CHAPTER

9.2

DIFFERENTIAL CALCULUS

1. If $f(x) = x^3 - 6x^2 + 11x - 6$ is on [1, 3], then the point $c \leftrightarrow]1, 3[$ such that $f'(c) = 0$ is given by

(A) $c = 2 \pm \frac{1}{\sqrt{2}}$ (B) $c = 2 \pm \frac{1}{\sqrt{3}}$

(C) $c = 2 \pm \frac{1}{2}$ (D) None of these

2. Let $f(x) = \sin 2x, 0 \leq x \leq \frac{\pi}{2}$ and $f'(c) = 0$ for $c \leftrightarrow]0, \frac{\pi}{2}[$. Then, c is equal to

(A) $\frac{\pi}{4}$ (B) $\frac{\pi}{3}$

(C) $\frac{\pi}{6}$ (D) None

3. Let $f(x) = x(x+3)e^{-\frac{x}{2}}, -3 \leq x \leq 0$. Let $c \leftrightarrow]-3, 0[$ such that $f'(c) = 0$. Then, the value of c is

(A) 3 (B) –3

(C) –2 (D) $-\frac{1}{2}$

4. If Rolle's theorem holds for $f(x) = x^3 - 6x^2 + kx + 5$ on [1, 3] with $c = 2 + \frac{1}{\sqrt{3}}$, the value of k is

(A) –3 (B) 3

(C) 7 (D) 11

5. A point on the parabola $y = (x-3)^2$, where the tangent is parallel to the chord joining A (3, 0) and B (4, 1) is

(A) (7, 1) (B) $\left(\frac{3}{2}, \frac{1}{4}\right)$

(C) $\left(\frac{7}{2}, \frac{1}{4}\right)$ (D) $\left(-\frac{1}{2}, \frac{1}{2}\right)$

6. A point on the curve $y = \sqrt{x-2}$ on [2, 3], where the tangent is parallel to the chord joining the end points of the curve is

(A) $\left(\frac{9}{4}, \frac{1}{2}\right)$ (B) $\left(\frac{7}{2}, \frac{1}{4}\right)$

(C) $\left(\frac{7}{4}, \frac{1}{2}\right)$ (D) $\left(\frac{9}{2}, \frac{1}{4}\right)$

7. Let $f(x) = x(x-1)(x-2)$ be defined in $[0, \frac{1}{2}]$. Then, the value of c of the mean value theorem is

(A) 0.16 (B) 0.20

(C) 0.24 (D) None

8. Let $f(x) = \sqrt{x^2 - 4}$ be defined in [2, 4]. Then, the value of c of the mean value theorem is

(A) $-\sqrt{6}$ (B) $\sqrt{6}$

(C) $\sqrt{3}$ (D) $2\sqrt{3}$

9. Let $f(x) = e^x$ in [0, 1]. Then, the value of c of the mean-value theorem is

(A) 0.5 (B) $(e-1)$

(C) $\log(e-1)$ (D) None

10. At what point on the curve $y = (\cos x - 1)$ in $]0, 2\pi[$, is the tangent parallel to x–axis ?

(A) $\left(\frac{\pi}{2}, -1\right)$ (B) $(\pi, -2)$

(C) $\left(\frac{2\pi}{3}, \frac{-3}{2}\right)$ (D) None of these

11. log sin $(x+h)$ when expanded in Taylor's series, is equal to

(A) $\log \sin x + h \cot x - \frac{1}{2} h^2 \operatorname{cosec}^2 x + \ldots$

(B) $\log \sin x + h \cot x + \frac{1}{2} h^2 \sec^2 x + \ldots$

(C) $\log \sin x - h \cot x + \frac{1}{2} h^2 \operatorname{cosec}^2 x + \ldots$

(D) None of these

12. sin x when expanded in powers of $\left(x - \frac{\pi}{2}\right)$ is

(A) $1 + \frac{\left(x-\frac{\pi}{2}\right)^2}{2!} + \frac{\left(x-\frac{\pi}{2}\right)^3}{3!} + \frac{\left(x-\frac{\pi}{2}\right)^2}{4!} + \ldots$

(B) $1 - \frac{\left(x-\frac{\pi}{2}\right)^2}{2!} + \frac{\left(x-\frac{\pi}{2}\right)^2}{4!} - \ldots$

(C) $\left(x-\frac{\pi}{2}\right)^2 + \frac{\left(x-\frac{\pi}{2}\right)^3}{3!} + \frac{\left(x-\frac{\pi}{2}\right)^5}{5!} + \ldots$

(D) None of these

13. $\tan\left(\frac{\pi}{4} + x\right)$ when expanded in Taylor's series, gives

(A) $1 + x + x^2 + \frac{4}{3} x^3 + \ldots$

(B) $1 + 2x + 2x^2 + \frac{8}{3} x^3 + \ldots$

(C) $1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots$

(D) None of these

14. If $u = e^{xyz}$, then $\frac{\partial^3 u}{\partial x \partial y \partial z}$ is equal to

(A) $e^{xyz}[1 + xyz + 3x^2y^2z^2]$

(B) $e^{xyz}[1 + xyz + x^3y^3z^3]$

(C) $e^{xyz}[1 + 3xyz + x^2y^2z^2]$

(D) $e^{xyz}[1 + 3xyz + x^3y^3z^3]$

15. If $z = f(x + ay) + \phi(x - ay)$, then

(A) $\frac{\partial^2 z}{\partial x^2} = a^2 \frac{\partial^2 z}{\partial y^2}$

(B) $\frac{\partial^2 z}{\partial y^2} = a^2 \frac{\partial^2 z}{\partial x^2}$

(C) $\frac{\partial^2 z}{\partial y^2} = -\frac{1}{a^2} \frac{\partial^2 z}{\partial x^2}$

(D) $\frac{\partial^2 z}{\partial x^2} = -a^2 \frac{\partial^2 z}{\partial y^2}$

16. If $u = \tan^{-1}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)$, then $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$ equals

(A) $2 \cos 2u$

(B) $\frac{1}{4} \sin 2u$

(C) $\frac{1}{4} \tan u$

(D) $2 \tan 2u$

17. If $u = \tan^{-1} \frac{x^3 + y^3 + x^2y - xy^2}{x^2 - xy + y^2}$, then the value of $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$ is

