SIGNALS AND SYSTEMS

Ch-5 LAPLACE TRANSFORM

We know that

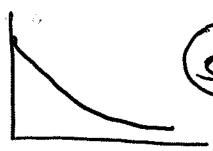
$$X[j\omega] = \int_{-\infty}^{\infty} x(t) \cdot e^{-j\omega t} \, dt$$

where $\int_{-\infty}^{\infty} |x(t)| \, dt < \infty$ [∵ Fourier Transform]

Now,

$$x(t) = e^{-at} u(t)$$

If $a > 0$



stable

$$x(t) = e^{at} u(t)$$

If $a > 0$

unstable

∴ To make the system stable i.e. finite

$x(t)\big|_{t \to \infty} = \text{finite}$ | $x(t)\big|_{t \to \infty} = \infty$ ⇓ unstable

$\sigma > 0$

$$x(t) \cdot e^{-\sigma t}\Big|_{t \to \infty} = 0$$

↑ unstable system $t > \infty$ $= \infty$

$$\therefore \int_{-\infty}^{\infty} |x(t) e^{-\sigma t}| \, dt < \infty$$

σ can be +ve & -ve

$\sigma > 0$

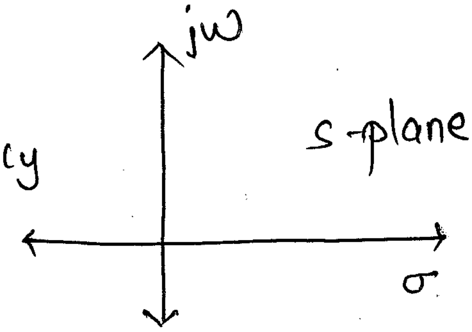
⇓

$X[s] < \infty \Rightarrow$ Laplace Transform exists

$$X[j\omega] = \int_{-\infty}^{\infty} x(t) \cdot e^{-\sigma t} \cdot e^{-j\omega t} \, dt$$

$$X[\sigma + j\omega] = \int_{-\infty}^{\infty} x(t) \cdot e^{-(\sigma + j\omega)t} \, dt$$

$s = \sigma + j\omega$ = complex frequency



$$\boxed{X[s] = \int_{-\infty}^{\infty} x(t) \cdot e^{-st} \, dt \Rightarrow \text{L.T. of } x(t)}$$

$$X[s] = \text{F.T.}\left\{ x(t) \cdot e^{-\sigma t} \right\}$$

Put $\sigma = 0$

$$\text{L.T.}\{x(t)\} = \text{F.T.}\left\{ x(t) \cdot e^{-\sigma t} \right\}$$

$\sigma = 0$

$$\text{L.T.}\{x(t)\} = \text{F.T.}\{x(t)\}$$

for $\sigma > 0$

$\text{Re}(s) > 0 \Rightarrow X[s] < \infty$

$\downarrow$

converges.

* Region of Convergence [ROC]

- L.T. converges to real value if $Re(s) > 0$

$$\int_{-\infty}^{\infty} |x(t) \cdot e^{-\sigma t}| \, dt < \infty$$

$\sigma > 0$

$$\Downarrow$$

$X(s) < \infty$ (finite & it's converges)

* Primary rule of L.T. in ENGINEERING is transient & stability analysis of causal LTI system.

* In addition to its simplicity many design techniques in circuits, filters and control systems develop in Laplace domain.

$$\boxed{\text{Generalization of F.T. is L.T.}}$$

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$$

$$y(t) = x(t) * h(t)$$

$$= \int_{-\infty}^{\infty} x(t-\tau) \cdot h(\tau) \, d\tau$$

$$= \int_{-\infty}^{\infty} e^{s(t-\tau)} h(\tau) \, d\tau$$

$$y(t) = e^{st} \int_{-\infty}^{\infty} h(\tau) \cdot e^{-s\tau} \, d\tau$$

Here, $\int_{-\infty}^{\infty} h(\tau)\cdot e^{-s\tau}\,d\tau$ = L.T. of $h(\tau)$

$$\boxed{y(t) = e^{st}\cdot H(s)} \qquad \boxed{y(t) = e^{j\omega t}\, H(j\omega)}$$

Now,

$$y(t) = e^{st}\, H(s)$$

$H(s)$ → eigen value

e^{st} → eigen ~~value~~ function

$$X(s) = \int_{-\infty}^{\infty} x(t)\cdot e^{-st}\,dt$$

B.L.T.
(bilateral laplace transform)
- causality
- stability
- frequency response

U.L.T.
(unilateral laplace transform)
- To solve D.E. (diff. eqⁿ) with I.C. (init. condi.).

* ROC (region of convergence) of L.T.

- ROC is set of values of s for which the laplace transform of signal converges. The L.T. is guaranteed to converges if

$$\int_{-\infty}^{\infty} |x(t)e^{-\sigma t}|\,dt < \infty$$

i.e. $X(s) < \infty$

Laplace Transform of some basic signals

(1) $x(t) = e^{-at} \cdot u(t)$

By definition,

$$X(s) = \int_{-\infty}^{\infty} x(t) \cdot e^{-st} \, dt$$

$$= \int_{-\infty}^{\infty} e^{-at} \cdot u(t) \cdot e^{-st} \, dt$$

$$= \int_{0}^{\infty} e^{-at} \cdot e^{-st} \, dt$$

$$= \int_{0}^{\infty} e^{-(s+a)t} \, dt$$

$$= \left[\frac{e^{-(s+a)t}}{-(s+a)} \right]_0^{\infty}$$

$$s = \sigma + j\omega$$

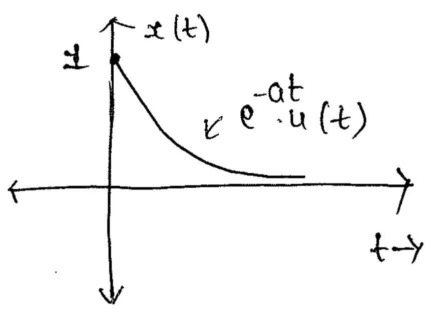
$$= \left[\frac{e^{-(\sigma + j\omega + a)t}}{-(\sigma + j\omega + a)} \right]_0^{\infty}$$

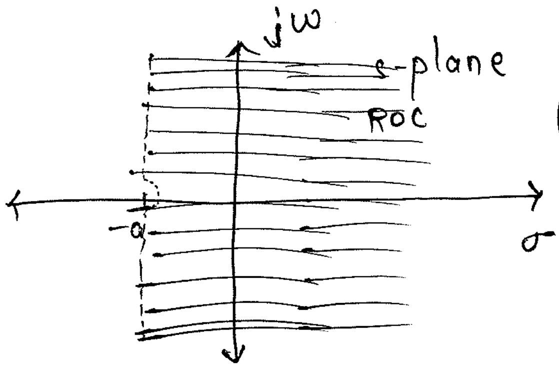
$$X(s) = \frac{1}{\sigma + j\omega + a} = \frac{1}{s+a} \quad ; \sigma + a > 0$$

$$X(s) = \frac{1}{s+a} \quad , \sigma > -a$$

$$X(s) = \frac{1}{s+a} \quad , Re(s) > -a$$

$$e^{-at} \cdot u(t) \xleftrightarrow{L.T.} \frac{1}{s+a} \quad ; Re(s) > -a$$





ROC

$\sigma > -a$

$Re(s) > -a$

(2) $x(t) = -e^{-at} \cdot u(-t)$

$$X(s) = \int_{-\infty}^{\infty} x(t) \cdot e^{-st} \, dt$$

$$= \int_{-\infty}^{\infty} -e^{-at} \cdot u(-t) \cdot e^{-st} \, dt$$

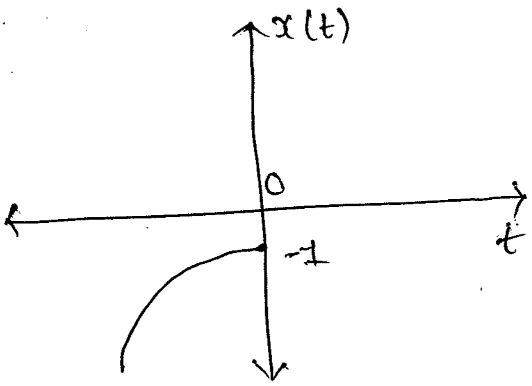
$$= -\int_{-\infty}^{0} e^{-at} \cdot e^{-st} \, dt$$

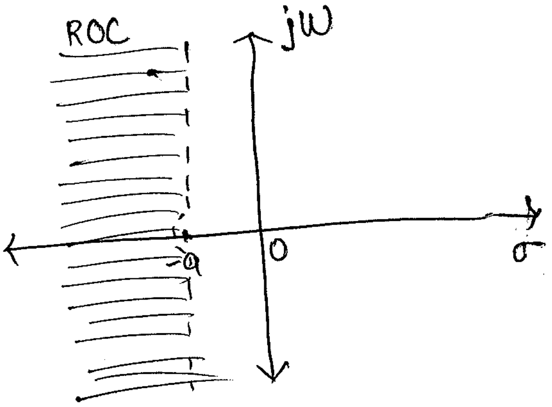
$$= \int_{0}^{-\infty} e^{-(a+s)t} \, dt$$

$$= \left[\frac{e^{-(a+s)t}}{-(a+s)} \right]_{0}^{-\infty}$$

$$X(s) = \frac{1}{s+a} \quad ; \sigma + a < 0$$

$$X(s) = \frac{1}{s+a} \quad ; Re(s) < -a$$





$$\boxed{\begin{array}{lll} e^{-at} \cdot u(t) \xleftrightarrow{L.T.} \dfrac{1}{s+a} & ; ROC: \sigma > -a \\ -e^{-at} u(-t) \xleftrightarrow{L.T.} \dfrac{1}{s+a} & ; ROC: \sigma < -a \end{array}}$$

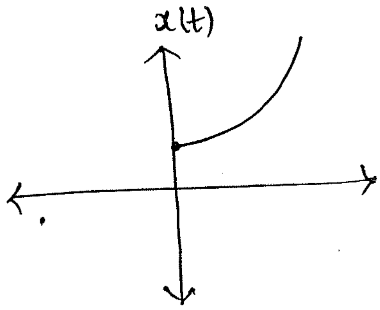
(iii) $x(t) = e^{at} \cdot u(t)$

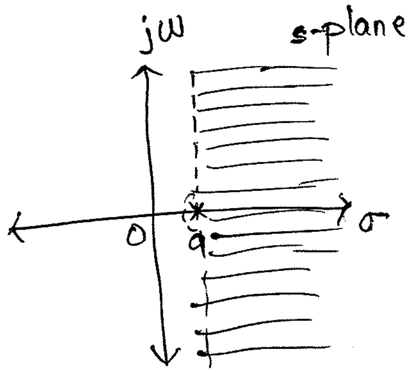
$$X(s) = \int_{-\infty}^{\infty} x(t) \cdot e^{-st} dt$$

$$= \int_{-\infty}^{\infty} e^{at} \cdot u(t) \cdot dt \; e^{-st}$$

$$= \int_{0}^{\infty} e^{-(s-a)t} dt$$

$$= \left[\frac{e^{-(s-a)t}}{-(s-a)} \right]_0^{\infty}$$





$$X(s) = \frac{1}{s-a} \quad ; \quad \sigma - a > 0$$

$$X(s) = \frac{1}{s-a} \quad ; \quad Re\{s\} > a$$

(iv) $x(t) = -e^{-at} \cdot u(-t)$

$$X[s] = \int_{-\infty}^{\infty} x(t) \cdot e^{-st} \cdot dt$$

$$= \int_{-\infty}^{\infty} -e^{-at} \cdot u(-t) \cdot e^{-st} dt$$

$$= \int_{-\infty}^{0} -e^{-at} \cdot e^{-st} dt$$

$$= -\int_{-\infty}^{0} e^{-at} \cdot e^{-st} dt$$

$$= -\left[\frac{e^{-(a+s)t}}{-(a+s)} \right]_{-\infty}^{0}$$

$$= \frac{+1}{(a+s)} - \infty \rightarrow \text{Infinite}$$

For making finite, we find ROC

Again,

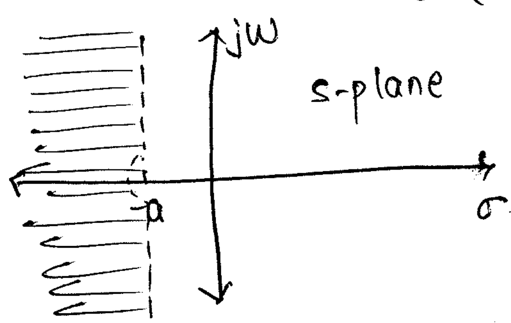
$$X[s] = \left[\frac{e^{-(s+a)t}}{-(s+a)} \right]_0^{-\infty}$$

$$= \left[\frac{e^{-(\sigma + j\omega + a)t}}{-(\sigma + j\omega + a)} \right]_0^{-\infty}$$

For finite, $\sigma + a < 0$

$\Rightarrow \sigma < -a$

$$X[s] = \frac{1}{s+a} \text{ if \& only if } \sigma < -a$$



(1) $e^{-at} \cdot u(t) \xleftrightarrow{L.T.} \dfrac{1}{s+a}$; ROC: $Re(s) > -a \Rightarrow \sigma > -a$

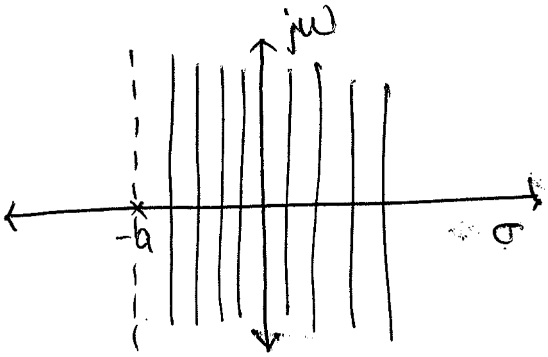
(2) $-e^{-at} u(-t) \xleftrightarrow{L.T.} \dfrac{1}{s+a}$; ROC: $Re(s) < -a \Rightarrow \sigma < -a$

(3) $e^{at} \cdot u(t) \xleftrightarrow{L.T.} \dfrac{1}{s-a}$; ROC: $Re(s) > a \Rightarrow \sigma > a$

(4) $-e^{at} u(-t) \xleftrightarrow{L.T.} \dfrac{1}{s-a}$; ROC: $Re(s) < a \Rightarrow \sigma < a$

ROC: s-plane

(1)



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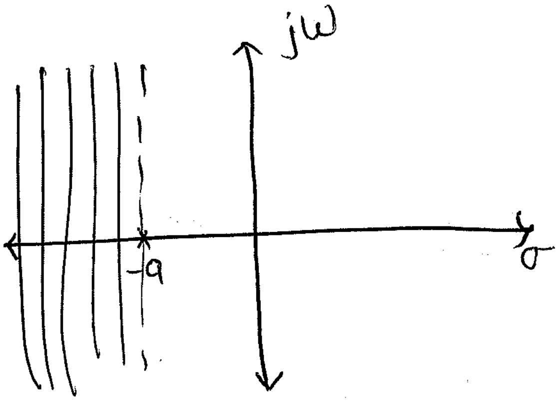

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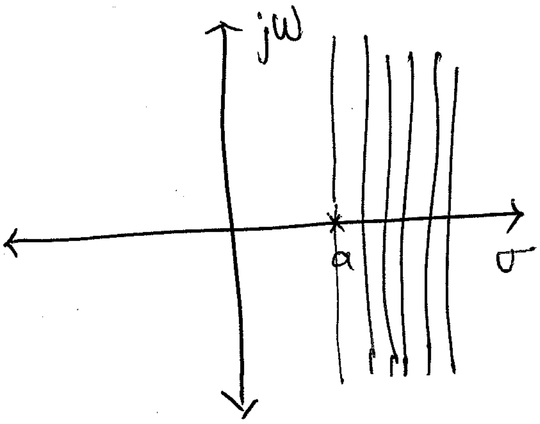

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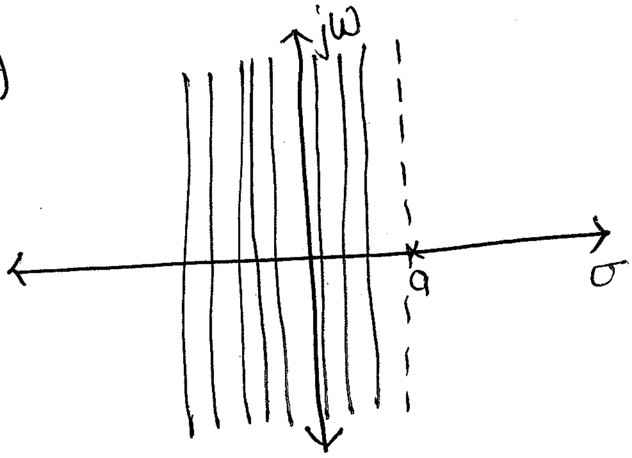


* ROC Observation

Sr.No.	jw	Stability
1)	✓	stable
2)	X	unstable
3)	X	unstable
4)	✓	stable

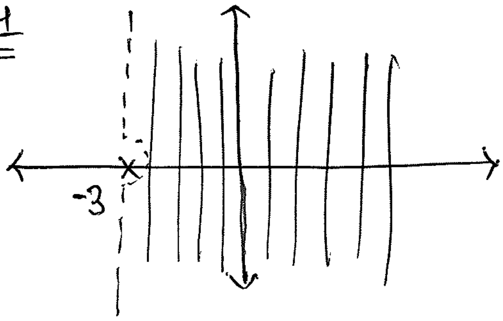
NOTE:- If ROC includes $j\omega$ axis, then $x(t)$ is said to be stable.

