Integrals

Integration as an Inverse Process of Differentiation

- Integration is the inverse of differentiation. It is also called as anti-differentiation.

- The integration of a function $f(x)$ with respect to x is denoted by $\int f(x)dx$.

- Example: We know that $\frac{d}{dx}\left(x^2\right) = 2x$

  Here, $2x$ is the derived function of x^2 and x^2 is primitive of $2x$ or we say x^2 is the anti-derivative (or an integral) of $2x$.

- If there is a function F such that $\frac{d}{dx}\left(F(x)\right) = f(x)$, &mnForE;$x \in I$ (interval), then for any $C \in \mathbf{R}$, $\frac{d}{dx}\left[F(x)+C\right]$ $= f(x)$, $x \in I$.

  $\therefore \{F + C, C \in \mathbf{R}\}$ is called the family of anti-derivatives of f.

  [C is called the constant of integration]

- Notation: If $\frac{dy}{dx} = f(x)$, then we write $y = \int f(x)dx$

- Formulae for integrals of some functions.

(i) $\int x^n dx = \frac{x^{n+1}}{n+1} + C,\ x \neq -1$

(ii) $\int \cos x dx = \sin x + C$

(iii) $\int \sin x dx = -\cos x + C$

(iv) $\int \sec^2 x dx = \tan x + C$

(v) $\int \operatorname{cosec}^2 x dx = -\cot x + C$

(vi) $\int \sec x \tan x = \sec x + C$

(viii) $\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x + C$ s

(ix) $\int \frac{dx}{\sqrt{1-x^2}} = -\cos^{-1} x + C$

(x) $\int \frac{dx}{1+x^2} = \tan^{-1} x + C$

(xi) $\int \frac{dx}{1+x^2} = -\cot^{-1} x + C$

(xii) $\int \frac{dx}{x\sqrt{x^2-1}} = \sec^{-1} x + C$

(xiii) $\int \frac{dx}{x\sqrt{x^2-1}} = -\text{cosec}^{-1} x + C$

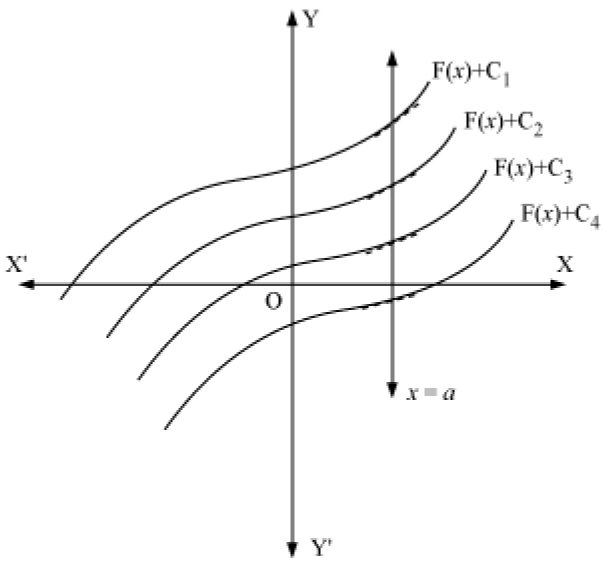
(xiv) $\int e^x dx = e^x + C$

(xv) $\int \frac{1}{x} dx = \log|x| + C$

(xvi) $\int a^x dx = \frac{a^x}{\log a} + C$

- **Geometrical interpretation of indefinite integral**

  The equation $\int f(x)dx = F(x) + C$ = y (say) represents a family of curves. For different values of *C*, there correspond different members of this family and these members can be obtained by shifting any one of the curves parallel to it. This can be diagrammatically represented as



Important properties of indefinite integral

- Two indefinite integrals with the same derivative lead to the same family of curves. Hence, they are equivalent.

  The equivalence of families $\{\int f(x)\, dx + C_1, C_1 \in \mathbf{R}\}$ and $\{\int g(x)\, dx + C_2, C_2 \in \mathbf{R}\}$ is denoted as

  $\int f(x)\, dx = \int g(x)\, dx$

- $\int [f(x) \pm g(x)]dx = \int f(x)dx \pm \int g(x)dx$
- $\int kf(x)dx = k\int f(x)dx$ where $k \in \mathbf{R}$
- $$\int \left[k_1 f_1(x) + k_2 f_2(x) + \ldots + k_x f_x(x)\right]dx = k_1 \int f_1(x)dx + k_2 \int f_2(x)dx + \ldots + k_n \int f_n(x)dx$$

Comparison between Differentiation and Integration

- Both satisfy the property of linearity
- All functions are not differentiable. Similarly, all functions are not integrable
- The derivative of a function, when it exists, is a unique function. However, it is not so in the case of integration.
- The derivative of a function at a point may exist. However, an integral at a point makes no sense. We usually find the integral of a function over an interval.

