CHAPTER 05

External Equilibrium of Structures

5.1 Equations of Equilibrium

A structure may be in rest or motion. Generally all Civil Engineering structures like, bridges, buildings or dam etc. are in rest so it may be said to be in static equilibrium.

The equations of equilibrium are based on Newton's Law's of motion. The equations of equilibrium may be written as,

$\Sigma F_x = 0,$ $\quad \Sigma F_y = 0$ $\quad \Sigma F_z = 0$ ...(i)

$\Sigma M_x = 0,$ $\quad \Sigma M_y = 0$ $\quad \Sigma M_z = 0$ ...(ii)

For plane structure, if all forces lies in the plane of structure then only three equations are sufficient for equilibrium.

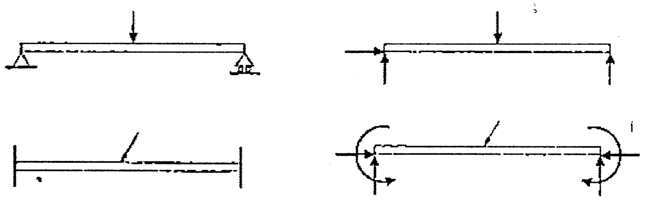
$\Sigma F_x = 0,$ $\quad \Sigma F_y = 0$ ...(iii)

$\Sigma M_z = 0,$ ...(iv)

5.2 Free Body Diagram

Free body diagram is a pictorial representation of all forces acting on a structural component or whole structure. If a structure is in static equilibrium then all the components will also be in static equilibrium. Hence equations of equilibrium are also applicable for each component or element of structure.

The free body diagram of any component or whole structure can be drawn by isolating the structure from its support and showing the support reactions and loading acted upon that component of structure.



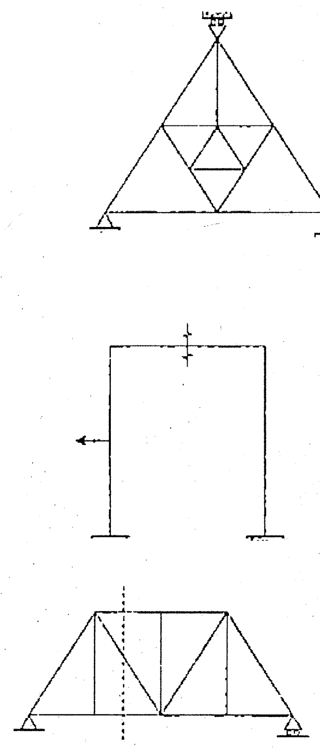


Fig. 5.1 Actual Structure

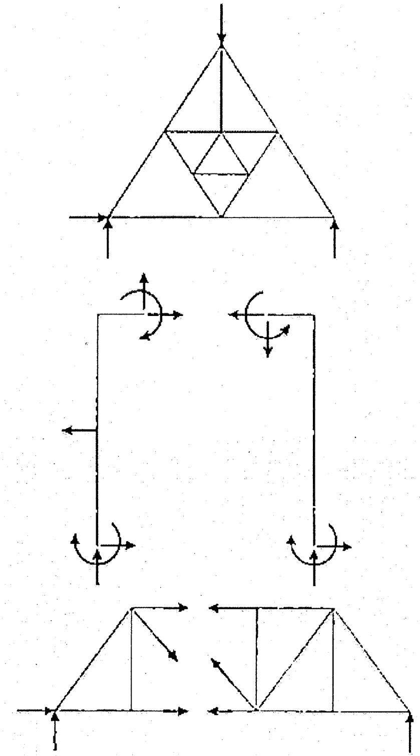
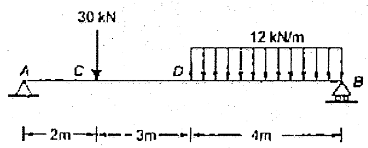
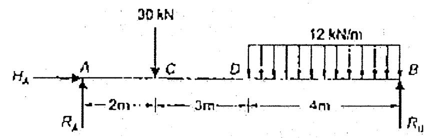


Fig. 5.2 Corresponding Free Body Diagram

Example 5.1 A simply supported beam of 9 m span is loaded as shown in figure. Find the support reactions for beam.



