Chapter 11

Three Dimensional Geometry

Basic Concepts of Three Dimensional

Distance Between Two Points

Let P and Q be two given points in space. Let the co-ordinates of the points P and Q be $(x_1, y_1\ z_1)$ and (x_2, y_2, z_2) with respect to a set OX, OY, OZ of rectangular axes.

The position vectors of the points P and Q are given by $\overrightarrow{OP} = x_1\hat{i} + y_1\hat{j} + z_1\hat{k}$ and $\overrightarrow{OQ} = x_2\hat{i} + y_2\hat{j} + z_2\hat{k}$

Now we have

$$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP}. = (x_2\hat{i} + y_2\hat{j} + z_2\hat{k}) - (x_1\hat{i} + y_1\hat{j} + z_1\hat{k})$$

$$= (x_2 - x_1)\ \hat{i} - (y_2 - y_1)\ \hat{j} - (z_2 - z_1)\ \hat{k}.$$

$$\therefore \quad PQ = |\overrightarrow{PQ}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

Distance (d) between two points (x_1, y_1, z_1) and (x_2, y_2, z_2) is

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

B. Section Formula

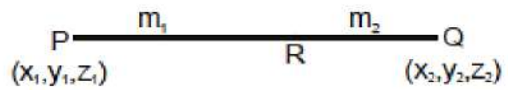
1. $x = \dfrac{m_2x_1 + m_1x_2}{m_1 + m_2}$
2. $y = \dfrac{m_2y_1 + m_1y_2}{m_1 + m_2}$
3. $z = \dfrac{m_2z_1 + m_1z_2}{m_1 + m_2}$

(for external division take –ve sign)

To determine the co-ordinates of a point R which divides the joining of two points P (x_1, y_1, z_1) and $Q(x_2, y_2, z_2)$ internally in the ratio m1 : m2. Let OX, OY, OZ be a set of rectangular axes.
The position vectors of the two given points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ are given by

$$\overrightarrow{OP} = x_1\hat{i} + y_1\hat{j} + z_1\hat{k} \quad ....(1) \text{ and}$$

$$\overrightarrow{OQ} = x_2\hat{i} + y_2\hat{j} + z_2\hat{k} \quad ....(2)$$



Also if the co-ordinates of the point R are (x, y, z), then $\overrightarrow{OR} = x\hat{i} + y\hat{j} + z\hat{k}$ ......(3)

Now the point R divides the join of P and Q in the ratio $m_1 : m_2$, so that

Hence $m_2\overrightarrow{PR} = m_1\overrightarrow{RQ}$

or $m_2(\overrightarrow{OR} - \overrightarrow{OP}) = m_1(\overrightarrow{OQ} - \overrightarrow{OR})$

or $\overrightarrow{OR} = \dfrac{m_1\overrightarrow{OQ} + m_2\overrightarrow{OP}}{m_1 + m_2}$

or $x\hat{i} + y\hat{j} + z\hat{k} = \dfrac{(m_1x_2 + m_2x_1)\hat{i} + (m_1y_2 + m_2y_1)\hat{j} + (m_1z_2 + m_2z_1)\hat{k}}{(m_1 + m_2)}$ [Using (1), (2) and (3)]

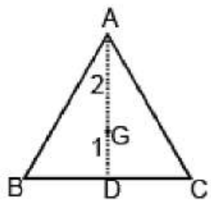
Remark : The middle point of the segment PQ is obtained by putting $m_1 = m_2$. Hence the co-ordinates of the middle point of PQ are $\left(\frac{1}{2}(x_1 + x_2), \frac{1}{2}(y_1 + y_2), \frac{1}{2}(z_1 + z_2)\right)$

Centroid Of A Triangle:

Let ABC be a triangle. Let the co-ordinates of the vertices A, B and C be (x_1, y_1, z_1), (x_2, y_2, z_2) and (x_3, y_3, z_3) respectively. Let AD be a median of the ΔABC. Thus D is the mid point of BC.

$\therefore$ The co-ordinates of D are $\left(\dfrac{x_2 + x_3}{2}, \dfrac{y_2 + y_3}{2}, \dfrac{z_2 + z_3}{2}\right)$

Now if G is the centroid of ΔABC, then G divides AD in the ratio 2 : 1. Let the co-ordinates of G be (x, y, z).



Then $x = \dfrac{2.\left(\dfrac{x_2 + x_3}{2}\right) + 1.x_1}{2+1}$, or $x = \dfrac{x_1 + x_2 + x_3}{3}$.

Similarly $y = \dfrac{1}{2}(y_1 + y_2 + y_3)$, $z = \dfrac{1}{2}(z_1 + z_2 + z_3)$.

Centroid Of A Tetrahedron :

Let ABCD be a tetrahedron, the co-ordinates of whose vertices are (x_r, y_r, z_r), $r = 1, 2, 3, 4$.

Let G_1 be the centroid of the face ABC of the tetrahedron. Then the co-ordinates of G_1 are

$$\left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}, \frac{z_1 + z_2 + z_3}{3}\right)$$

The fourth vertex D of the tetrahedron does not lie in the plane of ΔABC. We know from statics that the centroid of the tetrahedron divides the line DG_1 in the ratio 3 : 1. Let G be the centroid of the tetrahedron and if (x, y, z) are its co-ordinates, then

$$x = \frac{3.\dfrac{x_1 + x_2 + x_3}{3} + 1.x_4}{3+1} \text{ or } x = \frac{x_1 + x_2 + x_3 + x_4}{4}.$$

Similarly $y = \dfrac{1}{4}(y_1 + y_2 + y_3 + y_4)$, $z = \dfrac{1}{4}(z_1 + z_2 + z_3 + z_4)$.

Ex.1 P is a variable point and the co-ordinates of two points A and B are (–2, 2, 3) and (13, –3, 13) respectively. Find the locus of P if 3PA = 2PB.

Sol. Let the co-ordinates of P be (x, y, z).

$\therefore PA = \sqrt{(x+2)^2 + (y-2)^2 + (z-3)^2}$ ....(1)

and $PB = \sqrt{(x-13)^2 + (y+3)^2 + (z-13)^2}$ ....(2)

Now it is given that 3PA = 2PB i.e., $9PA^2 = 4PB^2$. ....(3)

Putting the values of PA and PB from (1) and (2) in (3), we get

$9\{(x+2)^2 + (y-2)^2 + (z-3)^2\} = 4\{(x-13)^2 + (y+3)^2 + (z-13)^2\}$

or $9\{x^2 + y^2 + z^2 + 4x - 4y - 6z + 17\} = 4\{x^2 + y^2 + z^2 - 26x + 6y - 26z + 347\}$

or $5x^2 + 5y^2 + 5z^2 + 140x - 60y + 50z - 1235 = 0$

or $x^2 + y^2 + z^2 + 28x - 12y + 10z - 247 = 0$

This is the required locus of P.

Ex.2 Find the ratio in which the xy-plane divides the join of (–3, 4, –8) and (5, –6, 4). Also find the point of intersection of the line with the plane.

Sol. Let the xy-plane (i.e., z = 0 plane) divide the line joining the points (–3,4, –8) and (5, –6, 4) in the ratio $\mu : 1$, in the point R. Therefore, the co-ordinates of the point R are

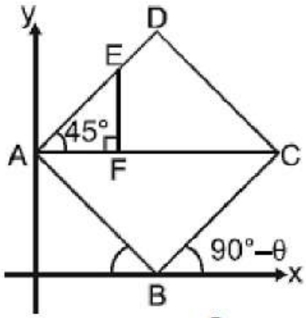
$\left(\frac{5\mu-3}{\mu+1}, \frac{-6\mu+4}{\mu+1}, \frac{4\mu-8}{\mu+1}\right)$ ....(1)

But on xy-plane, the z co-ordinate of R is zero

$(4\mu - 8) / (\mu + 1) = 0$, or $\mu = 2$. Hence $\mu : 1 = 2 : 1$. Thus the required ratio is 2 : 1. Again putting $\mu = 2$ in (1), the co-ordinates of the point R become (7/3, –8/3, 0).