Q:- Find L.T. $x(t) = e^{-3t} \cdot u(t) + e^{-2t} \cdot u(t)$

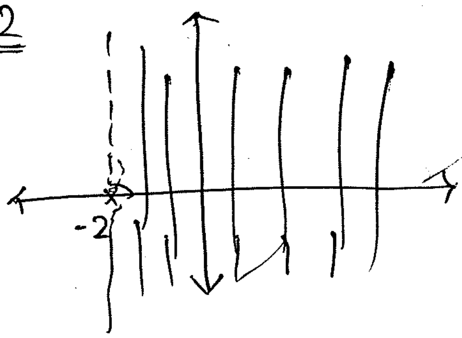
ROC $\sigma > -3$ ROC $\sigma > -2$

$$X[s] = \frac{1}{s+3} + \frac{1}{s+2}$$

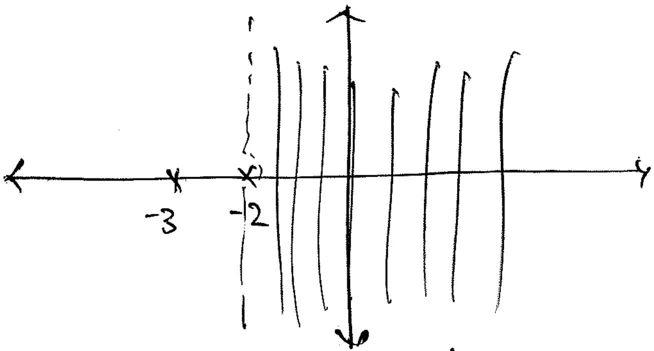
ROC:1



ROC:2



ROC: $ROC_1 \cap ROC_2$



Q:- Find L.T. of $-e^{-at} \cdot u(-t) + e^{-at} \cdot u(t)$

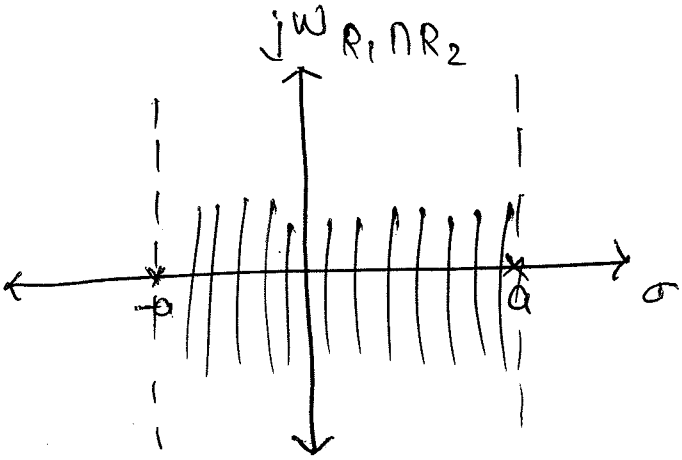
$$X[s] = \frac{1}{s-a} + \frac{1}{s+a}$$

$$X[s] = \frac{2s}{s^2 - a^2}$$

$ROC_1 : \sigma < a$

$ROC_2 : \sigma > -a$

$R_1 \cap R_2 : -a < \sigma < a$



Find L.T.

1) $u(t) \xleftrightarrow{L.T.}$

$e^{-at} \cdot u(t) \xleftrightarrow{L.T.} \frac{1}{s+a}$ ROC : $\sigma > -a$

$\lim_{a \to 0} e^{-at} \cdot u(t) \xleftrightarrow{L.T.} \lim_{a \to 0} \frac{1}{s+a}$; ROC : $\sigma > \lim_{a \to 0} -a$

$$\boxed{u(t) \xleftrightarrow{L.T.} \frac{1}{s} \; ; \text{ROC} : \sigma > 0}$$

2) $u(-t) \xleftrightarrow{L.T.} ?$

-We can do it by taking two standard signals

(1) $-e^{-at} \cdot u(-t) \xleftrightarrow{L.T.} \frac{1}{s+a}$; ROC : $\sigma < -a$

(2) $-e^{at} \cdot u(-t) \xleftrightarrow{L.T.} \frac{1}{s-a}$; ROC : $\sigma < a$

Now,

1) $e^{-at} \cdot u(t) \xleftrightarrow{L.T.} \frac{1}{s+a}$, ROC : $\sigma > -a$

2) $e^{-j\omega t} u(t) \xleftrightarrow{L.T.} \frac{1}{s+j\omega}$, ROC : $Re(s) > 0$
$\sigma > 0$

3) $e^{at} u(t) \xleftrightarrow{L.T.} \frac{1}{s-a}$, ROC : $\sigma > a$

4) $e^{j\omega t} u(t) \xleftrightarrow{L.T.} \frac{1}{s-j\omega}$, ROC : $\sigma > 0$

From above observation, find L.T. of $\cos \omega t \cdot u(t)$?

Now, $\cos \omega t \cdot u(t) = \frac{e^{j\omega t} \cdot u(t) + e^{-j\omega t} \cdot u(t)}{2}$

$= \left(\frac{1}{s-j\omega} + \frac{1}{s+j\omega}\right)\frac{1}{2}$

$$\cos\omega t \cdot u(t) \xleftrightarrow{L.T.} \frac{s}{s^2+\omega^2} \quad ; \text{ ROC : } Re(s) > 0$$

Similarly,

$$\sin\omega t \cdot u(t) = \frac{e^{j\omega t} \cdot u(t) - e^{-j\omega t} \cdot u(t)}{2j}$$

$$= \frac{1}{2j}\left[\frac{1}{s-j\omega} - \frac{1}{s+j\omega}\right]$$

$$\sin\omega t \cdot u(t) = \frac{2j\omega}{s^2+\omega^2} \cdot \frac{1}{2j} = \frac{\omega}{s^2+\omega^2}$$

$$\boxed{\sin\omega t \cdot u(t) \xleftrightarrow{L.T.} \frac{\omega}{s^2+\omega^2} \quad ; \text{ ROC : } \sigma > 0}$$

Properties of ROC:

1] ROC along with algebraic expression of $x[s]$ provides complete specification of laplace transform

2] In case of identical algebraic expression for $X[s]$. The laplace transforms are differentiated only by ROC.

3] ROC of $X[s]$ consists of strips (lines) parallel to $j\omega$ axis in s-plane

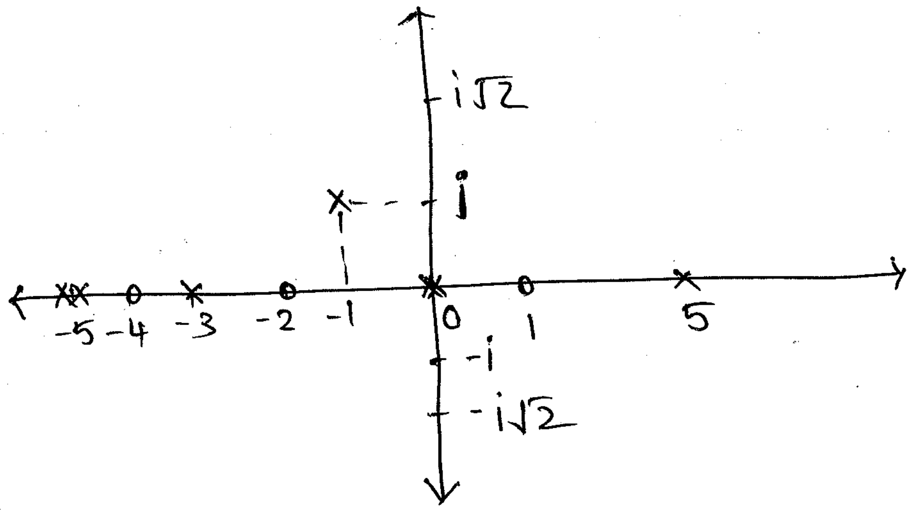
4] If signal $x(t)$ is of finite duration and absolutely integrable (i.e. stable) then ROC is in entire s-plane. Eg:- $\delta(t)$

5] For rational laplace transform, the ROC does not contains any pole.

6] For rational laplace transform, if $x(t)$ is left sided, the ROC is the region in s-plane to left of leftmost pole and if $x(t)$ is right sided The ROC is region in s-plane to right of rightmost pole.

7] If ROC for $X(s)$ includes $j\omega$ axis, than signal is said to be stable and fourier transform of $x(t)$ is possible.

*s-plane, poles and zeros



* s-plane

- Plot between σ(Real) and jω in s-plane

*Zeros

- When $F(s)=0$, we say that points as 0 (zeros)

Consider $F[s] = \dfrac{k\,N[s]}{D[s]} = \dfrac{k(s-T_1)(s-T_2)\cdots}{s^n(s-a)(s-b)\cdots\cdots}$

$\therefore N[s]=0$

$s = T_1, T_2, \ldots\ldots$ are zeros

* Poles

- When $F(s)=\infty$, we say that points as poles

$\therefore D(s)=0$

$s = a, b, c, d, \ldots\ldots$

Eg:- Plot on s-plane

$$F(s) = \frac{k(s-1)(s+2)(s+6)(s^2+2)}{s^2(s+3)(s-5)(s+7)^2(s^2+2s+2)}$$

* Properties of ~~Fourier~~ Laplace transform:-

① Linearity

If $x_1(t) \xleftrightarrow{L.T.} X_1(s)$: ROC : R_1

$x_2(t) \xleftrightarrow{L.T.} X_2(s)$ ROC : R_2

Then,

$$\alpha x_1(t) + \beta x_2(t) \xleftrightarrow{L.T.} \alpha X_1(s) + \beta X_2(s) \text{ ; ROC : } R_1 \cap R_2$$

② Time shifting property

$x(t) \xleftrightarrow{L.T.} X(s)$: ROC : R_1

$x(t-t_0) \xleftrightarrow{L.T.} e^{-st_0} X(s)$: ROC : R_1

③ Shifting s-plane

$x(t) \xleftrightarrow{L.T.} X(s)$: ROC : R_1

$e^{s_0 t} x(t) \xleftrightarrow{L.T.} X(s-s_0)$: ROC : $R_1 + Re(s_0)$

④ Time scaling

If $x(t) \xleftrightarrow{L.T.} X(s)$: ROC : R_1

then $x(at) \xleftrightarrow{L.T.} \frac{1}{|a|} X\left(\frac{s}{a}\right)$: ROC : $|a| R_1$

⑤ Time Reversal

$Re(s) > 0$

If $x(t) \xleftrightarrow{L.T.} X(s)$; ROC : R_1

$x(-t) \xleftrightarrow{L.T.} X(-s)$; ROC : $-R_1$

$Re(s) < 0$

(change inequality, change sign)

⑥ Differentiation in time

If $\frac{d}{dt} x(t) \xleftrightarrow{L.T.} sX(s)$; ROC: R_1

(remain unchanged)

⑦ Integration in time

If $x(t) \xleftrightarrow{L.T.} X(s)$; ROC: R_1

$$\int_{-\infty}^{t} x(t)\,dt \longleftrightarrow \frac{X(s)}{s} \quad , \text{ROC}: R_1$$

⑧ Convolution in time domain

If $x_1(t) \longleftrightarrow X_1(s)$; $R: R_1$

$x_2(t) \longleftrightarrow X_2(s)$; $R: R_2$

Then $x_1(t) * x_2(t) \xleftrightarrow{L.T.} X_1[s]\, X_2[s]$; $R: R_1 \cap R_2$

⑨ Multiplication in time domain

$$x_1(t) \cdot x_2(t) = \frac{1}{2\pi f}\left[X_1(s) * X_2(s)\right] \quad ; R: R_1 \cap R_2$$

⑩ Differentiation in s-domain

$$-t \cdot x(t) \longleftrightarrow \frac{d\,X[s]}{ds}$$

OR

$$t\,x(t) \longleftrightarrow -\frac{dX[s]}{ds}$$

General:

$$(-1)^n \cdot t\,x(t) \xleftrightarrow{L.T.} \frac{d^n X(s)}{ds^n} \quad ; \text{ROC: unchanged}$$

(11) Integration in s-domain

$$\frac{x(t)}{t} \xleftrightarrow{L.T.} \int_s^{\infty} X(s)\, ds \quad : \text{ROC : unchanged}$$

(12) Conjugate property

$$x^*(t) \xleftrightarrow{L.T.} X^*(s^*)$$

Q:- Find L.T. of following signals with ROC:

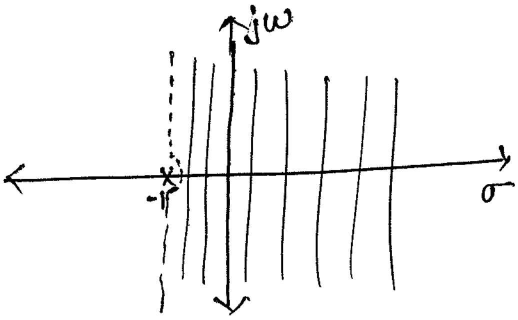
NOTE :-

ROC	
$u(t) \rightarrow$ Right	$>$
$-u(-t) \rightarrow$ Left	$<$

(1) $x_1(t) = e^{-t} \cdot u(t) + e^{-3t} \cdot u(t)$

$$X_1(s) = \frac{1}{s+1} + \frac{1}{s+3} = \frac{2s+4}{(s+1)(s+3)}$$

$\sigma > -1 \qquad \sigma > -3$



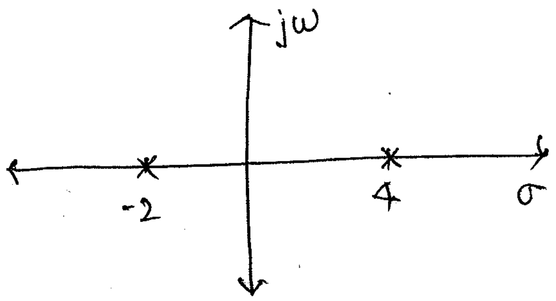
ROC: $\sigma > -1$

Stable (It covers $j\omega$ axis)

(2) $x_2(t) = e^{-2t} \cdot u(t) + e^{4t} \cdot u(-t)$

$$X(s) = \frac{1}{s+2} - \frac{1}{s-4}$$

$\sigma > -2 \qquad \sigma < 4$

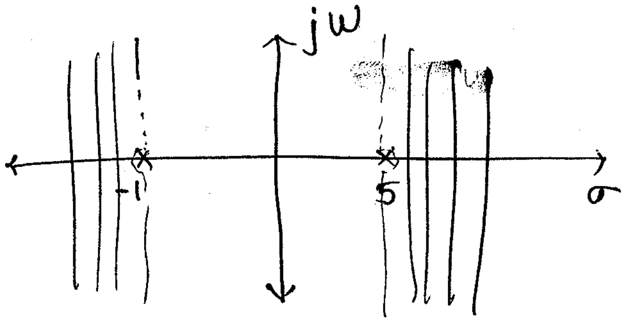


ROC: $-2 < \sigma < 4$

(3) $x_3(t) = e^{-t}u(-t) + e^{5t}u(t)$

$$X_3(s) = \frac{-1}{s+1} + \frac{1}{s-5}$$

$\sigma < -1 \qquad \sigma > 5$



No common ROC

L.T. doesn't exists.