Solution:



$$\Sigma F_x = 0$$

$\Rightarrow$ $H_A = 0$ ...(i)

Also, $\Sigma F_y = 0$

$\Rightarrow$ $R_A + R_B - 30 - 12 \times 4 = 0$

$\Rightarrow$ $R_A + R_B = 78$ ...(ii)

Also, $\Sigma M_z = 0$

$\Rightarrow$ $\Sigma M_B = 0$ (taking moment of all forces about B)

$\Rightarrow$ $R_A \times 9 - 30 \times 7 - 12 \times 4 \times 2 = 0$

$\Rightarrow$ $9 R_A = 30 \times 7 + 12 \times 4 \times 2$

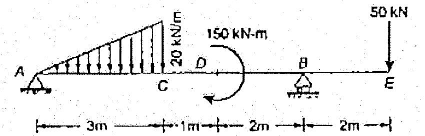
$\therefore$ $R_A = 34$ kN (↑)

From equation (ii), we have

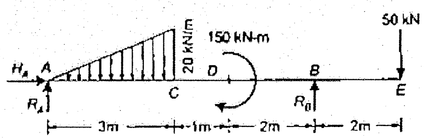
$R_B = 78 - R_A$

$\Rightarrow$ $R_B = 78 - 34 = 44$ kN (↑)

Example 5.2 Find the support reactions for beam shown in figure.



Solution:



$\Sigma F_x = 0$

$\Rightarrow$ $H_A = 0$ ...(i)

Also, $\Sigma F_y = 0$

$\Rightarrow$ $R_A + R_B - \left[\frac{1}{2} \times 3 \times 20\right] - 50 = 0$

$\Rightarrow$ $R_A + R_B = 80$ kN ...(ii)

Also, $\Sigma M_A = 0$

$\Rightarrow$ $\left(\frac{1}{2} \times 3 \times 20\right) \times \frac{2}{3} \times 3 + 150 - R_B \times 6 + 50 \times 8 = 0$

$\Rightarrow$ $60 + 150 + 400 - 6 R_B = 0$

$\Rightarrow$ $6 R_B = 610$

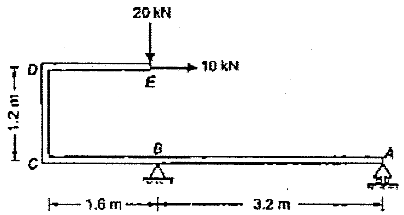
$\Rightarrow$ $R_B = 101.67$ kN (↑)

From equation (ii), we have

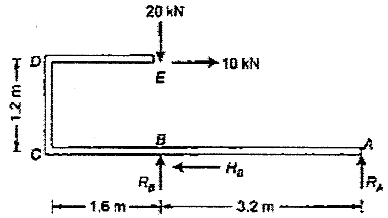
$R_A = 80 - R_B = 80 - 101.67$

$\therefore$ $R_A = -21.67$ kN (↓)

Example 5.3 Find the support reactions for the rigid jointed frame shown in the figure.



Solution:



$\Sigma F_x = 0$

$\Rightarrow$ $10 - H_B = 0$

$\Rightarrow$ $H_B = 10$ kN (←) ...(i)

Also, $\Sigma F_y = 0$

$\Rightarrow$ $R_A + R_B - 20 = 0$

$\Rightarrow$ $R_A + R_B = 20$ ...(ii)

Also, $\Sigma M_A = 0$

$\Rightarrow$ $R_B \times 3.2 + 10 \times 1.2 - 20 \times 3.2 = 0$

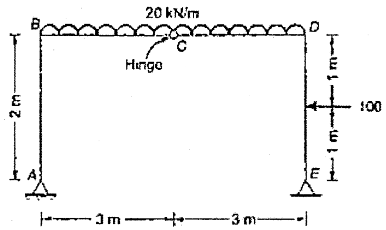
$\therefore$ $R_B = 16.25$ kN (↑)

From equation (ii), we have

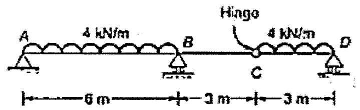
$R_A = 20 - R_B = 20 - 16.25$

$\Rightarrow$ $R_A = 3.75$ kN (↑)

Example 5.4 Calculate support reactions for the following frames shown in figures.