Solved Examples

Example 1

Find the anti-derivative of sin2*x*.

Solution:

$$\frac{d}{dx}(\cos 2x) = -2\sin 2x$$

$$\Rightarrow \sin 2x = -\frac{1}{2}.\frac{d}{dx}(\cos 2x) = \frac{d}{dx}\left(-\frac{1}{2}\cos 2x\right)$$

∴ The anti-derivative of sin2*x* is $-\frac{1}{2}\cos 2x$.

Example 2

Integrate $\int \frac{x^2+2}{x}dx$.

Solution:

$$\int \frac{x^2+2}{x}dx = \int\left(x+\frac{2}{x}\right)dx = \frac{x^2}{2} + 2\log|x| + C$$

Method of Integration by Substitution

- The given integral ∫*f*(*x*)*dx* can be transformed into another form by changing the independent variable *x* to *t* by substituting *x* = *g*(*t*)

  Put *x* = *g*(*t*) so that $\frac{dx}{dt} = g'(t) \Rightarrow dx = g'(t)\,dt$

  Then, $\int f(x)dx = \int f(g(t))g'(t)\,dt$

- For example, integrate cos (*mx* + 1) with respect to *x*.

  Let *t* = (*mx + 1*)

  ∴ *dt* = *m dx* ⇒ *dx* = $\frac{1}{m}dt$

  ∴ ∫cos(*mx* + 1) *dx* = $\int \frac{\cos t}{m}dt = \frac{1}{m}\int \cos t\,dt$

  $= \frac{1}{m}\sin t + C$

$$= \frac{\sin(mx+1)}{m} + C$$

Integration using trigonometric identities

- When the integrand involves some trigonometric functions, identities can be used to find the integral.
  - For example, solve $\int \cos^3 x dx$
    $\cos 3x = 4\cos^3 x - 3\cos x$
    $\Rightarrow \cos^3 x = \frac{1}{4}\cos 3x + \frac{3}{4}\cos x$

$\therefore$ $\int \cos^3 x dx = \frac{1}{4}\int \cos 3x dx + \frac{3}{4}\int \cos x dx$

$$= \frac{1}{4}\cdot\frac{\sin 3x}{3} + \frac{3}{4}\cdot\sin x + C$$

Integrals of Some Particular Functions

- $\int \frac{dx}{x^2 - a^2} = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + C$
- $\int \frac{dx}{a^2 - x^2} = \frac{1}{2a}\log\left|\frac{a+x}{a-x}\right| + C$
- $\int \frac{dx}{x^2 + a^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + C$
- $\int \frac{dx}{\sqrt{x^2 - a^2}} = \log\left|x + \sqrt{x^2 - a^2}\right| + C$
- $\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\frac{x}{a} + C$
- $\int \frac{dx}{\sqrt{x^2 + a^2}} = \log\left|x + \sqrt{x^2 + a^2}\right| + c$

- For finding $\int \frac{dx}{ax^2+bx+c}$, we first express
  $$ax^2+bx+c = a\left[x^2+\frac{b}{a}x+\frac{c}{a}\right]$$
  $$= a\left[\left(x+\frac{b}{2a}\right)^2+\left(\frac{c}{a}-\frac{b^2}{4a^2}\right)\right]$$
  Then, let t=x+b2at=x+b2a so that $dt = dx$ and write $\frac{c}{a}-\frac{b^2}{4a^2} = \pm k^2$
  Then, the integral is reduced to the form $\frac{1}{a}\int \frac{dt}{t^2 \pm k^2}$. Depending upon the sign of $\frac{c}{a}-\frac{b^2}{4a^2}$, the given integral can be evaluated.

- Integral of the type $\int \frac{dx}{\sqrt{ax^2+bc+c}}$ can also be evaluated by the previous method.

- To find the integral of the type $\int \frac{px+q}{ax^2+bx+c}dx$, where p, q, a, b and c are constants, we first find A, B and C such that

  $px+q = \mathrm{A}\frac{d}{dx}(ax^2 + bx + c) + \mathrm{B}$

- = A($2ax + b$) + B
  A and B are to be determined by equating the co-efficients of x and constant terms. Thus, the given integral reduces to some standard form, which can be further evaluated easily.

- Integral of type $\int \frac{px+q}{\sqrt{ax^2+bx+c}}dx$ can be evaluated by the previous method.

Solved Examples

Example 1:

Integrate $\sin^5 x \cos^3 x$ with respect to x.

