x\to 0^+}{Lt}\dfrac{\sin x.\sin^2 x}{x^3\cos x(1+\cos x)} = \left(\underset{x\to 0^+}{Lt}\dfrac{\sin x}{x}\right)^3.\underset{x\to 0^+}{Lt}\dfrac{1}{\cos x(1+\cos x)} = 1^3.\dfrac{1}{1(1+1)} = \dfrac{1}{2}$

For the function f to be continuous at $x=0$, we must have $\underset{x\to 0^-}{Lt} f(x) = \underset{x\to 0^+}{Lt} f(x) = f(0)$

$\Rightarrow a=\dfrac{1}{2}=a \quad \Rightarrow a=\dfrac{1}{2}$. Hence, $a=\dfrac{1}{2}$.

27. Prove that the function f given by $f(x)=|x-3|, x\in R$ is continuous but not differentiable at $x=3$

Sol. $f(x)=\begin{cases}5-x, & x\le 5\\ x-5, & x\ge 5\end{cases}$. Hence we have, $f(5)=5-5=0$

Now, $\lim\limits_{x\to 3^+} f(x) = \lim\limits_{h\to 0^+} f(3+h) = \lim\limits_{h\to 0^+}(3+h-5)=0$

$\lim\limits_{x\to 3^-} f(x) = \lim\limits_{h\to 0^+} f(3-h) = \lim\limits_{h\to 0^+} 3-(3-h) = \lim\limits_{h\to 0^+} h = 0$

$\therefore \lim\limits_{x\to 3^+} f(x) = \lim\limits_{x\to 3^-} f(x) = f(3)$. Thus, f is continuous at $x=3$.

Also, $Rf'(3) = $ (RHD at $x=3$) $= \lim\limits_{h\to 0^+}\dfrac{f(3+h)-f(3)}{h} = \lim\limits_{h\to 0^+}\dfrac{3+h-3-0}{h} = \lim\limits_{h\to 0^+}\dfrac{h}{h}=1$

$Lf'(3) = $ (LHD at $x=3$) $= \lim\limits_{h\to 0^+}\dfrac{f(3-h)-f(3)}{-h} = \lim\limits_{h\to 0^+}\dfrac{3-(3-h)-0}{-h} = \lim\limits_{h\to 0^+}\dfrac{h}{-h}=-1$

$\therefore Rf'(3)\ne Lf'(3)$. Hence, f is not differentiable at $x=3$.

EXERCISE 9B (Pg. No.: 358)

1. Show that the function $f(x)=\begin{cases}7x+5, & \text{when } x\ge 0\\ 5-3x, & \text{when } x<0\end{cases}$, is a continuous function.

Sol. Left hand limit at $x=0$,

$\lim\limits_{x\to 0^-} f(x) = \lim\limits_{h\to 0^+} f(0-h) = \lim\limits_{h\to 0^+} 5+3h = 5+3.0=5$

Right hand limit at $x=0$,

$\lim\limits_{x\to 0^+} f(x) = \lim\limits_{h\to 0^+} f(0+h) = \lim\limits_{h\to 0^+} 7h+5 = 7\times 0+5=5$

Value of function at $x=0$, $f(x)=7x+5$, $f(0)=7\times 0+5=5$

$\Rightarrow \lim\limits_{x\to 0^-} f(x) = \lim\limits_{x\to 0^+} f(x) = f(0)=5$. This function is continuous at $x=0$.

2. Show that the function $f(x)=\begin{cases}\sin x, & \text{if } x<0\\ x, & \text{if } x\ge 0\end{cases}$ is continuous

Sol. Left hand limit at $x=0$, $\lim\limits_{x\to 0^-} f(x) = \lim\limits_{h\to 0^+} f(-h) = \lim\limits_{h\to 0^+}\sin(-h) = -\lim\limits_{h\to 0^+}\sin h = 0$

Right hand limit at $x=0$, $\lim\limits_{x\to 0^+} f(x) = \lim\limits_{h\to 0^+} f(0+h) = \lim\limits_{h\to 0^+} h = 0$

Value of function at $x=0$, $f(x)=x$, $f(0)=0$

$\Rightarrow \lim\limits_{x\to 0^-} f(x) = \lim\limits_{x\to 0^+} f(x) = f(0)=0$. This function is continuous at $x=0$.

