10

Locus

10.01. Introduction

As certained, you have seen many unique scenes around you in your life many times. All certain have you ever seen some birds flying in the sky in a rare shape? Or have you ever seen an army of ants moving on the wall or on the floor in special or unusual shape? There is a particular condition which is in their nature in the motion of these tiny creatures. All of them follow these rules accordingly their nature and living conditions. If every creature of the two incident considered a point,then the unique figure made by them is a set of many points who obey these rules. In fact in geometrical figures, the set of the points, with a determination by one or more specified condition from the figures. i.e., for a particular figure, a set of all the necessary points in the space is called the locus.

10.02. Defination

Locus is a special set of the points which satisfies the certain given conditions. To understand it, we shall take some example :

(*a*) Let O be the fixed point in a plane and r be a positive integer. All points in the plane which are at a distance of 'r' units from 'O' describe a locus. This locus is a circle with centre O and radius r (Fig. 10.01)

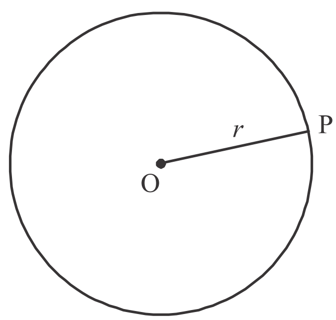

Fig. 10.01

(*b*) Take two parallel lnes *l* and *m*. Consider all those points which are equidistant from *l* and *m.* A figure of a line *n* is forward by all these points and the obtained is equidistant from *l* and *m.* see (Fig. 10.02).

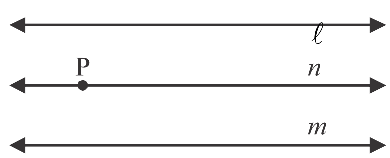

Fig. 10.02

(*c*) Let *l* be a line and *d* be a positive real number. consider all of the points which are at the *d* distance away form *l*. In this way two lines *m* and *n* parallel to *l* are obtained which are at *d* distant from *l* (Fig.10.03)

It is considerable that all three above situations the points follow the certain condition/conditions.

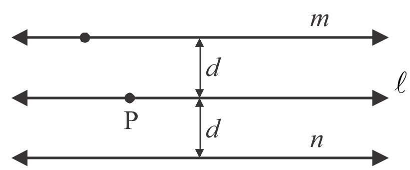

Fig. 10.03

Thus the loucs of the given points is the set of these points which satisfy the certain condition/conditions.

Remember in this defination there are two complimentary statements.

(i) The points which satisfies the given conditions is the point of the locus.

(ii) Every point of the locus has to satisfy the given conditions.

In this way the locus and its deciding points may be considered same. When one of two is described, other automatically cleared.

Let us study two important locii, which are very useful in other theorems and in some geometrical constructions.

10.03. Locus of points equidistant from two given points

Let A and B be two given points. Consider path of the point P which satisfies the condition $AP = BP$.

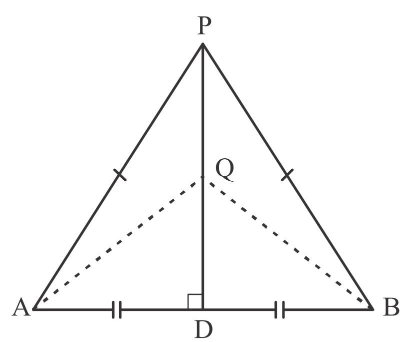

Fig. 10.04

If D is the mid point of AB, then AD=DB therefore D is also situated at the locus. Let besides D, P is another point such that AP = BP. Now is P is joined to D, then AP and BP become the sides of two triangles ADP and BDP respectively. What can we say about these two triangles? We observe that these follow the SSS congurency theorem. So, $\Delta ADP \cong \Delta BDP$ in which $\angle ADP = \angle BDP$, that $\angle ADP = 90^\circ$ or $PD \perp AB$ can be proved easily. So, is the perpendicular bisector of AB. PD is also a straight line. Since, P is equidistant from A and B. If another point Q is also taken at PD, it will also be proved equidistant from A and B. Similarly, all the points on are equidistant form A and B. So, above description gives the following results.

Theorem 10.1.

The locus of a point equidistant form two given points is the perpendicular bisector of the line segment joining the two points.