Ex.3 ABCD is a square of side length 'a'. Its side AB slides between x and y-axes in first quadrant. Find the locus of the foot of perpendicular dropped from the point E on the diagonal AC, where E is the midpoint of the side AD.

Sol. Let vertex A slides on y-axis and vertex B slides on x-axis coordinates of the point A are $(0, a\sin\theta)$ and that of C are $(a\cos\theta + a\sin\theta, a\cos\theta)$



In ΔAEF, AF = $\frac{a}{2}\cos 45° = \frac{a}{2\sqrt{2}}$ and FC = AC – AF = $\sqrt{2}a - \frac{a}{2\sqrt{2}} = \frac{3a}{2\sqrt{2}}$

$\Rightarrow$ AF : FC = $\frac{a}{2\sqrt{2}} : \frac{3a}{2\sqrt{2}}$ = 1 : 3

$\Rightarrow$ Let the coordinates of the point F are (x, y)

$\Rightarrow$ $x = \frac{3\times 0 + 1(a\cos\theta + a\sin\theta)}{4} = \frac{a(\sin\theta + \cos\theta)}{4}$

$\Rightarrow$ $\frac{4x}{a} = \sin\theta + \cos\theta$ ....(1)

$y = \frac{3a\sin\theta + a\cos\theta}{4} \Rightarrow \frac{4y}{a} = 3\sin\theta + \cos\theta$ ...(2)

Form (1) and (2),

$\sin\theta = \frac{2(y-x)}{a}$ and $\cos\theta = \frac{6x-2y}{a}$

$\Rightarrow$ $(y-x)^2 + (3x-y)^2 = \frac{a^2}{4}$ is the locus of the point F.

Direction Cosines and Direction Ratios of a Line

Direction Cosines Of A Line

If α, β, γ are the angles which a given directed line makes with the positive directions of the axes. of x, y and z respectively, then $\cos\alpha$, $\cos\beta$ $\cos\gamma$ are called the direction cosines (briefly written as d.c.'s) of the line. These d.c.'s are usually denote by l, m, n. Let AB be a given line. Draw a line OP parallel to the line AB and passing through the origin O. Measure angles α, β, γ, then $\cos\alpha$, $\cos\beta$ $\cos\gamma$ are the d.c.'s of the line AB. It can be easily seen that l, m, n, are the direction cosines of a line if and only

if $\ell\hat{i} + m\hat{j} + n\hat{k}$ is a unit vector in the direction of that line. Clearly OP'(i.e. the line through O and parallel to BA) makes angle 180° – α, 180°– β, 180° – γ with OX, OY and OZ respectively. Hence d.c.'s of the line BA are $\cos(180° - \alpha), \cos(180° - \beta), \cos(180° - \gamma)$ i.e., are $-\cos\alpha, -\cos\beta, -\cos\gamma$.
If the length of a line OP through the origin O be r, then the co-ordinates of P are (lr, mr, nr) where l, m, n are the d c.'s of OP.

If l, m, n are direction cosines of any line AB, then they will satisfy $\ell^2 + m^2 + n^2 = 1$.

Direction Ratios :

If the direction cosines l, m, n of a given line be proportional to any three numbers a, b, c respectively, then the numbers a, b, c are called direction ratios (briefly written as d.r.'s of the given line.

Relation Between Direction Cosines And Direction Ratios :

Let a, b, c be the direction ratios of a line whose d.c.'s are l, m, n. From the definition of d.r.'s. we have $l/a = m/b = n/c = k$ (say). Then $l = ka$, $m = kb$, $n = kc$.
But $\ell^2 + m^2 + n^2 = 1$.

$$k^2(a^2 + b^2 + c^2) = 1, \text{ or } k^2 = 1/(a^2 + b^2 + c^2)$$

$$\text{or } k = \pm \frac{1}{\sqrt{(a^2+b^2+c^2)}}.$$

.
Taking the positive value of k, we get l

$$= \frac{a}{\sqrt{(a^2+b^2+c^2)}},\ m = \frac{b}{\sqrt{(a^2+b^2+c^2)}},\ n = \frac{c}{\sqrt{(a^2+b^2+c^2)}}$$

Again taking the negative value of k, we get l

$$= \frac{-a}{\sqrt{(a^2+b^2+c^2)}},\ m = \frac{-b}{\sqrt{(a^2+b^2+c^2)}},\ n = \frac{-c}{\sqrt{(a^2+b^2+c^2)}}$$

Remark. Direction cosines of a line are unique. But the direction ratios of a line are by no means unique. If a, b, c are direction ratios of a line, then ka, kb, kc are also direction ratios of that line where k is any non-zero real number. Moreover if a, b, c are direction ratios of a line, then ai^ + bj^ + ck^ is a vector parallel to that line.

Ex.4 Find the direction cosines l + m + n of the two lines which are connected by the relation l + m + n = 0 and mn – 2nl –2lm = 0.

Sol. The given relations are $l + m + n = 0$ or $l = -m - n$ ....(1)

and $mn - 2nl - 2lm = 0$ ...(2)
Putting the value of l from (1) in the relation (2), we get
$mn - 2n(-m - n) - 2(-m - n)m = 0$
or $2m^2 + 5mn + 2n^2 = 0$ or $(2m + n)(m + 2n) = 0$.

Now when $\frac{m}{n} = -\frac{1}{2}$, (3) given $\frac{\ell}{n} = \frac{1}{2} - 1 = -\frac{1}{2}$.

$\therefore \quad \frac{m}{1} = \frac{n}{-2}$ and $\frac{\ell}{1} = \frac{n}{-2}$

i.e. $\frac{\ell}{1} = \frac{m}{1} = \frac{n}{-2} = \frac{\sqrt{(\ell^2 + m^2 + n^2)}}{\sqrt{\{1^2 + 1^2 + (-2)^2\}}} = \frac{1}{\sqrt{6}}$

$\therefore$ The d.c.'s of one line are $\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{-2}{\sqrt{6}}$.

Again when $\frac{m}{2} = -2$, (3) given $\frac{\ell}{n} = 2 - 1 = 1$.

i.e. $\frac{\ell}{1} = \frac{m}{-2} = \frac{n}{1} = \frac{\sqrt{(\ell^2 + m^2 + n^2)}}{\sqrt{\{1^2 + (-2)^2 + 1^2\}}} = \frac{1}{\sqrt{6}} \therefore$

The d.c.'s of the other line are $\frac{1}{\sqrt{6}}, \frac{-2}{\sqrt{6}}, \frac{1}{\sqrt{6}}$.

To find the projection of the line joining two points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ on the another line whose d.c.'s are l, m, n.

Let O be the origin. Then $\overrightarrow{OP} = x_1\hat{i} + y_1\hat{j} + z_1\hat{k}$ and $\overrightarrow{OQ} = x_2\hat{i} + y_2\hat{j} + z_2\hat{k}$.

$\therefore \quad \overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = (x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j} + (z_2 - z_1)\hat{k}$.
Now the unit vector along the line whose d.c.'s are ℓ, m, n $= \ell\hat{i} + m\hat{j} + n\hat{k}$.