(4) $x_4(t) = 1 \quad , \forall t$

$$X_4(s) = u(t) + u(-t)$$

$$= \frac{1}{s} - \frac{1}{s}$$

$\downarrow \qquad \downarrow$

$\sigma > 0 \qquad \sigma < 0$

No common ROC . L.T. doesn't exists

(5) $x_5(t) = sgn(t)$

$$x_5(t) = u(t) - u(-t)$$

$$= 1/s + 1/s$$

$\sigma > 0 \ , \ \sigma < 0$

No common ROC . L.T. doesn't exists

(6) Consider the signal $x(t) = e^{-5t} \cdot u(t) + e^{-\beta t} \cdot u(t)$ and its laplace transform is $X(s)$. What are the constraints placed on real and imaginary parts of β if ROC of $X(s)$ is real part of $s > -3$.

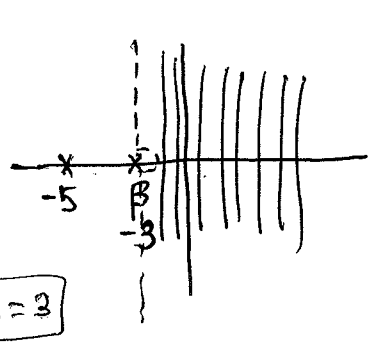
$$x(t) = e^{-5t} \cdot u(t) + e^{-\beta t} \cdot u(t)$$

$$\frac{1}{s+5} + \frac{1}{s+\beta}$$

$\sigma > -5 \qquad \sigma > -\beta$

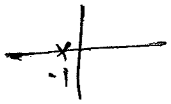
$Re(\beta) < 3$

Imag. any

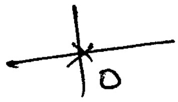


$\boxed{\beta = 3}$

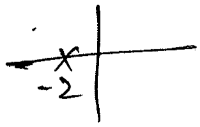
If $\beta = 1$ then $\frac{1}{s+1}$



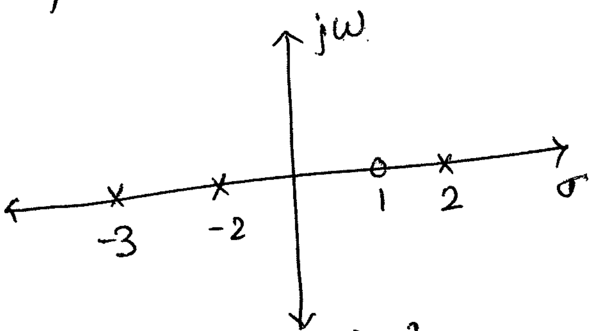
If $\beta = 0$ then $\frac{1}{s}$



If $\beta = 2$ $\frac{1}{s+2}$



(7) How many possible ROC are there for the pole zero plot shown in figure?



$$X(s) = \frac{(s-1)}{(s+3)(s+2)(s-2)}$$

$$X(s) = \frac{-4/5}{(s+3)} + \frac{3/4}{(s+2)} + \frac{1/20}{(s-2)}$$

$$x(t) = -\frac{4}{5}e^{-3t}\cdot u(t) + \frac{3}{4}e^{-2t}\cdot u(t) + \frac{1}{20}e^{2t}\cdot u(t)$$

		-3	-2	2
1]	$\sigma > 2$	(R	R	R)
2]	$\sigma < -2$	(L	L	L)
3]	$-3 < \sigma < -2$	(R	L	L)
4]	$-2 < \sigma < 2$	(R	R	L)

For case:-1 $\sigma > 2$

$$x(t) = -\frac{4}{5}e^{-3t}\cdot u(t) + \frac{3}{4}e^{-2t}\cdot u(t) + \frac{1}{20}e^{2t}u(t)$$

For case: 2 $\sigma < -2$

$$x(t) = \frac{4}{5}e^{-3t}\cdot u(-t) - \frac{3}{4}e^{-2t}\cdot u(-t) - \frac{1}{20}e^{2t}u(-t)$$

For case: 3 $-3 < \sigma < -2$

$$x(t) = -\frac{4}{5}e^{-3t}\cdot u(t) - \frac{3}{4}e^{-2t}\cdot u(-t) - \frac{1}{20}e^{2t}\cdot u(-t)$$

For case: 4 $-2 < \sigma < 2$

$$x(t) = -4/5\, e^{-3t}\cdot u(t) + 3/4\, e^{-2t} u(t) - 1/20\, e^{2t}\cdot u(-t)$$

(8) In what range should Re(s) remains so that laplace transform of the function $x(t) = e^{(a+2)t+5} \cdot u(t)$ exists.

$$x(t) = e^{(a+2)t+5} \cdot u(t)$$

$$= e^{5} \cdot e^{(a+2)t} \cdot u(t)$$

$$= \frac{e^{5}}{s-(a+2)}$$

$$\sigma > a+2$$

$$Re(s) > (a+2)$$

OR

$$X(s) = \int_{-\infty}^{\infty} e^{(a+2)t} \cdot e^{5} \cdot e^{-st} u(t)\, dt$$

$$= e^{5} \int_{0}^{\infty} e^{(a+2)t} \cdot e^{-\sigma t} \cdot e^{-j\omega t} dt$$

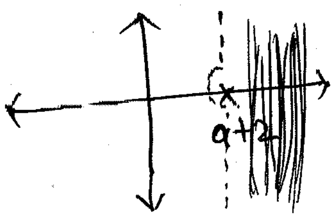
$$= e^{5} \int_{0}^{\infty} e^{-(\sigma-(a+2))t} \cdot e^{-j\omega t} dt$$

$$\sigma-(a+2) > 0$$

$$Re(s) > (a+2)$$

OR

$$e^{5} \cdot e^{(a+2)t} u(t) = \frac{e^{5} \cdot 1}{s-(a+2)}$$



(9) Find L.T. $x(t) = t \cdot u(t)$

$$\updownarrow$$

$$X(s) = \frac{1}{s^{2}}$$

OR $t\,x(t) \xleftrightarrow{L.T.} -\frac{dX(s)}{ds}$ [∵ Property]

Now, $x(t) = t \cdot u(t)$

$$= -\frac{d}{ds}\left(\frac{1}{s}\right)$$

$$= 1/s^{2}$$

$$x(t) = t^{2} u(t)$$

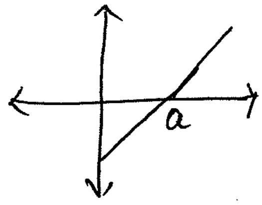
$$= (-1)^{2} \frac{d^{2}}{ds^{2}}\left(\frac{1}{s}\right)$$

$$= \frac{d}{ds}\left(\frac{-1}{s^{2}}\right)$$

$$x(t) = 2/s^{3}$$

$$\boxed{x(t) = t^n \cdot u(t) \xleftrightarrow{L.T.} X(s) = \frac{n!}{s^{n+1}}}$$

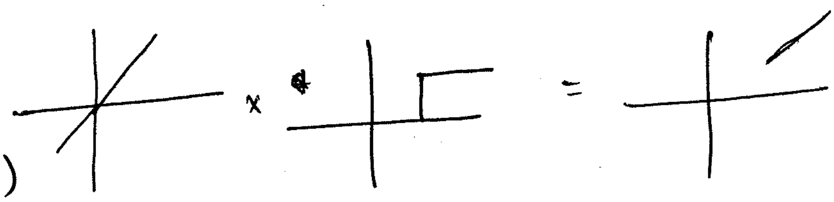
(10) $x(t) = (t-a)\,u(t)$



$= t \cdot u(t) - a u(t)$

$= \frac{1}{s^2} - \frac{a}{s} = \frac{1-as}{s^2}$

(11) $x(t) = t \cdot u(t-a)$



<u>Method: 1</u>

$x(t) = (t-a+a)\,u(t-a)$

$x(t) = (t-a)\,u(t-a) + a\,u(t-a)$

$\updownarrow$ L.T.

$X[s] = e^{-as} L\{t \cdot u(t)\} + a \cdot e^{-as} L\{u(t)\}$

$X[s] = \frac{e^{-as}}{s^2} + \frac{a \cdot e^{-as}}{s}$

<u>Method: 2</u>

$x(t) = t \cdot u(t-a)$

$t - a = \tau$

$t = \tau + a$

$x(\tau + a) = (\tau + a)\,u(\tau)$

$x(\tau + a) = \tau \cdot u(\tau) + a u(\tau)$

$\tau = t$

$x(t+a) = t \cdot u(t) + a u(t)$

$\updownarrow$ L.T.

$e^{as} X(s) = 1/s^2 + a/s$

$X(s) = \frac{e^{-as}}{s^2} + \frac{a e^{-as}}{s}$

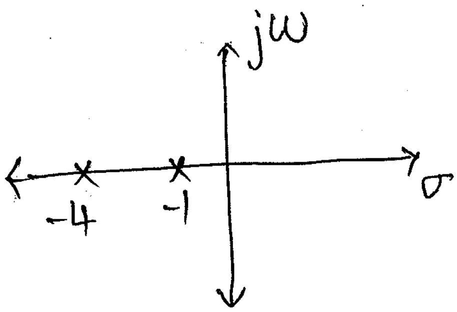
(12) Find inverse laplace with all possible ROC. Find $x(t)$

(1) If $x(t)$ is right sided

(2) If $x(t)$ is left sided

(3) If $x(t)$ is both sided

$$X(s) = \frac{1}{(s+1)(s+4)}$$



Sol:-
$$X(s) = \frac{1}{(s+1)(s+4)}$$

$$X(s) = \frac{A}{s+1} + \frac{B}{s+4}$$

$$X(s) = \frac{1/3}{(s+1)} + \frac{-1/3}{(s+4)}$$

$$X(s) = \frac{1}{3(s+1)} - \frac{1}{3(s+4)}$$

$$x(t) = \frac{1}{3} e^{-t} \cdot u(t) - \frac{1}{3} e^{-4t} \cdot u(t)$$

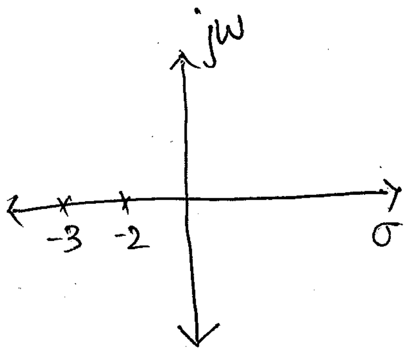
(1) $x(t) = \frac{1}{3} e^{-t} \cdot u(t) - \frac{1}{3} e^{-4t} \cdot u(t)$

(2) $x(t) = -\frac{1}{3} e^{-t} u(-t) + \frac{1}{3} e^{-4t} u(-t)$

(3) $x(t) = -\frac{1}{3} e^{-t} \cdot u(-t) - \frac{1}{3} e^{-4t} u(t)$ $\quad -4 < \sigma < -1$

(13) $X(s) = \dfrac{2s+5}{s^2+5s+6}$

$$X(s) = \frac{2s+5}{(s+2)(s+3)}$$



$$X(s) = \frac{1}{s+2} + \frac{1}{s+3}$$

$$x(t) = e^{-2t} \cdot u(t) + e^{-3t} \cdot u(t)$$

(1) $\sigma > -2$

$$x(t) = e^{-2t} \cdot u(t) + e^{-3t} \cdot u(t)$$

(2) $\sigma < -3$

$$x(t) = -e^{-2t} \cdot u(-t) - e^{-3t} \cdot u(-t)$$

(3) $-3 < \sigma < -2$

$$x(t) = e^{-3t} u(t) - e^{-2t} \cdot u(-t)$$

(14) $x(t) = u(t-5)$

$$\updownarrow \text{L.T.}$$

$$X(s) = e^{-5s} L\{u(t)\}$$

$$X(s) = \frac{e^{-5s}}{s}$$

(15) $x(t) = e^{5t} \cdot u(-t+3)$

$$= e^{5t} \cdot u(-(t-3))$$

$$L\left\{e^{5t}\left[e^{-3s} \cdot \frac{1}{|-1|}\left(-\frac{1}{s}\right)\right]\right\}$$

$$= \frac{e^{-3(s-5)}}{-(s-5)}$$

$$x(at+b) \xleftrightarrow{\text{L.T.}}$$

$$x(a(t \pm b/a)) \xleftrightarrow{\text{L.T.}} e^{\pm b/a\, s} \cdot \frac{1}{|a|} X(s/a)$$

ROC: $a\sigma$

$$x(t) \cdot e^{-at} \xleftrightarrow{\text{L.T.}} X[s+a]$$

(16) Consider a signal $x(t) = e^{-5t} \cdot u(t-1)$

(i) Find L.T. of $x(t)$

$$x(t) = e^{-5t} \cdot u(t-1)$$

$$X(s) = e^{-s} \cdot L\{e^{-5t} \cdot u(t)\}$$

$$X(s) = e^{-s} \cdot \frac{1}{s}\Big|_{s \Rightarrow s+5}$$

$$X(s) = \frac{e^{-(s+5)}}{s+5} \qquad \text{ROC: } \sigma > -5 \text{ (Right sided signal)}$$

(ii) Find the values of a and to such that g(s) of $g(t) = A \cdot e^{-5t} \cdot u(-t-t_0)$ has same algebraic form as $x(s)$. What is the ROC of corresponding to g(s)?

$$g(t) = A \cdot e^{-5t} \cdot u(-t-t_0)$$

$$= A e^{-5t} u(-1(t+t_0))$$

$$= A e^{-5t} L\left\{e^{+t_0 s}\left(-\frac{1}{s}\right)\right\}$$

$$g(s) = \frac{A(-1)e^{+(s+5)t_0}}{(s+5)}$$

Now, $g(s) \longrightarrow X(s)$ algebraic form

$$t_0 = -1 \quad , \quad A = -1$$

$$\text{ROC: } \sigma < -5$$

<u>ROC</u>

Right $u(t) \longrightarrow \sigma >$ Pole location

Left $u(t) \longrightarrow \sigma <$ Pole location

(17) Find inverse L.T. $Y[s] = \dfrac{e^{-3s}}{(s+1)(s+2)}$, $\sigma > -1$

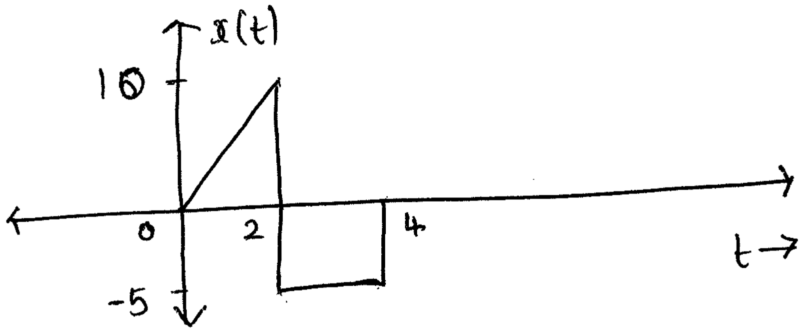
Sol:- $\hat{Y}(s) = \dfrac{e^{-3s}}{s+1} - \dfrac{e^{-3s}}{s+2}$

$\Big\downarrow$ I.L.T.