(A) $\frac{1}{2} \sin 2u$

(B) $\sin 2u$

(C) $\sin u$

(D) 0

18. If $u = \phi\left(\frac{y}{x}\right) + x\psi\left(\frac{y}{x}\right)$, then the value of

$$x^2 \frac{\partial^2 u}{dx^2} + 2xy \frac{\partial^2 u}{dx\, dy} + y^2 \frac{\partial^2 u}{\partial y^2}, \text{ is}$$

(A) 0

(B) u

(C) $2u$

(D) $-u$

19. If $z = e^x \sin y$, $x = \log_e t$ and $y = t^2$, then $\frac{dz}{dt}$ is given by the expression

(A) $\frac{e^x}{t}(\sin y - 2t^2 \cos y)$

(B) $\frac{e^x}{t}(\sin y + 2t^2 \cos y)$

(C) $\frac{e^x}{t}(\cos y + 2t^2 \sin y)$

(D) $\frac{e^x}{t}(\cos y - 2t^2 \sin y)$

20. If $z = z(u, v)$, $u = x^2 - 2xy - y^2$, $v = a$, then

(A) $(x + y)\frac{\partial z}{\partial x} = (x - y)\frac{\partial z}{\partial y}$

(B) $(x - y)\frac{\partial z}{\partial x} = (x + y)\frac{\partial z}{\partial y}$

(C) $(x + y)\frac{\partial z}{\partial x} = (y - x)\frac{\partial z}{\partial y}$

(D) $(y - x)\frac{\partial z}{\partial x} = (x + y)\frac{\partial z}{\partial y}$

21. If $f(x, y) = 0$, $\phi(y, z) = 0$, then

(A) $\frac{\partial f}{\partial y} \cdot \frac{\partial \phi}{\partial z} = \frac{\partial f}{\partial x} \cdot \frac{\partial \phi}{\partial y} \cdot \frac{dz}{dx}$

(B) $\frac{\partial f}{\partial y} \cdot \frac{\partial \phi}{\partial z} \cdot \frac{\partial f}{\partial x} = \frac{\partial f}{\partial x} \cdot \frac{dz}{dx}$

(C) $\frac{\partial f}{\partial y} \cdot \frac{\partial \phi}{\partial z} \cdot \frac{dz}{dx} = \frac{\partial f}{\partial x} \cdot \frac{\partial \phi}{\partial y}$

(D) None of these

22. If $z = \sqrt{x^2 + y^2}$ and $x^3 + y^3 + 3axy = 5a^2$, then at $x = a$, $y = a$, $\frac{dz}{dx}$ is equal to

(A) $2a$

(B) 0

(C) $2a^2$

(D) a^3

23. If $x = r\cos\theta$, $y = r\sin\theta$ where r and θ are the functions of x, then $\frac{dx}{dt}$ is equal to

(A) $r\cos\theta\frac{dr}{dt} - r\sin\theta\frac{d\theta}{dt}$ (B) $\cos\theta\frac{dr}{dt} - r\sin\theta\frac{d\theta}{dt}$

(C) $r\cos\theta\frac{dr}{dt} + \sin\theta\frac{d\theta}{dt}$ (D) $r\cos\theta\frac{dr}{dt} - \sin\theta\frac{d\theta}{dt}$

24. If $r^2 = x^2 + y^2$, then $\frac{\partial^2 r}{dx^2} + \frac{\partial^2 r}{\partial y^2}$ is equal to

(A) $r^2\left\{\left(\frac{\partial r}{\partial x}\right)^2 + \left(\frac{\partial r}{\partial y}\right)^2\right\}$ (B) $2r^2\left\{\left(\frac{\partial r}{\partial x}\right)^2 + \left(\frac{\partial r}{\partial y}\right)^2\right\}$

(C) $\frac{1}{r^2}\left\{\left(\frac{\partial r}{\partial x}\right)^2 + \left(\frac{\partial r}{\partial y}\right)^2\right\}$ (D) None of these

25. If $x = r\cos\theta$, $y = r\sin\theta$, then the value of $\frac{\partial^2\theta}{\partial x^2} + \frac{\partial^2\theta}{\partial y^2}$ is

(A) 0 (B) 1

(C) $\frac{\partial r}{\partial x}$ (D) $\frac{\partial r}{\partial y}$

26. If $u = x^m y^n$, then

(A) $du = mx^{m-1}y^n + nx^m y^{n-1}$ (B) $du = mdx + ndy$

(C) $udu = mxdx + nydy$ (D) $\frac{du}{u} = m\frac{dx}{x} + n\frac{dy}{y}$

27. If $y^3 - 3ax^2 + x^3 = 0$, then the value of $\frac{d^2y}{dx^2}$ is equal to

(A) $-\frac{a^2x^2}{y^5}$ (B) $\frac{2a^2x^2}{y^5}$

(C) $-\frac{2a^2x^4}{y^5}$ (D) $-\frac{2a^2x^2}{y^5}$

28. $z = \tan^{-1}\frac{y}{x}$, then

(A) $dz = \frac{xdy - ydx}{x^2 + y^2}$ (B) $dz = \frac{xdy + ydx}{x^2 + y^2}$

(C) $dz = \frac{xdx - ydy}{x^2 + y^2}$ (D) $dz = \frac{xdx - ydy}{x^2 + y^2}$

29. If $u = \log\frac{x^2 + y^2}{x + y}$, then $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$ is equal to

(A) 0 (B) 1

(C) u (D) eu

30. If $u = x^{n-1}yf\left(\frac{y}{x}\right)$, then $x\frac{\partial^2 u}{\partial x^2} + y\frac{\partial^2 y}{\partial y\,\partial x}$ is equal to

(A) nu (B) $n(n-1)u$

(C) $(n-1)\frac{\partial u}{\partial x}$ (D) $(n-1)\frac{\partial u}{\partial y}$

31. Match the List–I with List–II.

List–I

(i) If $u = \frac{x^2y}{x + y}$ then $x\frac{\partial^2 u}{\partial x^2} + y\frac{\partial^2 u}{\partial x\,\partial y}$

(ii) If $u = \frac{x^{\frac{1}{2}} - y^{\frac{1}{2}}}{x^{\frac{1}{4}} + y^{\frac{1}{4}}}$ then $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\,\partial y} + y^2\frac{\partial^2 u}{\partial y^2}$