3. Show that the function $f(x)=\begin{cases}\dfrac{x^n-1}{x-1}, & \text{when } x\neq 1\\ n, & \text{when } x=1\end{cases}$ is continuous.

Sol. Left hand limit at $x=1$, $\lim\limits_{x\to 1^-} f(x)=\lim\limits_{h\to 0^+} f(1-h)=\lim\limits_{h\to 0^+}\dfrac{(1-h)^n-1^n}{(1-h)-1}=n(1)^{n-1}=n$

Right hand limit at $x=1$, $\lim\limits_{x\to 1^+} f(x)=\lim\limits_{h\to 0^+} f(1+h)=\lim\limits_{h\to 0^+}\dfrac{(1+h)^n-1^n}{(1+h)-1}=n(1)^{n-1}=n$

Value of function at $x=1,\ f(x)=n,\ f(1)=n$

$\Rightarrow \lim\limits_{x\to 1^-} f(x)=\lim\limits_{x\to 1^+} f(x)=f(1)=n$. This function is continuous at $x=n$

4. Show that $\sec x$ is a continuous function.

Sol. Let $f(x)=\sec x$. Let $x=c$ be any real number.

$\lim\limits_{x\to c} f(x)=\lim\limits_{h\to 0^+} f(c+h)=\lim\limits_{h\to 0^+}\sec(c+h)=\sec(c)$

Also, $f(x)=\sec c \quad \therefore \lim\limits_{x\to c} f(x)=f(c)$

$\therefore\ f(x)$, i.e., $\sec x$ is continuous at $x=c$. But $x=c$ is any real number.

$\therefore\ \sec x$ is continuous.

5. Show that $\cos|x|$ is a continuous function.

Sol. Let $f(x)=|x|$ and $g(x)=\cos x$, then $(gof)(x)=g\{f(x)\}=g\{|x|\}=\cos|x|$

Now, f and g being continuous it follows that their composite (gof) is continuous.

Hence, $\cos|x|$ is continuous.

6. Show that the function $f(x)=\begin{cases}\dfrac{\sin x}{x}, & \text{when } x\neq 0\\ 2, & \text{when } x=0\end{cases}$; is continuous at each point except 0.

Sol. Left hand limit at $x=0 \Rightarrow \lim\limits_{x\to 0^-} f(x)=\lim\limits_{h\to 0^+} f(-h)=\lim\limits_{h\to 0^+}\dfrac{\sin(-h)}{(-h)}=1$

Right hand limit at $x=0=\lim\limits_{x\to 0^+} f(x)=\lim\limits_{h\to 0^+} f(h)=\lim\limits_{h\to 0^+}\dfrac{\sin h}{h}=1$

Value of function at $x=0,\ f(x)=2,\ f(0)=2$

$\therefore\ \lim\limits_{x\to 0^-} f(x)=\lim\limits_{x\to 0^+} f(x)\neq f(0)$

7. Discuss the continuity of $f(x)=[x]$.

Sol. Let the integral value of $x=n,\ n\in I$

Let $f(x)=[x]$, is continuous at $x=n,\ n\in I$

At $x=n,\ f(x)=[n]=n$

LHL : $\lim\limits_{x\to n^-} f(x)=\lim\limits_{h\to 0^+} f(n-h)=\lim\limits_{h\to 0^+}[n-h]=n-1$

RHL : $\lim\limits_{x\to n^+} f(x)=\lim\limits_{h\to 0^+} f(n+h)=\lim\limits_{h\to 0^+}[n+h]=n$

So, LHL $\neq$ RHL. Hence, $f(x)$ is discontinuous at $x=n,\ n\in I$.

