EXERCISE 7.10

Evaluate the Integrals using substitution:

Q No 1 $\int_0^1 \frac{x}{x^2+1}\,dx.$

Sol. Substitute $x^2+1=t$

$\Rightarrow \quad 2x\,dx = dt \quad \Rightarrow x\,dx = dt/2.$

When $x=0$; $t = 0^2+1 = 1$

When $x=1$; $t = 1^2+1 = 2$

$\therefore \; I = \int_1^2 \frac{dt/2}{t} = \frac{1}{2}\int \frac{1}{t}\,dt = \frac{1}{2}\left[\log|t|\right]_1^2$

$= \frac{1}{2}\left[\log 2 - \log 1\right] = \frac{1}{2}\log 2.$

Q No 2 : $\int_0^{\pi/2} \sqrt{\sin\phi}\,\cos^5\phi\,d\phi$

Sol. Substitute $\sin\phi = t$

$\Rightarrow \quad \cos\phi\,d\phi = dt$

When $\phi = 0$; $t = \sin 0^\circ = 0$

When $\phi = \pi/2$; $t = \sin \pi/2 = 1$

$\therefore \; I = \int_0^{\pi/2} \sqrt{\sin\phi}\cdot\cos^4\phi\cdot\cos\phi\cdot d\phi = \int_0^{\pi/2} \sqrt{\sin\phi}\,(1-\sin^2\phi)^2\cos\phi\,d\phi$

$= \int_0^1 \sqrt{t}\,(1-t^2)^2\,dt = \int_0^1 \sqrt{t}\,(1+t^4-2t^2)\,dt$

$= \int_0^1 \left(\sqrt{t} + t^{4+\frac{1}{2}} - 2\,t^{2+\frac{1}{2}}\right)dt = \int_0^1 \left(t^{9/2} - 2t^{5/2} + t^{1/2}\right)dt$

$= \left[\frac{t^{11/2}}{11/2} - 2\,\frac{t^{\frac{7}{2}}}{\frac{7}{2}} + \frac{t^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^1 = \left(\frac{2}{11}(1)^{\frac{11}{2}} - \frac{4}{7}(1)^{7/2} + \frac{2}{3}(1)^{3/2}\right) - (0)$

$= \frac{2}{11} + \frac{2}{3} - \frac{4}{7} = \frac{64}{231}.$

Q.No. 3 $\int_0^1 \sin^{-1}\left(\frac{2x}{1+x^2}\right) dx$

Sol. Put $x = \tan\theta$

$\Rightarrow dx = \sec^2\theta \cdot d\theta$

When $x = 0$; $\tan\theta = 0 \Rightarrow \theta = 0$

When $x = 1$; $\tan\theta = 1 \Rightarrow \theta = \frac{\pi}{4}$

$$\therefore I = \int_0^{\pi/4} \sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right) \cdot \sec^2\theta \, d\theta$$

$$= \int_0^{\pi/4} \sin^{-1}(\sin 2\theta) \cdot \sec^2\theta \cdot d\theta$$

$$= \int_0^{\pi/4} 2\theta \cdot \sec^2\theta \cdot d\theta$$

Integrating by parts taking θ as first part.

$$I = 2\left[\left[\theta \cdot \tan\theta\right]_0^{\pi/4} - \int_0^{\pi/4} 1 \cdot \tan\theta \, d\theta\right]$$

$$= 2\left[\left(\frac{\pi}{4}\tan\frac{\pi}{4} - 0\right) + \left[\log|\cos\theta|\right]_0^{\pi/4}\right]$$

$$= 2\left[\frac{\pi}{4}\tan\frac{\pi}{4} + \left(\log\left(\cos\frac{\pi}{4}\right) - \log(\cos 0)\right)\right]$$

$$= \frac{\pi}{2}\tan\frac{\pi}{4} + 2\log\left(\frac{1}{\sqrt{2}}\right) - 2\log 1$$

$$= \frac{\pi}{2} \times 1 + 2\log 1 - 2\log\sqrt{2} + 0$$

$$= \frac{\pi}{2} - \log\left[(2)^{1/2}\right]^2 = \frac{\pi}{2} - \log 2.$$

Q No. 4 $\int_0^2 x\sqrt{x+2}\,dx$

Sol. Put $x+2 = t^2 \Rightarrow x = t^2-2$

$\Rightarrow dx = 2t\,dt$

When $x=0$, $t^2 = 0+2 \Rightarrow t = \sqrt{2}$

When $x=2$, $t^2 = 2+2 \Rightarrow t = \sqrt{4} = 2.$

$\therefore I = \int_{\sqrt{2}}^{2} (t^2-2)(t)\,2t\,dt = 2\int_{\sqrt{2}}^{2} (t^4 - 2t^2)\,dt$