(iii) If $u = x^{\frac{1}{2}} + y^{\frac{1}{2}}$ then $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\,\partial y} + y^2\frac{\partial^2 u}{\partial y^2}$

(iv) If $u = f\left(\frac{y}{x}\right)$ then $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$

List–II

(1) $-\frac{3}{16}u$ (2) $\frac{\partial u}{\partial x}$

(3) 0 (4) $-\frac{1}{4}u$

Correct match is—

	(I)	(II)	(III)	(IV)
(A)	1	2	3	4
(B)	2	1	4	3
(C)	2	1	3	4
(D)	1	2	4	3

32. If an error of 1% is made in measuring the major and minor axes of an ellipse, then the percentage error in the area is approximately equal to

(A) 1% (B) 2%

(C) π% (D) 4%

33. Consider the Assertion (A) and Reason (R) given below:

Assertion (A): If $u = xyf\left(\frac{y}{x}\right)$, then $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2u$

Reason (R): Given function u is homogeneous of degree 2 in x and y.

Of these statements

(A) Both A and R are true and R is the correct explanation of A

(B) Both A and R are true and R is not a correct explanation of A

(C) A is true but R is false

(D) A is false but R is true

34. If $u = x \log xy$, where $x^3 + y^3 + 3xy = 1$, then $\frac{du}{dx}$ is equal to

(A) $(1 + \log xy) - \frac{x}{y}\left(\frac{x^2 + y}{y^2 + x}\right)$

(B) $(1 + \log xy) - \frac{y}{x}\left(\frac{y^2 + x}{x^2 + y}\right)$

(C) $(1 - \log xy) - \frac{x}{y}\left(\frac{x^2 + y}{y^2 + x}\right)$

(D) $(1 - \log xy) - \frac{y}{x}\left(\frac{y^2 + x}{x^2 + y}\right)$

35. If $z = xyf\left(\frac{y}{x}\right)$, then $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y}$ is equal to

(A) z (B) $2z$

(C) xz (D) yz

36. $f(x) = 2x^3 - 15x^2 + 36x + 1$ is increasing in the interval

(A) $]\,2, 3\,[$ (B) $]-\infty, 3\,[$

(C) $]-\infty, 2\,[\cup]\,3, \infty$ (D) None of these

37. $f(x) = \frac{x}{(x^2 + 1)}$ is increasing in the interval

(A) $]-\infty, -1\,[\cup]\,1, \infty\,[$ (B) $]-1, 1\,[$

(C) $]-1, \infty\,[$ (D) None of these

38. $f(x) = x^4 - 2x^2$ is decreasing in the interval

(A) $]-\infty, -1\,[\cup]\,0, 1\,[$ (B) $]-1, 1\,[$

(C) $]-\infty, -1\,[\cup]\,1, \infty\,[$ (D) None of these

39. $f(x) = x^9 + 3x^7 + 6$ is increasing for

(A) all positive real values of x

(B) all negative real values of x

(C) all non-zero real values of x

(D) None of these

40. If $f(x) = kx^3 - 9x^2 + 9x + 3$ is increasing in each interval, then

(A) $k < 3$ (B) $k \le 3$

(C) $k > 3$ (D) $k \ge 3$

41. If $a < 0$, then $f(x) = e^{ax} + e^{-ax}$ is decreasing for

(A) $x > 0$ (B) $x < 0$

(C) $x > 1$ (D) $x < 1$

42. $f(x) = x^2 e^{-x}$ is increasing in the interval

(A) $]-\infty, \infty\,[$ (B) $]-2, 0\,[$

(C) $]\,2, \infty\,[$ (D) $]\,0, 2\,[$

43. The least value of a for which $f(x) = x^2 + ax + 1$ is increasing on $]\,1, 2, [$ is

(A) 2 (B) –2

(C) 1 (D) –1

44. The minimum distance from the point (4, 2) to the parabola $y^2 = 8x$, is

(A) $\sqrt{2}$ (B) $2\sqrt{2}$

(C) 2 (D) $3\sqrt{2}$

45. The co-ordinates of the point on the parabola $y = x^2 + 7x + 2$ which is closest to the straight line $y = 3x - 3$, are

(A) (–2, –8) (B) (2, –8)

(C) (–2, 0) (D) None of these

46. The shortest distance of the point (0, c), where $0 \le c < 5$, from the parabola $y = x^2$ is

(A) $\sqrt{4c + 1}$ (B) $\frac{\sqrt{4c + 1}}{2}$

(C) $\frac{\sqrt{4c - 1}}{2}$ (D) None of these

47. The maximum value of $\left(\frac{1}{x}\right)^x$ is

(A) e (B) $e^{-\frac{1}{e}}$

(C) $\left(\frac{1}{e}\right)^e$ (D) None of these

48. The minimum value of $\left(x^2 + \frac{250}{x}\right)$ is

(A) 75 (B) 50

(C) 25 (D) 0

49. The maximum value of $f(x) = (1 + \cos x)\sin x$ is

(A) 3 (B) $3\sqrt{3}$

(C) 4 (D) $\frac{3\sqrt{3}}{4}$

50. The greatest value of

$$f(x) = \frac{\sin 2x}{\sin\left(x + \frac{\pi}{4}\right)}$$

on the interval $[0, \frac{\pi}{2}]$ is

(A) $\frac{1}{\sqrt{2}}$ (B) $\sqrt{2}$

(C) 1 (D) $-\sqrt{2}$

51. If $y = a\log x + bx^2 + x$ has its extremum values at $x = -1$ and $x = 2$, then

(A) $a = -\frac{1}{2},\ b = 2$ (B) $a = 2,\ \ b = -1$

(C) $a = 2,\ \ b = -\frac{1}{2}$ (D) None of these

52. The co-ordinates of the point on the curve $4x^2 + 5y^2 = 20$ that is farthest from the point (0, –2) are

(A) $(\sqrt{5}\ ,\ 0)$ (B) $(\sqrt{6}\ ,\ 0)$

(C) (0, 2) (D) None of these

53. For what value of $x\left(0 \le x \le \frac{\pi}{2}\right)$, the function $y = \frac{x}{(1 + \tan x)}$ has a maxima ?