Solution:

Put $\cos x = t$

$\therefore -\sin x\, dx = dt$

$\Rightarrow dx = -\frac{dt}{\sin x}$

$$I = \int \sin^5 x \cos^3 x dx = -\int \sin^5 x \cdot t^3 \frac{dt}{\sin x} = -\int \sin^4 x t^3 dt$$

$\sin^4 x = (\sin^2 x)^2$

$= (1 - \cos^2 x)^2$

$= (1 - t^2)^2$

$= 1 + t^4 - 2t^2$

$\therefore \sin^4 x\, t^3 = (1 + t^4 - 2t^2)\, t^3 = t^3 + t^7 - 2t^5$

$$\therefore I = -\int \left(t^7 - 2t^5 + t^3\right) dt = -\left[\int t^7 dt - \int 2t^5 + dt + \int t^3 dt\right]$$

$$= -\left[\frac{t^8}{8} - \frac{2t^6}{6} + \frac{t^4}{4}\right] + C$$

$$= -\frac{\cos^8 x}{8} + \frac{\cos^6 x}{3} - \frac{\cos^4 x}{4} + C$$

Example 2:

Solve: $\int \sin 6x \sin 4x\, dx$

Solution:

$-2\sin 6x \sin 4x = \cos(6x + 4x) - \cos(6x - 4x)$

$= \cos 10x - \cos 2x$

$$\int \sin 6x \ \sin 4x\, dx = \int -\frac{1}{2}[\cos 10x - \cos 2x]\, dx$$

$$= \frac{1}{2}\left[\int \cos 2x\, dx - \int \cos 10x\, dx\right]$$

$$= \frac{1}{2}\left[\frac{\sin 2x}{2} - \frac{\sin 10x}{10}\right] + C$$

$$= \frac{1}{4}\left(\sin 2x - \frac{1}{5}\sin 10x\right) + C$$

Example 3:

Solve: $\int \frac{dx}{x^2-4}$

Solution:

$$\int \frac{dx}{x^2-4} = \int \frac{dx}{x^2-2^2} = \frac{1}{2\times 2}\log\left|\frac{x-2}{x+2}\right| + C$$

Example 4:

Solve: $\int \frac{dx}{x^2+3}$

Solution:

$$\int \frac{dx}{x^2+3} = \int \frac{dx}{x^2+\left(\sqrt{3}\right)^2} = \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{x}{\sqrt{3}}\right) + C$$

Example 5:

Solve: $\int \frac{x-1}{3x^2-2x+4}$

Solution:

$(x - 1) = A \frac{d}{dx} (3x^2 - 2x + 4) + B$

$\Rightarrow x - 1 = A (6x - 2) + B$

$= 6Ax - 2A + B$

Here, $6A = 1$, $2A - B = 1$

$\Rightarrow A = \frac{1}{6}$, $B = -\frac{2}{3}$

$$\therefore \int \frac{x-1}{3x^2-2x+4}\, dx = \frac{1}{6} \int \frac{(6x-2)}{3x^2-2x+4} - \frac{2}{3} \int \frac{dx}{3x^2-2x+4}$$

$$= \frac{1}{6} I_1 - \frac{2}{3} I_2 \text{ (say)}$$

In I_1, put $3x^2 - 2x + 4 = t$ so that $(6x - 2)dx = dt$

$I_1 = \int \frac{dt}{t} = \log|t| + C_1$

$= \log|3x^2 - 2x + 4| + C_1$

$$I_2 = \int \frac{dx}{3x^2 - 2x + 4} = \frac{1}{3}\int \frac{dx}{\left(x - \frac{1}{3}\right)^2 + \frac{11}{9}}$$

$$= \frac{1}{3}\int \frac{dx}{\left(x - \frac{1}{3}\right)^2 + \left(\frac{\sqrt{11}}{3}\right)^2} = \frac{1}{3} \times \frac{3}{\sqrt{11}} \tan^{-1}\left\{\left(x - \frac{1}{3}\right)\frac{3}{\sqrt{11}}\right\} + C$$

$$= \frac{1}{\sqrt{11}} \tan^{-1} \frac{3x - 1}{\sqrt{11}} + C_2$$

$\therefore I = \frac{1}{6}\log|3x^2 - 2x + 4| - \frac{2}{3\sqrt{11}} \tan^{-1}\left(\frac{3x - 1}{\sqrt{11}}\right) + C$ (where $C = C_1 + C_2$)

Integration by Partial Fractions

- Integral of rational function $\frac{p(x)}{q(x)}$ (where $p(x)$ and $q(x)$ are polynomials in x and $q(x) \neq 0$) can be performed by expressing the integral as a sum of simple rational functions, and then apply known method. In this method, if $\frac{p(x)}{q(x)}$ is improper, then we first convert it into proper fraction (i.e., $\frac{p(x)}{q(x)} = T(x) + \frac{p_1(x)}{Q_1(x)}$, where $T(x)$ is a polynomial in x and $\frac{p_1(x)}{Q_1(x)}$ is a proper rational function), by long division process.
- Types of simpler of partial fractions that are to be associated with various kind of rational functions can be listed as:

Form of Rational Function	Form of the Partial Fraction

$\frac{p(x)+q}{(x-a)(x-b)} a \neq b$	$\frac{A}{x-a}+\frac{B}{x-b}$
$\frac{p(x)+q}{(x-a)^2}$	$\frac{A}{x-a}+\frac{B}{(x-a)^2}$
$\frac{px^2+qx+r}{(x-a)(x-b)(x-c)}$	$\frac{A}{x-a}+\frac{B}{x-b}+\frac{C}{x-c}$
$\frac{px^2+qx+r}{(x-a)^2(x-b)}$	$\frac{A}{x-a}+\frac{B}{(x-a)^2}+\frac{C}{x-b}$
$\frac{px^2+qx+r}{(x-a)(x^2+bx+c)}$ Where, $x^2 + bx + c$ cannot be factorised further.	$\frac{A}{x-a}+\frac{Bx+C}{x^2+bx+c}$ A, B, C are real numbers that are to be determined.

Some Related Solved Problems

Example 1:

Solve: $\int \frac{1-2x}{x^2-3x-4} dx$

Solution:

$x^2 - 3x - 4 = (x + 1)(x - 4)$

Let $\frac{1-2x}{x^2-3x-4} = \frac{1-2x}{(x+1)(x-4)} = \frac{A}{x+1}+\frac{B}{x-4}$

$\therefore 1 - 2x = A(x - 4) + B(x + 1)$

$\Rightarrow 1 - 2x = (A + B)x + B - 4A$

Comparing coefficient of x and constant term,

$A + B = -2$ and $B - 4A = 1$

On solving the above two equations, we get A = $-\frac{3}{5}$, B = $-\frac{7}{5}$.

$$\therefore \int \frac{1-2x}{x^2-3x-4}\,dx = \int \frac{-3/5}{x+1}\,dx + \int \frac{-7/5}{x-4}dx$$

$$= -\frac{3}{5}\log|x+1| - \frac{7}{5}\log|x-4| + C$$

$$= -\frac{1}{5}\left[3\log|x+1| + 7\log|x-4|\right] + C$$

Example 2:

Solve: $\int \frac{2x+1}{(x-1)(x^2+x-2)}dx$

Solution:

$$\frac{2x+1}{(x-1)(x^2+x-2)} = \frac{2x+1}{(x-1)(x-1)(x+2)} = \frac{2x+1}{(x-1)^2(x+2)}$$

$$= \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+2}$$

$\Rightarrow 2x + 1 = A(x - 1)(x + 2) + B(x + 2) + C(x - 1)^2$

$\Rightarrow 2x + 1 = A(x^2 + x - 2) + B(x + 2) + C(x^2 - 2x + 1)$

$\Rightarrow 2x + 1 = (A + C)x^2 + (A + B - 2C)x - 2A + 2B + C$

Comparing coefficients and constant terms,

$A + C = 0$, $A + B - 2C = 2$, $-2A + 2B + C = 1$

On solving, we obtain A = $\frac{1}{3}$, B = 1, C = $-\frac{1}{3}$

$$I = \int \frac{1/3}{x-1}dx + \int \frac{1}{(x-1)^2}dx + \int \frac{-1/3}{x+2}dx$$

$$= \frac{1}{3}\log|x-1| - \frac{1}{(x-1)} - \frac{1}{3}\log|x+2| + C$$

$$= \frac{1}{3}\log\left|\frac{x-1}{x+2}\right| - \frac{1}{x-1} + C$$

Example 3:

Solve: $\int \frac{x+1}{(x-3)(x^2+2)}dx$

Solution:

Let $\int \frac{x+1}{(x-3)(x^2+2)} = \frac{A}{x-3} + \frac{Bx+C}{x^2+2}$

$$= \frac{A(x^2+2)+(Bx+C)(x-3)}{(x-3)(x^2+2)}$$

$\Rightarrow x + 1 = (A + B)x^2 + (C - 3B)\,x + 2A - 3C$

Comparing coefficients of x^2, x, and constant terms.