$= 2\left[\frac{t^5}{5} - 2\frac{t^3}{3}\right]_{\sqrt{2}}^{2} = \left[\frac{2}{5}t^5 - \frac{4}{3}t^3\right]_{\sqrt{2}}^{2}$

$= \frac{2}{5}\left[t^5\right]_{\sqrt{2}}^{2} - \frac{4}{3}\left[t^3\right]_{\sqrt{2}}^{2}$

$= \frac{2}{5}\left[(2)^5 - (\sqrt{2})^5\right] - \frac{4}{3}\left[(2)^3 - (\sqrt{2})^3\right]$

$= \frac{2}{5}\left[32 - 4\sqrt{2}\right] - \frac{4}{3}\left[8 - 2\sqrt{2}\right]$

$= \frac{64}{5} - \frac{8\sqrt{2}}{5} - \frac{32}{3} + \frac{8\sqrt{2}}{3}$

$= \frac{192-160}{15} + \frac{-24\sqrt{3}+40\sqrt{2}}{15} = \frac{32}{15} + \frac{16}{15}\sqrt{2}$

Q No 5. $\int_0^{\pi/2} \frac{\sin x}{1+\cos^2 x}\,dx.$

Sol. Put $\cos x = t \Rightarrow -\sin x\,dx = dt$

When $x = 0$, $t = \cos 0 = 1$

When $x = \pi/2$ $t = \cos \pi/2 = 0.$

$\therefore I = \int_1^0 \frac{-dt}{1+t^2} = -\int_1^0 \frac{1}{1+t^2}\,dt = -\left[\tan^{-1}t\right]_1^0$

$= -\left[\tan^{-1}(0) - \tan^{-1}(1)\right] = -\left(0 - \pi/4\right) = \pi/4.$

Q.No 6. $\int_0^2 \frac{dx}{x+4-x^2}$

$$I = \int_0^2 \frac{dx}{4-(x^2-x)} = \int_0^2 \frac{dx}{4-\left(x^2-x+\frac{1}{4}-\frac{1}{4}\right)} = \int_0^2 \frac{dx}{\left(4+\frac{1}{4}\right)-\left(x-\frac{1}{2}\right)^2}$$

$$= \int_0^2 \frac{dx}{\left(\frac{\sqrt{17}}{2}\right)^2-\left(x-\frac{1}{2}\right)^2}$$

Put $x-\frac{1}{2} = t \Rightarrow dx = dt$

When $x=0$, $t = -\frac{1}{2}$

When $x=2$, $t = 3/2$.

$$I = \int_{-\frac{1}{2}}^{3/2} \frac{dt}{\left(\frac{\sqrt{17}}{2}\right)^2-t^2} = \frac{1}{2\times\frac{\sqrt{17}}{2}}\left[\log\left|\frac{\frac{\sqrt{17}}{2}+t}{\frac{\sqrt{17}}{2}-t}\right|\right]_{-\frac{1}{2}}^{3/2} \quad \left[\because \int\frac{1}{a^2-x^2}dx = \frac{1}{2a}\log\left|\frac{a+x}{a-x}\right|\right.$$

$$= \frac{1}{\sqrt{17}}\left[\log\left|\frac{\frac{\sqrt{17}}{2}+\frac{3}{2}}{\frac{\sqrt{17}}{2}-\frac{3}{2}}\right| - \log\left|\frac{\frac{\sqrt{17}}{2}-\frac{1}{2}}{\frac{\sqrt{17}}{2}+\frac{1}{2}}\right|\right].$$

$$= \frac{1}{\sqrt{17}}\left[\log\left|\frac{\sqrt{17}+3}{\sqrt{17}-3}\right| - \log\left|\frac{\sqrt{17}-1}{\sqrt{17}+1}\right|\right.$$

$$= \frac{1}{\sqrt{17}}\log\left|\frac{(\sqrt{17}+3)(\sqrt{17}+1)}{(\sqrt{17}-3)(\sqrt{17}-1)}\right| = \frac{1}{\sqrt{17}}\log\left|\frac{20+4\sqrt{17}}{20-4\sqrt{17}}\right|$$