$$y(t) = e^{-t}\cdot u(t)\Big|_{t\to t-3} - e^{-2t}u(t)\Big|_{t\to t-3}$$

$$y(t) = e^{-(t-3)}u(t-3) - e^{-2(t-3)}u(t-3)$$

(18) Find the L.T. as shown in figure:-



$$x(t) = \frac{10}{2}r(t) - \frac{10}{2}r(t-2) - 15u(t-2) + 5u(t-4)$$

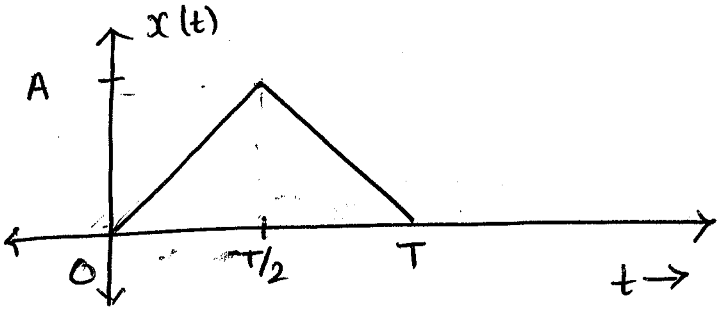
$$x(t) = 5r(t) - 5r(t-2) - 15u(t-2) + 5u(t-4)$$

$\Big\downarrow$ L.T.

$$X(s) = \frac{5\cdot 1}{s^2} - \frac{5e^{-2s}}{s^2} - \frac{15e^{-2s}}{s} + \frac{5e^{-4s}}{s}$$

NOTE:-

$\boxed{\text{For finite duration signal, ROC is entire s-plane}}$



$$x(t) = \frac{2A}{T} r(t) - \frac{4A}{T} r(t - T/2) + \frac{2A}{T} r(t-T)$$

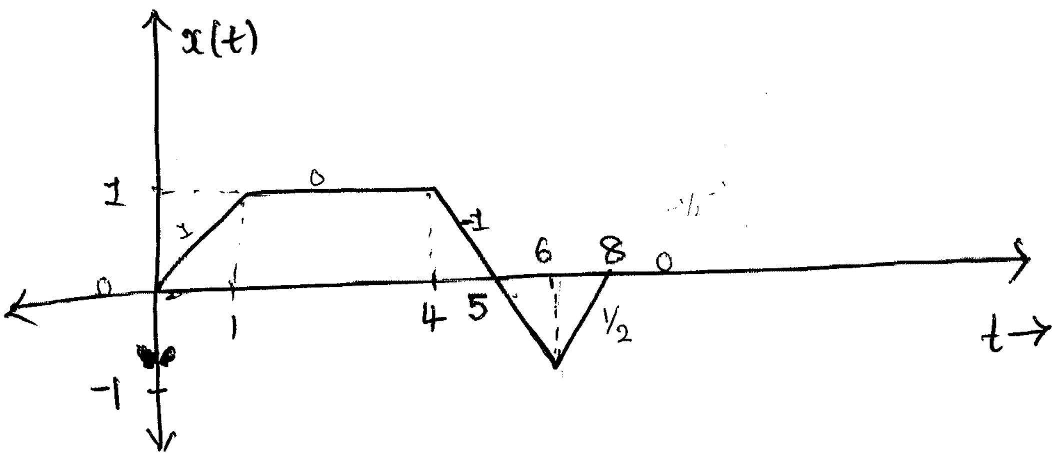
$\updownarrow$ L.T.

$$X(s) = \frac{2A}{T} \cdot \frac{1}{s^2} - \frac{4A}{T} \cdot \frac{e^{-T/2\, s}}{s^2} + \frac{2A}{T} \frac{e^{-sT}}{s^2}$$

ROC: entire s-plane

(19) Find A, B, C, D. L.T. of waveform shown in figure is

$$X(s) = \frac{1}{s^2}\left[1 + Ae^{-s} + Be^{-4s} + Ce^{-6s} + De^{-8s}\right]$$



$$x(t) = 1 \cdot r(t) - r(t-1) - 1\, r(t-4) + \frac{3}{2} r(t-6) - \frac{1}{2} r(t-8)$$

$A = -1$, $B = -1$, $C = 3/2$, $D = -1/2$

(20) Find L.T. of $x(t) = t \cdot e^{-3t} \cdot u(t)$

$$x(t) = t \cdot e^{-3t} \cdot u(t)$$

$$X(s) = L\left\{ t \cdot \left(\frac{1}{s+3} \right) \right\}$$

$$= (-1) \frac{d}{ds} \left(\frac{1}{s+3} \right)$$

$$= -1 \; \frac{0 - 1(1)}{(s+3)^2}$$

$$X(s) = \frac{1}{(s+3)^2}$$

(21) $X(t) = e^{-at} \sin \omega_0 t \; u(t)$

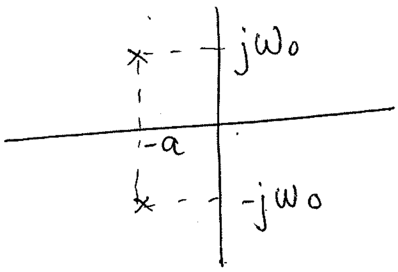
$$\cancel{L\{u(t)\} = 1/s}$$

$$L\{\sin \omega_0 t \overset{u(t)}{}\} = \frac{\omega_0}{s^2 + \omega_0^2}$$

$$L\{e^{-at} \cdot u(t) \sin \omega_0 t\} = \frac{\omega_0}{(s+a)^2 + \omega_0^2}$$

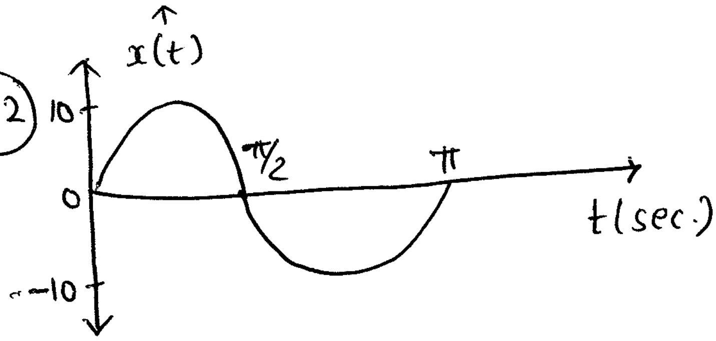
$s = -a \pm j\omega_0$

ROC:



ROC: $Re(s) > -a$
$\sigma > -a$

(22)



$$x(t) = A \sin \omega t$$

$$x(t) = 10 \sin \frac{2\pi}{T} \cdot t \qquad = 10 \sin 2t \; (u(t) - u(t - \pi))$$

$$x(t) = 10\sin 2t\, u(t) - 10\sin 2t\, u(t-\pi)$$

$$\Big\updownarrow \text{L.T.}$$

$$X(s) = \frac{10 \cdot 2}{s^2+4} - \frac{10 \cdot e^{-\pi s} \cdot 2}{s^2+4}$$

$$X(s) = \frac{20}{s^2+4}\left[1 - e^{-\pi s}\right]$$

Method:2

$$x(t) = 10\sin 2t \cdot u(t) - 10\sin 2t \cdot u(t-\pi)$$

$$= 10\sin 2t \cdot u(t) - 10\sin(2(t-\pi) + 2\pi)\, u(t-\pi)$$

$$= 10\sin 2t \cdot u(t) - 10\sin(2(t-\pi))\, u(t-\pi)$$

$$X(s) = 10 \cdot \frac{2}{s^2+4} - 10 \cdot \frac{2}{s^2+4} \cdot e^{-\pi s}$$

$$X(s) = \frac{20}{s^2+4}\left[1 - e^{-\pi s}\right]$$

(23) Let $x(t)$ be a signal that has a rational laplace transform with exactly 2 poles located at $s=-1$ & $s=-3$. If $g(t) = e^{2t} \cdot x(t)$ and $g(w)$ converges determine whether $g(t)$ is

(a) Right sided
(b) Left sided
(c) Both sided
(d) Finite duration

Sol:-

$$\cancel{x(t) = \frac{A}{(t+1)(t+3)}}$$

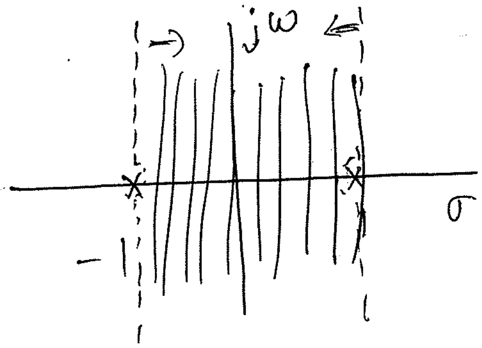
$$X(s) = \frac{A}{(s+1)(s+3)}$$

$$\updownarrow$$

$$g(t) = e^{2t} x(t)$$

$$\Big\updownarrow \text{L.T.}$$

$$G(s) = X[s-2] = \frac{A}{(s-1)(s+1)}$$



Signal will converges when signal having pole -1 is Right sided and 1 pole is ~~left~~ sided.

$\therefore$ Both sided

$-1 < \sigma < 1$

(24) Let $g(t) = x(t) + \alpha x(-t)$ and $x(t) = \beta e^{-t} \cdot u(t)$ and

$G(s) = \dfrac{s}{s^2-1}$; $-1 < Re\{s\} < 1$. Find α & β.

Sol^n:-

$$g(t) = \beta e^{-t} \cdot u(t) + \alpha \beta e^{t} \cdot u(-t)$$

$\Big\updownarrow$ L.T.

$$G(s) = \underset{\sigma > -1}{\frac{\beta}{s+1}} + \underset{\sigma < 1}{\frac{\alpha \beta(-1)}{(s-1)}} \quad ; \text{ROC: } -1 < \sigma < 1$$

$$= \frac{(s-1)\beta + -(\alpha\beta)(s+1)}{s^2-1}$$

$$= \frac{s\beta - \beta - \alpha\beta s - \alpha\beta}{s^2-1}$$

$$G(s) = \frac{\beta s(1-\alpha) - \beta(1+\alpha)}{s^2-1}$$

$\boxed{\alpha = -1}$, $\boxed{\beta = 1/2}$

(25) $Y[s] = \dfrac{s^2-s+1}{(s+1)^2}$; $\sigma > -1$. Find $y(t)$

$$\frac{s^2-s+1}{(s+1)^2} = \frac{A}{s+1} + \frac{B}{(s+1)^2}$$

$$s^2-s+1 = A(s^2+2s+1) + B(s+1) + \cancel{C(s+1)}$$

$$1 = A + B \quad ; \quad -1 = 2A + B + C \quad ; \cancel{1 = A + C}$$

$$Y(s) = \frac{s^2+2s+1-3s}{s^2+2s+1}$$

$$Y(s) = 1 - \frac{3s+3-3}{(s+1)^2}$$

$$Y(s) = 1 - \frac{3}{(s+1)} + \frac{3}{(s+1)^2}$$

I.L.T.

$$y(t) = \delta(t) - 3e^{-t}u(t) + 3e^{-t}t\,u(t)$$

(26) Find $X(s)$ & $Y(s)$ with ROC:

$$\frac{dx(t)}{dt} = -2y(t) + \delta(t)$$

$$\frac{dy(t)}{dt} = 2x(t)$$

$$\frac{dx(t)}{dt} = -2y(t) + \delta(t)$$

$\Big\updownarrow$ L.T.

$$sX(s) = -2Y(s) + 1$$

$$s \cdot \frac{sY(s)}{2} = -2Y(s) + 1$$

$$s^2 Y(s) = -4Y(s) + 2$$

$$Y(s) = \frac{2}{s^2+4}$$

$\Big\updownarrow$ L.T.

$$y(t) = \cancel{2} \sin 2t \cdot u(t)$$

$$\frac{dy(t)}{dt} = 2x(t)$$

$\Big\updownarrow$ L.T.

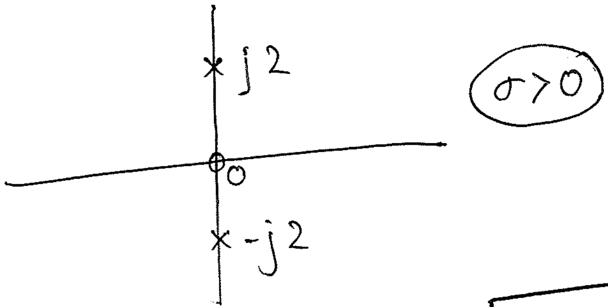
$$sY(s) = 2X(s)$$

$$s\,\cancel{\sigma}\,\frac{\cancel{2}}{s^2+4} = \cancel{2}X(s)$$

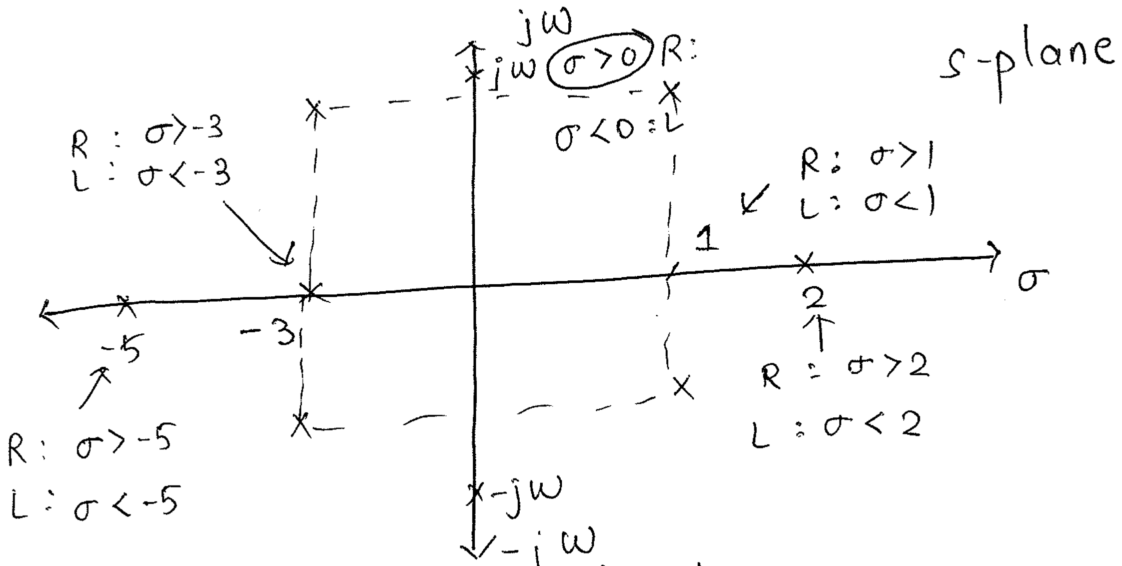
$$X(s) = \frac{\cancel{2}s}{s^2+4}$$

$\Big\updownarrow$ L.T.

$$x(t) = \cos 2t \cdot u(t)$$



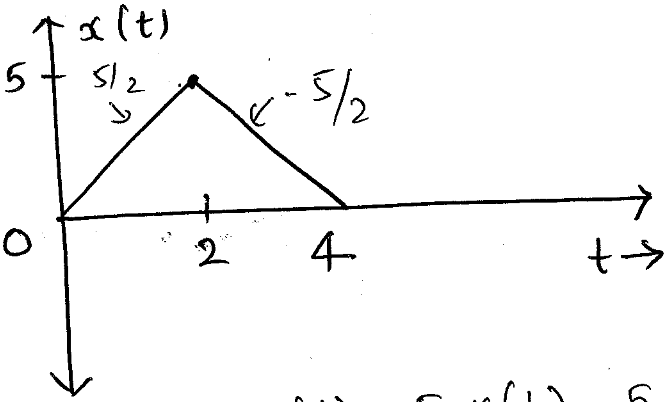
$\boxed{\text{NOTE}}$



Right : ROC : Re(s) > Real part of pole

Left : ROC : Re(s) < Real part of pole

(27) By using derivative signal find L.T. of following signals.



Sol^n:

$$x(t) = \frac{5}{2} r(t) - \frac{5}{\cancel{2}} r(t-2) + \frac{5}{2} r(t-4)$$

$$\frac{dx(t)}{dt} = \frac{5}{2} u(t) - \frac{5}{\cancel{2}} u(t-2) + \frac{5}{2} u(t-4)$$

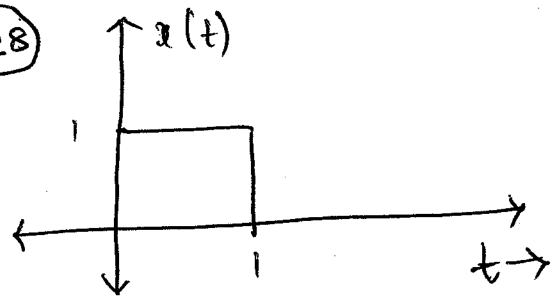
$\updownarrow$ L.T.

$$sX(s) = \frac{5}{2s} - \frac{5e^{-2s}}{\cancel{2}s} + \frac{5e^{-4s}}{2s}$$

$$X(s) = \frac{5}{2s^2} - \frac{5e^{-2s}}{\cancel{2}s^2} + \frac{5e^{-4s}}{2s^2}$$

ROC: Entire s-plane

(28)



$$x(t) = u(t) - u(t-1)$$

$\updownarrow$ L.T.