Theorem 10.2. (Converse of theorem : 10.1)

Every point on the perpendicular bisector of line joining the two points is equidistant form these points.

Example 1 : ΔPBC, ΔQBC and ΔRBC three isoceles triangles are formed at the same base BC.Prove that the points P, Q and R are collinear.

Solution : Given : The $\Delta PBC, \Delta QBC$ and ΔRBC are such that PB = PC, QB = QC, RB = RC.

Proof: ΔPBC is an isosceles triangle

∴ $PB = PC$

And the locus of the point equidistant from B and C is the perpendicular bisector of B. Let it is l.

∴ P lies on l . . . (1)

Similarly Q and R also lie on l . . . (2)

From 1, and 2 we get

Hence, points P, Q and R collinear.

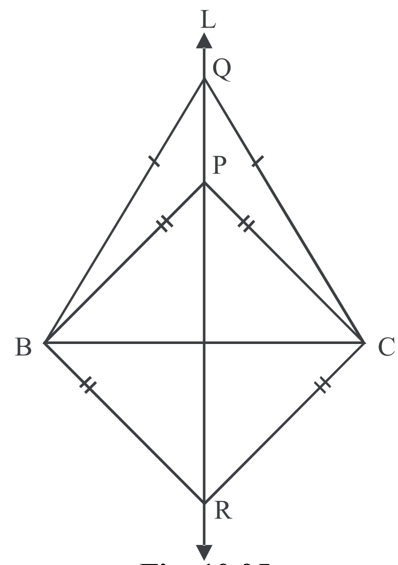

Fig. 10.05

10.04. Locus of points equidistant from two intersecting lines

Let two straight lines m and l intersect each other at O. And we are to find the locus equidistant from line l and m. If a point P which is not on l and m and its distance from p will be equal to the length of perpendicular drawn on l and m from P. At the other hand. If P lies on the sides l and m both then its distance from these lines will be zero.

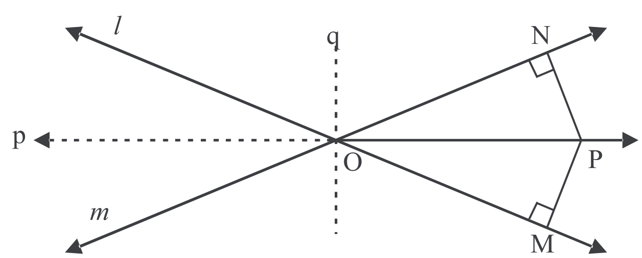

Fig. 10.06

If $d = 0$ then P is on l and m both *i.e.,* P will be coincidental to O and will be on the locus.

If d ? 0 then P will be neither on l nor on m. So it will be some where in four the angle formed by l and m.

Now, if $PM \perp \ell$ and $PN \perp m$, then

$PM = PN = d$ (by given conditions)

In ΔOPM and ΔOPN

$\angle M = \angle N$ ($90°$ each)

$OP = OP$ (Common)

$PM = PN (PM = PN = d)$

$\therefore$ $\Delta OPM \cong \Delta OPN$ (right angle, hypontaneous, side)

$\therefore$ $\angle POM = \angle PON$

Clearly, P is located inside $\angle MON$ andOP is the bisector of $\angle MON$ or P is on the bisector of $\angle MON$. Similarly, the point P may lie on the bisector of other three angles. The bisectors of these four angles form a straight lines. Let these lines be p and q. P will be an eliments of all points located on p and q. In this way we can say lines p and q are the locus of point P.

Hence, we obtain the following result form above explantion.

Theorem 10.2

The locus of a point equidistant from two intersecting lines is the pair of bisectors of the angles at that lines.

Example 2: The bisectors of $\angle B$ and $\angle C$ of a quadrilateral $ABCD$ meet each other at P. Prove that point P is equidistant form opposite sides AB and CD.

Solution :

Given : In quadrilateral $ABCD$ in which bisectors of $\angle B$ and $\angle C$ meet at P.

Also $PM \perp AB$ and $PN \perp CD$

To prove : $PM=PN$.

Construction : Draw $PL \perp BC$.