8. Show that $f(x)=\begin{cases}(2x-1), & \text{if } x<2\\ \dfrac{3x}{2}, & \text{if } x\ge 2\end{cases}$ is continuous.

Sol. Left hand limit at $x=2$, $\lim_{x\to 2^-} f(x)=\lim_{h\to 0^+} f(2-h)=\lim_{h\to 0^+} 2(2-h)-1$

$=\lim_{h\to 0^+} 4-2h-1=\lim_{h\to 0^+} 3-2h=3-2\times 0=3$

Right hand limit at $x=2$, $\lim_{x\to 2^+} f(x)=\lim_{h\to 0^+} f(2+h)=\lim_{h\to 0^+} 3\dfrac{(2+h)}{2}=\dfrac{3(2+0)}{2}=\dfrac{6}{2}=3$

Value of function at $x=2$, $f(x)=\dfrac{3x}{2}, f(2)=\dfrac{3.2}{2}=3$

$\therefore \lim_{x\to 2^-} f(x)=\lim_{x\to 2^+} f(x)=f(2)=3$. This function is continuous at $x=2$

9. Show that $f(x)=\begin{cases}x, & \text{if } x\ne 0\\ 1, & \text{if } x=0\end{cases}$ is continuous at each point except 0.

Sol. Let hand limit at $x=0$, $\lim_{x\to 0^-} f(x)=\lim_{x\to 0^+} f(-h)=\lim_{h\to 0^+}(-h)=0$

Right hand limit at $x=0$, $\lim_{x\to 0^+} f(x)=\lim_{h\to 0^+} f(h)=\lim_{h\to 0^+} h=0$

Value of function at $x=0$, $f(x)=1, f(0)=1$

$\lim_{x\to 0^-} f(x)=\lim_{x\to 0^+} f(x)\ne f(0)$

This function is discontinuous at $x=0$. Again $f(x)=\begin{cases}x, & \text{if } x\ne 1\\ 1, & \text{if } x=1\end{cases}$

Left hand limit at $x=1$, $\lim_{x\to 1^-} f(x)=\lim_{h\to 0^+}(1-h)=\lim_{h\to 0^+}(1-h)=1-0=1$

Right hand limit at $x=1$, $\lim_{x\to 1^+} f(x)=\lim_{h\to 0^+} f(1+h)=\lim_{h\to 0^+}(1+0)=1$

Value of function at $x=1$, $f(x)=x, f(1)=1$

$\lim_{x\to 1^-} f(x)=\lim_{x\to 1^+} f(x)=f(1)=1$

$\therefore$ This function is continuous at $x=1$.

10. Locate the point of discontinuity of the function $f(x)=\begin{cases}(x^3-x^2+2x-2), & \text{if } x\ne 1\\ 4, & \text{if } x=1\end{cases}$

Sol. The only doubtful point is $x=1$. Hence, we check the continuity at $x=1$

Left hand limit at $x=1$,

$\lim_{x\to 1^-} f(x)=\lim_{h\to 0^+} f(1-h)=\lim_{h\to 0^+}(1-h)^3-(1-h)^2+2(1-h)-2$

$=\lim_{h\to 0^+}(1-h)^3-(1-h)^2+2(1-h)-2=(1-0)^3-(1-0)^2+2(1-0)-2=0$

Right hand limit at $x=1$, $\lim_{x\to 1^+} f(x)=\lim_{h\to 0^+} f(1+h)=\lim_{h\to 0^+}(1+h)^3-(1+h)^2+2(1+h)-2$

$=(1+0)^3-(1+0)^2+2(1+0)-2=0$

Value of function at $x=1$, $f(x)=4$, $f(1)=4$

$\Rightarrow \lim_{x\to 1^-} f(x)=\lim_{x\to 1^+} f(x)\ne f(1)$. This function is discontinuous at $x=0$.

11. Discuss the continuity of the function $f(x)=|x|+|x-1|$ in the interval $[-1, 2]$.

Sol. Given $\Rightarrow f(x)=\begin{cases}-x-x-1 & \text{if } x\le 0\\ x-x-1 & \text{if } 0\le x\le 1\\ x+x-1 & \text{if } x\ge 1\end{cases}$. Hence, only doubtful point.