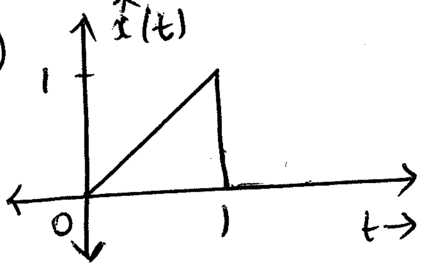
$$X(s) = \frac{1}{s} - \frac{e^{-s}}{s}$$

ROC:

$$\lim_{s\to 0} \frac{1}{s} - \frac{e^{-s}}{s}$$

$$\lim_{s\to 0} \frac{1-e^{-s}}{s} = \underline{\underline{1}}$$ entire s-plane

(29)



$$x(t) = \cancel{r(t)}\; r(t) - r(t-1) \cancel{-u(t-1)} - u(t-1)$$

$$x(t) = 1/s^2 - \frac{e^{-s}}{s^2} \cancel{\frac{e^{-s}}{s}} - \frac{e^{-s}}{s}$$

$$\lim_{s\to 0} \frac{1 - e^{-s} - se^{-s}}{s^2} \quad \left(\frac{0}{0}\right)$$

$$\lim_{s\to 0} \frac{+e^{-s} + se^{-s} - e^{-s}}{2s} \quad \left(\frac{0}{0}\right)$$

$$\lim_{s\to 0} \frac{-\cancel{e^{-s}} + e^{-s} + \cancel{e^{-s}} - se^{-s}}{2} = 1/2$$ entire s-plane: ROC

Method:2 $x(t) = r(t)\,[u(t) - u(t-1)]$

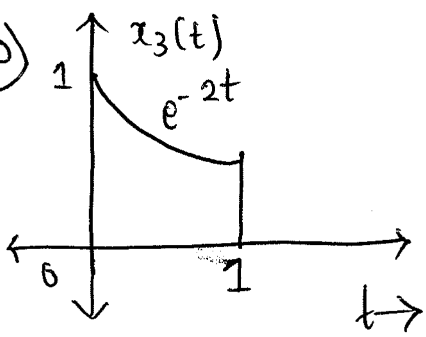
$$x(t) = t\cdot u(t)\,[u(t) - u(t-1)]$$

$$x(t) = t\,u(t) - t\cdot u(t)\cdot u(t-1)$$

$$x(t) = t\,u(t) - ((t-1)+1)\,u(t-1)$$

$$x(t) = t\,u(t) - (t-1)\,u(t-1) - u(t-1)$$

(30)



$$x_3(t) = e^{-2t}\cdot[u(t) - u(t-1)]$$

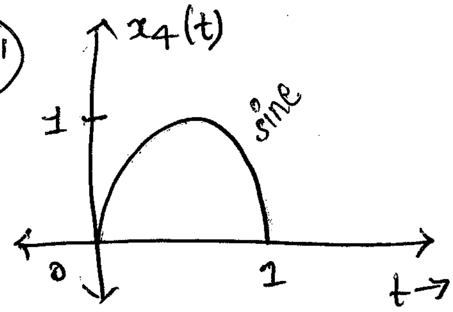
$$= e^{-2t}\cdot u(t) - e^{-2t}\cdot u(t-1)$$

$$x_3(t) = e^{-2t}\cdot u(t) - e^{-2(t-1)}\cdot e^{-2}\cdot u(t-1)$$

$\updownarrow$ L.T.

$$X_3(s) = 1/(s+2) - \frac{e^{-2}\cdot e^{-s}}{(s+2)}$$

(31)



$T=2$

$x(t) = A \sin \omega t$

$x(t) = 1 \cdot \sin \frac{2\pi}{2} \cdot t$

$x(t) = \sin \pi t$

$x_4(t) = \sin \pi t \, (u(t) - u(t-1))$

$= \sin \pi t \cdot u(t) - u(t-1) \sin \pi t$

$= \sin \pi t \cdot u(t) - u(t-1) \sin(\pi(t-1) + \pi)$

$x_4(t) = \sin \pi t \, u(t) + u(t-1) \sin \pi (t-1)$

$\updownarrow$ L.T.

$$X_4(s) = \frac{\pi}{s^2+\pi^2} + \frac{e^{-s} \cdot \pi}{s^2+\pi^2}$$

(32) Find I.L.T. of $X(s) = \log\left[\frac{s+5}{s+6}\right]$

$X(s) = \log\left(\frac{s+5}{s+6}\right)$

$\cancel{= \log 1 - \log\left(\frac{-1}{s+6}\right)}$

Hint: $t^n x(t) \longleftrightarrow (-1)^n \frac{d^n X(s)}{ds^n}$

$$\frac{dX(s)}{ds} = \frac{s+6}{s+5}\left[\frac{(s+6)-(s+5)}{(s+6)^2}\right]$$

$$\frac{dX(s)}{ds} = \frac{\cancel{s+6}}{s+5}\left[\frac{1}{(s+6)^{\cancel{2}}}\right]$$

$$\frac{dX(s)}{ds} = \frac{1}{(s+5)(s+6)}$$

$$\frac{dX(s)}{ds} = \frac{1}{s+5} - \frac{1}{s+6}$$

$$\Big\updownarrow \text{I. L.T.}$$

$$(-1)\, t \cdot x(t) = e^{-5t} \cdot u(t) - e^{-6t} \cdot u(t)$$

$$x(t) = \frac{e^{-6t} \cdot u(t) - e^{-5t} \cdot u(t)}{t}$$

$$\Big\updownarrow \text{L.T.}$$

$$\log\left(\frac{s+5}{s+6}\right)$$

$$\boxed{\frac{e^{-\alpha t} \cdot u(t) - e^{-\beta t} u(t)}{t} \xleftrightarrow{\text{L.T.}} \log\left(\frac{s+\beta}{s+\alpha}\right)}$$

$$\alpha = 0, \beta = 0 \longleftrightarrow 0$$

Inverse Laplace Transform

$$X[s] = \frac{k \cdot N[s]}{D[s]}$$

(1) Simple Pole:-

$$X[s] = \frac{2(s+1)}{(s+2)(s+3)(s+4)}$$

$$X[s] = \frac{A}{s+2} + \frac{B}{s+3} + \frac{C}{s+4}$$

$$x(t) = A e^{-2t} \cdot u(t) + B e^{-3t} \cdot u(t) + C e^{-4t} \cdot u(t)$$

(2) Repeated real pole:-

$$X[s] = \frac{2(s+1)}{(s+2)^3(s+3)(s+4)}$$

$$= \frac{A}{s+3} + \frac{B}{s+4} + \frac{C}{s+2} + \frac{D}{(s+2)^2} + \frac{E}{(s+2)^3}$$

$$x(t) = Ae^{-3t}u(t) + Be^{-4t}u(t) + Ce^{-2t}\cdot u(t) + De^{-2t}\cdot t\,u(t) + De^{-2t}\cdot \frac{t^2}{2}u(t)$$

(3) Complex Pole

$$X[s] = \frac{2(s+1)}{s^2+2s+3}$$

$$= \frac{2(s+1)}{s^2+2s+1+2}$$

$$= \frac{2(s+1)}{(s+1)^2+(2j)^2}$$

$$x(t) = 2e^{-t}\cos\sqrt{2}t \cdot u(t)$$

$$\boxed{x(t)\left[\frac{e^{\alpha t}-e^{\beta t}}{t}\right]u(t) \xleftrightarrow{L.T.} X(s)\cdot \log\left[\frac{s-\beta}{s-\alpha}\right]}$$

Q:- Find inverse laplace transform of $X(s) = \frac{4}{(s+2)(s+1)^3}$

$$X(s) = \frac{4}{(s+2)(s+1)^3} = \frac{A}{s+2} + \frac{B}{(s+1)} + \frac{C}{(s+1)^2} + \frac{D}{(s+1)^3}$$

$$4 = A(s+1)^3 + B(s+2)(s+1)^2 + C(s+2)(s+1) + D(s+2)$$

At $s=-2$

$\boxed{A=-4}$

At $s=-1$

$4 = D(1)$

$\boxed{D=4}$

At $s=0$

$4 = A+2B+2C+2D$

$4 = -4+2(B+C)+8$

$$2(B+C)=0$$

$$B+C=0$$

Comparing co-efficient of s^3

$$0=A+B$$

$$B=-A=4$$

$$\boxed{B=4} \qquad \boxed{C=-4}$$

$$X(s)=\frac{-4}{s+2}+\frac{4}{(s+1)}-\frac{4}{(s+1)^2}+\frac{4}{(s+1)^3}$$

$$x(t)=\left(-4e^{-2t}+4e^{-t}-4.te^{-t}+\frac{4t^2e^{-t}}{2}\right)u(t)$$

Q:- $X[s]=e^{-2s}\frac{d}{ds}\left(\frac{1}{(s+1)^2}\right)$

$$X[s]=e^{-2s}\frac{-2}{(s+1)^3}$$

$$X[s]=\frac{-2e^{-2s}}{(s+1)^3}$$

$$e^{-2s}\longleftrightarrow(t-2) \quad [x(t-t_0)\longleftrightarrow e^{-st_0}]$$

$$\frac{1}{(s+1)^3}\longleftrightarrow\frac{t^2}{2}e^{-t}\cdot u(t)$$

$$x(t)=-2\frac{(t-2)^2}{2}e^{-(t-2)}\cdot u(t-2)=-(t-2)^2e^{-(t-2)}u(t-2)$$

Q:- $X[s] = \dfrac{s^2+12}{s(s+2)(s+3)}$

$$\frac{s^2+12}{s(s+2)(s+3)} = \frac{A}{s} + \frac{B}{s+2} + \frac{C}{s+3}$$

$$s^2+12 = A(s+2)(s+3) + Bs(s+3) + C(s)(s+2)$$

At $s=0$,
$12 = A \cdot 6$
$\boxed{A=2}$

At $s=-2$
$16 = B(-2)(1)$
$\boxed{B=-8}$

At $s=-3$
$21 = C(-3)(-1)$
$\boxed{C=7}$

$$X[s] = \frac{2}{s} - \frac{8}{s+2} + \frac{7}{s+3}$$

$$= \left(2 - 8e^{-2t} + 7e^{-3t}\right)u(t)$$

For Quadratic eq^n.

NOTE:

$$\frac{20}{s^2+5s+30} = \frac{20(Bs+C)}{s^2+5s+30}$$

Q:- $X[s] = \dfrac{2}{s^2+4s+13}$

$$X[s] = \frac{2}{s^2+4s+4+9}$$

$$X[s] = \frac{2}{(s+2)^2+(3)^2}$$

$$X[s] = \frac{2}{3}\left[\frac{3}{(s+2)^2+(3)^2}\right]$$

$$x(t) = \frac{2}{3}e^{-2t} \cdot \sin 3t\, u(t)$$

Q:- $X[s] = \dfrac{20}{(s+3)(s^2+8s+25)}$

$$\frac{20}{(s+3)(s^2+8s+25)} = \frac{A}{s+3} + \frac{Bs+C}{s^2+8s+25}$$

$$20 = A(s^2+8s+25) + (Bs+C)(s+3)$$

At $s=-3$

$$20 = A(9-24+25)$$

$$\boxed{A = 2}$$

co-efficient of s^2

$$0 = A + B$$

$$\boxed{B = -2}$$

At $s = 0$

$$20 = A(25) + 3C$$

$$20 = 50 + 3C$$

$$\boxed{C = -10}$$

$$X[s] = \frac{2}{s+3} \quad \frac{-2s-10}{(s^2+8s+25)}$$

$$X[s] = \frac{2}{s+3} - 2\left[\frac{(s+5)}{(s+4)^2+(3)^2}\right]$$

$$X[s] = \frac{2}{s+3} - 2\left[\frac{(s+4)+1}{(s+4)^2+(3)^2}\right]$$

$$X[s] = \frac{2}{s+3} - \frac{2(s+4)}{(s+4)^2+(3)^2} - \frac{2}{(s+4)^2+(3)^2}$$

$$x(t) = \left[2e^{-3t} - 2\cos 3t \cdot e^{-4t} - \frac{2}{3}\sin 3t \cdot e^{-4t}\right]u(t)$$

Convolution

$$\boxed{x(t) * h(t) \xleftrightarrow{L.T.} X[s] \cdot H[s]}$$

Q:- Solve the following eq^n:-

(1) $y(t) + \int_0^\infty y(\tau) \cdot x(t-\tau)\, d\tau = x(t) + \delta(t)$

$y(t) + y(t) * x(t) = x(t) + \delta(t)$

$Y[s] + Y[s] \cdot X[s] = X[s] + 1$

$Y[s]\,[1 + \cancel{X[s]}] = \cancel{X[s]} + 1$

$Y[s] = 1$

$$\boxed{y(t) = \delta(t)}$$

Q:- Consider a signal $y(t) = x_1(t-2) * x_2(t+3)$, $x_1(t) = e^{-2t}u(t)$ & $x_2(t) = e^{-3t}u(t)$. Find $Y[s]$ with ROC.

Sol^n:- $y(t) = x_1(t-2) * x_2(t+3)$

$\cancel{= e^{-2t}}$

$X_1(s) = \frac{1}{s+2}$, $\sigma > -2$

$X_2(s) = \frac{1}{s+3}$, $\sigma > -3$

$Y[s] = e^{-2s} \cdot X_1(s) \cdot e^{-3s} \cdot X_2(s)$

$Y[s] = \frac{e^{-2s}}{s+2} \cdot \frac{e^{-3s}}{(-s+3)}$

$\sigma > -2$ $\sigma < 3$

So, ROC is

$$\boxed{-2 < \sigma < 3}$$

$$Y[s] = \frac{e^{-5s}}{5}\left[\frac{1}{s+2} - \frac{1}{s-3}\right]$$

$$y(t) = \frac{e^{-5s}}{5}\left\{\left[e^{-2t}u(t) - e^{3t}u(t)\right]\right\}$$

$$y(t) = \frac{1}{5}\left[e^{-2(t-5)}u(t-5) - e^{3(t-5)}u(t-5)\right]$$

Q:- Find the transfer function and impulse response of the filter whose i/p and o/p relation is describe by $y(t) = x(t) + \int_{-\infty}^{t} y(\lambda) \cdot e^{-3(t-\lambda)} \cdot u(t-\lambda)\, d\lambda$

$$y(t) = x(t) + \left(y(t) * e^{-3t} \cdot u(t)\right)$$

$$Y(s) = X[s] + Y[s] \cdot \frac{1}{s+3}$$

$$Y[s]\left[1 - \frac{1}{s+3}\right] = X[s]$$

$$\boxed{\frac{Y[s]}{X[s]} = \frac{s+3}{s+2}}$$

$$Y[s]\left[\frac{s+2}{s+3}\right] = X[s]$$

For impulse response,

$x(t) = \delta(t)$

$X[s] = 1$

$$Y[s] = \left[\frac{s+2}{s+3}\right]^{-1} \cdot 1 = \frac{s+3}{s+2}$$

$$Y[s] = 1 + \frac{1}{s+2}$$

$$\boxed{y(t) = \delta(t) + e^{-2t} \cdot u(t)}$$

Q:- An i/p $x(t) = e^{-2t} \cdot u(t) + \delta(t-6)$ is applied to an LTI system with impulse response $h(t) = u(t)$ the o/p is ____.