Proof : Point P lies on the bisector of $\angle B$

$\therefore$ $PM = PL$ . . . (1)

Point P lies on the bisector of $\angle C$

$\therefore$ $PL = PN$ . . . (2)

from (i) and (ii) we get

PM=PN **Proved.**

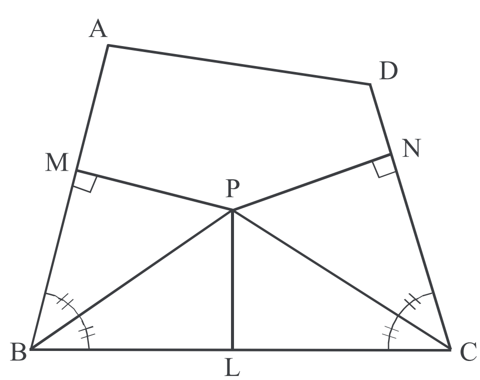

Fig. 10.07

Exercise 10.1

1. State whether following statements are true or false and justify your answer.
   (i) The set of the points equidistant form the given line is also a line.
   (ii) A circle is a locus of the points which are equidistant form a certain point.
   (iii) Three points are collinear only when they are not the elements of the set of the points of a line.
   (vi) The locus of the points equidistant from two lines will be a parallel line.
   (v) The locus of a point equidistant from two given points, the perpendicular bisector of the line joining the two points.
2. The diagonals of a quadrilateral bisect each other. Prove that the given quadrilateral is a parallelogram.
3. What will be the locus of a point equiditant from three non-collinear points A, B and C ? Justify your answer.
4. What will the locus of a points, equidistant form three collinear points? Justify your answer.
5. Prove that the locus of the centres of the circles passing through the points A and B is the bisector of line segment AB.
6. In the fig. 10.08. ΔPBC and ΔQBC are situated on the opposite sides of common base BC. Prove that the line joining P and Q. is a perpendicular bisector of BC.

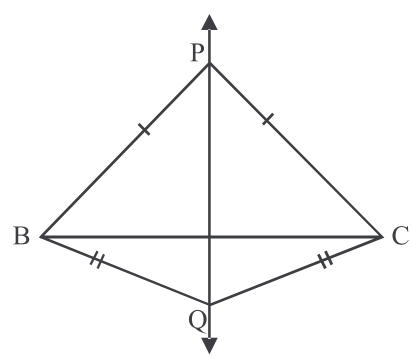

Fig. 10.08

7. In the fig.10.09, ΔPQR and ΔSQR are on the same side of common base QR. Prove that line SP is the perpendicular bisector of QR.

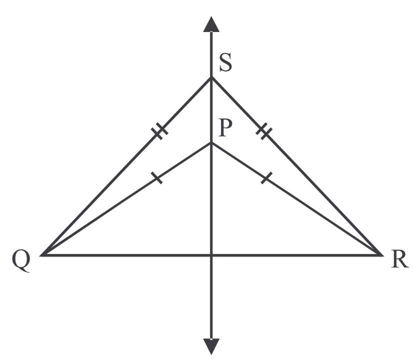

Fig. 10.09

8. In the given fig. 10.10 PS is the bisector of the angle $\angle P$ intersect QR at S. $SN \perp PQ$ and $SM \perp PR$ are drawn. Prove that $SN = SM$.

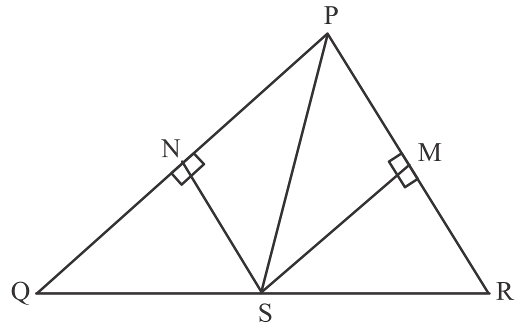

Fig. 10.10

9. In the fig. 10.11, find the locus of the points inside the $\angle ABC$ and equidistant form two sides BA and BC.

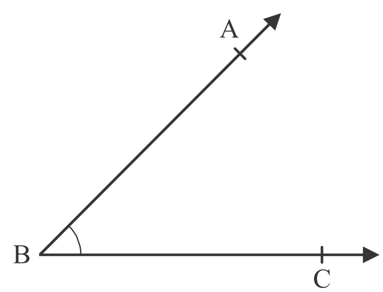

Fig. 10.11

10.5. Concurrent lines

In the previous class you have studied some terms related to triangles, which will be used in this section of the chapter. It is necesary to remind them.