EXERCISE 9C (Pg.No.: 364)

1. Show that $f(x)=x^3$, is continuous as well as differentiable at $x=3$.

Sol. Left hand limit at $x=3$, $\lim\limits_{x\to 3^-} f(x)=\lim\limits_{h\to 0^+} f(3-h)=\lim\limits_{h\to 0^+}(3-h)^3=(3-0)^3=27$

Right hand limit at $x=3$, $\lim\limits_{x\to 3^+} f(x)=\lim\limits_{h\to 0^+} f(3+h)=\lim\limits_{h\to 0^+}(3+h)^3=(3+0)^3=27$

Value of function at $x=3$, $f(x)=x^3=f(3)=(3)^3=27$

$\lim\limits_{x\to 3^-} f(x)=\lim\limits_{x\to 3^+} f(x)=f(3)=27$

Left hand derivative at $x=3$

$$Lf'(3)=\lim_{x\to 3^-}\frac{f(x)-f(3)}{x-3}=\lim_{h\to 0^+}\frac{f(3-h)-f(3)}{0-h}=\lim_{h\to 0^+}\frac{(3-h)^3-(3)^3}{-h}$$

$$=\lim_{h\to 0^+}\frac{(3)^3-(h)^3-3(3)^2.h+3(h)^2.3-27}{-h}=\lim_{h\to 0^+}\frac{-h^3-27h+9h^2}{-h}$$

$$=\lim_{h\to 0^+}\frac{-h(h^2+27-9h)}{-h}=(0)^2+27-9\times 0=27$$

Right hand derivative at $x=3$

$$Rf'=\lim_{x\to 3^+}\frac{f(x)-f(3)}{x-3}=\lim_{h\to 0^+}\frac{f(3+h)-f(3)}{h}=\lim_{h\to 0^+}\frac{(3+h)^3-(3)^3}{h}$$

$$=\lim_{h\to 0^+}\frac{(3)^3+h^3+3(3)^2h+3(h)^2.3-27}{h}$$

$$=\lim_{h\to 0^+}\frac{h^3+27h+9h^2}{h}=\lim_{h\to 0^+}\frac{h(h^2+27+9h)}{h}=0+27+9\times 0=27$$

$\Rightarrow Lf'(3)=Rf'(3)$. Hence, the function is continuous as well as differentiable at $x=3$.

2. Show that $f(x)=(x-1)^{1/3}$ is differentiable at $x=1$.

Sol. Left hand derivative at $x=1$

$$Lf'(1)=\lim_{x\to 1^-}\frac{f(x)-f(1)}{x-1}=\lim_{h\to 0^+}\frac{f(1-h)-f(1)}{(1-h-1)}=\lim_{h\to 0^+}\frac{(1-h-1)^{1/3}-(1-1)^{1/3}}{-h}$$

$$=\lim_{h\to 0^+}\frac{(-h)^{1/3}-0}{-h}=\lim_{h\to 0^+}\frac{h^{1/3}}{h}=\lim_{h\to 0^+}\frac{1}{h^{2/3}}=\infty$$

Right hand derivative at $x=1$

$$Rf'(1)=\lim_{x\to 1^+}\frac{f(x)-f(1)}{x-1}=\lim_{h\to 0^+}\frac{f(1+h)-f(1)}{(1+h-1)}=\lim_{h\to 0^+}\frac{(1+h-1)^{1/3}-(1-1)^{1/3}}{h}$$

$$= \lim_{h\to 0^+} \frac{h^{1/3}-0}{h} = \lim_{h\to 0^+} \frac{h^{1/3}}{h} = \infty$$

Hence $Lf'(1) \neq Rf'(1)$. This function is not differentiable at $x=1$.