(A) $\tan x$ (B) 0

(C) $\cot x$ (D) $\cos x$

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SOLUTIONS

1. (B) A polynomial function is continuous as well as differentiable. So, the given function is continuous and differentiable.

$f(1) = 0$ and $f(3) = 0$. So, $f(1) = f(3)$.

By Rolle's theorem Ec such that $f'(c) = 0$.

Now, $f'(x) = 3x^2 - 12x + 11$

$\Rightarrow \quad f'(c) = 3c^2 - 12c + 11.$

Now, $f'(c) = 0 \quad \Rightarrow \quad 3c^2 - 12c + 11 = 0$

$\Rightarrow \quad c = \left(2 \pm \frac{1}{\sqrt{3}}\right).$

2. (A) Since the sine function is continuous at each $x \leftrightarrow R$, so $f(x) = \sin 2x$ is continuous in $\left[0, \frac{\pi}{2}\right]$.

Also, $f'(x) = 2\cos 2x$, which clearly exists for all $x \leftrightarrow]0, \frac{\pi}{2}[$ .So, $f(x)$ is differentiable in $x \leftrightarrow]0, \frac{\pi}{2}[$.

Also, $f(0) = f\left(\frac{\pi}{2}\right) = 0$. By Rolle's theorem, there exists $c \leftrightarrow]0, \frac{\pi}{2}[$ such that $f'(c) = 0$.

$2\cos 2c = 0 \quad \Rightarrow \quad 2c = \frac{\pi}{2} \quad \Rightarrow \quad c = \frac{\pi}{4}$.

3. (C) Since a polynomial function as well as an exponential function is continuous and the product of two continuous functions is continuous, so $f(x)$ is continuous in [–3, 0].

$$f'(x) = (2x + 3)\cdot e^{-\frac{x}{2}} - \frac{1}{2}e^{-\frac{x}{2}}(x^2 + 3x) = e^{-\frac{x}{2}}\left[\frac{x + 6 - x^2}{2}\right]$$

which clearly exists for all $x \leftrightarrow]-3, 0[$.

$f(x)$ is differentiable in] –3, 0 [.

Also, $f(-3) = f(0) = 0$.

By Rolle's theorem $c \leftrightarrow]-3, 0[$ such that $f'(c) = 0$.

Now, $f'(c) = 0 \quad \Rightarrow \quad e^{-\frac{c}{2}}\left[\frac{c + 6 - c^2}{2}\right] = 0$

$c + 6 - c^2 = 0$ i.e. $c^2 - c - 6 = 0$

$\Rightarrow \quad (c + 2)(3 - c) = 0 \quad \Rightarrow \quad c = -2,\ c = 3.$

Hence, $c = -2 \leftrightarrow]-3, 0[$.

4. (D) $f'(x) = 3c^2 - 12x + k$

$f'(c) = 0 \quad \Rightarrow \quad 3c^2 - 12c + k = 0$

$f\left(\frac{\pi}{2}\right)=1,\ f'\left(\frac{\pi}{2}\right)=0,\ f''\left(\frac{\pi}{2}\right)=-1,$

$f'''\left(\frac{\pi}{2}\right)=0,\ f''''\left(\frac{\pi}{2}\right)=1,\ ....$

13. (B) Let $f(x)=\tan x$ Then,

$$f\left(\frac{\pi}{4}+x\right)=f\left(\frac{\pi}{4}\right)+xf'\left(\frac{\pi}{4}\right)+\frac{x^2}{2!}\cdot f''\left(\frac{\pi}{4}\right)+\frac{x^3}{3!}f'''\left(\frac{\pi}{4}\right)+...$$

$f'(x)=\sec^2,\ f''(x)=2\sec^2 x\tan x,$

$f'''(x)=2\sec^4 x+4\sec^2 x\tan^2 x$ etc.

Now,

$f\left(\frac{\pi}{4}\right)=1,\ f'\left(\frac{\pi}{4}\right)=2,\ f''\left(\frac{\pi}{4}\right)=4,\ f'''\left(\frac{\pi}{4}\right)=16,\ ...$

Thus $\tan\left(\frac{\pi}{4}+x\right)=1+2x+\frac{x^2}{2}\cdot 4+\frac{x^3}{6}\cdot 16+...$

$=1+2x+2x^2+\frac{8}{3}x^3+...$

14. (C) Here $u=e^{xyz} \Rightarrow \frac{\partial u}{\partial x}=e^{xyz}\cdot yz$

$\frac{\partial^2 u}{\partial x\partial y}=ze^{xyz}+yze^{xyz}\cdot xz=e^{xyz}(z+xyz^2)$

$\frac{\partial^3 u}{\partial x\partial y\partial z}=e^{xyz}\cdot(1+2xyz)+(z+xyz^2)e^{xyz}\cdot xy$

$=e^{xyz}(1+3xyz+x^2y^2z^2)$

15. (B) $z=f(x+ay)+\phi(x-ay)$

$\frac{\partial z}{\partial x}=f'(x+ay)+\phi'(x-ay)$

$\frac{\partial^2 z}{dx^2}=f''(x+ay)+\phi''(x-ay)$....(1)

$\frac{\partial z}{\partial y}=af'(x+ay)-a\phi'(x-ay)$

$\frac{\partial^2 z}{\partial y^2}=a^2f''(x+ay)+a^2\phi''(x-ay)$....(2)

Hence from (1) and (2), we get $\frac{\partial^2 z}{\partial y^2}=a^2\frac{\partial^2 z}{\partial x^2}$

16. (B) $u=\tan^{-1}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)$

$\Rightarrow \tan u=\frac{x+y}{\sqrt{x}+\sqrt{y}}=f$ (say)

Which is a homogeneous equation of degree 1/2

By Euler's theorem. $x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}=\frac{1}{2}f$

$\Rightarrow x\frac{\partial(\tan u)}{\partial x}+y\frac{\partial(\tan u)}{\partial y}=\frac{1}{2}\tan u$

$x\sec^2 u\frac{\partial u}{\partial x}+y\sec^2 u\frac{\partial u}{\partial y}=\frac{1}{2}\tan u$

$\Rightarrow x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\frac{1}{2}\sin u\cos u=\frac{1}{4}\sin 2u$

17. (A) Here $\tan u=\frac{x^3+y^3+x^2y-xy^2}{x^2-xy+y^2}=f$ (say)