Sol:- $x(t) = e^{-2t} \cdot u(t) + \delta(t-6)$, $h(t) = u(t)$

$$y(t) = x(t) * h(t)$$

$$y(t) = (e^{-2t} \cdot u(t) + \delta(t-6)) * u(t)$$

$$Y[s] = \left[\frac{1}{s+2} + e^{-6s}\right] \cdot \frac{1}{s}$$

$$= \frac{1}{s(s+2)} + \frac{e^{-6s}}{s}$$

$$= \frac{1}{2}\left[\frac{1}{s} - \frac{1}{s+2}\right] + \frac{e^{-6s}}{s}$$

$$Y[s] = \frac{1}{2} u(t) - \frac{1}{2} e^{-2t} \cdot u(t) + u(t-6)$$

OR

$$y(t) = (e^{-2t} \cdot u(t) + \delta(t-6)) * u(t)$$

$$= e^{-2t} \cdot u(t) * u(t) + \delta(t-6) * u(t)$$

$$= e^{-2t} \qquad + u(t-6)$$

Q:- Let L.T. of function $f(t)$ which exists for $t>0$ be $F_1(s)$ & the L.T. of its delay version $f(t-\tau)$ be $F_2(s)$. Let $F_1^*(s)$ be complex conjugate of $F_1(s)$ with $s=\sigma+j\omega$ if $G(s) = \frac{F_2(s) \cdot F_1^*(s)}{|F_1(s)|^2}$, then the I.L.T. of $G(s)$ is

(a) $\delta(t)$ (b) $\delta(t-\tau)$ (c) $u(t)$ (d) $u(t-\tau)$

Soln:- $F_1(t) \longleftrightarrow F_1(s)$

$$F_1(t-\tau) \longleftrightarrow F_2(s) = e^{-s\tau} \cdot F_1(s)$$

$$F_2(s) = F_1(t-\tau)$$

$$\Rightarrow F_2(s) = F_1(s) \cdot e^{-s\tau}$$

$$G(s) = \frac{e^{-s\tau} \cdot F_1(s) \cdot F_1^*(s)}{|F_1(s)|^2}$$

$$G(s) = e^{-s\tau}$$

$$g(t) = \delta(t-\tau)$$

NOTE:-

$x = 2+j3 \rightarrow |x| = \sqrt{4+9}$

$x^* = 2-j3 \quad |x|^2 = 4+9$

$|x \cdot x^*| = |4+9|$

$$\boxed{|x^2| = |x \cdot x^*| = |x| \cdot |x^*| = x \cdot x^*}$$

$$\boxed{\frac{x(t)}{t} \xleftrightarrow{L.T.} \int_s^{\infty} X[S]\, ds}$$

Q:- Find L.T. of $\frac{\sin\omega_0 t \cdot u(t)}{t}$

$$= \int_s^{\infty} \frac{\omega_0}{s^2+\omega_0^2}\, ds = \frac{\omega_0}{\omega_0}\left[\tan^{-1}\left(\frac{s}{\omega_0}\right)\right]_s^{\infty}$$

$$= \tan^{-1}\infty - \tan^{-1} s/\omega_0$$

$$= \pi/2 - \tan^{-1} s/\omega_0 = \cot^{-1} s/\omega_0$$

*LTI System & Laplace Transform

$$x(t) \longrightarrow \boxed{\text{LTI system}} \longrightarrow y(t)$$

$$h(t)$$

$$y(t) = x(t) * h(t)$$

$$Y[s] = X[s] \cdot H[s]$$

$$\boxed{H[s] = \frac{Y[s]}{X[s]} \Bigg|_{\substack{\text{initial}\\ \text{conditions}}=0.}}$$

Here, $H[s]$ is known as transfer function of an LTI system by considering all initial condition are zeros.

*Impulse & Step Response

* Impulse Response

$$x(t) = \delta(t) \Rightarrow X(s) = 1$$

$$\boxed{\text{I.R.} = Y[s] = H[s] = \text{I.R.} \Rightarrow y(t) = h(t)}$$

* Step Response

$$x(t) = u(t) = 1/s$$

$$\boxed{\text{S.R.} = Y[s] = H[s] \cdot 1/s \Rightarrow \underbrace{H(s)}_{\text{I.R.}} = s\, \underbrace{Y[s]}_{\text{S.R.}}}$$

By I.L.T.,

$$\underbrace{y(t)}_{\text{S.R.}} = \int_{-\infty}^{t} \underbrace{h(\tau)}_{\text{I.R.}}\, d\tau$$

$$\underbrace{\text{I.R.}}_{\text{I.R.} = d/dt\,(\text{S.R.})} = h(t) = \frac{d}{dt} y(t)$$

Steady State Response (IMP.)

Steady state response of an LTI system has two sinusoidal input $x(t) = A\cos(\omega_0 t + \theta)$

$$x(t) \longrightarrow \boxed{\begin{array}{c}\text{LTI}\\ H(s)\end{array}} \longrightarrow y_{ss}(t)$$

steady state response

$$x(t) = A\cos(\omega_0 t + \theta)$$

$$H[s] = |H(s)|\, e^{\angle H(s)}$$

$$|H(s)|\Big|_{s=j\omega_0} = K$$

$$\angle H(s)\Big|_{s=j\omega_0} = \phi$$

$$\boxed{y_{ss} = A \cdot \underline{K} \cos(\omega_0 t + \theta + \underline{\phi})}$$

Q:- Consider a system with T.F. $H(s) = \dfrac{s-2}{s^2+4s+4}$. Find the steady state response when i/p applied to $8\cos 2t$.

Sol:- $x(t) = 8\cos 2t$

$$H[j\omega] = \frac{j\omega - 2}{(j\omega)^2 + 4j\omega + 4}$$

$$H[s]\Big|_{s=j2} = \frac{j2-2}{\cancel{-4} + 8j + \cancel{4}} = \frac{-2+j2}{8j}$$

$$|H(s)| = K = \frac{\sqrt{8}}{8} = \frac{1}{\sqrt{8}}$$

$$\phi = 180^\circ - \tan^{-1} 1 - 90^\circ$$

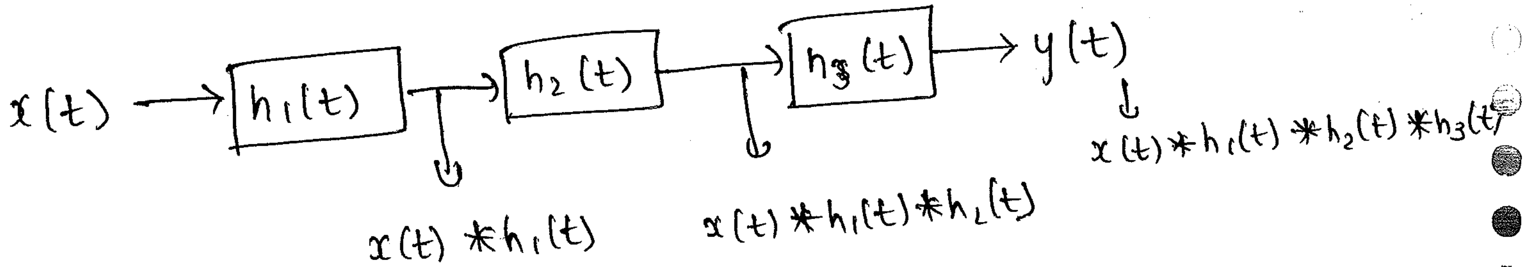
$$\phi = 45^\circ$$

$$y_{ss} = 8 \cdot \frac{1}{\sqrt{8}} \cos(2t + 45^\circ)$$

$$\boxed{y_{ss} = \sqrt{8}\cos(2t + 45^\circ)}$$

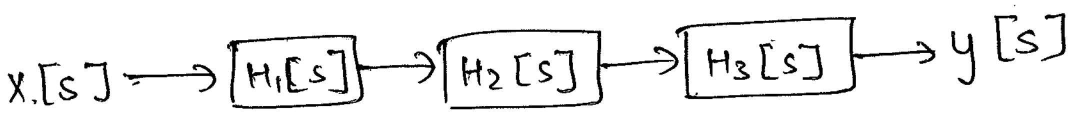
Interconnection of LTI systems

① Series OR Cascade Connection



$$Y[s] = X[s]\left[H_1(s) \cdot H_2(s) \cdot H_3(s)\right]$$

$$Y[s] = X[s]\, H[s]$$



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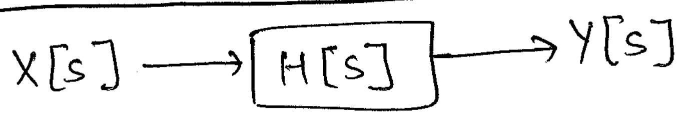
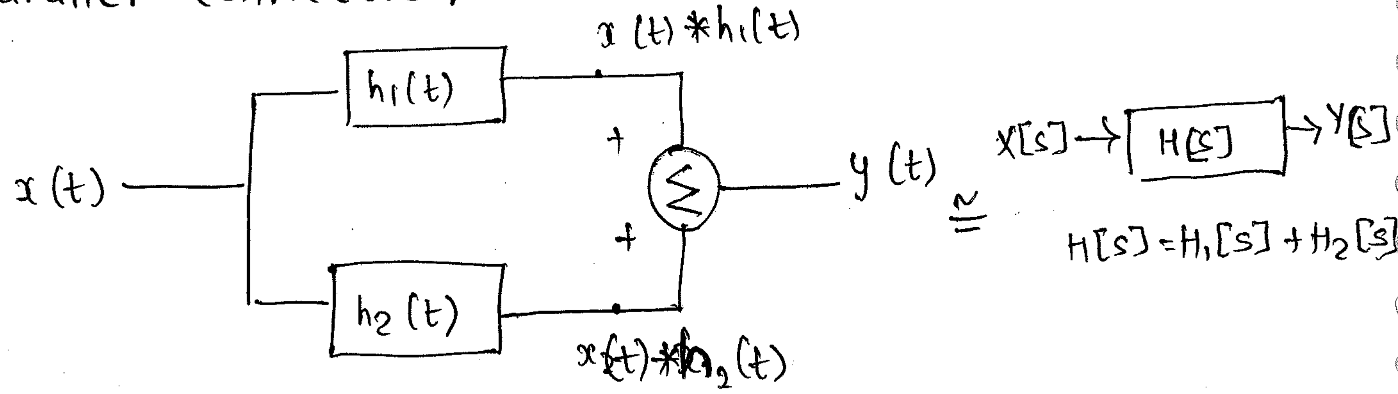


$$H[s] = H_1[s] \cdot H_2[s] \cdot H_3[s]$$

② Parallel connection



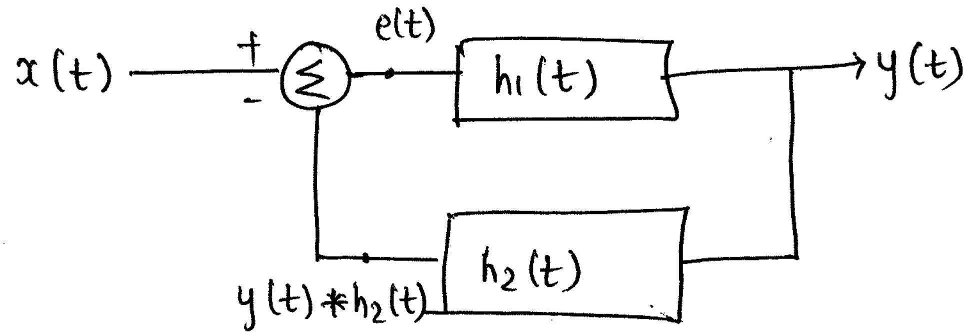
$$H[s] = H_1[s] + H_2[s]$$

$$y(t) = x(t) * h_1(t) + x_2(t) * h_2(t)$$

$$= x(t)\left[h_1(t) * h_2(t)\right]$$

$$Y[s] = X[s]\left[H_1[s] \cdot H_2[s]\right]$$

Feedback connection



$$e(t) = x(t) - y(t) * h_2(t)$$

$$y(t) = e(t) * h_1(t)$$

$$y(t) = (x(t) - y(t) * h_2(t)) * h_1(t)$$

$$y(t) = x(t) * h_1(t) - y(t) * h_2(t) * h_1(t)$$

$$y(t)\left[1 + h_1(t) * h_2(t)\right] = x(t) * h_1(t)$$

$$y(t) = \frac{h_1(t) * x(t)}{1 + h_1(t) * h_2(t)}$$

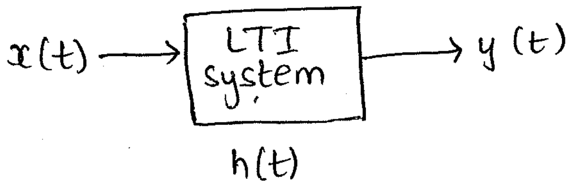
$$Y[s] = \frac{H_1[s] \cdot X[s]}{1 + H_1[s] H_2[s]}$$

$$\boxed{\frac{Y[s]}{X[s]} = \frac{H_1[s]}{1 + H_1[s] H_2[s]}}$$

*Causality and Stability

-For an LTI system to be causal ; if impulse response $h(t) = 0, t < 0$. Thus it is right sided and corresponding ROC is right of the right most pole. However the converge of this statement is not necessarily be true.

-An ROC to the right most pole doesn't guarantee that system is causal.



Causal.

$\Rightarrow h(t) = 0 \quad , t < 0$

$\neq 0 \quad , t > 0$

$\Downarrow$

ROC: right sided

$$X[s] = \frac{e^{s}}{s+1}$$

$\downarrow$

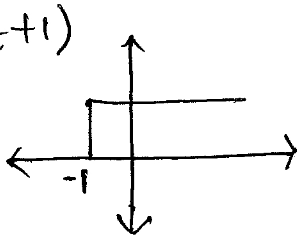
$x(t) = e^{-t} u(t)$ ROC: $Re\{s\} > -1$

$\downarrow$

$t \to t+1$

$x(t) = e^{-(t+1)} u(t+1)$

NC



Causal and Stable

NOTE:

Causality implies that ROC is through right of the rightmost pole but the converge is not in general true, unless the system function is rational.

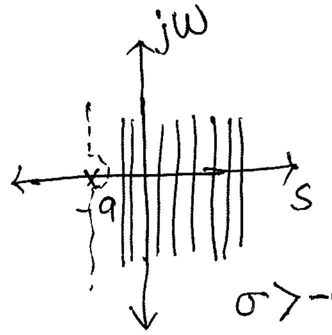
Stability: For a LTI system, to be stable ROC must includes $j\omega$ axis.

Stability of Causal System:

- A causal syetem with rational system function with $H(s)$ is stable if and only if all the poles of $H(s)$ lies on left half of the s-plane

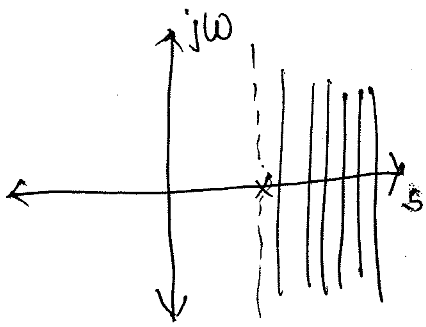
$x(t) \cdot u(t)$

① $\overset{+ve}{e^{-at}} \cdot u(t)$ — stable $\xleftrightarrow{L.T.}$ $\frac{1}{s+a}$
$\downarrow$
$t>0$



$\sigma > -a$

② $\overset{-ve}{e^{+at}} u(t)$ — unstable $\xleftrightarrow{L.T.}$ $\frac{1}{s-a}$
$\downarrow$
$t>0$

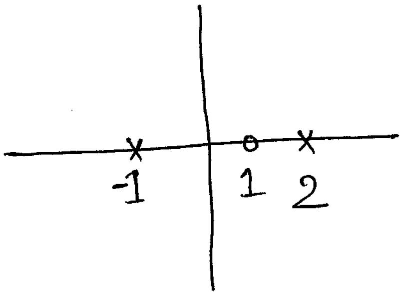


ROC: $\sigma > a$

Q:- Given $H(s) = \frac{s-1}{(s+1)(s-2)}$. Find $h(t)$ for each of the following cases.