3. Show that a constant function is always differentiable.

Sol. Let the function $f(x)=\lambda$, where λ is constant

We will discuss differentiability at $x=a$.

$$\therefore \quad f'(a^-) = \lim_{h\to 0^+} \frac{f(a-h)-f(a)}{-h} = \lim_{h\to 0^+} \frac{\lambda-\lambda}{-h} = 0$$

and $f'(a^+) = \lim_{h\to 0^+} \frac{f(a+h)-f(a)}{h} = \lim_{h\to 0^+} \frac{\lambda-\lambda}{h} = 0$

$\therefore$ LHD = RHD. Hence, the function is always differentiable.

4. Show that $f(x)=|x-5|$ is continuous but not differentiable at $x=5$.

Sol. Left hand limit at $x=5$, $\lim_{x\to 5^-} f(x) = \lim_{h\to 0^+} (5-h) = \lim_{h\to 0^+} |5-h-5| = \lim_{h\to 0^+} h = 0$

Right hand limit at $x=5$, $\lim_{x\to 5^+} f(x) = \lim_{h\to 0^+} f(5+h) = \lim_{h\to 0^+} |5+h-5| = \lim_{h\to 0^+} |h| = 0$

Value of function at $x=5$, $f(x)=|x-5|$, $f(5)=|5-5|=0$

$$\lim_{x\to 5^-} f(x) = \lim_{x\to 5^+} f(x) = f(5) = 0$$

Left hand derivative at $x=5$

As, $Lf'(5) = \lim_{x\to 5^-} \frac{f(x)-f(5)}{x-5} = \lim_{h\to 0^+} \frac{f(5-h)-f(5)}{-h} = \lim_{h\to 0^+} \frac{|5-h-5|-|5-5|}{-h}$

$$= \lim_{h\to 0^+} \frac{|-h|}{-h} = \lim_{h\to 0^+} \frac{h}{-h} = -1$$

Right hand derivative at $x=5$

$$Rf'(5) = \lim_{x\to 5^+} \frac{f(x)-f(5)}{x-5} = \lim_{h\to 0^+} \frac{f(5+h)-f(5)}{h} = \lim_{h\to 0^+} \frac{|5+h-5|-|5-5|}{h} = \lim_{h\to 0^+} \frac{|h|}{h} = \lim_{h\to 0^+} \frac{h}{h} = 1$$

$Lf'(5) \neq Rf'(5)$. Hence, the function is continuous but not differentiable at $x=5$.

5. Let $f(x) = \begin{cases} (2-x), & \text{when } x \geq 1 \\ x, & \text{when } 0 \leq x \leq 1 \end{cases}$. Show that $f(x)$ is continuous but not differentiable at $x=1$.

Sol. Left hand limit at $x=1$, $\lim_{x\to 1^+} f(x) = \lim_{h\to 0^+} f(1-h) = \lim_{h\to 0^+} (1-h) = (1-0) = 1$

Right hand limit at $x=1$, $\lim_{x\to 1^+} f(x) = \lim_{h\to 0^+} f(1+h) = \lim_{h\to 0^+} 2-(1+h) = \lim_{h\to 0^+} 1-h = 1-0 = 1$

Value of function at $x=1$, $f(x)=2-x$ or x, $f(1)=2-1$ or 1, $f(1)=1$

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x) = f(1) = 1$$

Left hand derivative at $x=1$

$$Lf'(1) = \lim_{x\to 1^+} \frac{f(x)-f(1)}{x-1} = \lim_{h\to 0^+} \frac{f(1-h)-f(1)}{-h} = \lim_{h\to 0^+} \frac{1-h-1}{-h} = \lim_{h\to 0^+} \frac{-h}{-h} = 1$$

Right hand derivative at $x=1$

$$Rf'(1) = \lim_{x\to 1^+} \frac{f(x)-f(1)}{x-1} = \lim_{h\to 0^+} \frac{f(1+h)-f(1)}{h} = \lim_{h\to 0^+} \frac{\{2-(1+h)\}-(2-1)}{h} = \lim_{h\to 0^+} \frac{(-h)}{h} = -1$$

$Lf'(1) \neq Rf'(1)$. Hence, the $f(x)$ is continuous but not differentiable at $x=1$.