$\therefore$ projection of PQ on the line whose d.c.'s are l, m, n

$= [(x_2 - x_1)\,\hat{i} + (y_2 - y_1)\,\hat{j} + (z_2 - z_1)\,\hat{k}]\,.\,(\ell\hat{i} + m\hat{j} + n\hat{k})$

$= \ell(x_2 - x_1) + m\,(y_2 - y_1) + n(z_2 - z_1).$

The angle θ between these two lines is given by $\cos\theta = \dfrac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{(a_1^2 + b_1^2 + c_1^2)}\sqrt{(a_2^2 + b_2^2 + c_2^2)}}$

If l_1, m_1, n_1 and l_2 , m_2, n_2 are two sets of real numbers, then

$(l_1^2 + m_1^2 + n_1^2)(l_2^2 + m_2^2 + n_2^2) - (l_1l_2 + m_1m_2 + n_1n_2)^2 = (m_1n_2 - m_2n_1)^2 + (n_1l_2 - n_2l_1)^2 + (l_1m_2 - l_2m_1)^2$

Now, we have

$\sin2\theta = 1 - \cos2\theta = 1 - (l_1l_2 + m_1m_2 + n_1n_2)^2 = (l_1^2 + m_1^2 + n_1^2)(l_2^2 + m_2^2 + n_2^2) - (l_1l_2 + m_1m_2 + n_1n_2)^2$

$= (m_1n_2 - m_2n_1)^2 + (n_1l_2 - n_2l_1)^2 + (l_1m_2 - l_2m_1)^2$ $= \begin{vmatrix} m_1 & n_1 \\ m_2 & n_2 \end{vmatrix}^2 + \begin{vmatrix} \ell_1 & n_1 \\ \ell_2 & n_2 \end{vmatrix}^2 + \begin{vmatrix} \ell_1 & m_1 \\ \ell_2 & m_2 \end{vmatrix}^2$

Condition for perpendicularity $\Rightarrow l_1l_2 + m_1m_2 + n_1n_2 = 0$.

Condition for parallelism $\Rightarrow$ $l_1 = l_2, m_1 = m_2, n_1 = n_2. \Rightarrow \dfrac{a_1}{a_2} = \dfrac{b_1}{b_2} = \dfrac{c_1}{c_2}$

Ex.5 Show that the lines whose d.c.'s are given by $l + m + n = 0$ and $2mn + 3ln - 5lm = 0$ are at right angles.

Sol. From the first relation, we have $l = -m - n$....(1)

Putting this value of l in the second relation, we have

$2mn + 3(-m - n)n - 5(-m - n)m = 0$ or $5m^2 + 4mn - 3n^2 = 0$ or $5(m/n)^2 + 4(m/n) - 3 = 0$ ....(2)

Let l_1, m_1, n_1 and l_2, m_2, n_2 be the d,c, s of the two lines. Then the roots of (2) are m_1/n_1 and m_2/n_2.

product of the roots = $\dfrac{m_1}{n_1}.\dfrac{m_2}{n_2} = -\dfrac{3}{5}$ or $\dfrac{m_1m_2}{3} = \dfrac{n_1n_2}{-5}$ ...(3)

Again from (1), $n = -l - m$ and putting this value of n in the second given relation, we have

$2m(-l - m) + 3l(-l - m) - 5lm = 0$ or $3(l/m)^2 + 10(l/m) + 2 = 0$.

$$\therefore \quad \frac{\ell_1}{m_1}\cdot\frac{\ell_2}{m_2} = \frac{2}{3} \text{ or } \frac{\ell_1\ell_2}{2} = \frac{m_1m_2}{3}$$

From (3) and (4) we have $\frac{\ell_1\ell_2}{2} = \frac{m_1m_2}{3}\ \frac{n_1n_2}{-5} = k$ (say)

$l_1l_2 + m_1m_2 + n_1n_2 = (2 + 3 - 5)k = 0 \cdot k = 0. \Rightarrow$ The lines are at right angles.

Remarks :

(a) Any three numbers a, b, c proportional to the direction cosines are called the direction ratios i.e. $\frac{\ell}{a} = \frac{m}{b} = \frac{n}{c} = \pm\frac{1}{\sqrt{a^2+b^2+c^2}}$ same sign either +ve or –ve should be taken throughout.

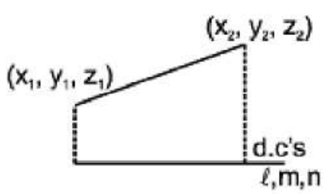
Note that d.r's of a line joining x_1, y_1, z_1 and x_2, y_2, z_2 are proportional to $x_2 - x_1$, $y_2 - y_1$ and $z_2 - z_1$

(b) If θ is the angle between the two lines whose d.c's are l_1, m_1, n_1 and l_2, m_2, n_2

$\cos\theta = l_1l_2 + m_1m_2 + n_1n_2$

Hence if lines are perpendicular then $l_1l_2 + m_1m_2 + n_1n_2 = 0$.

if lines are parallel then $\frac{\ell_1}{\ell_2} = \frac{m_1}{m_2} = \frac{n_1}{n_2}$



Note that if three lines are coplanar then $\begin{vmatrix} \ell_1 & m_1 & n_1 \\ \ell_2 & m_2 & n_2 \\ \ell_3 & m_3 & n_3 \end{vmatrix} = 0$

(c) Projection of the join of two points on a line with d.c's l, m, n are $l(x_2 - x_1) + m(y_2 - y_1) + n(z_2 - z_1)$

(d) If l_1, m_1, n_1 and l_2, m_2, n_2 are the d.c.'s of two concurrent lines, show that the d.c's of two lines bisecting the angles between them are proportional to $l_1 \pm l_2, m_1 \pm m_2, n_1 \pm n_2$

Equation of Straight Line

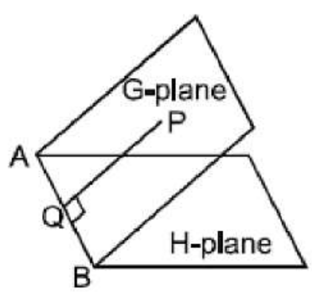
Straight Line

(i) Equation of a line through $A(x_1, y_1, z_1)$ and having direction cosines l , m , n are $\frac{x - x_1}{\ell} = \frac{y - y_1}{m} = \frac{z - z_1}{n}$ and the lines through (x_1, y_1, z_1) and (x_2, y_2, z_2) $\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}$

(ii) Intersection of two planes $a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$ together represent the unsymmetrical form of the straight line.

(iii) General equation of the plane containing the line $\frac{x - x_1}{\ell} = \frac{y - y_1}{m} = \frac{z - z_1}{n}$ is $A(x - x_1) + B(y - y_1) + c(z - z_1) = 0$ where $Al + bm + cn = 0$.

(iv) Line of Greatest Slope



AB is the line of intersection of G-plane and H is the horizontal plane. Line of greatest slope on a given plane, drawn through a given point on the plane, is the line through the point 'P' perpendicular to the line of intersection of the given plane with any horizontal plane.

Ex.15 Show that the distance of the point of intersection of the line $\frac{x-2}{3}=\frac{y+1}{4}=\frac{z-2}{12}$ and the plane x – y + z = 5 from the point (–1, –5, –10) is 13.

Sol. The equation of the given line are $\frac{x-2}{3}=\frac{y+1}{4}=\frac{z-2}{12}$ = r (say). ....(1)

The co-ordinates of any point on the line (1) are (3r + 2, 4r - 1, 12 r + 2). If this point lies on the plane x – y + z = 5, we have 3r + 2 – (4r – 1) + 12r + 2 = 5, or 11r = 0, or r = 0.
Putting this value of r, the co-ordinates of the point of intersection of the line (1) and the given plane are (2, –1, 2).