(1) stable (2) causal (3) NCNS

<u>Ans</u>: $H(s) = \frac{s-1}{(s+1)(s-2)}$



$$H(s) = \frac{A}{s+1} + \frac{B}{s-2}$$

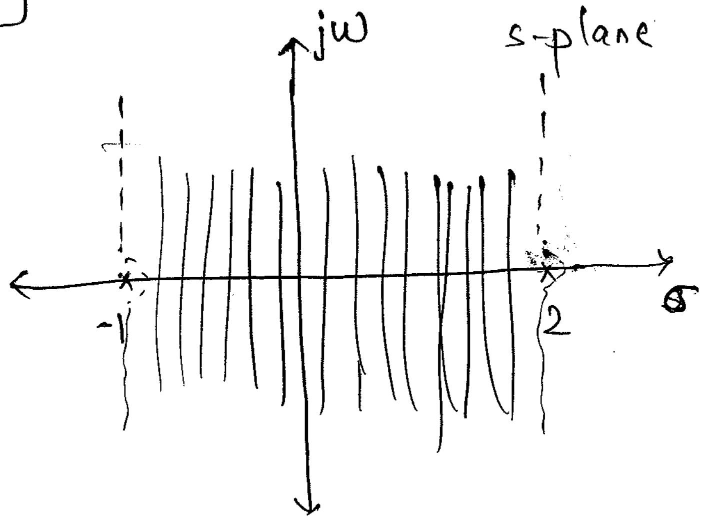
$$s-1 = A(s-2) + B(s+1)$$

At $s=-1$ At $s=2$

$-2 = A(-3)$ $1 = B(3)$

$\boxed{A = 2/3}$ $\boxed{B = 1/3}$

$$H(s) = \frac{2/3}{s+1} + \frac{1/3}{s-2}$$



(1) stable

(covers jw axis)

$\sigma > -1$ & $\sigma < 2$

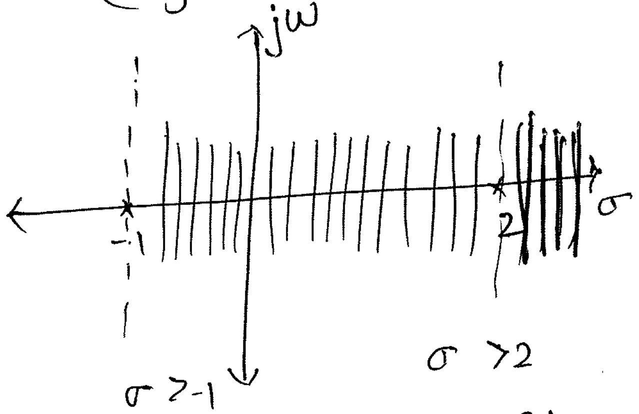
$\downarrow$ right $\downarrow$ left

$$h(t) = \frac{2}{3} e^{-t} \cdot u(t) + \frac{-1}{3} e^{+2t} \cdot u(-t)$$

$$h(t) = \frac{2}{3} e^{-t} u(t) - \frac{1}{3} e^{2t} \cdot u(-t)$$

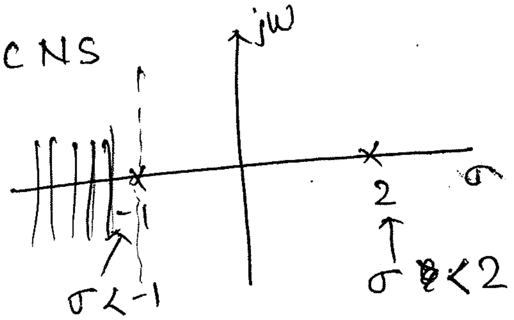
(2) ~~Stable~~ causal

(right sided)



$$h(t) = \frac{2}{3}e^{-t}\cdot u(t) + \frac{1}{3}e^{2t}u(t)$$

(3) NCNS



$$h(t) = -\frac{2}{3}e^{-t}\cdot u(-t) - \frac{1}{3}e^{2t}\cdot u(-t)$$

Above H(s) is not both causal & stable simutaneously as one pole is lie on right side

Q:- $X(s) = \dfrac{5-s}{s^2-s-6}$ and if F.T. of signal is not defined then x(t) is ____.

$$X(s) = \frac{5-s}{s^2-3s+2s-6}$$

$$= \frac{5-s}{s(s-3)+2(s-3)}$$

$$X(s) = \frac{5-s}{(s-3)(s+2)}$$

$$\frac{5-s}{(s-3)(s+2)} = \frac{A}{s+2} + \frac{B}{s-3}$$

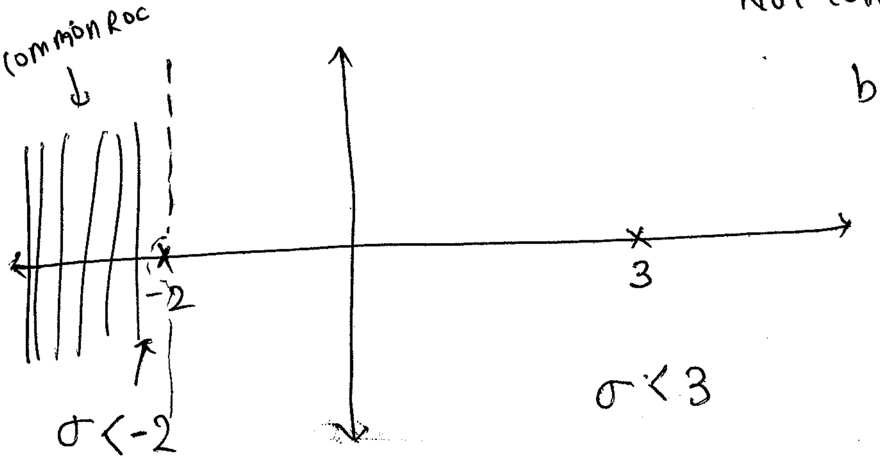
$5 - s = A(s-3) + B(s+2)$

$2 = B(5)$ $\qquad$ $7 = A(-5)$

$\boxed{B = 2/5}$ $\qquad$ $\boxed{A = \frac{-7}{5}}$

$$X(s) = \frac{-7}{5(s+2)} + \frac{2}{5(s-3)}$$

Not cover $j\omega$ axis
bcz. Fourier transform
doesn't exists.



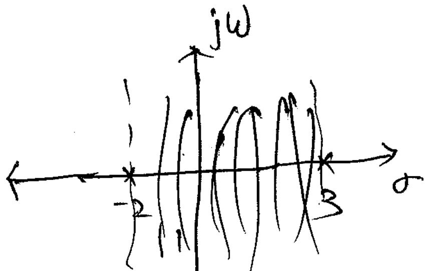
$$x(t) = \frac{2}{5} e^{3t} \cdot u(t) - \frac{7}{5} e^{-2t} \cdot u(t)$$

OR $\quad x(t) = \frac{-2}{5} e^{3t} \cdot u(-t) + \frac{7}{5} e^{-2t} \cdot u(-t)$

For stable :- (Fourier Transform is defined)

Covers $j\omega$ axis $\quad -2 < \sigma < 3$
$\qquad\qquad\qquad\qquad$ ↓ right $\quad$ ↓ left

$$x(t) = \frac{-7}{5} e^{-2t} \cdot u(t) - \frac{2}{5} e^{3t} \cdot u(-t)$$



★ Q:- Consider an LTI system for which we are given following
★ information $X(s) = \frac{s+2}{s-2}$ & $\underline{x(t) = 0, t > 0}$ and
output is $y(t) = \frac{-2}{3} e^{2t} u(-t) + \frac{1}{3} e^{-t} \cdot u(t)$

(a) Find transfer function and ROC

(b) Find the output if input is $x(t) = e^{3t}$ using part (A)

$$y(t) = \frac{-2}{3} e^{2t} \cdot u(-t) + \frac{1}{3} e^{-t} \cdot u(t)$$

$$\updownarrow$$

$$Y(s) = \frac{+2}{3(s+2)} + \frac{1}{3(s-1)}$$

$$\frac{Y(s)}{X(s)} = \frac{\frac{2}{3(s+2)} + \frac{1}{3(s-1)}}{\left(\frac{s+2}{s-2}\right)} = \frac{\frac{2(s-1) + (s+2)}{3(s+2)(s-1)}}{\frac{(s+2)}{(s-2)}}$$

$$= \frac{[2(s-1) + (s+2)](s-2)}{[3(s+2)(s-1)][s+2]}$$

$$\frac{Y(s)}{X(s)} = \frac{2(s-1)(s-2) + (s^2-4)}{3(s-1)(s+2)(s+2)}$$

$$= \frac{2[s^2-3s+2] + s^2-4}{3(s-1)(s+2)(s+2)}$$

$$= \frac{2s^2-6s+4+s^2-4}{3(s-1)(s+2)^2}$$

$$= \frac{3s^2-6s}{3(s-1)(s+2)^2}$$

$$\frac{Y(s)}{X(s)} = \frac{\cancel{3}s(s-2)}{\cancel{3}(s-1)(s+2)^2}$$

$$\frac{Y[s]}{X[s]} = \frac{s}{(s-2)(s+1)}$$

$$\sigma < 2 \qquad \sigma > -1$$

$$\boxed{-1 < \sigma < 2}$$

$$H[s] = \frac{s\,\cancel{(s-2)}}{\cancel{(s-2)}(s+1)(s+2)}$$

$$H[s] = \frac{s}{(s+1)(s+2)} \quad ; \text{ROC } \sigma > -1$$

$$x(t) \longrightarrow \boxed{\text{LTI system}} \longrightarrow y(t)$$

left sided signal (given)

$-1 < \sigma < 2$

left side ROC

$\sigma < 2$

$\sigma > -1$

$$Y(s) = X[s] \cdot H[s]$$

$$\text{ROC}: \ R_y = R_x \cap R_H$$

(b) $x(t) = e^{3t}$

$$y(t) = e^{3t} \cdot H(s)\big|_{s=3}$$

$$\boxed{y(t) = \frac{3}{20} e^{3t}}$$

Q:-Which of the following signals is NOT TRUE for continuous time causal and stable LTI system

(a) All the poles of the system must lies on left side of $j\omega$ axis

(b) Zeros of the system can lie anywhere in s-plane

(c) All the poles must lie within $|s| = 1$ ✓

(d) All the roots of characteristics eq^n must be located on the left side of $j\omega$ axis.

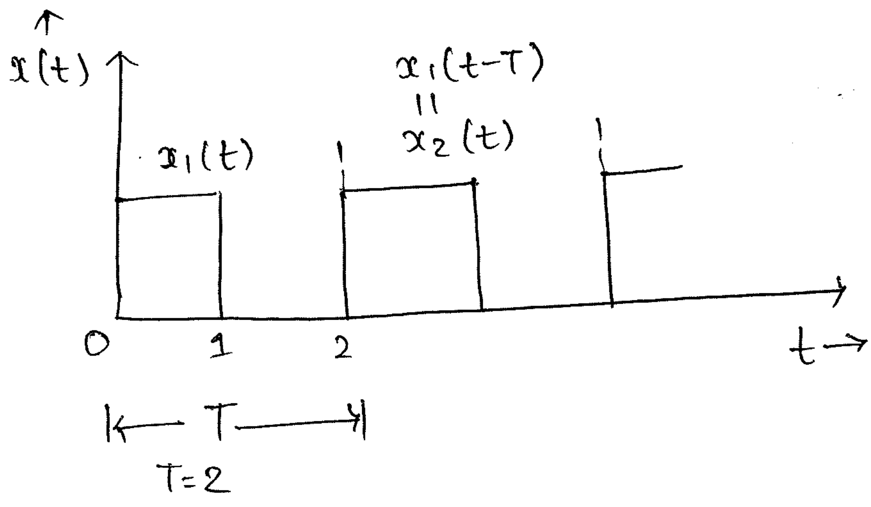
*Laplace Transform of Causal Periodic Signal

⇒ Periodic signal

$-\infty \leq t \leq \infty$

⇒ Causal Periodic signal

$0 \leq t < \infty$



$$x(t) = x(t - kT)$$

$$x(t) = x_1(t) + x_1(t-T) + x_1(t-2T) + x_1(t-3T) + \cdots$$

$\updownarrow$ L.T.

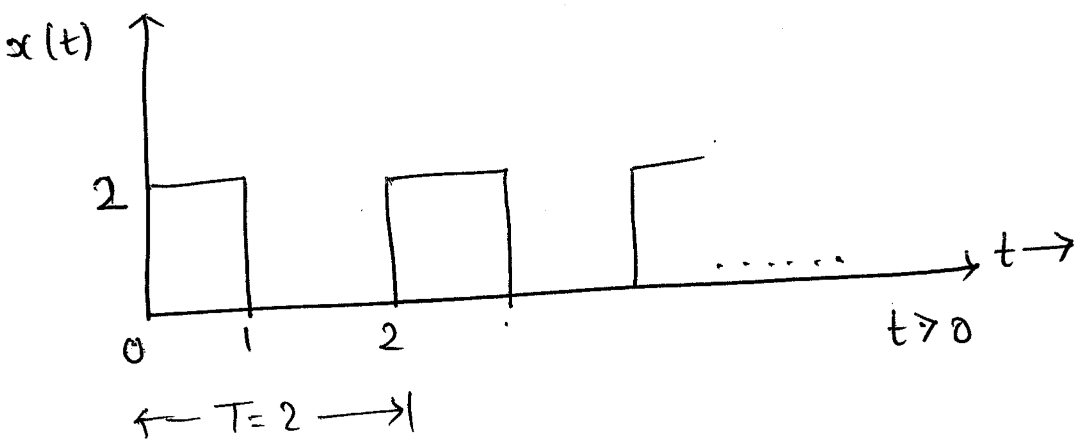
$$X[s] = X_1[s] + X_1[s]e^{-sT} + X_1[s]e^{-2sT} + X_1[s]e^{-3sT} + \cdots$$

$$X[s] = X_1[s]\left[1 + e^{-sT} + e^{-2sT} + e^{-3sT} + \cdots\right]$$

$$X[s] = \frac{X_1[s]}{1-e^{-sT}} \rightarrow \text{L.T. of } x_1(t) \rightarrow 0<t<T$$

$$\boxed{X[s] = \frac{\int_0^T x_1(t)\cdot e^{-st}\cdot dt}{1-e^{-sT}}}$$

Q:- Find L.T. of $x(t)$ which is periodic.



$$X[s] = \frac{\int_0^1 2\cdot e^{-st}\,dt}{1-e^{-sT}} \qquad \left[\int_0^2 x(t)e^{-st}dt = \int_0^1 x(t)e^{-st}dt + 0\right]$$

$$= \frac{2\left[\frac{e^{-s}-e^{0}}{-s}\right]}{1-e^{-sT}}$$

$$X[s] = \frac{2(1-e^{-s})}{s(1-e^{-2s})}$$

*Unilateral laplace Transform

$t>0 \rightarrow u(t)$

$$X[s] = \int_0^\infty x(t)e^{-st}dt$$

$$X(s) = \int_0^\infty x(t)e^{-st}dt$$

ROC right side always

Bilateral L.T.

① $X[s] = \int_{-\infty}^{\infty} x(t) \cdot e^{-st} dt$

② gives insight (in detail) of system characteristics like frequency response, causality and sensitivity

③ Causal, Non-causal

Unilateral L.T.

① $X[s] = \int_{0}^{\infty} x(t) \cdot e^{-st} dt$

② To solve D.E. with initial conditions

③ Causal.

Q:- Evaluate bilateral and unilateral laplace transform of

$x(t) = e^{-a(t+1)} u(t+1)$

Bilateral

$$X[s] = \int_{-1}^{\infty} e^{-a(t+1)} \cdot e^{-st} dt$$

$$= e^{-a} \int_{-1}^{\infty} e^{-(s+a)t} dt$$

$$= e^{-a} \left[\frac{e^{-(s+a)t}}{-(s+a)} \right]_{-1}^{\infty}$$

$$= e^{-a} \left[0 + \frac{e^{+(s+a)}}{s+a} \right]$$

$$X[s] = \frac{e^{-a} \cdot e^{+(s+a)}}{s+a} = \frac{e^{-s} \cdot}{s+a}, \quad \sigma > -a$$

Unilateral

$$X[s] = \int_{0}^{\infty} e^{-a(t+1)} \cdot e^{-st} dt$$

$$X[s] = e^{-a} \int_{0}^{\infty} e^{-(s+a)t} dt$$

$$= e^{-a} \left[\frac{e^{-(s+a)}}{+(s+a)} \right]_{\infty}^{0}$$

$$= e^{-a} \left[\frac{1}{s+a} - 0 \right]$$

$$= \frac{e^{-a}}{s+a}, \quad \sigma > -a$$

Properties of Unilateral L.T.