1. **Median:** The line joining the vertex and the mid point of the opposite side is called the median of triangle.
2. **Perpendicular bisector:** Perpendicular drawn at the mid-point of the line segment is called the perpendicular bisector.
3. **Angle bisector:** Angle bisector is line which divides an angle into two equal parts.
4. **Altitude:** The line-segment which is obtained by drawing the perpendicular form a vertex of a triangle to its opposite side.
5. **Concurrent lines:** When three or more straight lines pass through a same point, they are called the concurrent lines. In this condition the common point of these lines is called the (point of concurrency). Let us discuss the points of cocurrency of above lines with which we get some certain results. That can be proved through the following theorems. All of these results are very useful in geometry.

Theorem 10.3

Perpendicular bisectors of the sides of a triangle are concurrent.

Given : In ΔABC, the perpendicular bisectors of AB and AC meet at O and $OD \perp BC$

To prove : OD is the perpendicular bisector of BC

Construction : Join OA, OB and OC

Proof :

OE and OF are the respectively perpendicular bisectors.

$\therefore$ $OA = OB = OC$ (from converse of 10.1)

$\because$ $OD \perp BC$

$\therefore$ OB = OC (By theorem 10.1)

OD is perpendicular bisector of BC

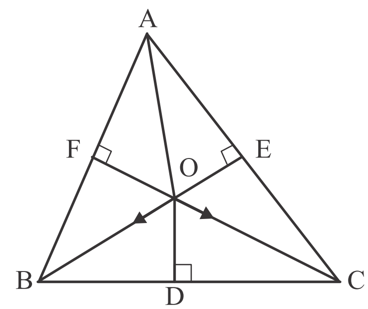

Fig. 10.12

Circumcentre : The perpendicular bisectors of three side of a triangle intersect each other at the same point, this point of intersection is called circumcentre of the triangle.

Theorem 10.4

Bisectors of three angles of a triangle are concurrent.

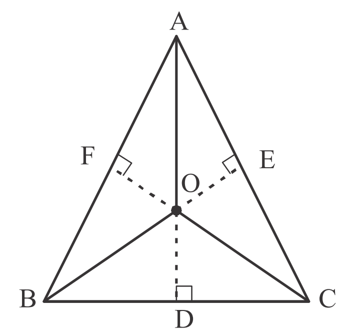

Fig. 10.13

Given : In ΔABC, bisector of $\angle B$ and $\angle C$ meet at O.

To prove : OA bisects $\angle A$

Construction : From O, draw $OD \perp BC, OE \perp AC$

and $OF \perp AB$

Proof : OB bisects $\angle B$ and OC bisects $\angle C$

$\therefore$ $OD = OE$ . . . (i) (from 10.2)

and $OD = OF$ . . . (ii)

from (i) and (ii) $OE = OF$

Thus O is equidistant form AB and AC.

Therefore OA bisects $\angle A$ **Proved.**

Circumcenter : The perpendicular bisectors of three side of a triangle intersect each other at the same point, this point of intersection is called circumcenter of the triangle.

Theorem 10.5

Three altitudes of a triangle are concurrent.

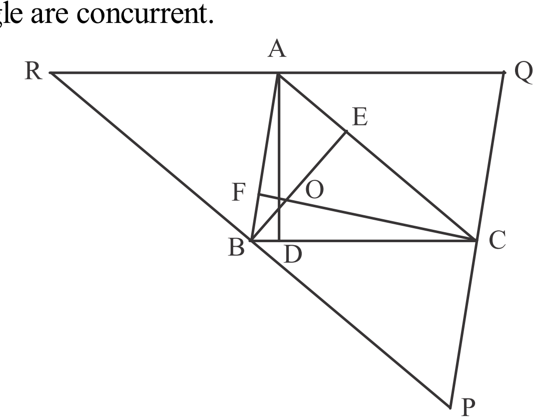

Fig. 10.14

Given : In ΔABC AD, CF and BE are vertex altitudes.

To prove : AD, CF and BE pass through a certain point.

Construction : From each vertex draw $QP \parallel AB$, $RQ \parallel BC$ and $RP \parallel AC$ (see fig. 10.14)

Proof : In BCAR, ∵ $AC \parallel RB$ and $BC \parallel RA$ (By construction)

∴ BCAR is a parallelogram.