∴ The required distance = distance between the points (2, –1, 2) and (–1, –5, –10)

$= \sqrt{(2+1)^2+(-1+5)^2+(2+10)^2}$

$= \sqrt{(9+16+144)}=\sqrt{(169)}=13$

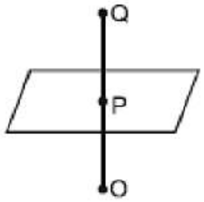
Ex.16 Find the co-ordinates of the foot of the perpendicular drawn from the origin to the plane 3x + 4y – 6z + 1 = 0. Find also the co-ordinates of the point on the line which is at the same distance from the foot of the perpendicular as the origin is.

Sol. The equation of the plane is 3x + 4y – 6z + 1 = 0. ....(1)
The direction ratios of the normal to the plane (1) are 3, 4, –6. Hence the line normal to the plane (1) has d.r.'s 3, 4, –6, so that the equations of the line through (0, 0, 0) and perpendicular to the plane (1) are x/3 = y/4 = z/–6 = r ....(2)

The co-ordinates of any point P on (2) are (3r, 4r, – 6r) ....(3)

If this point lies on the plane (1), then 3(3r) + r(4r) – 6(–6r) + 1 = 0, or r = –1/61.

Putting the value of r in (3), the co-ordinates of the foot of the perpendicular P are (–3/61, –4/61, 6/61).
Now let Q be the point on the line which is at the same distance from the foot of the perpendicular as the origin. Let (x_1, y_1, z_1) be the co-ordinates of the point Q. Clearly P is the middle point of OQ.



Hence we have $\frac{x_1+0}{2} = -\frac{3}{61}, \frac{y_1+0}{2} = \frac{4}{61}, \frac{z_1+0}{2} = \frac{6}{61}$

or $x_1 = 6/61, y_1 = -8/61, z_1 = 12/61$.

$\therefore$ The co-ordinates of Q are $(-6/61, -8/61, 12/61)$.

Ex.17 Find in symmetrical form the equations of the line 3x + 2y – z – 4 = 0 & 4x + y – 2z + 3 = 0 and find its direction cosines.

Sol. The equations of the given line in general form are $3x + 2y - z - 4 = 0$ & $4x + y - 2z + 3 = 0$ ..(1)

Let l, m, n be the d.c.'s of the line. Since the line is common to both the planes, it is perpendicular to the normals to both the planes. Hence we have $3l + 2m - n = 0$, $4l + m - 2n = 0$.

Solving these, we get

$$\frac{\ell}{-4+1} = \frac{m}{-4+6} = \frac{n}{3-8} \text{ or } \frac{\ell}{-3} = \frac{m}{2} = \frac{n}{-5} = \frac{\sqrt{(\ell^2+m^2+n^2)}}{\sqrt{(9+4+25)}} = \frac{1}{\sqrt{38)}}$$

$\therefore$ the d.c.'s of the line are $-\frac{3}{\sqrt{(38)}}, \frac{2}{\sqrt{(38)}}, -\frac{5}{\sqrt{(38)}}$.

Now to find the co-ordinates of a point on the line given by (1), let us find the point where it meets the plane z = 0. Putting z = 0 i the equations given by (1), we have $3x + 2y - 4 = 0, 4x + y + 3 = 0$.

Solving these, we get

$\frac{x}{6+4} = \frac{y}{-16-9} = \frac{1}{3-8}$, or $x = -2, y = 5$.

Therefore the equation of the given line in symmetrical form is $\frac{x+2}{-3} = \frac{y-5}{2} = \frac{z-0}{-5}$.

Ex.18 Find the equation of the plane through the line 3x – 4y + 5z = 10, 2x + 2y – 3z = 4 and parallel to the line x = 2y = 3z.

Sol. The equation of the given line are 3x – 4y + 5z = 10, 2x + 2y – 3z = 4 ...(1)
The equation of any plane through the line (1) is (3x – 4y + 5z – 10) + λ (2x + 2y – 3z – 4) = 0
or (3 + 2λ)x + (–4 +2λ) y + (5 – 3λ) z – 10 – 4λ = 0. ...(2)

The plane (1) will be parallel to the line x = 2y = 3z i.e. $\frac{x}{6}=\frac{y}{3}=\frac{z}{2}$ if
(3 + 2λ) . 6 + (–4 + 2λ). 3 + (5 – 3λ).2 = 0 or

λ(12 + 6 – 6) + 18 – 12 + 10 = 0 or $\lambda = -\frac{4}{3}$.

Putting this value of λ in (2), the required equation of the plane is given by

$$\left(3-\frac{8}{3}\right)x+\left(-4-\frac{8}{3}\right)y+(5+4)z-10+\frac{16}{3}=0$$

or $x - 20y + 27z = 14$.

Ex.19 Find the equation of a plane passing through the line $\frac{x-1}{1}=\frac{y-2}{1}=\frac{z-2}{-2}$ and making an angle of 30° with the plane x + y + z = 5.
Sol. The equation of the required plane is (x – y + 1) + λ(2y + z – 6) = 0 ⇒ x + (2λ – 1) y + λz + 1 – 6λ = 0

Since it makes an angle of 30° with x +y + z = 5

$$\Rightarrow \frac{|1+(2\lambda-1)+\lambda|}{\sqrt{3}.\sqrt{1+\lambda^2+(2\lambda-1)^2}}=\frac{\sqrt{3}}{2} \Rightarrow$$

$$\Rightarrow \quad 4\lambda^2 = 5\lambda^2 - 4\lambda + 2$$

$$\Rightarrow \lambda^2 - 4\lambda + 2 = 0 \Rightarrow \lambda = (2 \pm \sqrt{2}):$$

$\Rightarrow (x - y + 1) + (2 \pm \sqrt{2})(2y + x - 6) = 0$ are two required planes.

Ex.20 Prove that the lines 3x + 2y + z – 5 = 0 = x + y – 2z – 3 and 2x – y – z = 0 = 7x + 10y – 8z – 15 are perpendicular.

Sol. Let l_1, m_1, n_1 be the d.c. s of the first line. Then $3l_1 + 2m_1 + n_1 = 0$, $l_1 + m_1 - 2n_1 = 0$. Solving, we get

$$\frac{\ell_1}{-4-1} = \frac{m_1}{1+6} = \frac{n_1}{3-2} \text{ or } \frac{\ell_1}{-5} = \frac{m_1}{7} = \frac{n_1}{1}.$$

Again let l_2, m_2, n_2 be the d.c.'s of the second line, then $2l_2 - m_2 - n_2 = 0$, $7l_2 + 10m_2 - 8n_2 = 0$.

Solving, $\frac{\ell_2}{8+10} = \frac{m_2}{-7+16} = \frac{n_2}{20+7}$ or $\frac{\ell_2}{2} = \frac{m_2}{1} = \frac{n_2}{3}$.

Solving, $\frac{\ell_2}{8+10} = \frac{m_2}{-7+16} = \frac{n_2}{20+7}$ or $\frac{\ell_2}{2} = \frac{m_2}{1} = \frac{n_2}{3}$.

Hence the d.c.'s of the two given lines are proportional to −5, 7, 1 and 2, 1, 3. We have

$-5.2 + 7.1 + 1.3 = 0$ ∴the given lines are perpendicular.