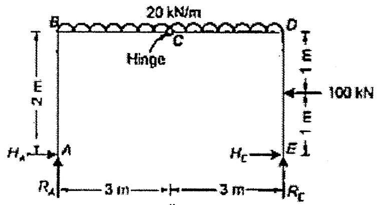
(i)



(ii)

Solution:

(i)



(ii)

$\Sigma F_x = 0$

$\Rightarrow$ $H_A + H_E = 100$ ...(i)

$\Sigma F_y = 0$

$\Rightarrow$ $R_A + R_E = 20 \times 6 = 120$ ...(ii)

$\Sigma M_E = 0$

$\Rightarrow$ $R_A \times 6 - 20 \times 6 \times 3 - 100 \times 1 = 0$

$\Rightarrow \qquad 6R_A = 460$

$\Rightarrow \qquad R_A = 76.67 \text{ kN } (\uparrow)$

From eq. (ii), we get, $\qquad R_E = 120 - 76.67 = 43.33 \text{ kN } (\uparrow)$

Also at hinge C, $\qquad \Sigma M_C = 0$

(Taking moment of all the forces from the left side of Hinge C)

$\Rightarrow \qquad R_A \times 3 - H_A \times 2 - 20 \times 3 \times 1.5 = 0$

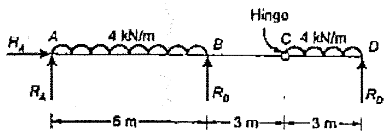
$\Rightarrow \qquad 76.67 \times 3 - 2H_A - 90 = 0$

$\Rightarrow \qquad H_A = 70 \text{ kN } (\rightarrow)$

From eq. (i), we get

$$H_E = 100 - 70 = 30 \text{ kN } (\rightarrow)$$

(ii)



Free body diagram

$\Sigma F_x = 0$

$\Rightarrow \qquad H_A = 0$

Also, $\qquad \Sigma F_y = 0$

$\Rightarrow \qquad R_A + R_B + R_D = (4 \times 6) + (4 \times 3)$

$\Rightarrow \qquad R_A + R_B + R_D = 36 \qquad ...(i)$

There is a hinge at C. Hence BM at C will be zero.

$M_C = 0 \qquad$ (From left)

$\Rightarrow \quad R_A \times 9 + R_B \times 3 - 4 \times 6 \times 6 = 0$

$\Rightarrow \qquad 9R_A + 3R_B = 144 \qquad ...(ii)$

$M_C = 0 \qquad$ (From right)

$\Rightarrow \qquad R_D \times 3 - 4 \times 3 \times 1.5 = 0$

$\Rightarrow \qquad R_D = 6 \text{ kN } (\uparrow)$

Substituting value of R_D in equation (*i*), we get

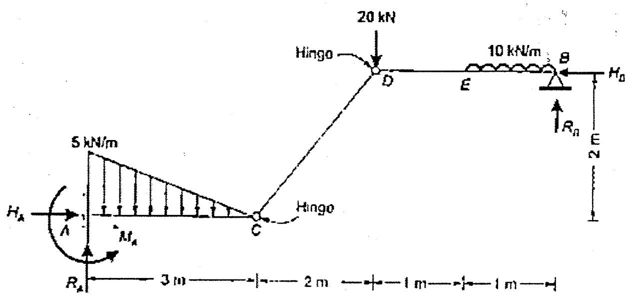
$R_B + R_A = 30 \qquad ...(iii)$

Solving equation (*ii*) and (*iii*), we get

$R_A = 9 \text{ kN } (\uparrow)$

$R_B = 21 \text{ kN } (\uparrow)$

Example 5.5 **Determine all the reaction components at supports *A* (fixed) and *B* (hinged). Joint *C* and *D* are hinged. Effect of axial forces in various beams parts *AC*, *CD*, *DB* need not to be considered.**



Solution:

$\Sigma F_x = 0$

$\Rightarrow \qquad H_A = H_B \qquad ...(i)$

Also. $\qquad \Sigma F_y = 0$

$\Rightarrow \qquad R_A + R_B = 10 \times 1 + 20 + \frac{1}{2} \times 3 \times 5$

$\Rightarrow \qquad R_A + R_B = 37.5 \qquad ...(ii)$

Since D is hinge, hence $\qquad M_D = 0$ (from right side)

$\Rightarrow \qquad -R_B \times 2 + 10 \times 1 \times 1.5 = 0$

$\Rightarrow \qquad R_B = 7.5 \text{ kN } (\uparrow)$

From eq. (ii), we get

$R_A = 37.5 - 7.5 = 30 \text{ kN } (\uparrow)$

Also, $\qquad M_C = 0$ (From right side)

$\Rightarrow \quad -R_B \times 4 + 10 \times 1 \times 3.5 + 20 \times 2 - H_B \times 2 = 0$

$\Rightarrow \qquad -4 \times 7.5 + 35 + 40 - 2H_B = 0$

$H_B = 22.5 \text{ kN } (\leftarrow)$

From eq. (i), $\qquad H_A = 22.5 \text{ kN } (\rightarrow)$

Also, $\qquad M_C = 0$ (from left hand side)

$\Rightarrow \qquad R_A \times 3 - M_A - \frac{1}{2} \times 3 \times 5 \times \frac{2}{3} \times 3 = 0$

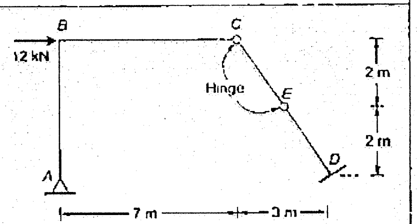
$\Rightarrow \qquad 30 \times 3 - M_A - 15 = 0$

$\Rightarrow \qquad M_A = 75 \text{ kNm } (\circlearrowright)$

Thus reactions are

$R_A = 30 \text{ kN } (\uparrow)$	$R_B = 7.5 \text{ kN } (\uparrow)$
$H_A = 22.5 \text{ kN } (\leftarrow)$	$H_B = 22.5 \text{ kN } (\rightarrow)$
$M_A = 75 \text{ kNm } (\circlearrowright)$	

Example 5.6 A portal frame *ABCD* as shown in the figure has hinged end at *A* and fixed end at *D*. Vertical leg *AB* = 4 m, horizontal *BC* = 7 m. Joint *C* and mid-point *E* of inclined leg *CD* are acting as hinges. *A* and *D* are at same level. Calculate the component of reactions at *A* and *D* when a horizontal force of 12 kN is applied at joint *B*.



Solution:

Let vertical reactions at A and D be $R_A (\uparrow)$ and $R_D (\uparrow)$ respectively. Also let horizontal reactions at A and D be $H_A (\leftarrow)$ and $H_D (\leftarrow)$ respectively. Also, let the reactive moment at D be M_D

Let, $\qquad \angle CDA = \theta$

$\therefore \qquad \tan\theta = \frac{4}{3}$

$$\Sigma F_x = 0$$

$\Rightarrow$ $H_A + H_D = 12$ ...(i)

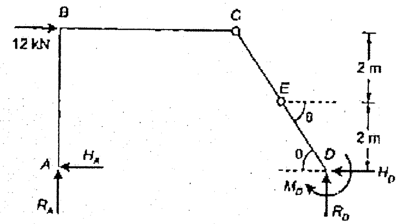
Also, $\Sigma F_y = 0$

$\Rightarrow$ $R_A + R_D = 0$ ...(ii)

Also, $\Sigma M_A = 0$

$\Rightarrow$ $(R_D \times 10) - (12 \times 4) - M_D = 0$

$\Rightarrow$ $10\,R_D - M_D = 48$ ...(iii)



Since, C is a hinge, therefore moments about C either from right or left will be zero.

$$M_C = 0 \text{ (From left)}$$

$\Rightarrow$ $R_A \times 7 + H_A \times 4 = 0$

$\Rightarrow$ $4\,H_A + 7\,R_A = 0$ ...(iv)