Which is homogeneous of degree 1

Thus $x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}=f$

As above question number 16 $x\frac{\partial f}{\partial x}+y\frac{\partial u}{\partial y}=\frac{1}{2}\sin 2u$

18. (A) Let $v=\phi\left(\frac{y}{x}\right)$ and $w=x\Psi\left(\frac{y}{x}\right)$

Then $u=v+w$

Now v is homogeneous of degree zero and w is homogeneous of degree one

$\Rightarrow x^2\frac{\partial^2 v}{\partial x^2}+2xy\frac{\partial^2 v}{\partial x\,\partial y}+y^2\frac{\partial^2 v}{\partial y^2}=0$....(1)

and $x^2\frac{\partial^2 w}{\partial x^2}+2xy\frac{\partial^2 w}{\partial x\,\partial y}+y^2\frac{\partial^2 w}{\partial y^2}=0$....(2)

Adding (1) and (2), we get

$x^2\frac{\partial^2}{\partial x^2}(v+w)+2xy\frac{\partial^2}{\partial x\,\partial y}(v+w)+y^2\frac{\partial^2}{\partial y^2}(v+w)=0$

$\Rightarrow x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\,\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=0$

19. (B) $z=e^x\sin y \Rightarrow \frac{\partial z}{\partial x}=e^x\sin y$

And $\frac{\partial z}{\partial y}=e^x\cos y,\ x=\log_e t \Rightarrow \frac{dx}{dt}=\frac{1}{t}$

And $y=t^2 \Rightarrow \frac{dy}{dt}=2t$

$\frac{dz}{dt}=\frac{\partial z}{\partial x}\cdot\frac{dx}{dt}+\frac{\partial z}{\partial y}\cdot\frac{dy}{dt}$

$=e^x\sin y\cdot\frac{1}{t}+e^x\cos y\cdot 2t=\frac{e^x}{t}(\sin y+2t^2\cos y)$

20. (C) Given that

$z=z(u,v),\ u=x^2-2xy-y^2,\ v=a$....(i)

$\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x}$....(ii)

and $\frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y}$....(iii)

From (i),

$\frac{\partial u}{\partial x}=2x-2y,\ \frac{\partial u}{\partial y}=-2x-2y,\ \frac{\partial v}{\partial x}=0,\ \frac{\partial v}{\partial y}=0$

Substituting these values in (ii) and (iii)

$$\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}(2x-2y)+\frac{\partial z}{\partial v}\cdot 0....(iv)$$

and $\frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}\cdot(-2x-2y)+\frac{\partial z}{\partial v}\cdot 0....(v)$

From (iv) and (v), we get

$$(x+y)\frac{\partial z}{\partial x}=(y-x)\frac{\partial z}{\partial y}$$

21. (C) Given that $f(x, y)=0, \quad \phi(y, z)=0$

These are implicit functions

$$\frac{dy}{dx}=-\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}, \quad \frac{dz}{dy}=-\frac{\frac{\partial \phi}{\partial y}}{\frac{\partial \phi}{\partial z}}$$

$$\frac{dy}{dx}\cdot\frac{dz}{dy}=\left(\frac{-\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}\right)\times\left(\frac{-\frac{\partial \phi}{\partial y}}{\frac{\partial \phi}{\partial z}}\right)$$

or, $\frac{\partial f}{\partial y}\cdot\frac{\partial \phi}{\partial z}\cdot\frac{dz}{dx}=\frac{\partial f}{\partial x}\cdot\frac{\partial \phi}{\partial y}$

22. (B) Given that $z=\sqrt{x^2+y^2}$

and $x^3+y^3+3axy=5a^2...(i)$

$$\frac{dz}{dx}=\frac{\partial z}{\partial x}+\frac{\partial z}{\partial y}\cdot\frac{dy}{dx}....(ii)$$

from (i), $\frac{\partial z}{\partial x}=\frac{1}{2\sqrt{x^2+y^2}}\cdot 2x$, $\frac{\partial z}{\partial y}=\frac{1}{2\sqrt{x^2+y^2}}\cdot 2y$

and $3x^2+3y^2\frac{dy}{dx}+3ax\frac{dy}{dx}+3ay.1=0$

$$\Rightarrow \quad \frac{dy}{dx}=-\left(\frac{x^2+ay}{y^2+ax}\right)$$

Substituting these value in (ii), we get

$$\frac{dz}{dx}=\frac{x}{\sqrt{x^2+y^2}}+\frac{y}{\sqrt{x^2+y^2}}\left(-\frac{x^2+ay}{y^2+ax}\right)$$

$$\left(\frac{dz}{dx}\right)_{(a,a)}=\frac{a}{\sqrt{a^2+a^2}}+\frac{a}{\sqrt{a^2+a^2}}\left(-\frac{a^2+aa}{a^2+a.a}\right)=0$$

23. (B) Given that $x=r\cos\theta,\ y=r\sin\theta....(i)$

$$\frac{dx}{dt}=\frac{\partial x}{\partial r}\cdot\frac{dr}{dt}+\frac{\partial x}{\partial \theta}\cdot\frac{d\theta}{dt}....(ii)$$

From (i), $\frac{\partial x}{\partial r}=\cos\theta,\ \frac{\partial x}{\partial \theta}=-r\sin\theta$

Substituting these values in (ii), we get

$$\frac{dx}{dt}=\cos\theta\,\frac{dr}{dt}-r\sin\theta\cdot\frac{d\theta}{dt}$$

24. (C) $r^2=x^2+y^2 \quad \Rightarrow \quad \frac{\partial r}{\partial x}=2x$ and $\frac{\partial r}{\partial y}=2y$

and $\frac{\partial^2 r}{\partial x^2}=2$ and $\frac{\partial^2 r}{\partial y^2}=2 \quad \Rightarrow \quad \frac{\partial^2 r}{\partial x^2}+\frac{\partial^2 r}{\partial y^2}=2+2+4$

and $\left(\frac{\partial r}{\partial x}\right)^2+\left(\frac{\partial r}{\partial y}\right)^2=4x^2+4y^2=4r^2$

$$\Rightarrow \quad \frac{\partial^2 r}{\partial x^2}+\frac{\partial^2 y}{\partial y^2}=\frac{1}{r^2}\left\{\left(\frac{\partial r}{\partial x}\right)^2+\left(\frac{\partial r}{\partial y}\right)^2\right\}$$