Ex.21 Find the equation of the plane which contains the two parallel lines

$$\frac{x+1}{3} = \frac{y-2}{2} = \frac{z}{1} \text{ and } \frac{x-3}{3} = \frac{y+4}{2} = \frac{z-1}{1}.$$

Sol. The equation of the two parallel lines are

$(x + 1)/3 = (y - 2)/2 = (z - 0)/1$ ....(1) and $(x - 3)/3 = (y + 4)/2 = (z - 1)/1$. ....(2)

The equation of any plane through the line (1) is

$a(x + 1) + b(y - 2) + cz = 0$, ...(3)

where $3a + 2b + c = 0$. ....(4)

The line (2) will also lie on the plane (3) if the point (3, −4, 1) lying on the line (2) also lies on the plane (3), and for this we have $a(3 + 1) + b(-4 - 2) + c.1 = 0$ or $4a - 6b + c = 0$. ....(5)

Solving (4) and (5), we get $a/8 = b/1 = c/{-24}$

Putting these proportionate values of a, b, c in (3), the required equation of the plane is

8(x + 1) + 1.(y – 2) – 26z = 0, or 8x + y – 26 + 6 = 0.

Ex.22 Find the distance of the point P(3, 8, 2) from the line $\frac{x-1}{2}=\frac{y-3}{4}=\frac{z-2}{3}$ measured parallel to the plane 3x + 2y – 2z + 17 = 0.

Sol . The equation of the given line are (x – 1)/2 = (y – 3)/4 = (z – 2)/3 = r, (say). ...(1)

Any point Q on the line (1) is (2r + 1, 4r + 3, 3r + 2)

Now P is the point (3, 8, 2) and hence d.r.'s of PQ are

2r + 1 – 3, 4r + 3 – 8, 3r + 2 – 2 i.e. 2r – 2, 4r – 5, 3r.

It is required to find the distance PQ measured parallel to the plane 3x + 2y – 2z + 17 = 0 ...(2)

Now PQ is parallel to the plane (2) and hence PQ will be perpendicular to the normal to the plane (2).

Hence we have (2r – 2) (3) + (4r – 5) (2) + (2r) (–2) = 0 or 8r – 16 = 0, or r = 2.

Putting the value of r, the point Q is (5, 11, 8) =

$\sqrt{[(3-5)^2+(8-11)^2+(2-8)^2]}=\sqrt{(4+9+36)}=7$.

Ex.23 Find the projection of the line 3x – y + 2z = 1, x + 2y – z = 2 on the plane 3x + 2y + z = 0.

Sol. The equations of the given line are 3x – y + 2z = 1, x + 2y – z = 2. ...(1)

The equation of the given plane is 3x + 2y + z = 0. ...(2)

The equation of any plane through the line (1) is (3x – y + 2z – 1) + l(x + 2y - z - 2) = 0

or (3 + l) x + (-1 + 2l) y + (2 - l) z - 1 - 2l = 0 ....(3)

The plane (3) will be perpendicular to the plane (2), if 3(3 + l)+ 2(-1 + 2l) + 1 (2 - l) = 0 or l = -3/3.

Putting this value of l in (3), the equation of the plane through the line (1) and perpendicular to the plane (2) is given by

$$\left(3-\frac{3}{2}\right)x + (-1-3)\,y + \left(2+\frac{3}{2}\right)z - 1 + 3 = 0$$ or $3x - 8y + 7z + 4 = 0$. ....(4)

The projection of the given line (1) on the given plane (2), is given by the equations (2) and (4) together.

Ex.25 Find the foot and hence the length of the perpendicular from the point (5, 7, 3) to the line (x – 15)/3 = (y – 29)/8 = (z – 5)/(–5). Find the equations of the perpendicular. Also find the equation of the plane in which the perpendicular and the given straight line lie.

Sol. Let the given point (5, 7, 3) be P.

The equations of the given line are $(x - 15)/3 = (y - 29)/8 = (z - 5)/(-5) = r$ (say). ...(1)

Let N be the foot of the perpendicular from the point P to the line (1). The co-ordinates of N may be taken as $(3r + 15, 8r + 29, -5r + 5)$. ...(2)

the direction ratios of the perpendicular PN are

$3r + 15 - 5, 8r + 29 - 7, -5r + 5 - 3$, i.e. are $3r + 10, 8r + 22, -5r + 2$. ...(3)

Since the line (1) and the line PN are perpendicular to each other, therefore

$3(3r + 10) + 8(8r + 22) - 5(-5r + 2) = 0$ or $98r + 196 = 0$ or $r = -2$

Putting this value of r in (2) and (3), the foot of the perpendicular N is (9, 13, 15) and the direction ratios of the perpendicular PN are 4, 6, 12 or 2, 3, 6.

the equations of the perpendicular PN are $(x - 5)/2 = (y - 7)/3 = (z - 3)/6$. ...(4)

Length of the perpendicular PN

= the distance between P(5, 7, 3) and N(9, 13, 15)

$= \sqrt{(9-5)^2 + (13-7)^2 + (15-3)^2} = 14.$

Lastly the equation of the plane containing the given line (1) and the perpendicular (4) is given by $\begin{vmatrix} x-15 & y-29 & z-5 \\ 3 & 8 & -5 \\ 2 & 3 & 6 \end{vmatrix} = 0$

or (x – 15) (48 + 15) – (y – 29) (18 + 10) + (z – 5) (9 – 16) = 0 or 9x – 4y – z =– 14 = 0.

Ex.26 Show that the planes 2x – 3y – 7z = 0, 3x – 14y – 13z = 0, 8x – 31y – 33z = 0 pass through the one line find its equations.

Sol .The rectangular array of coefficient is $\begin{vmatrix} 2 & -3 & -7 & 0 \\ 3 & -14 & -13 & 0 \\ 8 & -31 & -33 & -0 \end{vmatrix}$.

We have, $\Delta_4 = \begin{vmatrix} 2 & -3 & -7 \\ 3 & -14 & -13 \\ 8 & -31 & -33 \end{vmatrix} = \begin{vmatrix} 2 & -1 & -1 \\ 3 & -11 & -4 \\ 8 & -23 & -9 \end{vmatrix}$ (by $C_2 + C_1$, $C_2 + 3C_1$)

$= \begin{vmatrix} 0 & 0 & -1 \\ -5 & -7 & -4 \\ -10 & -14 & -9 \end{vmatrix} = -1(70 - 70) = 0,$ (by $C_1 + 2C_2$, $C_2 - C_2$)

since $\Delta_4 = 0$, therefore, the three planes either intersect in a line or form a triangular prism.

Now $\Delta_3 = \begin{vmatrix} 2 & -3 & 0 \\ 3 & -14 & 0 \\ 8 & -31 & 0 \end{vmatrix} = 0$ Similarly $\Delta_2 = 0$ and $\Delta_1 = 0$,

Hence the three planes intersect in a common line.

Clearly the three planes pass through (0, 0, 0) and hence the common line of intersection will pass through (0, 0, 0). The equations of the common line are given by any of the two given planes. Therefore the equations of the common line are given by 2x – 3y – 7z = 0 and 3x – 14y – 13z = 0.

∴ the symmetric form of the line is given by $\frac{x}{39-98} = \frac{y}{-21+26} = \frac{z}{-28+9}$ or $\frac{x}{-59} = \frac{y}{5} = \frac{z}{-19}$.

Ex.27 For what values of k do the planes x – y + z + 1 = 0, kx + 3y + 2z – 3 = 0, 3x + ky + z – 2 = 0 (i) intersect in a point ; (ii) intersect in a line ; (iii) form a triangular prism ?