Since, E is hinge, therefore moments about E either from right or left will be zero

$$M_E = 0 \quad \text{(From right)}$$

$\Rightarrow$ $R_D \times \dfrac{2}{\tan\theta} - H_D \times 2 - M_D = 0$

$\Rightarrow$ $R_D \times \dfrac{2}{4/3} - 2H_D - M_D = 0$

$\Rightarrow$ $1.5\,R_D - 2H_D - M_D = 0$ ...(v)

Substituting values of H_D, R_D and M_D from equation (i), (ii) and (iii) respectively in (v), we get

$\Rightarrow$ $1.5\,(-R_A) - 2(12 - H_A) - (10\,R_D - 48) = 0$

$\Rightarrow$ $-1.5\,R_A - 24 + 2\,H_A - 10\,R_D + 48 = 0$

$\Rightarrow$ $-1.5\,R_A - 10\,(-R_A) + 2\,H_A + 24 = 0$

$\Rightarrow$ $8.5\,R_A + 2\,H_A = -24$ ...(vi)

Solving, equation (iv) and (vi), we get

$H_A = 8.4$ kN

$R_A = -4.8$ kN

$R_D = -R_A = -(-4.8) = 4.8$ kN

$H_D = 12 - H_A = 12 - 8.4 = 3.6$ kN

$M_D = 10\,R_D - 48 = 10 \times 4.8 - 48 = 0$ kN-m

Thus the reactions are

$H_A = 8.4$ kN ($\leftarrow$)

$R_A = 4.8$ kN ($\downarrow$)

$H_D = 3.6$ kN ($\leftarrow$)

$R_D = 4.8$ kN ($\uparrow$)

$M_D = 0$ kN-m

Summary

- The equations of equilibrium are based on Newton's Law of motion.
- For space structure, the equations of static equilibrium are

  $\Sigma F_x = 0, \quad \Sigma F_y = 0, \quad \Sigma F_z = 0$ ...(i)

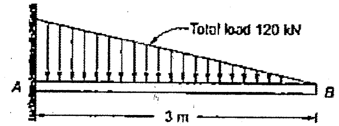
  and $\Sigma M_x = 0, \quad \Sigma M_y = 0, \quad \Sigma M_z = 0$ ...(ii)
- For plane structure, the three equations of static equilibrium are enough

  $\Sigma F_x = 0, \quad \Sigma F_y = 0$

  and $\Sigma M_z = 0$
- Free body diagram is a pictorial representation of all forces acting on a structural component or whole structure.
- The free body diagram of any component or whole structure can be drawn by isolating the structure from its support and showing support reactions and loading acted upon that component of structure.

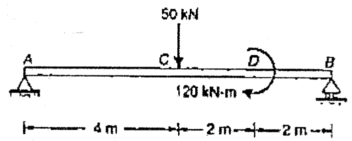
Objective Brain Teasers

Q.1 What are the support reactions at the fixed end of the cantilever beam shown in the figure below?



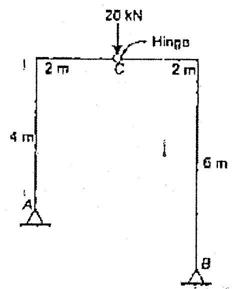
(a) 120 kN, 120 kN-m
(b) 120 kN, 240 kN-m
(c) 240 kN, 120 kN-m
(d) 120 kN, 60 kN-m

Q.2 The beam shown in the figure given below is subjected to concentrated load and clockwise couple. What is the vertical reaction at A?



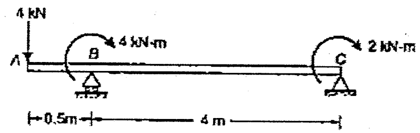
(a) 10 kN (b) 40 kN
(c) 50 kN (d) 30 kN

Q.3 For the frame shown below, what is the horizontal thrust at A?