∴ $RA = BC$ . . . (i)

Similarly ABCQ is a parallelogram.

∴ $AQ = BC$ . . . (ii)

From (i) and (ii) $AR = AQ$ . . . (iii)

And $AD \perp BC$, also $BC \parallel QR$

∴ $AD \perp QR$ . . . (iv)

AD is perpendicular bisector of QR similarly BE and CF are respectively perpendicualr bisectors of PR and PQ.

Since perpendicular bisectors of sides of the triangles are concurrent

∴ AD,CF and BE pass through a fixed point.

Ortho centre : The concurrent point of the altitudes of a triangles is called ortho centre.

Theorem 10.6

The medians of a triangle pass through a fixed point which divides every median in 2 : 1.

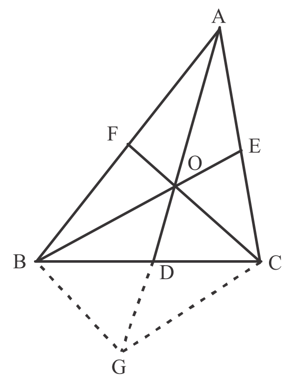

Fig. 10.15

Given : The medians BE and CF of a ΔABC intersect each other at O.

To prove : (i) The line AO is produced which bisects BC at D. *i.e.,* $BD = DC$.

(ii) $AO : OD = BO : OE = CO : OF = 2 : 1$

Construction : produce AD to G such that we get

$AO = OG$. Join BG and CG.

Proof : We know that the line joining of mid points of two sides in a triangle is parallel and half of the third side.

∴ In ΔABG, F mid point of AB, (given)

And O is the mid point of AG. (by constructiuon)

∴ OF BG and CO BG (since CO and OF are the part of CF) . . . (i)

and $OF = \frac{1}{2} BG$ . . . (ii)

Similarly in ΔACG , E and O are respectively mid points of AC and AG ,

So, OE GC and BO GC (Since, BO and OE are parts of BE) . . . (iii) and $OE = \frac{1}{2} GC$ . . . (iv)

From (i) and (iii) BOCG is a parallelogram. Since, diagonals of a parallelogram bisect each other.

$\therefore$ BD=DC

Hence, the line AD form the vertex A is also a median of ΔABC.

(ii) Since D is the point of intersection of diagonal of the parallelogram BOCG

$\therefore$ $OD = DG$ . . . (v)

and $OD = \frac{1}{2} OG$ . . . (vi)

Also AO = OG (by construction)

from (v) and (vi)

$$\text{OD} = \frac{1}{2}\text{AO}$$

or $\frac{AO}{OD} = \frac{2}{1} \Rightarrow$ AO : OD = 2 : 1

Similarly, we can prove that

BO : OE = 2 : 1 and CO : OF = 2 : 1

Hence AO : OD = BO : OE = CO : OF = 2 : 1 **Hence Proved.**

Illustrative Examples

Example 1 : The medians AD, BE and CF of a triangle ABC pass through the point G. If $AG = 6$ cm, BE = 12.6 cm and FG = 3cm then find the length of AD, GE and GC.

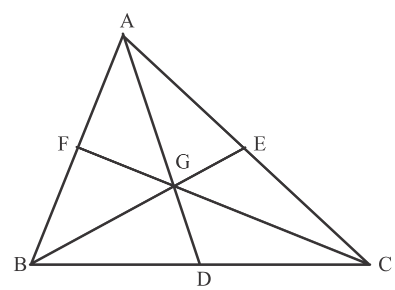

Fig. 10.16

Solution : We know that centroid of a triangle divides its medians in the ratio of 2 : 1

$$\therefore \quad \frac{AG}{GD} = \frac{2}{1} \quad \Rightarrow \quad \frac{GD}{AG} = \frac{1}{2}$$

Adding 1 to both sides

$$\frac{GD}{AG} + 1 = \frac{1}{2} + 1 \Rightarrow \quad \frac{GD + AG}{AG} = \frac{1+2}{2}$$

$$\Rightarrow \quad \frac{AD}{AG} = \frac{3}{2} \quad \Rightarrow \quad \frac{AD}{6} = \frac{3}{2}$$

$\Rightarrow \quad AD = \frac{3}{2} \times 6 \qquad \Rightarrow \quad AD = 9$ cm.