25. (A) $x=r\cos\theta, \quad y=r\sin\theta$

$$\Rightarrow \quad \tan\theta=\frac{y}{x} \quad \Rightarrow \quad \theta=\tan^{-1}\left(\frac{y}{x}\right)$$

$$\Rightarrow \quad \frac{\partial\theta}{\partial x}=\frac{1}{1+(y/x)^2}\left(\frac{-y}{x^2}\right)=\frac{-y}{x^2+y^2}$$

and $\frac{\partial^2\theta}{\partial x^2}=\frac{-2xy}{(x^2+y^2)^2}$

Similarly $\frac{\partial^2\theta}{\partial y^2}=\frac{2xy}{(x^2+y^2)^2}$ and $\frac{\partial^2\theta}{\partial x^2}+\frac{\partial^2\theta}{\partial y^2}=0$

26. (D) Given that $u=x^m y^n$

Taking logarithm of both sides, we get

$\log u=m\log x+n\log y$

Differentiating with respect to x, we get

$\frac{1}{u}\frac{du}{dx}=m\cdot\frac{1}{x}+n\cdot\frac{1}{y}\frac{dy}{dx}$ or, $\frac{du}{u}=m\frac{dx}{x}+n\cdot\frac{dy}{y}$

27. (D) Given that $f(x, y)=y^3-3ax^2+x^3=0$

$f_x=-6ax+3x^2$, $f_y=3y^2$, $f_{xx}=-6a+6x$,

$f_{yy}=6y$, $f_{xy}=0$

$$\frac{d^2y}{dx^2}=-\left[\frac{f_{xx}(f_y)^2-2f_xf_yf_{xy}+f_{yy}(f_x)^2}{(f_y)^3}\right]$$

$$=-\left[\frac{(6x-6a(3y^2)^2-0+6y(3x^2-6ax)^2}{(3y^2)^3}\right]$$

$$=-\frac{2}{y^5}(-ax^3-ay^3+4a^2x^2)$$

$$=-\frac{2}{y^5}[-a(a^3+y^3)+4a^2x^2]$$

$$=-\frac{2}{y^5}[-a(3ax^2)+4a^2x^2]\ [\therefore x^3+y^3-3ax^2=0]$$

$$=-\frac{2a^2x^2}{y^5}$$

28. (A) Given that $z=\tan^{-1}\frac{y}{x}....(i)$

$$\frac{dz}{dx}=\frac{\partial z}{\partial x}+\frac{\partial z}{\partial y}\cdot\frac{dy}{dx}....(ii)$$

From (i) $\frac{\partial z}{\partial x}=\frac{1}{1+\left(\frac{y}{x}\right)^2}\cdot\left(\frac{-y}{x^2}\right)=\frac{-y}{x^2+y^2}$

$$\frac{\partial z}{\partial y}=\frac{1}{1+\left(\frac{y}{x}\right)^2}\cdot\left(\frac{1}{x}\right)=\frac{x}{x^2+y^2}$$

Substituting these in (ii), we get

$$\frac{dz}{dx}=\frac{-y}{x^2+y^2}+\frac{x}{x^2+y^2}\cdot\frac{dy}{dx},\ dz=\frac{xdy-ydx}{x^2+y^2}$$

29. (B) $u=\log\frac{x^2+y^2}{x+y},\ e^u=\frac{x^2+y^2}{x+y}=f$ (say)

f is a homogeneous function of degree one

$$x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}=f\ \Rightarrow\ x\frac{\partial e^u}{\partial x}+y\frac{\partial e^u}{\partial y}=e^u$$

or $xe^u\frac{\partial u}{\partial x}+ye^u\frac{\partial u}{\partial y}=e^u$

or, $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=1$

30. (C) Given that $u=x^{n-1}yf\left(\frac{y}{x}\right)$.

It is a homogeneous function of degree n

Euler's theorem $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=nu$

Differentiating partially w.r.t. x, we get

$$x\frac{\partial^2 u}{\partial x^2}+\frac{\partial u}{\partial x}+y\frac{\partial^2 u}{\partial y\,\partial x}=\frac{n\partial u}{\partial x}$$

$$\Rightarrow\quad x\frac{\partial^2 u}{\partial x^2}+y\frac{\partial^2 u}{\partial y\,\partial x}=(n-1)\frac{\partial u}{\partial x}$$

31. (B) In (a) $u=\frac{x^2y}{x+y}$ It is a homogeneous function of degree 2.

$$x\frac{\partial^2 u}{\partial x^2}+y\frac{\partial^2 u}{\partial x\,\partial y}=(n-1)\frac{\partial u}{\partial x}=\frac{\partial u}{\partial x}\text{ (as in question 30)}$$

In (b) $u=\frac{x^{1/2}-y^{1/2}}{x^{1/4}+y^{1/4}}$. It is a homogeneous function of degree $\left(\frac{1}{2}-\frac{1}{4}\right)=\frac{1}{4}$

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\,\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=n(n-1)u$$

$$=\frac{1}{4}\left(\frac{1}{4}-1\right)u=-\frac{3}{16}u$$

In (c) $u=x^{1/2}+y^{1/2}$ It is a homogeneous function of degree $\frac{1}{2}$.

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\,dy}+y^2\frac{\partial^2 u}{\partial y^2}=n(n-1)u$$

$$=\frac{1}{2}\left(\frac{1}{2}-1\right)u=-\frac{1}{4}u$$

In (d)$u=f\left(\frac{y}{x}\right)$ It is a homogeneous function of degree zero.

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=0.u=0$$

Hence correct match is

a	b	c	d
2	1	3	4

32. (B) Let $2a$ and $2b$ be the major and minor axes of the ellipse

Area $A=\pi ab$

$\Rightarrow\ \log A=\log\pi+\log a+\log b$

$\Rightarrow\ \partial(\log A)=\partial(\log\pi)+\partial(\log a)+\partial(\log b)$

$\Rightarrow\ \frac{\partial A}{A}=0+\frac{\partial a}{a}+\frac{\partial b}{b}$

$\Rightarrow\ \frac{100}{A}\partial A=\frac{100}{a}\partial a+\frac{100}{b}\cdot\partial b$

But it is given that $\frac{100}{a}\partial a=1$, and $\frac{100}{b}\partial b=1$

$$\frac{100}{A}\partial A=1+1=2$$

Thus percentage error in A =2%

33. (A) Given that $u=xyf\left(\frac{y}{x}\right)$. Since it is a homogeneous function of degree 2.