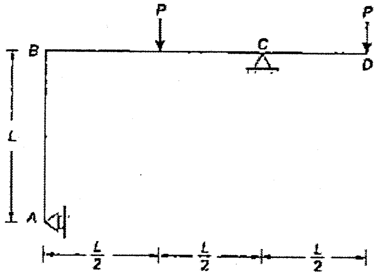
(a) 2 kN (b) 4 kN
(c) 5 kN (d) zero

Q.4 In the beam shown in the given figure, the vertical reactions at B and C respectively are



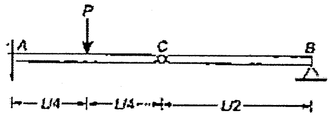
(a) 2 kN, 2 kN (b) 4 kN, 0 kN
(c) 3 kN, 1 kN (d) 1 kN, 3 kN

Q.5 A frame $ABCD$ is supported by roller at A and is on a hinge at C as shown in the figure. The reaction at the roller end A is given by



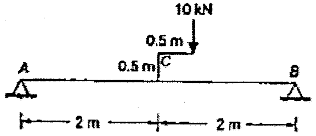
(a) P (b) $2P$
(c) $\dfrac{P}{2}$ (d) zero

Q.6 For the propped cantilever shown in figure, the vertical reaction at the fixed end is



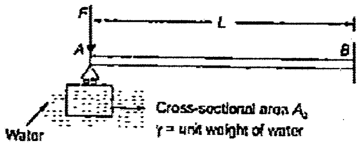
(a) $\dfrac{4P}{3}$ (b) P
(c) $\dfrac{3P}{4}$ (d) $\dfrac{P}{2}$

Q.7 The reaction at support '*B*' of the statically determine beam shown below is



(a) 3.75 kN (b) 4.25 kN
(c) 5.75 kN (d) 6.25 kN

Q.8 A cantilever beam *AB* as shown in figure is clamped at B and supported through a hinge by a partially submerged (in water) platform at *A*. The beam has flexural rigidity *EI* and length *L*. It is loaded by a vertical concentrated force *F* at *A*. (Neglect the weight of platform).



The reaction developed at *A* is

(a) $\dfrac{\frac{FL^3}{3}}{\left(\frac{L^3}{2}+\frac{EI}{A_0\gamma}\right)}$ (b) $\dfrac{\frac{FL^3}{2}}{\left(\frac{L^3}{2}+\frac{EI}{A_0\gamma}\right)}$

(c) $\dfrac{\frac{FL^3}{3}}{\left(\frac{L^3}{3}+\frac{EI}{A_0\gamma}\right)}$ (d) $\dfrac{\frac{FL^2}{2}}{\left(\frac{L^3}{2}+\frac{EI}{A_0\gamma}\right)}$

Answers

1. (a) 2. (a) 3. (c) 4. (b) 5. (d)
6. (b) 7. (d) 8. (c)

Hints and Explanations:

1. (a)

Vertical reaction at *A*,

$\Sigma F_y = 0$

$\Rightarrow R_A - 120\text{ kN} = 0$

$\Rightarrow R_A = 120\text{ kN}(\uparrow)$

The centroid of the load is at 1 m from the support. Therefore the moment at support will be

$= 120 \times 1$

$= 120\text{ kN-m}$ (Clockwise)

Hence, reactive moment at support

$M_A = 120\text{kNm}$ (Anticlockwise)

2. (a)

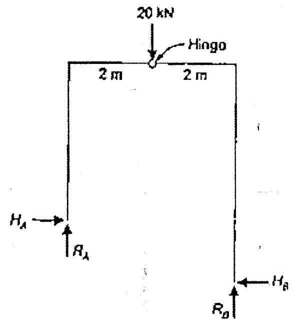
Let R_A be the vertical reaction at *A*.