A + B = 0, C − 3B = 1, 2A − 3C = 1

$$\Rightarrow A = \frac{4}{11},\ B = -\frac{4}{11},\ C = \frac{-1}{11}$$

$$\therefore I = \frac{4}{11}\int \frac{1}{x-3}dx + \int \frac{-\frac{4}{11}x - \frac{1}{11}}{x^2+2}dx$$

$$= \frac{4}{11}\log|x-3| - \frac{1}{11}\int \frac{4x+1}{x^2+2}dx + C_1$$

$$\int \frac{4x+1}{x^2+2}dx = \int \frac{2(2x)+1}{x^2+2}dx$$

$$= 2\int \frac{2x}{x^2+2}dx + \int \frac{1}{x^2+2}dx$$

$$= 2\log|x^2+2| + \frac{1}{\sqrt{2}}\tan^{-1}\frac{x}{\sqrt{2}} + C_2$$

$$\therefore I = \frac{4}{11}\log|x-3| + 2\ \log|x^2+2| + \frac{1}{\sqrt{2}}\tan^{-1}\frac{x}{\sqrt{2}} + C \qquad (\text{where } C_1 + C_2 = C)$$

Integration by Parts

- The function of the form $f(x)\,g(x)$ can be integrated by using the method of integration by parts.

  $$\int f(x)g(x)\,dx = f(x)\int g(x)\,dx - \int\left[f'(x)\int g(x)\,dx\right]dx$$

  Integral of the product of two functions = (First function) × (Integral of the second function) − Integral of [(Differential coefficient of the first function) × (Integral of the second function)]

- Generally, a polynomial function is taken as first function. In cases where other function is inverse trigonometric or a logarithmic function, then they are taken as first function.
- In finding the integral of second function, the constant of integration is not added.

- **The integral of the type** $\int e^x\left[f(x)+f'(x)\right]dx$ **can be evaluated by the method of integration by parts and can be concluded as**

  $$\int e^x\left[f(x)+f'(x)\right]dx = e^x f(x) + C$$

- **For example, consider** $\int\left(\frac{x-1}{x^2}\right)e^x\,dx$

  $$I = \int\left(\frac{x-1}{x^2}\right)e^x\,dx$$

  $$= \int e^x\left(-\frac{1}{x^2}+\frac{1}{x}\right)dx \qquad \left\{\int e^x\left[f(x)+f'(x)\right]dx \text{ where } f(x)=\frac{1}{x}\right\}$$

  $$\therefore I = \frac{e^x}{x} + c$$

- **Integrals of the form** $\sqrt{x^2-a^2}, \sqrt{x^2+a^2}, \sqrt{a^2-x^2}$ **can also be integrated by the method of integration by parts.**

- $\int\sqrt{x^2-a^2}\,dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\log\left|x+\sqrt{x^2-a^2}\right| + C$
- $\int\sqrt{x^2+a^2} = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\log\left|x+\sqrt{x^2+a^2}\right| + C$
- $\int\sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + C$

- **For example, consider** $\int\sqrt{x^2+6x+5}\;dx$

$$\int \sqrt{x^2+6x+5}\ dx = \int \sqrt{(x+3)^2 - 2^2}\, dx$$

Put $x + 3 = t$, so that $dx = dt$
Then,

$$I = \int \sqrt{t^2 - 2^2}\, dt$$

$$I = \frac{t}{2}\sqrt{t^2-2^2} - \frac{2^2}{2}\log\left|t + \sqrt{t^2-2^2}\right| + C$$

$$I = \frac{1}{2}(x+3)\sqrt{x^2+6x+5} - 2\log\left|x+3+\sqrt{x^2+6x+5}\right| + C$$

Solved Examples

Example 1: Find the integral of:

(i) $x \sin x$ (ii) $e^x x$

Solution:

(i) $\int x \sin x dx$

$$= x\int \sin x dx - \int\left[\frac{d}{dx}(x)\int \sin x dx\right] dx$$

$$= -x\cos x + \int \cos x dx$$

$$= -x\cos x + \sin x + C$$

(ii) $\int e^x x dx$

$$= x\int e^x dx - \int\left[\frac{d}{dx}(x)\int e^x dx\right] dx$$

$$= xe^x - e^x + C$$

$$= e^x(x-1) + C$$

Example 2: Find the integral of $\int e^x\left(\frac{\sqrt{1-x^2}\sin^{-1}x + 1}{\sqrt{1-x^2}}\right) dx$.

Solution:

$$\int e^x \left(\frac{\sqrt{1-x^2}\sin^{-1}x+1}{\sqrt{1-x^2}} \right) dx = \int e^x \left[\sin^{-1}x + \frac{1}{\sqrt{1-x^2}} \right] dx$$

Now, $f(x) = \sin^{-1}x$ and $f'(x) = \frac{1}{\sqrt{1-x^2}}$

Therefore, the given integrand is of the form $e^x\left[f(x)+f'(x)\right]$**.**

$$\therefore I = \int e^x \left[\sin^{-1} + \frac{1}{\sqrt{1-x^2}} \right] dx = e^x \sin^{-1}x + C$$