$$= \frac{1}{\sqrt{17}}\log\left|\frac{20+4\sqrt{17}}{20-4\sqrt{17}}\times\frac{20+4\sqrt{17}}{20+4\sqrt{17}}\right|.$$

$$= \frac{1}{\sqrt{17}}\log\left|\frac{400+272+160\sqrt{17}}{400-272}\right| = \frac{1}{\sqrt{17}}\log\left|\frac{672+160\sqrt{17}}{128}\right|$$

$$= \frac{1}{\sqrt{17}}\log\left[\frac{21+5\sqrt{17}}{4}\right]$$

Q No 7. $\int_{-1}^{1} \frac{dx}{x^2+2x+5}$

Sol

$$I = \int_{-1}^{1} \frac{dx}{x^2+2x+1-1+5} = \int_{-1}^{1} \frac{dx}{(x+1)^2+(2)^2}$$

Put $x+1 = t \Rightarrow dx = dt$.

When $x = -1$; $t = -1+1 = 0$
When $x = 1$; $t = 1+1 = 2$

$$\therefore I = \int_0^2 \frac{dt}{t^2+(2)^2} = \frac{1}{2}\left[\tan^{-1}\frac{t}{2}\right]_0^2 = \frac{1}{2}\left[\tan^{-1}(1) - \tan^{-1}(0)\right]$$

$$= \frac{1}{2}\left[\frac{\pi}{4} - 0\right] = \frac{\pi}{8}$$

Q No 8. $\int_1^2 \left(\frac{1}{x} - \frac{1}{2x^2}\right) e^{2x}\, dx.$

Sol Put $2x = t \Rightarrow 2dx = dt$ and $x = \frac{t}{2}$

When $x = 1$; $t = 2$
$x = 2$, $t = 4$

$$\therefore I = \int_2^4 \left(\frac{1}{t/2} - \frac{1}{2\left(\frac{t}{2}\right)^2}\right) \cdot e^t \frac{dt}{2}$$

$$= \int_2^4 \left(\frac{2}{t} - \frac{2}{t^2}\right) e^t \frac{dt}{2} = \int_2^4 \left(\frac{1}{t} - \frac{1}{t^2}\right) e^t\, dt$$

$$= \left[e^t \cdot \frac{1}{t}\right]_2^4 \qquad \left[\because \int e^x\left(f(x) + f'(x)\right) dx = e^x f(x) + c\right]$$

$$= \frac{e^4}{4} - \frac{e^2}{2} = \frac{e^2}{2}\left[\frac{e^2}{2} - 1\right]$$

Choose the correct answer.

Q.No 9 The Value of $\int_{\frac{1}{3}}^{1} \frac{(x-x^3)^{1/3}}{x^4}\,dx$ is

(A) 6 (B) 0 (C) 3 (D) 4

Sol. $\int_{1/3}^{1} \frac{(x^3)^{1/3}\left[\frac{x}{x^3}-\frac{x^3}{x^3}\right]^{\frac{1}{3}}}{x^4}\,dx. = \int_{\frac{1}{3}}^{1} \frac{\left(\frac{1}{x^2}-1\right)^{1/3}}{x^3}\,dx.$

Put $\frac{1}{x^2} = t \Rightarrow -\frac{2}{x^3}\,dx = dt$

When $x = 1$, $t = 1$

When $x = \frac{1}{3}$; $t = 9$

$\therefore I = \int_{9}^{1} (t-1)^{1/3}\left(-\frac{1}{2}\,dt\right) = -\frac{1}{2}\left[\frac{(t-1)^{\frac{4}{3}}}{4/3}\right]_9^1$

$= -\frac{1}{2}\times\frac{3}{4}\left[(1-1)^{4/3} - (9-1)^{4/3}\right] = -\frac{3}{8}\left[0-(2^3)^{4/3}\right]$

$= -\frac{3}{8}\times(-16) = 6$

$\therefore$ (A) is correct answer.

Q.No 10: If $f(x) = \int_0^x t\sin t\,dt$, $f'(x)$ is

(A) $\cos x + \sin x$ (B) $x\sin x$ (C) $x\cos x$ (D) $\sin x + x\cos x$.

Sol: Given $f(x) = \int_0^x t\sin t\,dt \Rightarrow f'(x) = x\sin x$.

(By first fundamental theorem of Integral Calculus)

$\therefore$ (B) is correct answer.

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