Similarly, $\frac{BG}{GE} = \frac{2}{1}$ or $\frac{BG}{GE} + 1 = \frac{2}{1} + 1$

or $\frac{BG + GE}{GE} = \frac{2+1}{1}$ or $\frac{BE}{GE} = \frac{3}{1}$

or $GE = \frac{1}{3} BE$ or $GE = \frac{12.6}{3}$

or $GE = 4.2$ and $\frac{FG}{GC} = \frac{1}{2}$

or $2FG = GC$ or $GC = 2 \times 3 = 6$

Hence $AD = 9$ cm, $GE = 4.2$ cm and $GC = 6$ cm.

Example 2. If the medians of a triangle are equal, then it is equilateral.

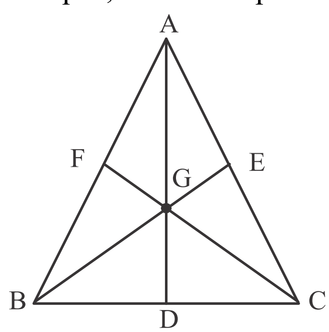

Fig. 10.17

Solution :

Given : In ΔABC, medians *AD, BE* and *CF* meet at *G* and *AD = BE = CF.*

To Prove : ΔABC is equilateral triangle.

Proof : We know that the centroid divides the medians of a triangle in 2 : 1.

So, $AD = BE = CF$ (given)

$\therefore \quad \frac{2}{3} AD = \frac{2}{3} BE = \frac{2}{3} CF$

Or $AG = BG = CG$ . . . (i)

Similarly, $\frac{1}{3} AD = \frac{1}{3} BE = \frac{1}{3} CF$

or $GD = GE = GF$ . . . (ii)

Now in ΔBGF and ΔCGE

BG = CG (from (i))

GF = GE (form (ii))

and $\angle BGF = \angle CGE$ (vertically opposite angle)

$\therefore \quad \Delta BGF \cong \Delta CGE$ (side, angle side congruency)

And corresponding sides of two congruent triangles are equal.

$\therefore \quad BF = CE$

And $\quad 2BF = 2CE$

$\Rightarrow \quad AB = AC$ . . . (iii)

Similarly, $\quad \Delta CGD \cong \Delta AGF$

So $\quad BC = AB$

from (iii) and (iv) . . . (iv)

$AD = BC = CF$

Hence ΔABC is a equilateral triangle. **Proved.**

Example 3 : If two medians of a triangle are equal in length, the triangle is an isosceles triangle.

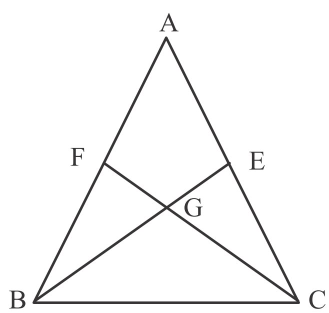

Fig. 10.18

Given : In ΔABC, $BE = CF$

and F and E are the mid-points of AB and AC

To prove : ΔABC is an isosceles triangle

Proof : In ΔABC, G is the centroid.

$\therefore \quad BG : GE = CG : GF = 2 : 1$ (given)

$\therefore \quad BG = \frac{2}{3} BE$ . . . (i)

$GE = \frac{1}{3} BE$ . . . (ii)

and $\quad CG = \frac{2}{3} CF$ . . . (iii)

$GF = \frac{1}{3} CF$ . . . (iv)

but $\quad BE = CF$ (given)

From (i) and (iii)

$BG = CG$

and from (ii) and (iv)

$GE = GF$

Now in ΔBGF and ΔCGE

BG = CG (Proved)

GE = GF (Proved)

$\angle BGF = \angle CGE$ (Vertically opposite angles)

$\therefore$ $\Delta FBG \cong \Delta ECG$ (SAS congurency)

Thus BF = CE or $2BF = 2CE$

$\therefore$ $AB = AC$

Hence *ABC* is an isosceles triangle **Proved.**

Exercise 10.2

1. Find the locus of a point equidistant form the three vertices and also find the locus of a pointed equidistant from the sides of a triangle.
2. In the ΔABC, medians *AD, BE* and *CF* intersect each other at G. If AG = 6 cm, BE = 9 cm, and GF = 4.5 cm, find the length of *GD*, *BG*, and *CF*.
3. In a triangle medians *AD, BE* and *CF* interesect each other at *G* prove that $AD + BE > \frac{3}{2}AB$.