Taking, $\Sigma M_B = 0$

$\Rightarrow R_A \times 8 - 50 \times 4 + 120 = 0$

$\Rightarrow R_A = \dfrac{200-120}{8} = 10\text{ kN}$

3. (b)



$\Sigma F_y = 0$

$\Rightarrow R_A + R_B = 20\text{ kN}$ ...(*i*)

$\Sigma F_x = 0$

$\Rightarrow H_A = H_B = H$ (Say)

$\Rightarrow M_C = 0$ (From left)

$\Rightarrow R_A \times 2 - H \times 4 = 0$

$\Rightarrow R_A = 2H$ ...(*ii*)

Also, $M_C = 0$ (From right)

$\Rightarrow -R_B \times 2 + H \times 6 = 0$

$\Rightarrow R_B = 3H$ ...(*iii*)

Substituting value of R_A and R_B from equation (*ii*) and (*iii*) into (*i*), we get

$\Rightarrow 2H + 3H = 20$

$\Rightarrow 5H = 20$

$\Rightarrow H = 4\text{ kN}$

4. (c)

Let R_B and R_C be the vertical reactions at *B* and *C* respectively.

$\Sigma M_C = 0$

$\Rightarrow -4 \times 4.5 + 4 + 2 + R_B \times 4 = 0$

$\Rightarrow 4R_B = 12$

$\Rightarrow R_B = 3\text{ kN}(\uparrow)$

Also, $\Sigma F_y = 0$

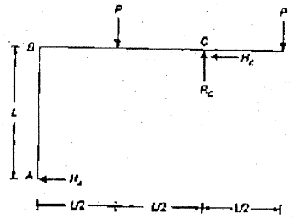
$\Rightarrow R_B + R_C = 4$

$\Rightarrow R_C = 4 - R_B$

$= 4 - 3 = 1\text{ kN}$

5. (d)

Let H_A and H_C be the horizontal reactions at supports *A* and *B* respectively. Also let vertical reaction at *C* be R_C

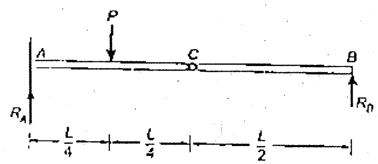


$\Sigma M_C = 0$

$\Rightarrow H_A \times L - P \times \dfrac{L}{2} + P \times \dfrac{L}{2} = 0$

$\Rightarrow H_A = 0$

6. (b)



$\Sigma M_C = 0$ (From right)

$\therefore -R_B \times \dfrac{L}{2} = 0$

$\therefore R_B = 0$

Also $\Sigma F_y = 0$

$R_A + R_B = P$

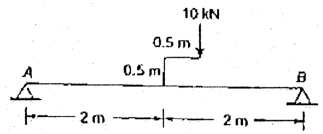
$\therefore R_A + 0 = P$

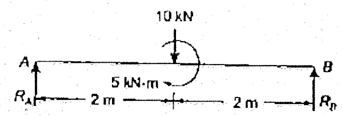
$R_A = P$

7. (d)

$\Sigma F_y = 0; R_A + R_B = 10$ ...(*i*)

$\Sigma M_B = 0; R_A \times 4 + 5 - 10 \times 2 = 0$





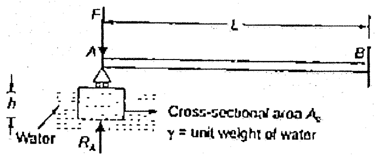
$R_A \times 4 = 15$

$R_A = \dfrac{15}{4} = 3.75\text{ kN}(\uparrow)$

From (*i*) $R_B = 10 - R_A = 10 - 3.75$

$R_B = 6.25\text{ kN}(\uparrow)$

8. (c)



Reaction developed at *A* = Force of buoyancy = weight of the fluid displaced $= \gamma A_0 h$

$\therefore R_A = \gamma A_0 h$

Here h = depth of immersion

It is also equal to deflection of support at *A*.

Now Net force at

A = $(F - \gamma A_0 h)$ downward

Compatibility condition at *A*,

$(F - R_A)\dfrac{L^3}{3EI} = h$

$\Rightarrow F - \gamma A_0 h = 3EI\dfrac{h}{L^3}$

$\Rightarrow h = \dfrac{F}{\left(\gamma A_0 + \frac{3EI}{L^3}\right)}$

Now, $R_A = \gamma A_0 \dfrac{F}{\left(\gamma A_0 + \frac{3EI}{L^3}\right)} = \dfrac{\frac{FL^3}{3}}{\left(\frac{L^3}{3} + \frac{EI}{\gamma A_0}\right)}$