6. Show that $f(x)=[x]$ is neither continuous nor deriviable at $x=2$.

Sol. L.H.L $= \lim_{x\to 2^-} f(x) = \lim_{x\to 2^-}[x] = \lim_{h\to 0^+}[2-h] = \lim_{h\to 0^+}(1) = 1$

R.H.L $= \lim_{x\to 2^+} f(x) = \lim_{x\to 2^+}[x] = \lim_{h\to 0^+}[2+h] = \lim_{h\to 0^+}(2) = 2$

Again left hand derivative at $x=2$. $\therefore \lim_{x\to 2^-} f(x) \neq \lim_{x\to 2^+} f(x)$

$$Lf'(2) = \lim_{x\to 2^-}\frac{f(x)-f(2)}{x-2} = \lim_{x\to 2^-}\frac{[x]-[2]}{x-2} = \lim_{h\to 0^+}\frac{[2-h]-2}{2-h-2} = \lim_{h\to 0^+}\frac{1-2}{-h} = \lim_{h\to 0^+}\frac{1}{h} = \infty$$

$\therefore$ $Lf'(2)$ does not exist.

Right hand derivative at $x=2$

$$Rf'(2) = \lim_{x\to 2^+}\frac{f(x)-f(2)}{x-2} = \lim_{x\to 2^+}\frac{[x]-[2]}{x-2} = \lim_{h\to 0^+}\frac{[2+h]-2}{2+h-2} = \lim_{h\to 0^+}\frac{2-2}{h} = 0$$

$\therefore$ $Lf'(2) \neq Rf'(2)$ $\Rightarrow f$ is not derivable at $x=2$

Hence, the $f(x)$ is neither continuous nor derivable at $x=2$.

7. Show that the function $f(x)=\begin{cases}(1-x), & \text{when } x<1\\ (x^2-1), & \text{when } x\geq 1\end{cases}$ is continuous but not differentiable at $x=1$.

Sol. Left hand limit at $x=1$, $\lim_{x\to 1^-} f(x) = \lim_{h\to 0^+} f(1-h) = \lim_{h\to 0^+} 1-(1-h) = \lim_{h\to 0^+} h = 0$

Right hand limit at $x=1$, $\lim_{x\to 1^+} f(x) = \lim_{h\to 0^+} f(1+h) = \lim_{h\to 0^+}(1+h)^2 - 1 = 0$

Value of function at $x=1$, $f(x)=x^2-1$, $f(1)=(1)^2-1=0$

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x) = f(1) = 0$$

Again, left hand derivative at $x=1$

$$Lf'(1) = \lim_{x\to 1^-}\frac{f(x)-f(1)}{x-1} = \lim_{h\to 0^+}\frac{f(1-h)-f(1)}{-h} = \lim_{h\to 0^+}\frac{\{1-(1-h)\}-\{1-1\}}{-h}$$

$= \lim_{h\to 0^+}\frac{h-0}{-h} = \lim_{h\to 0^+}\frac{h}{-h} = -1$ Right hand derivative at $x=1$

$$Rf'(1) = \lim_{x\to 1^+}\frac{f(x)-f(1)}{x-1} = \lim_{h\to 0^+}\frac{f(1+h)-f(1)}{h} = \lim_{h\to 0^+}\frac{\{(1+h)^2-1\}-\{1-1\}}{h}$$

$$= \lim_{h\to 0^+}\frac{(1+2h+h^2-1)}{h} = \lim_{h\to 0^+}\frac{h(2+h)}{h} = \frac{0(2+0)}{0} = 0$$

$Lf'(1) \neq Rf'(1)$. Hence the $f(x)$ is continuous but not differentiable at $x=1$.

8. Let $f(x)=\begin{cases}(2+x), & \text{if } x\geq 0\\ (2-x), & \text{if } x<0\end{cases}$. Show that $f(x)$ is not derivable at $x=0$.