(1) $\frac{d\,x(t)}{dt} \xleftrightarrow{L.T.} sX[s] - \underbrace{x(0)}_{I.C.}$

$$\frac{d^2x(t)}{dt^2} \xleftrightarrow{L.T.} s^2X[s] - sX[0] - x'(0) \quad ; \; x'(0) = \left.\frac{dx(t)}{dt}\right|_{x=0}$$

$$\frac{d^3x(t)}{dt^3} \xleftrightarrow{L.T.} s^3X[s] - s^2X[0] - sX'[0] - x''[0]$$

(2) $$\boxed{\int_{-\infty}^{t} x(t)\,dt \xleftrightarrow{L.T.} \frac{X[s]}{s} + \frac{\int_{-\infty}^{0} x(t)\,dt}{s}}$$

(3) Initial Value Theorem (I.V.T.)

$$\boxed{\left. x(t)\right|_{\substack{t=0 \\ s\to\infty}} = \lim_{t\to 0} x(t) = \lim_{s\to\infty} sX[s]}$$

(4) Final Value Theorem (F.V.T.)

$$\boxed{\left. x(t)\right|_{\substack{t=\infty \\ s\to 0}} = \lim_{t\to \infty} x(t) = \lim_{s\to 0} sX[s]}$$

⇓

Steady state value

I.V.T. applicable only for strictly proper laplace transform

i.e. $N(s) < D(s)$

NOTE:

⇒ I.V.T. applys only if $X[s]$ is strictly proper (i.e. Power of N(s) < Power of D(s))

⇒ For improper $X[s]$ using long division method $X[s]$ is expressed as polynomial in s + strictly proper function and in this case value of remainder fraction gives initial value.

Q:- Find I.V. and F.V. of following functions:-

(1) $X[s] = \dfrac{2s+5}{s^2+5s+6}$

→ strictly proper (both will be there)

$$I.V. = \lim_{s\to\infty} s\,X[s]$$

$$= \lim_{s\to\infty} \frac{s(2s+5)}{s^2+5s+6}$$

$$= \lim_{s\to\infty} \frac{2s^2+5}{s^2+5s+6}$$

$$= \lim_{s\to\infty} \frac{4s}{2s+5}$$

$$= \lim_{s\to\infty} \frac{4}{2}$$

$$I.V. = 2$$

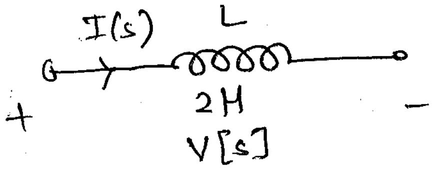
$$F.V. = \lim_{s\to 0} s\,X[s]$$

$$= \lim_{s\to 0} \frac{s(2s+5)}{s^2+5s+6}$$

$$= \lim_{s\to 0} \frac{2s^2+5s}{s^2+5s+6}$$

$$F.V. = 0$$

Q:- A voltage having L.T. $\dfrac{4s^2+3s+2}{7s^2+6s+5}$ is applied across a 2H inductor having 0 initial current. What is the current in inductor at $t=\infty$?



$$v(t) = L\frac{di(t)}{dt}$$

$$\downarrow$$

$$V[s] = sL\,I_L[s]$$

$$X_L = sL$$

$$I_L[s] = \frac{V_L[s]}{sL}$$

$$i_L[\infty] = \lim_{t\to\infty} i_L(\infty) = \lim_{s\to 0} s\,I_L[s]$$

$$= \lim_{s\to 0} \cancel{s}\cdot\frac{V_L[s]}{\cancel{s}L}$$

$$= \lim_{s\to 0} \frac{4s^2+3s+2}{7s^2+6s+5}\cdot\frac{1}{(2)}$$

$$= \frac{\cancel{2}}{5\cdot\cancel{2}}$$

$$i_L(\infty) = 1/5 \text{ A}$$

Q:- Calculate the inital value of the following function?

(1) $X[s] = \frac{4s+5}{2s+1}$

Here, Power of $N(s)$ = Power of $D(s)$

$$X[s] = \frac{4s+2+3}{2s+1}$$

$$= \frac{2(2s+1)+3}{2s+1}$$

$$X[s] = 2 + \frac{3}{2s+1}$$

Now,
$$\lim_{s\to\infty} s\left(\frac{3}{2s+1}\right)$$
$$\lim_{s\to\infty} \frac{3}{2+1/s}$$
$$\boxed{x(0) = 3/2}$$

OR

$$\begin{array}{r|l} & 2 \\ \hline 2s+1 & 4s+5 \\ & 4s+2 \\ & \overline{\quad 3} \end{array}$$

$$2 + \frac{3}{2s+1}$$
↓
I.V. find of this term

Q:- $X[s] = \dfrac{s^3+3s^2+s+1}{s^2+2s+1}$

$$\begin{array}{r|l} & s+1 \\ \hline s^2+2s+1 & s^3+3s^2+s+1 \\ & s^3+2s^2+s \\ & \overline{\quad s^2+1} \\ & s^2+2s+1 \\ & \overline{\quad -2s} \end{array}$$

$$s+1 \quad \frac{-2s}{(s^2+2s+1)}$$
strictly proper

$$x(0) = \lim_{s\to\infty} s\frac{(-2s)}{s^2+2s+1}$$

$$\boxed{\text{I.V.} = -2}$$

NOTE:

⇒ Final Value Theorem apply only if poles of X(s) lies in left half of s-plane including s = 0 (single pole).

⇒ Final Value Theorem is not applicable if poles lies on imaginary axis & right half of s-plane.

Q:- Consider the function $f(t)$ having laplace transform

$$F[s] = \frac{\omega_0}{s^2+\omega_0^2}, \quad \mathrm{Re}\{s\}>0$$

The final value of $f(t)$ is

(a) 0 (c) $-1 \le f(\infty) \le 1$ ✓

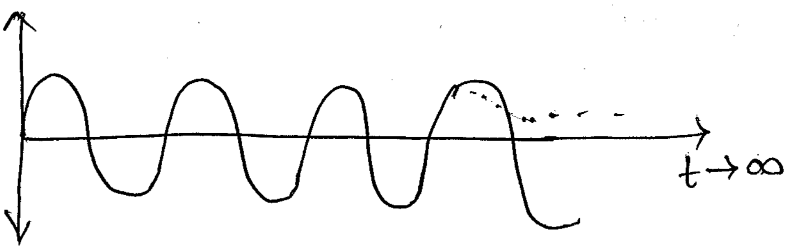
(b) 1 (d) ∞

Sol:-

$$F[s] = \frac{\omega_0}{s^2+\omega_0^2}$$

$$\Big\downarrow \text{I.L.T.}$$

$$f(t) = \sin\omega_0 t \cdot u(t)$$



At $t \to \infty$

$[-1, 1]$

Q:- If laplace transform of signal $y(t)$ is $\frac{1}{s(s-1)}$

Then it's final value is

(a) -1 (c) 0

(b) 1 (d) unbounded ✓

$$Y[s] = \frac{1}{s(s-1)}$$

$$= -\left[\frac{1}{s} - \frac{1}{s-1}\right]$$

$$Y[s] = \frac{1}{s-1} - \frac{1}{s}$$

$$Y[t] = e^{+t}u(t) - u(t)$$

unbounded

Q:- Find I.V. and F.V.

(1) $F[s] = \dfrac{2(s+1)}{s^2+2s+5}$

$$f(0) = \lim_{s\to\infty} s \cdot F(s)$$

$$= \lim_{s\to\infty} s \cdot \frac{2(s+1)}{s^2+2s+5}$$

$$= \lim_{s\to\infty} \frac{2s^2+2s}{s^2+2s+5}$$

$f(0) = 2$

$$f(\infty) = \lim_{s\to 0} s \cdot F(s)$$

$$= \lim_{s\to 0} s \cdot \frac{2(s+1)}{s^2+2s+5}$$

$$= \lim_{s\to 0} \frac{2s^2+2s}{s^2+2s+5}$$

$f(\infty) = 0$

* Application of Laplace Transform

Q:- Find the sol^n for 2^{nd} order differential eq^n ~~and~~ for initial conditions and determine $y(t)$ for input $x(t) = u(t)$, $\dfrac{d^2y(t)}{dt^2} + 6\dfrac{dy(t)}{dt} + 8y(t) = 2u(t)$. $y(0^-) = 1$, $y'(0^-) = 2$.

$$s^2 y(s) + 6sy(s) + 8y(s) = 2 \cdot \frac{1}{s}$$

$$(s^2+6s+8)\,y(s) = \frac{2}{s}$$

$$y(s) = \frac{2}{s(s^2+6s+8)}$$

$$y(s) = \frac{s}{s(s+4)(s+2)} = \frac{A}{s} + \frac{B}{s+2} + \frac{C}{s+4}$$

$$\cancel{s = A(s+2)(s+4) + Bs(s+4)}$$

Applying L.T.

$$s^2Y[s] - sy(0) - y'(0) + 6sY[s] - 6y(0) + 8y(s) = 2X(s)$$

$$Y[s]\left[s^2+6s+8\right] - s - 2 - 6(1) = 2X(s)$$

$$Y[s]\left[s^2+6s+8\right] = \frac{2}{s} + s + 8$$

$$Y[s] = \frac{s^2+8s+2}{s(s^2+6s+8)}$$

$$Y[s] = \frac{s^2+8s+2}{s(s+2)(s+4)}$$

$$s^2+8s+2 = \frac{A}{s} + \frac{B}{s+2} + \frac{C}{s+4}$$

At $s=0$

$$2 = A(2)(4)$$

$$\boxed{A = 1/4}$$

At $s=-2$

$$4-16+2 = B(-2)(2)$$

$$-10 = B(-4)$$

$$\boxed{B = 5/2}$$

At $s=-4$

$$16-32+2 = C(-4)(-2)$$

$$-14 = C8$$

$$\boxed{C = \frac{-7}{4}}$$

$$Y[s] = \frac{1}{4s} + \frac{5}{2(s+2)} - \frac{7}{4(s+4)}$$

$$y(t) = \left[\frac{1}{4} + \frac{5}{2}e^{-2t} - \frac{7}{4}e^{-4t}\right]u(t)$$

Q:- Solve the inte-gro and differential eq^n

$$\frac{dy(t)}{dt} + 2y(t) + 5\int_{-\infty}^{t} y(\tau)\, d\tau = 10x(t)$$ for the initial

conditions $y(0^-) = 2$, $\int_{-\infty}^{0} y(\tau)\, d\tau = 2$, $x(t) = u(t)$

Ans: Property of unilateral L.T.

$$s\,y(s) - y(0) + 2y(s) + 5\left[\frac{y(s)}{s} + \frac{\int_{-\infty}^{0} y(\tau)\, d\tau}{s}\right] = 10X(s)$$

$$sy(s) - 2 + 2y(s) + 5\left[\frac{y(s)}{s} + \frac{2}{s}\right] = 10x(s)$$

$$sy(s) + 2y(s) + \frac{5y(s)}{s} + \frac{10}{s} - 2 = 10X(s)$$

$$y(s)\left[s + 2 + \frac{5}{s}\right] = \frac{10}{s} - \frac{10}{s} + 2$$

$$y(s) = \frac{2}{s + 2 + 5/s} = \frac{2s}{s^2 + 2s + 5}$$

$$Y[s] = \frac{2s}{(s+1)^2 + 4}$$

$$Y[s] = \frac{2s + 2 - 2}{(s+1)^2 + 4}$$

$$Y[s] = \frac{2(s+1)}{(s^2+1)^2 + (2)^2} - \frac{2}{(s+1)^2 + (2)^2}$$

$$y(t) = \left[2e^{-t}\cdot\cos 2t - e^{-t}\sin 2t\right]u(t)$$

Q:- A continuous time LTI system is described by $\frac{d^2y(t)}{dt^2} + 4\frac{dy(t)}{dt} + 3y(t) = 2\frac{dx(t)}{dt} + 4x(t)$. Assuming zero initial conditions the response $y(t)$ of above system for the input $x(t) = e^{-2t}u(t)$.

Sol^n:-

$$s^2Y(s) + 4sY[s] + 3Y(s) = 2sX(s) + 4X(s)$$

$$Y[s](s^2+4s+3) = X(s)[2s+4] \qquad \frac{Y(s)}{X(s)} = \text{T.F.}$$

$$Y[s][s^2+4s+3] = \frac{1}{\cancel{s+2}}\,2\cancel{(s+2)}$$

$$Y[s] = \frac{2}{(s^2+4s+3)}$$

$$Y[s] = \frac{2}{\cancel{(s+2)^2-(1)^2}\ (s+3)(s+1)}$$

$$= 2\left[\frac{-1/2}{s+3} + \frac{1/2}{s+1}\right]$$

$$Y[s] = \frac{1}{s+1} - \frac{1}{s+3}$$

$$\boxed{y(t) = (e^{-t} - e^{-3t})\,u(t)}$$

* Zero input response and zero state response

Total Response

Zero input response

Zero state response

Zero input response

↓

input = 0

↓

By considering initial conditions

↓

natural response

↓

transient response

↓

free forced response

(Complementary function)
C.F.

Zero state response

↓

initial con. = 0

↓

only input

↓

steady state response

↓

forced response

↓

$$\left.\frac{Y[s]}{X[s]}\right|_{I.C.=0} = T.F.$$

(Particular function)
P.I.

Q:- $\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 3y = x(t)$

A system described by above given differential eq^n is initially at rest and then excited by $x(t) = 3u(t)$

Then output $y(t)$ is ____ .

Sol^n:- initially at rest ⇒ I.C. = 0

So, zero state response

$$s^2Y(s) + 4sY(s) + 3Y(s) = X(s)$$

$$Y(s)\cdot(s^2+4s+3) = X(s)$$

$$Y(s) = \frac{3}{s(s+1)(s+3)}$$

$$= 3\left[\frac{1}{3s} - \frac{1}{(s+1)2} + \frac{1}{6(s+3)}\right]$$

$$y(t) = 3\left[\frac{1}{3}u(t) - \frac{e^{-t}}{2}\cdot u(t) + \frac{1}{6}e^{-3t}\cdot u(t)\right]$$

$$= \left(1 - 1.5e^{-t} + 0.5e^{-3t}\right)u(t)$$

Q:- $\frac{d^2y(t)}{dt^2} + 6\frac{dy(t)}{dt} + 8y(t) = 2x(t)$

$y(0^-) = 2, \; y'(0^-) = -2 \quad , x(t) = u(t)$

Sol^n:- $s^2Y(s) - sy(0) - y'(0) + 6Y(s) - 6y(0) + 8Y(s) = 2X(s)$

$$Y(s)[s^2+6s+8] = 2X(s) + 2s - 2 + 12$$

$$Y(s)[s^2+6s+8] = \underbrace{2s+10}_{\text{initial condition}} + \underbrace{2X(s)}_{\text{input}}$$

$$Y(s) = \underbrace{\frac{2s+10}{(s^2+6s+8)}}_{\text{zero input response}} + \underbrace{\frac{2}{s(s^2+6s+8)}}_{\text{zero state responce}}$$

$$= \frac{2s+10}{(s+2)(s+4)} + \frac{2}{s(s+2)(s+4)}$$

$2s+10 = A(s+4) + B(s+2)$

At $s = -2$ | At $s = -4$

$6 = A(2)$ | $2 = B(-2)$

$\boxed{A = 3}$ | $\boxed{B = -1}$

$$= \frac{3}{s+2} - \frac{1}{s+4}$$

$$= 3e^{-2t}u(t) - e^{-4t}u(t)$$

$$+ \frac{1}{4s} - \frac{1}{2(s+2)} + \frac{1}{4\,s+4}$$

$$+ \frac{1}{4}u(t) - \frac{1}{2}e^{-2t}u(t) + \frac{1}{4}e^{-4t}u(t)$$

$$y(t) = \left(3e^{-2t} - e^{-4t}\right)u(t) + \left(\frac{1}{4} - \frac{1}{2}e^{-2t} + \frac{1}{4}e^{-4t}\right)u(t)$$