Example 3: Find the integral of $\int\sqrt{8+2x-x^2}\,dx$

Solution:

$$I = \int\sqrt{8+2x-x^2}\,dx = \int\sqrt{3^2-(1-x)^2}\,dx$$

Put1 − *x* = *t* so that *dx* = −*dt*

Then,

$$I = -\int\sqrt{3^2-t^2}\,dt$$
$$I = -\frac{1}{2}t\sqrt{3^2-t^2} - \frac{3^2}{2}\sin^{-1}\frac{t}{3} + C$$
$$I = -\frac{1}{2}(1-x)\sqrt{8+2x-x^2} - \frac{9}{2}\sin^{-1}\left(\frac{1-x}{3}\right) + C$$

Definite Integral as Limit of Sums

- A definite integral is denoted by $\int_a^b f(x)\,dx$, where '*a*' is called the lower limit of the integral and '*b*' is called upper limit of the integral.

Definite integral of a function *f*(*x*) over an interval [*a, b*] can be calculated as:

$$\int_a^b f(x)\,dx = (b-a)\lim_{n\to\infty}\frac{1}{n}\left[f(a)+f(a+h)+\ldots+f(a+(n-1)h)\right]$$

where, $h = \frac{b-a}{n} \to 0$ as $n \to \infty$.

Solved Examples

Example 1:

Find $\int_0^3 x^3 dx$ as the limit of a sum.

Solution:

$$\int_a^b f(x)dx = (b-a)\lim_{n\to\infty}\frac{1}{n}\left[f(a)+f(a+h)+\ldots+f\left(a+(n-1)h\right)\right]$$

Here, $a = 0$, $b = 3$, $f(x) = x^3$, $h = \frac{3-0}{n} = \frac{3}{n}$

$$\therefore \int_0^3 x^3 dx = 3\lim_{n\to\infty}\frac{1}{n}\left[f(0)+f\left(\frac{3}{n}\right)+f\left(\frac{6}{n}\right)+\ldots+f\left(\frac{3(n-1)}{n}\right)\right]$$

$$= 3\lim_{n\to\infty}\frac{1}{n}\left[0+\frac{3^3}{n^3}+\frac{6^3}{n^3}+\ldots+\frac{\left[3(n-1)\right]^3}{n^3}\right]$$

$$= 3\lim_{n\to\infty}\frac{1}{n}\left[\frac{1}{n^3}3^3\left\{1^3+2^3+\ldots+(n-1)^3\right\}\right]$$

$$= 3\lim_{n\to\infty}\frac{27}{n^4}\frac{\left[(n-1)n\right]^2}{4} \qquad \left[\sum_{k=1}^{n}k^3 = \frac{\left[k(k+1)\right]^2}{4}\right]$$

$$= \frac{27\times 3}{4}\lim_{n\to\infty}\left[\frac{(n-1)}{n}\right]^2$$

$$= \frac{81}{4}\left[\lim_{n\to\infty}\left(1-\frac{1}{n}\right)\right]^2$$

$$= \frac{81}{4}$$

Example 2:

Find: $\int_0^1 e^{2x} dx$

Solution:

Here, $a = 0$, $b = 1$, $f(x) = e^{2x}$, $h = \frac{1-0}{n} = \frac{1}{n}$

$$\int_0^1 e^{2x}dx = (1-0)\lim_{n\to\infty} \frac{1}{n}\left[e^0 + e^{\frac{2}{n}} + e^{\frac{4}{n}} + \ldots + e^{\frac{2(n-1)}{n}}\right]$$

$$= \lim_{n\to\infty}\frac{1}{n}\left[1 + e^{\frac{2}{n}} + e^{\frac{4}{n}} + \ldots + e^{\frac{2n-2}{n}}\right]$$

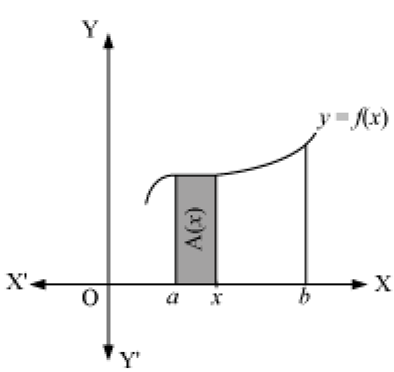
$$= \lim_{n\to\infty}\frac{1}{n}\left[\frac{\left(e^{\frac{2}{n}}\right)^n - 1}{e^{\frac{2}{n}} - 1}\right] \qquad \left[S_n = \frac{a(r^n-1)}{r-1}, a = 1, r = e^{\frac{2}{n}}\right]$$

$$= \lim_{n\to\infty}\frac{1}{n}\left(\frac{e^2 - 1}{e^{\frac{2}{n}} - 1}\right)$$

$$= \frac{e^2 - 1}{\lim_{n\to\infty}\left[\frac{e^{\frac{2}{n}} - 1}{\frac{2}{n}}\right] \times 2} = \frac{1}{2}\left(e^2 - 1\right)$$

Fundamental Theorem of Calculus

- The area function A (x) is defined as $A(x) = \int_a^x f(x)dx$, where f is a continuous function defined on the interval [a, b] and it represents the area of the shaded region as shown below.