Sol. The rectangular array of coefficients is $\begin{vmatrix} 1 & -1 & 1 & 1 \\ k & 3 & 2 & 3 \\ 3 & k & 1 & -2 \end{vmatrix}$

Now we calculate the following determinants

$\Delta_4 = \begin{vmatrix} 1 & -1 & 1 \\ k & 3 & 2 \\ 3 & k & 1 \end{vmatrix} = \begin{vmatrix} 0 & -1 & 0 \\ k+3 & 3 & 5 \\ 3+k & k & k+1 \end{vmatrix}$ (adding 2nd column to 1st and 3rd)

$= (k+3) \begin{vmatrix} 0 & -1 & 0 \\ 1 & 3 & 5 \\ 1 & k & k+1 \end{vmatrix} = (k+3)(k+1-5) = (k+3)(k-4).$

$\Delta_2 = \begin{vmatrix} 1 & -1 & 1 \\ k & 3 & -3 \\ 3 & k & -2 \end{vmatrix} = \begin{vmatrix} 0 & -1 & 0 \\ k+3 & 3 & 0 \\ 3+k & k & k-2 \end{vmatrix} = (k+3)(k-2),$ (adding 2nd column to 1st and 3rd)

$\Delta_2 = \begin{vmatrix} 1 & 1 & 1 \\ k & 2 & -3 \\ 3 & 1 & -2 \end{vmatrix} = \begin{vmatrix} 0 & 1 & 0 \\ k-2 & 2 & -5 \\ 2 & 1 & -3 \end{vmatrix}$ (adding (–1) times 2nd column to 1st and 3rd)

$= -\{(k-2)(-3) + 10\} = 3k -$
16, and $\Delta_1 = \begin{vmatrix} -1 & 1 & 1 \\ 3 & 2 & -3 \\ k & 1 & -2 \end{vmatrix} = \begin{vmatrix} 0 & 1 & 1 \\ 0 & 2 & -3 \\ k-2 & 1 & -2 \end{vmatrix} = -5(k-2)$ (adding 3rd column to 1st)

(i) The given planes will intersect in a point if $\Delta_4 \neq 0$ and so we must have $k \neq -3$ and $k \neq 4$. Thus the given planes will intersect in a point for all real values of k other than -3 and 4.

(ii) If $k = -3$, we have $\Delta_4 = 0$, $\Delta_3 = 0$ but D2 $\neq 0$. Hence the given planes will form a triangular prism if $k = -3$.

(iii) If $k = 4$, we have $\Delta_4 = 0$ but $\Delta_3 \neq 0$. Hence the given planes will form a triangular prism if $k = 4$.
We observe that for no value of k the given planes will have a common line of intersection.

Ex.28 Find the equation of the line passing through (1, 1, 1) and perpendicular to the line of intersection of the planes $x + 2y - 4z = 0$ and $2x - y + 2z = 0$.

Sol. Equation of the plane through the lines $x + 2y - 4z = 0$ and $2x - y + 2z = 0$ is

$x + 2y - 4z + l(2x - y + 2z) = 0$ ...(1)

If (1, 1, 1) lies on this plane, then $-1 + 3l = 0$

$\Rightarrow l = 1/3$, so that the plane becomes $3x + 6y - 12z + 2x - y + 2z = 0 \Rightarrow x + y - 2z = 0$ ....(2)

Also (1) will be perpendicular to (2) if 1 + 2l + 2 - l - 2(-4 + 2l) = ⇒ l = 11/3

Equation of plane perpendicular to (2) is $5x - y + 2z = 0$. ...(3)

Therefore the equation of line through (1, 1, 1) and perpendicular to the given line is parallel to the normal to the plane (3). Hence the required line is $\frac{x-1}{5} = \frac{y-1}{-1} = \frac{z-1}{2}$

Alternate :

Solving the equation of planes $x + 2y - 4z = 0$ and $2x - y + 2z = 0$, we get $\frac{x}{0} = \frac{y}{-10} = \frac{z}{-5}$ ...(1)

Any point P on the line (1) can be written as (0, −10l, -5l).

Direction ratios of the line joining P and Q(1, 1, 1) is (1, 1, + 10l, 1 + 5l).

Line PQ is perpendicular to line (1) ⇒ 0(1) − 10(1 + 10l) - 5(1 + 5l) = 0

⇒ 0 - 10 - 100l - 5 - 25x = 0 or 125l + 15 = 0 ⇒ $= \frac{-15}{125} = \frac{-3}{25} \Rightarrow P = \left(0, \frac{6}{5}, \frac{3}{5}\right)$

Direction ratios of PQ = $\left(-1, \frac{1}{5}, \frac{-2}{5}\right)$

Hence equations of lien are $\frac{x-1}{5} = \frac{y-1}{-1} = \frac{z-1}{2}$.

Ex.29 Find the shortest distance (S.D.) between the lines $\frac{x-3}{3} = \frac{y-8}{-1} = \frac{z-3}{1}, \frac{x+3}{-3} = \frac{y+7}{2} = \frac{z-6}{4}$. Find also its equations and the points in which it meets the given lines.

Sol. The equations of the given lines are $(x - 3)/3 = (y - 8)/-1 = (z - 3)/1 = r_1$ (say) ...(1)

and $(x + 3)/(-3) = (y + 7)/2 = (z - 6)/4 = r_2$ (say) ..(2)

Any point on line (1) is $(3r_1 + 3, -r_1 + 8, r_1 + 3)$, say P. ...(3)

any point on line (2) is $(-3r_2 - 3, 2r2 - 7, 4r_2 + 6)$, say Q. ..(4)

The d.r.'s of the line PQ are $(-3r_2 - 3) - (3r_1 + 3), (2r_2 - 7) - (-r_1 + 8), (4r_2 + 6) - (r_1 + 3)$

or $-3r_2 - 3r_1 - 6, 2_{r2} + r_1 - 15, 4r_2 - r_1 + 3$. ..(5)

Let the line PQ be the lines of S.D., so that PQ is perpendicular to both the given lines (1) and (2), and so we have

$3(-3r_2 - 3r_1 - 6) - 1(2r_2 + r_1 - 15) + 1.(4r_2 - r_1 + 3) = 0$

and $-3(-3r_2 - 3r_1 - 6) + 2.(2R_2 + r_1 - 15) + 4(4r_2 - r_1 + 3) = 0$

or $-7r_2 - 11r_1 = 0$ and $11r_2 + 7r_1 = 0$

Solving these equations, we get $r_1 = r_2 = 0$.

Substituting the values of r_1 and r_2 in (3), (4) and (5), we have P(3, 8, 3), Q(−3, −7, 6) And the d.r.'s of PQ (the line of S.D.) are −6, −15, 3 or −2, −5, 1.

The length of S.D. = the distance between the points P and

$$= \sqrt{(-3-3)^2 + (-7-8)^2 + (6-3)^2} = 3\sqrt{30}$$

Q

Now the line PQ of shortest distance is the line passing through P(3, 8, 3) and having d.r.'s −2, −5, 1 and hence its equations are given by

$$\frac{x-3}{-2} = \frac{y-8}{-5} = \frac{z-3}{1} \text{ or } \frac{x-3}{2} - \frac{y-8}{5} = \frac{z-3}{1}.$$

Angle Between Two Intersecting Lines and Shortest Distance

Angle Between Two Intersecting Lines

If $l(x_1, m_1, n_1)$ and $l(x_2, m_2, n_2)$ be the direction cosines of two given lines, then the angle θ between them is given by

$\cos\theta = l_1 1_2 + m_1 m_2 + n_1 n_2$

(i) The angle between any two diagonals of a cube is $\cos^{-1}(1 / 3)$.

(ii) The angle between a diagonal of a cube and the diagonal of a face (of the cube is $\cos^{-1}(\sqrt{2} / 3)$

Straight Line in Space

The two equations of the line ax + by + cz + d = 0 and a' x + b' y + c' z + d' = 0 together represents a straight line.