By Euler's theorem $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=nu$ (where $n=2$)

Thus $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=2u$

34. (A) Given that $u=x\log xy....(i)$

$x^3+y^3+3xy=1....(ii)$

we know that $\frac{\partial u}{\partial x}=\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}\frac{dy}{dx}....(ii)$

From (i) $\frac{\partial u}{\partial x}=x\cdot\frac{1}{xy}\cdot y+\log xy=1+\log xy$

and $\frac{\partial u}{\partial y}=x\cdot\frac{1}{xy}\cdot x=\frac{x}{y}$

From (ii), we get

$$3x^2 + 3y^2 \frac{dy}{dx} + 3\left(x\frac{dy}{dx} + y\cdot 1\right) = 0 \Rightarrow \frac{dy}{dx} = -\left(\frac{x^2+y}{y^2+x}\right)$$

Substituting these in (A), we get

$$\frac{du}{dx} = (1+\log xy) + \frac{x}{y}\left\{-\left(\frac{x^2+y}{y^2+x}\right)\right\}$$

35. (B) The given function is homogeneous of degree 2.

Euler's theorem $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 2z$

36. (C) $f'(x) = 6x^2 - 30x + 36 = 6(x-2)(x-3)$

Clearly, $f'(x) > 0$ when $x < 2$ and also when $x > 3$.

$f(x)$ is increasing in $]-\infty, 2[\ \cup\ ]3, \infty[$.

37. (B) $f'(x) = \frac{(x^2+1)-2x^2}{(x^2+1)^2} = \frac{1-x^2}{(x^2+1)^2}$

Clearly, $(x^2+1)^2 > 0$ for all x.

So, $f'(x) > 0 \Rightarrow (1-x^2) > 0$

$\Rightarrow (1-x)(1+x) > 0$

This happens when $-1 < x < 1$.

So, $f(x)$ is increasing in $]-1, 1[$.

38. (A) $f'(x) = 4x^3 - 4x = 4x(x-1)(x+1)$.

Clearly, $f'(x) < 0$ when $x < -1$ and also when $x > 1$.

Sol. $f(x)$ is decreasing in $]-\infty, -1[\ \cup\ ]1, \infty[$.

39.(C) $f'(x) = 9x^8 + 21x^6 > 0$ for all non-zero real values of x.

40. (C) $f'(x) = 3kx^2 - 18x + 9 = 3[kx^2 - 6x + 3]$

This is positive when $k > 0$ and $36 - 12k < 0$ or $k > 3$.

41. (A) $f(x) = (e^{ax} + e^{-ax}) = 2\cosh ax$.

$f'(x) = 2a\sinh ax < 0$ When $x > 0$ because $a < 0$

42. (D) $f'(x) = -x^2e^{-x} + 2xe^{-x} = e^{-x}x(2-x)$.

Clearly, $f'(x) > 0$ when $x > 0$ and $x < 2$.

43. (B) $f'(x) = (2x + a)$

$1 < x < 2 \Rightarrow 2 < 2x < 4 \Rightarrow 2 + a < 2x + a < 4 + a$

$\Rightarrow (2+a) < f'(x) < (4+a)$.

For $f(x)$ increasing, we have $f'(x) > 0$.

$\therefore 2 + a \geq 0$ or $a \geq -2$. So, least value of a is -2.

44. (B) Let the point closest to (4, 2) be $(2t^2, 4)$.

Now, $D = \sqrt{(2t^2-4)^2 + (4t-2)^2}$ is minimum when

$E = (2t^2-4)^2 + (4t-2)^2$ is minimum.

Now, $E = 4t^4 - 16t + 20$

$\Rightarrow \frac{dE}{dt} = 16t^3 - 16 = 16(t-1)(t^2+t+1)$

$\frac{dE}{dt} = 0 \Rightarrow t = 1$

$\frac{d^2E}{dt^2} = 48t^2$. So, $\left[\frac{d^2E}{dt^2}\right]_{(t=1)} = 48 > 0$.

So, $t = 1$ is a point of minima.

Thus Minimum distance $= \sqrt{(2-4)^2 + (4-2)^2} = 2\sqrt{2}$.

45. (A) Let the required point be $P(x, y)$. Then, perpendicular distance of $P(x, y)$ from $y - 3x + 3 = 0$ is

$$p = \frac{|y-3x+3|}{\sqrt{10}} = \frac{|x^2+7x+2-3x+3|}{\sqrt{10}}$$

$$= \frac{|x^2+4x+5|}{\sqrt{10}} = \frac{|(x+2)^2+1|}{\sqrt{10}} \text{ or } p = \frac{(x+2)^2+1}{\sqrt{10}}$$

So, $\frac{dp}{dx} = \frac{2(x+2)}{\sqrt{10}}$ and $\frac{d^2p}{dx^2} = \frac{2}{\sqrt{10}}$

$\frac{dp}{dx} = 0 \Rightarrow x = -2$, Also, $\left(\frac{d^2p}{dx^2}\right)_{x=-2} > 0$.

So, $x = -2$ is a point of minima.

When $x = -2$, we get $y = (-2)^2 + 7\times(-2) + 2 = -8$.

The required point is $(-2, -8)$.

46. (C) Let $A(0, c)$ be the given point and $P(x, y)$ be any point on $y = x^2$.

$D = \sqrt{x^2 + (y-c)^2}$ is shortest when $E = x^2 + (y-c)^2$ is shortest.

Now,

$E = x^2 + (y-c)^2 = y + (y-c)^2 \Rightarrow E = y^2 + y - 2cy + c^2$

$\frac{dE}{dy} = 2y + 1 - 2c$ and $\frac{d^2E}{dy^2} = 2 > 0$.