- Let f be a continuous function on the closed interval [a, b] and let A (x) be the area function. Then, $A'(x) = f(x)$ for all $x \in [a, b]$. This is the first fundamental theorem of calculus.

- Let f be a continuous function defined on the closed interval $[a, b]$ and F be an anti-derivative of f. Then, $\int_a^b f(x)dx = \left[F(x)\right]_a^b = F(b) - F(a)$. This is the second fundamental theorem of calculus.
- In $\int_a^b f(x)dx$, the function f needs to be well-defined and continuous in $[a, b]$.

Solved Examples

Example 1

Evaluate the integral $\int_{-2}^{2}(x+1)^2 dx$.

Solution:

$$\int (x+1)^2 dx = \int \left(x^2+2x+1\right) dx = \frac{x^3}{3}+\frac{2x^2}{2}+x = \frac{x^3}{3}+x^2+x$$

$$\therefore \int_{-2}^{2}(x+1)^2 dx = \left[\frac{x^3}{3}+x^2+x\right]_{-2}^{2} = \left(\frac{8}{3}+4+2\right)-\left(-\frac{8}{3}+4-2\right) = \frac{16}{3}+4 = \frac{28}{3}$$

Example 2

Evaluate the integral $\int_0^{\frac{\pi}{8}} \sin^2 2x \, dx$.

Solution:

$$\int_0^{\frac{\pi}{8}} \sin^2 2x \, dx$$

$$\sin^2 2x = \frac{1-\cos 4x}{2}$$

$$\int \sin^2 2x dx = \frac{1}{2}\int (1-\cos 4x) dx = \frac{1}{2}\left[x - \frac{\sin 4x}{4}\right] = \frac{1}{8}(4x - \sin 4x)$$

$$\therefore \int_0^{\frac{\pi}{8}} \sin^2 2x \, dx = \left[\frac{1}{8}(4x-\sin 4x)\right]_0^{\frac{\pi}{8}} = \frac{1}{8}\left[\left(\frac{\pi}{2}-1\right)-0\right] = \frac{1}{16}(\pi - 2)$$

Example 3

Evaluate the integral $\int_0^1 \frac{1}{1+3x^2}dx$.

Solution:

$$\int \frac{1}{1+3x^2}dx = \frac{1}{3}\int \frac{1}{\frac{1}{3}+x^2}dx = \frac{1}{3}\int \frac{dx}{\left(\frac{1}{\sqrt{3}}\right)^2 + x^2} = \frac{1}{3}\frac{1}{\left(\frac{1}{\sqrt{3}}\right)} \cdot \tan^{-1}\left(\frac{x}{\frac{1}{\sqrt{3}}}\right) = \frac{1}{\sqrt{3}}\tan^{-1}\left(\sqrt{3}x\right)$$

$$\therefore \int_0^1 \frac{1}{1+3x^2}dx = \left[\frac{1}{\sqrt{3}}\tan^{-1}\sqrt{3}x\right]_0^1 = \frac{1}{\sqrt{3}}\left(\tan^{-1}\sqrt{3} - 0\right) = \frac{1}{\sqrt{3}}\tan^{-1}\sqrt{3}$$

Evaluating Definite Integrals by Substitution Method

- The steps for evaluating $\int_a^b f(x)dx$ by substitution method can be listed as:

Step l: Considering the integral without limits, substitute $y = f(x)$ or $x = g(y)$ to reduce the given integral to a known form and the limits of integral are accordingly changed.
Step 2: Integrate the new integrand with respect to the new variable, and then find the difference of the values at the obtained upper and lower limits.

Solved Examples

Example 1:

Evaluate: $\int_1^2 \frac{3x^2}{1+x^3}dx$

Solution:

Put $1 + x^3 = t$

Then, $3x^2dx = dt$

When $x = 1$, $t = 2$

$x = 2$, $t = 9$

$$\therefore \int_1^2 \frac{3x^2}{4+x^3}\,dx = \int_2^9 \frac{dt}{t} = [\log t]_2^9 = \log 9 - \log 2 = \log\frac{9}{2}$$

Example 2:

Evaluate : $\int_0^{\frac{\pi}{6}} \sqrt{\cos 3x}\ \sin 3x\ dx$

Solution:

Put $\cos 3x = t$

Then, $-3\sin 3x dx = dt$

$x = 0 \Rightarrow t = 1$ and $x = \frac{\pi}{6} \Rightarrow t = 0$

$$\therefore \int_0^{\frac{\pi}{6}} \sqrt{\cos 3x}\sin 3x dx = -\int_1^0 t^{\frac{1}{2}}.\frac{dt}{3}$$

$$= \left[-\frac{1}{3}.t^{\frac{3}{2}} \times \frac{2}{3}\right]_1^0 = -\frac{2}{9}[0-1] = \frac{2}{9}$$

Example 3:

Evaluate: $\int_{-2}^{3} \frac{1}{x\log x^2}dx$

Solution:

Put $\log x^2 = t$ so that $\frac{1}{x^2} \times 2x dx = dt$

$$\Rightarrow \frac{2}{x}dx = dt$$

$x = -2 \Rightarrow t = \log 4$

$x = 3 \Rightarrow t = \log 9$

$$\int_1^2 \frac{1}{x\log x^2}dx = \frac{1}{2}\int_{\log 4}^{\log 9}\frac{dt}{t} = \frac{1}{2}\left[\log t\right]_{\log 4}^{\log 9}$$

$$= \frac{1}{2}\left[\log(\log 9) - \log(\log 4)\right]$$

$$= \frac{1}{2}\log\left(\frac{\log 9}{\log 4}\right)$$

Properties of Definite Integrals

- $\int_a^b f(x)\,dx = \int_a^b f(t)\,dt$
- $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$. In particular: $\int_a^a f(x)\,dx = 0$
- $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$.
- $\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$.In particular: $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$
- $\int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx + \int_0^a f(2a-x)\,dx$
- $\int_0^{2a} f(x)\,dx = \begin{cases} 2\int_0^a f(x)\,dx, & \text{if } f(2a-x) = f(x) \\ 0 & \text{if } f(2a-x) = -f(x) \end{cases}$
- In $\int_{-a}^a f(x)\,dx = 2\int_0^a f(x)\,dx,$ if f is an even function i.e., $f(-x) = f(x)$
- $\int_{-a}^a f(x)\,dx = 0$, if f is an odd function i.e., $f(-x) = -f(x)$

Solved Examples

Example 1: Evaluate: $\int_{-6}^{4}|x+2|dx$

Solution:

$$|x+2|=\begin{cases}x+2 & \text{if } x\geq -2\\ -(x+2) & \text{if } x<-2\end{cases}$$

$$\int_{-6}^{4}|x+2|\,dx=-\int_{-6}^{-2}(x+2)\,dx+\int_{-2}^{4}(x+2)\,dx \qquad \left[\because \int_{a}^{b}f(x)dx=\int_{a}^{c}f(x)dx+\int_{c}^{b}f(x)dx\right]$$

$$\int_{-6}^{4}|x+2|\,dx=-\left[\frac{x^2}{2}+2x\right]_{-6}^{-2}+\left[\frac{x^2}{2}+2x\right]_{-2}^{4}$$

$$\int_{-6}^{4}|x+2|\,dx=-[-2-6]+[16-(-2)]$$

$$\int_{-6}^{4}|x+2|\,dx=8+18=26$$

Example 2: Evaluate: $\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}}\cos x\,dx$

Solution:

It can be seen that *f* (*x*) is an even function. Therefore, $\int_{-a}^{a}f(x)\,dx=2\int_{0}^{a}f(x)\,dx$

$$\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}}\cos x\,dx=2\int_{0}^{\frac{\pi}{2}}\cos x\,dx=2[\sin x]_{0}^{\frac{\pi}{2}}=2\times(1-0)=2$$

Example 3: Evaluate: $I=\int_{0}^{\pi}x\tan x\sec x\,dx$

Solution:

$$\int_0^{\pi} x\tan x\sec x\ dx = \int_0^{\pi} (\pi - x)\tan(\pi - x)\sec(\pi - x)dx \qquad \left[\int_0^{a} f(x)\,dx = \int_0^{a} f(a-x)\,dx\right]$$

$$\int_0^{\pi} x\tan x\sec x\ dx = \int_0^{\pi} (\pi - x)(-\tan x)(-\sec x)dx$$

$$\int_0^{\pi} x\tan x\sec x\ dx = \int_0^{\pi} (\pi - x)\tan x\sec x dx$$

$$\int_0^{\pi} x\tan x\sec x\ dx = \int_0^{\pi} \pi\tan x\sec x dx - \int_0^{\pi} x\tan x\sec x dx$$

$$\int_0^{\pi} x\tan x\sec x\ dx = \int_0^{\pi} \pi\tan x\sec x dx - \mathrm{I}$$

$$\therefore \mathrm{I} = \frac{\pi}{2}\int_0^{\pi} \tan x\sec x dx = \frac{\pi}{2}[\sec x]_0^{\pi} = \frac{\pi}{2}(-1-1) = \frac{\pi}{2}\times -2 = -\pi$$