1. Equation of a straight line passing through a fixed point $A(x_1, y_1, z_1)$ and having direction ratios a, b, c is given by

$x - x_1 / a = y - y_1 / b = z - z_1 / c$, it is also called the symmetrically form of a line.

Any point P on this line may be taken as $(x_1 + \lambda a, y_1 + \lambda b, z_1 + \lambda c)$, where $\lambda \in R$ is parameter. If a, b, c are replaced by direction cosines 1, m, n, then λ, represents distance of the point P from the fixed point A.

2. Equation of a straight line joining two fixed points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ is given by

$x - x_1 / x_2 - x_1 = y - y_1 / y_2 - y_1 = z - z_1 / z_2 - z_1$

3. Vector equation of a line passing through a point with position vector a and parallel to vector b is r = a + λ b, where A, is a parameter.

4. Vector equation of a line passing through two given points having position vectors a and b is r = a + λ (b – a) , where λ is a parameter.

5. (a) The length of the perpendicular from a point $P(\vec{\alpha})$ on the line r – a + λ b is given by

$$\sqrt{|\vec{\alpha} - \mathbf{a}|^2 - \left\{\frac{(\vec{\alpha} - \mathbf{a}) \cdot \mathbf{b}}{|\mathbf{b}|}\right\}^2}$$

(b) The length of the perpendicular from a point $P(x_1, y_1, z_1)$ on the line

$$\frac{x-a}{l} = \frac{y-b}{m} = \frac{z-c}{n} \text{ is given by}$$

$$\sqrt{\{(a-x_1)^2 + (b-y_1)^2 + (c-z_1)^2\} - \{(a-x_1)\,l + (b-y_1)\,m + (c-z_1)\,n\}^2}$$

where, 1, m, n are direction cosines of the line.

6. **Skew Lines** Two straight lines in space are said to be skew lines, if they are neither parallel nor intersecting.

7. **Shortest Distance** If l_1 and l_2 are two skew lines, then a line perpendicular to each of lines 4 and 12 is known as the line of shortest distance.

If the line of shortest distance intersects the lines l_1 and l_2 at P and Q respectively, then the distance PQ between points P and Q is known as the shortest distance between l_1 and l_2.

8. The shortest distance between the lines

$$\frac{x-x_1}{l_1}=\frac{y-y_1}{m_1}=\frac{z-z_1}{n_1}$$

and $$\frac{x-x_2}{l_2}=\frac{y-y_2}{m_2}=\frac{z-z_2}{n_2} \text{ is given by}$$

$$d=\frac{\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix}}{\sqrt{(m_1n_2-m_2n_1)^2+(n_1l_2-n_2l_1)^2+(l_1m_2-l_2m_1)^2}}$$

9. The shortest distance between lines $r = a_1 + \lambda b_1$ and $r = a_2 + \mu b_2$ is given by

$$d=\left|\frac{(\mathbf{b}_1\times\mathbf{b}_2)\cdot(\mathbf{a}_2-\mathbf{a}_1)}{|\mathbf{b}_1\times\mathbf{b}_2|}\right|$$

10. The shortest distance parallel lines $r = a_1 + \lambda b_1$ and $r = a_2 + \mu b_2$ is given by

$$d=\left|\frac{(\mathbf{a}_2-\mathbf{a}_1)\times\mathbf{b}}{|\mathbf{b}|}\right|$$

11. Lines $r = a_1 + \lambda b_1$ and $r = a_2 + \mu b_2$ are intersecting lines, if $(b_1 * b_2) * (a_2 - a_1) = 0$.

12. The image or reflection (x, y, z) of a point (x_1, y_1, z_1) in a plane $ax + by + cz + d = 0$ is given by

$x - x_1 / a = y - y_1 / b = z - z_1 / c = - 2 (ax_1 + by_1 + cz_1 + d) / a^2 + b^2 + c^2$

13. The foot (x, y, z) of a point (x_1, y_1, z_1) in a plane $ax + by + cz + d = 0$ is given by

$x - x_1 / a = y - y_1 / b = z - z_1 / c = -(ax_1 + by_1 + cz_1 + d) / a^2 + b^2 + c^2$

14. Since, x, y and z-axes pass through the origin and have direction cosines (1, 0, 0), (0, 1, 0) and (0, 0, 1), respectively. Therefore, their equations are

x – axis : $x - 0 / 1 = y - 0 / 0 = z - 0 / 0$

y – axis : $x - 0 / 0 = y - 0 / 1 = z - 0 / 0$

z – axis : $x - 0 / 0 = y - 0 / 0 = z - 0 / 1$

Area of a Triangle and Equation of a Plane

D. AREA OF A TRIANGLE

Show that the area of a triangle whose vertices are the origin and the points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ is $\frac{1}{2}\sqrt{(y_1z_2 - y_2z_1)^2 + (z_1x_2 - z_2x_1)^2 + (x_1y_2 - x_2y_1)^2}$.

The direction ratios of OA are x_1, y_1, z_1 and those of OB are x_2, y_2, z_2.

Also OA $= \sqrt{(x_1 - 0)^2 + (y_1 - 0)^2 + (z_1 - 0)^2} = \sqrt{(x_1^2 + y_1^2 + z_1^2)}$

and OB $= \sqrt{(x_2 - 0)^2 + (y_2 - 0)^2 + (z_2 - 0)^2} = \sqrt{(x_2^2 + y_2^2 + z_2^2)}$.

∴ the d.c.' s of OA are $\frac{x_1}{\sqrt{(x_1^2 + y_1^2 + z_1^2)}}, \frac{y_1}{\sqrt{(x_1^2 + y_1^2 + z_1^2)}}, \frac{z_1}{\sqrt{(x_1^2 + y_1^2 + z_1^2)}}$

and the d.c.'s of OB are $\frac{x_2}{\sqrt{(x_2^2 + y_2^2 + z_2^2)}}, \frac{y_2}{\sqrt{(x_2^2 + y_2^2 + z_2^2)}}, \frac{z_2}{\sqrt{(x_2^2 + y_2^2 + z_2^2)}}$

Hence if θ is the angle between the line OA and OB, then

$$\sin\theta = \frac{\sqrt{\{\Sigma(y_1z_2 - y_2z_2)^2\}}}{\sqrt{(x_1^2 + y_1^2 + z_1^2)}\sqrt{(x_2^2 + y_2^2 + z_2^2)}} = \frac{\sqrt{\{\Sigma(y_1z_2 - y_2z_1)^2\}}}{OA.OB}$$

Hence the area of ΔOAB

$= \frac{1}{2} \cdot OA \cdot OB \sin\theta \quad [\because \angle AOB = \theta]$

$= \frac{1}{2} \cdot OA. OB. \frac{\sqrt{\{\Sigma(y_1z_2 - y_2z_2)^2\}}}{OAOB} = \frac{1}{2}\sqrt{\{\Sigma(y_1z_2 - y_2z_2)^2\}}$.

Ex.6 Find the area of the triangle whose vertices are A(1, 2, 3), B(2, –1, 1)and C(1, 2, –4).