Sol. Left hand derivative at $x=0$, $\lim_{x\to 0^-}\frac{f(x)-f(0)}{x-0} = \lim_{h\to 0^+}\frac{f(-h)-f(0)}{-h}$

$$= \lim_{h\to 0^+}\frac{\{2-(-h)\}-\{2-0\}}{-h} = \lim_{h\to 0^+}\frac{2+h-2}{-h} = \lim_{h\to 0^+}\frac{h}{-h} = -1$$

Right hand derivative at $x=0$, $\lim_{x\to 0^+}\frac{f(x)-f(0)}{x-0}$

$$=\lim_{h\to 0^+}\frac{f(h)-f(0)}{h}=\lim_{h\to 0^+}\frac{(2+h)-(2+0)}{h}=\lim_{h\to 0^+}\frac{2+h-2}{h}=\lim_{h\to 0^+}\frac{h}{h}=1$$

$Lf'(0)\neq Rf'(0)$. Hence, the $f(x)$ is not derivable at $x=0$.

9. If $f(x)=|x|$, show that $f'(2)=1$.

Sol. Left hand derivative at $x=2$

$$Lf'(2)=\lim_{h\to 0^+}\frac{f(2-h)-f(2)}{-h}=\lim_{h\to 0^+}\frac{|2-h|-|2|}{-h}=\lim_{h\to 0^+}\frac{2-h-2}{-h}=\lim_{h\to 0^+}\frac{-h}{-h}=1$$

Right hand derivative at $x=2$

$$Rf'(2)=\lim_{h\to 0^+}\frac{f(2+h)-f(2)}{h}=\lim_{h\to 0^+}\frac{2+h-2}{h}=\lim_{h\to 0^+}\frac{h}{h}=1$$

Hence, $Lf'(2)=Rf'(2)=1$

10. Find the values of a and b so that the function $f(x)=\begin{cases}(x^2+3x+a), & \text{when } x\le 1\\ (bx+2), & \text{when } x>1\end{cases}$ is differentiable at each $x\in R$.

Sol. Since $f(x)$ is derivable for every x:

$\therefore$ $f(x)$ is derivative at $x=1$ $\Rightarrow$ f is continuous at $x=1$

$\Rightarrow \lim_{x\to 1^-}f(x)=f(1)=\lim_{x\to 1^+}f(x)$ $\Rightarrow \lim_{x\to 1^-}(x^2+3x+a)=1+3+a=\lim_{x\to 1^+}(bx+2)$

$\Rightarrow 1+3+a=1+3+a=b+2$ $\Rightarrow 4+a=4+a=b+2$

$\Rightarrow 4+a=b+2$ $\Rightarrow b=a+2$ ...(1)

Now, $Lf'(1)=\lim_{x\to 1^-}\frac{f(x)-f(1)}{x-1}=\lim_{x\to 1^-}\frac{(x^2+3x+a)-(1+3+a)}{x}$

$$=\lim_{x\to 1^-}\frac{x^2+3x-4}{x-1}=\lim_{x\to 1^-}\frac{(x+4)(x-1)}{(x-1)}=\lim_{x\to 1^-}(x+4)=\lim_{h\to 0^+}(1-h+4)=5$$

$$Rf'(1)=\lim_{x\to 1^+}\frac{f(x)-f(1)}{x-1}=\lim_{x\to 1^+}\frac{bx+2-(1+3+a)}{x-1}$$

$$=\lim_{x\to 1^+}\frac{bx-a-2}{x-1}=\lim_{x\to 1^+}\frac{(a+2)x-a-2}{(x-1)}\quad\text{[Using (1)]}$$

$$=\lim_{x\to 1^+}\frac{a(x-1)+2(x-1)}{(x-1)}=\lim_{x\to 1^+}\frac{(a+2)(x-1)}{(x-1)}=\lim_{x\to 1^+}(a+2)=\lim_{h\to 0^+}(a+2)=(a+2)$$

Since $Lf'(1)=Rf'(1)$ $\therefore$ $5=a+2$, here $b=a+2$ $\therefore$ $b=3+2=5$ Hence, $a=3$, $b=5$.