$\frac{dE}{dy} = 0 \Rightarrow y = \left(c - \frac{1}{2}\right)$

Thus E minimum, when $y = \left(c - \frac{1}{2}\right)$

Also, $D = \sqrt{\left(c-\frac{1}{2}\right) + \left(c - \frac{1}{2} - c\right)^2}$ $\left[\because x^2 = y = \left(c - \frac{1}{2}\right)\right]$

$= \sqrt{c - \frac{1}{4}} = \frac{\sqrt{4c-1}}{2}$

47. (B) Let $y = \left(\frac{1}{x}\right)^x$ then, $y = x^{-x}$

$$\Rightarrow \quad \frac{dy}{dx} = -x^{-x}(1 + \log x)$$

$$\frac{d^2y}{dx^2} = x^{-x}(1 + \log x)^2 - x^{-x} \cdot \frac{1}{x}$$

$$\frac{dy}{dx} = 0 \quad \Rightarrow \quad 1 + \log x = 0 \quad \Rightarrow \quad x = \frac{1}{e}$$

$$\left[\frac{d^2y}{dx^2}\right]_{\left(x = \frac{1}{e}\right)} = -\left(\frac{1}{e}\right)^{-\frac{1}{e}-1} < 0.$$

So, $x = \frac{1}{e}$ is a point of maxima. Maximum value $= e^{1/e}$.

48. (A) $f'(x) = 2x - \frac{250}{x^2}$ and $f''(x) = \left(2 + \frac{500}{x^3}\right)$

$$f'(x) = 0 \quad \Rightarrow \quad 2x - \frac{250}{x^2} = 0 \quad \Rightarrow \quad x = 5.$$

$f''(5) = 6 > 0$. So, $x = 5$ is a point of minima.

Thus minimum value $= \left(25 + \frac{250}{5}\right) = 75.$

49. (D) $f'(x) = (2\cos x - 1)(\cos x + 1)$ and

$f''(x) = -\sin x(1 + 4\cos x)$.

$f'(x) = 0 \quad \Rightarrow \quad \cos x = \frac{1}{2}$ or $\cos x = -1 \quad \Rightarrow \quad x = \pi/3$ or

$x = \pi$.

$f''\left(\frac{\pi}{3}\right) = \frac{-3\sqrt{3}}{2} < 0$. So, $x = \pi/3$ is a point of maxima.

Maximum value $= \left(\sin\frac{\pi}{3}\right)\left(1 + \cos\frac{\pi}{3}\right) = \frac{3\sqrt{3}}{4}$.

50. (C) $f(x) = \dfrac{2\sin x\cos x}{\frac{\sin x + \cos x}{\sqrt{2}}}$

$= \frac{2\sqrt{2}}{(\sec x + \operatorname{cosec} x)} = \frac{2\sqrt{2}}{z}$ (say),

where $z = (\sec x + \operatorname{cosec} x)$.

$$\frac{dz}{dx} = \sec x\tan x - \operatorname{cosec} x\cot x = \frac{\cos x}{\sin^2 x}(\tan^3 x - 1).$$

$$\frac{dz}{dx} = 0 \quad \Rightarrow \quad \tan x = 1 \quad \Rightarrow \quad x = \frac{\pi}{4} \text{ in } \left[0, \frac{\pi}{2}\right].$$

Sign of $\frac{dz}{dx}$ changes from $-ve$ to $+ve$ when x passes through the point $\pi/4$. So, z is minimum at $x = \pi/4$ and therefore, $f(x)$ is maximum at $x = \pi/4$.

Maximum value $= \frac{2\sqrt{2}}{[\sec(\pi/4) + \operatorname{cosec}(\pi/4)]} = 1.$

51. (C) $\frac{dy}{dx} = \frac{a}{x} + 2bx + 1$

$$\left[\frac{dy}{dx}\right]_{(x=-1)} = 0 \quad \Rightarrow \quad -a - 2b + 1 = 0 \quad \Rightarrow \quad a + 2b = 1 \text{....(i)}$$

$$\left[\frac{dy}{dx}\right]_{(x=2)} = 0 \quad \Rightarrow \quad \frac{a}{2} + 4b + 1 = 0$$

$\Rightarrow \quad a + 8b = -2$....(ii)

Solving (i) and (ii) we get $b = -\frac{1}{2}$ and $a = 2$.

52. (C) The given curve is $\frac{x^2}{5} + \frac{y^2}{4} = 1$ which is an ellipse.

Let the required point be $(\sqrt{5}\cos\phi, 2\sin\phi)$. Then,

$D = \sqrt{(\sqrt{5}\cos\phi - 0)^2 + (2\sin\phi + 2)^2}$ is maximum

when $z = D^2$ is maximum

$z = 5\cos^2\phi + 4(1 + \sin\phi)^2$

$$\Rightarrow \quad \frac{dz}{d\phi} = -10\cos\phi\sin\phi + 8(1 + \sin\phi)\cos\phi$$

$$\frac{dz}{d\phi} = 0 \quad \Rightarrow \quad 2\cos\phi(4 - \sin\phi) = 0$$

$$\Rightarrow \quad \cos\phi = 0 \quad \Rightarrow \quad \phi = \frac{\pi}{2}.$$

$$\frac{dz}{d\phi} = -\sin 2\phi + 8\cos\phi \quad \Rightarrow \quad \frac{d^2z}{d\phi^2} = -2\cos 2\phi - 8\sin\phi$$

when $\phi = \frac{\pi}{2}$, $\frac{d^2z}{d\phi^2} < 0$.

z is maximum when $\phi = \frac{\pi}{2}$. So, the required point is $\left(\sqrt{5}\cos\frac{\pi}{2}, \sin\frac{\pi}{2}\right)$ i.e. (0, 2).

53. (D) Let $z = \frac{1 + \tan x}{x} = \frac{1}{x} + \frac{\tan x}{x}$

Then, $\frac{dz}{dx} = -\frac{1}{x^2} + \sec^2 x$ and $\frac{d^2z}{dx^2} = \frac{2}{x^3} + 2\sec^2 x\tan x$

$$\frac{dz}{dx} = 0 \quad \Rightarrow \quad -\frac{1}{x^2} + \sec^2 x = 0 \quad \Rightarrow \quad x = \cos x.$$

$$\left[\frac{d^2z}{dx^2}\right]_{x = \cos x} = 2\cos^3 x + 2\sec^2 x\tan x > 0.$$

Thus z has a minima and therefore y has a maxima at $x = \cos x$.

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