   [Hint $AG + BG > AB$]
4. In a triangle ABC, the sum of two medians is greater that third one.
5. In a ΔABC, medians *AD*, *BE*, and *CF* passses through the point G.

   $4(AD + BE + CF) > 3(AB + BC + CA)$
6. In a ΔABC, *P* is the orthocentre. Prove that orthocentre of ΔPBC is *A*.
7. In ΔABC, the medians *AD*, *BE* and *CF* passses through the point *G*.
   (a) If GF = 4 cm then, find *GC*.
   (b) If *AD*=7.5 cm, then find *GD*.
8. In an isosceles ΔABC, $AB = AC$, and *D* is the mid-point of BC. Prove that circum centre, in centre, ortho centre and centroid all are collinear.
9. In a triangle *ABC, H* is the orthocentre. *X, Y* and *Z* are mid-points of *AH*, *BH* and *CH* respectively prove that orthocentre of ΔXYZ is *H*.
10. How will you find a point *P* inside *BC* of ΔABC which is equidistant form *AB* and *AC*.

Miscellaneous Exercise 10

Objective Questions (from 1 to 7)

1. The point equidistant from the vertices of triangle is called :

   (a) Centre of gravitation (b) Cicrcumcentre (c) Orthocentre (d) Incentre
2. Gravitation centre of a triangle :
   (a) Point of concurrency of prependicular bisectors of the sides of the triangle .
   (b) Concurrent point of angles bisectors of the triangle.
   (c) Point of concurrency of medians of the triangle.
   (d) The orthocentre.

3. Locus of the centre of rolling circle in a plane is :

(a) Circle (b) Curve

(c) a line parallel to the plane (d) perpendicular to the plane.

4. If two medians of a triangle are equal, then the triangle is :

(a) right triangle (b) isoceles triangle (c) equilateral triangle (d) none of these.

5. If *AB* and *CD* are two non-parallel lines, then the locus of the point *P* equidistant form these lines will be

(a) The line parallel to AB, and passing through the point P.

(b) The bisector of the angle subtended by lines *AB* and *CD* passing through point *P*.

(c) A parallel line to AB and CD both and passing through point P.

(d) The altitiude drawn on the sides *AB* and *CD* both and passsing through point *P*.

6. The triangle whose orthocentre, circumcentre and incentre coincides is known as

(a) equilateral triangle (b) right triangle (c) isosceles triangle (d) none of these

7. The triangle whose ortho centre is its vertex point is called.

(a) Right triangle (b) equilateral triangle (c) isosceles triangle (d) none of these

8. Find the locus of an end of a pendulam of the clock.

9. In a triangle *ABC, D,E* and *F* are the mid-points of side *BC, CA* and respectively then prove that *EF* bisects *AD*.

Important Points

1. Under some conditions the locus of a point is such a geometrical figure whose all points satisfy the given conditions.
2. The locus of a point equidistant form two points is the perpendicular bisector of line joining two given points.
3. The locus of the points equidistant form two inntersecting lines is the bisector of the angle formed by two given lines.
4. The perpendicluar bisectors of the sides are concurrent and this point of concurrency is called the circumcentre of triangle.
5. The angle bisector of a triangle are concurrent and this point of concurrency is called the incentre of triangle.
6. The altitudes of a triangle are concurrent and this point of concurrency is called the orthocentre of triangle.
7. Three medians of a triangle are concurrent this point of concurrency divides the median in to the ratio of 2 : 1 and called the orthocentre.

Answer Sheet

Exercise 10.1

(i) False, The locus of the points equidistant from the given line is the parallels either side this line.

(ii) True

(iii) False, three points may colinear only when they are the elements of the set of points which lies on that line.

(iv) False, it depends on the location of two lines (If two given lines are parallel, then it will parallel to them and if they intrersect each other then it will on the bisector of angle formed by them)

(v) True.

Exercise 10.2

1. Circumcentre, incentre 2. 3 cm, 6 cm, 13.5 cm, 7.8 cm, 2.5 cm

Miscellaneous Exercise 10

1. b 2. c 3. c 4. b 5. b 6. c 7. a

8. an arc