Sol. Let Δx, Δy, Δz be the areas of the projections of the area Δ of triangle ABC on the yz, zx and xy-planes respectively. We have

$$\Delta x = \frac{1}{2}\begin{vmatrix} y_1 & z_1 & 1 \\ y_2 & z_2 & 1 \\ y_3 & z_3 & 1 \end{vmatrix} = \frac{1}{2}\begin{vmatrix} 2 & 3 & 1 \\ -1 & 1 & 1 \\ 2 & -4 & 1 \end{vmatrix} = \frac{21}{2} \; ;$$

$$\Delta y = \frac{1}{2}\begin{vmatrix} x_1 & z_1 & 1 \\ x_2 & z_2 & 1 \\ x_3 & z_3 & 1 \end{vmatrix} = \frac{1}{2}\begin{vmatrix} 1 & 3 & 1 \\ 2 & 1 & 1 \\ 1 & -4 & 1 \end{vmatrix} = \frac{7}{2}$$

$$\Delta z = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \frac{1}{2}\begin{vmatrix} 1 & 2 & 1 \\ 2 & -1 & 1 \\ 1 & 2 & 1 \end{vmatrix} = 0$$

∴ the required area $\Delta = \sqrt{[\Delta_x^2 + \Delta_y^2 + \Delta_z^2]} = \frac{7\sqrt{10}}{2}$ sq. units.

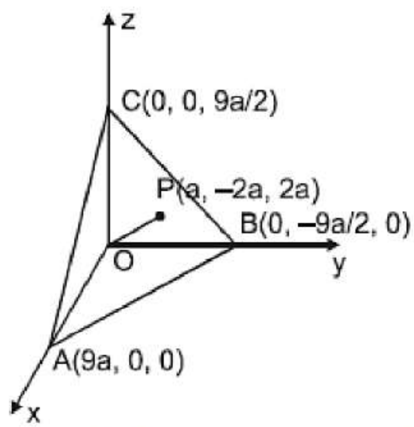
Ex.7 A plane is passing through a point P(a, –2a, 2a), $a \neq 0$, at right angle to OP, where O is the origin to meet the axes in A, B and C. Find the area of the triangle ABC.

Sol. OP $= \sqrt{a^2 + 4a^2 + 4a^2} = |3a|$.

Equation of plane passing through P(a, –2a, 2a) is
A(x – a) + B(y + 2a) + C(z – 2a) = 0.
∵ the direction cosines of the normal OP to the plane ABC are proportional to a – 0, –2a – 0, 2a – 0 i.e. a, –2a, 2a. ⇒ equation of plane ABC is

$a(x - a) - 2a(y + 2a) + 2a(z - 2a) = 0$ or $ax - 2ay + 2az = 9a^2$ ....(1)

Now projection of area of triangle ABC on ZX, XY and YZ planes are the triangles AOC, AOB and BOC respectively.



$\therefore$ (Area ΔABC)^2 = (Area ΔAOC)^2 + (Area ΔAOB)^2 + (Area ΔBOC)^2

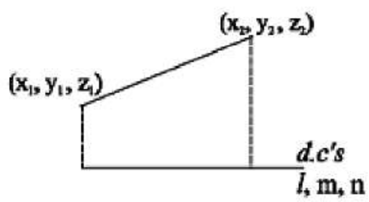
$$= \left(\frac{1}{2}.AO.OC\right)^2 + \left(\frac{1}{2}.AO.BO\right)^2 + \left(\frac{1}{2}.BO.OC\right)^2$$

$$= \frac{1}{4}\left[\left(9a.\frac{9}{2}a\right)^2 + \left(9a.\frac{-9}{2}a\right)^2 + \left(\frac{-9}{2}a.\frac{9}{2}a\right)^2\right] = \frac{1}{4},\frac{81^2a^4}{4}\left(1+1+\frac{1}{4}\right)$$

Important Formulas - 3D Geometry

3 -D Coordinate Geometry

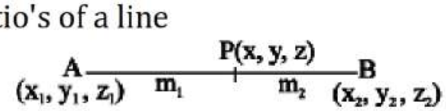
(1) Distance (d) between two points (x_1, y_1, z_1) and (x_2, y_2, z_2).



$$= \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$$

(2) Direction Cosine and direction ratio's of a line

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$



Section Fomula

$$x = \frac{m_2x_1 + m_1x_2}{m_1 + m_2} \quad ; \quad y = \frac{m_2y_1 + m_1y_2}{m_1 + m_2} \quad ; \quad z = \frac{m_2z_1 + m_1z_2}{m_1 + m_2}$$

(3) Direction cosine of a line has the same meaning as d.c's of a vector.

(a) Any three numbers a, b, c proportional to the direction cosines are called the direction ratio s i.e.

$$\frac{l}{a} = \frac{m}{b} = \frac{n}{c} = \pm \frac{1}{\sqrt{a^2 + b^2 + c^2}}$$

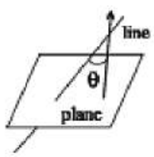
same sign either +ve or –ve should be taken throughout.

note that d.r's of a line joining x_1, y_1, z_1 and x_2, y_2, z_2 are proportional to $x_2 - x_1$, $y_2 - y_1$ and $z_2 - z_1$

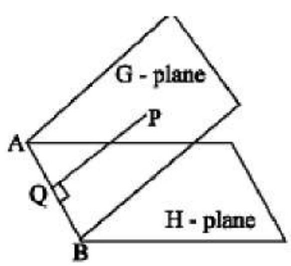
(b) If θ is the angle between the two lines whose d.c's are l_1, m_1, n_1 and l_2, m_2, n_2 $\cos\theta = l_1 l_2 + m_1 m_2 + n_1 n_2$ hence if lines are perpendicular then $l_1 l_2 + m_1 m_2 + n_1 n_2 = 0$

if lines are parallel then

$$\frac{l_1}{l_2} = \frac{m_1}{m_2} = \frac{n_1}{n_2}$$



note that if three lines are coplanar then $\begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix} = 0$



(4)Projection of join of 2 points on line with d.c's l, m, n are $l(x_2 - x_1) + m(y_2 - y_1) + n(z_2 - z_1)$ B PLANE

(i) General equation of degree one in x, y, z i.e. $ax + by + cz + d = 0$ represents a plane. (ii) Equation of a plane passing through (x_1, y_1, z_1) is $a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$ where a, b, c are the direction ratios of the normal to the plane.

(iii) Equation of a plane if its intercepts on the co-ordinate axes are x1 , y1 , z1 is z 1 z

x_1, y_1, z_1 is $\frac{x}{x_1} + \frac{y}{y_1} + \frac{z}{z_1} = 1$.

(iv) Equation of a plane if the length of the perpendicular from the origin on the plane is p and d.c's of the perpendicular as l , m, , n is $lx + my + nz = p$

(v) Parallel and perpendicular planes – Two planes $a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$ are perpendicular if $a_1a_2 + b_1b_2 + c_1c_2 = 0$,

parallel if $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$ and coincident if $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2} = \frac{d_1}{d_2}$

(vi) Angle between a plane and a line is the compliment of the angle between the normal to the plane and the line . If $\left.\begin{array}{l}\text{Line} : \vec{r} = \vec{a} + \lambda \vec{b} \\ \text{Plane} : \vec{r}.\vec{n} = d\end{array}\right]$ then $\cos(90 - \theta) = \sin\theta = \frac{b.n}{|\vec{b}|.|\vec{n}|}$.

where θ is the angle between the line and normal to the plane.

(vii) Length of the perpendicular from a point (x_1, y_1, z_1) to a plane $ax + by + cz + d = 0$ is $p = \left|\frac{ax_1 + by_1 + cz_1 + d}{\sqrt{a^2 + b^2 + c^2}}\right|$

(viii) Distance between two parallel planes $ax + by + cz + d1 = 0$ and $ax + by + cz + d_2 = 0$ is $\left|\frac{d_1 - d_2}{\sqrt{a^2 + b^2 + c^2}}\right|$

(ix) Planes bisecting the angle between two planes $a_1x + b_1y + c_1z + d1 = 0$ and $a_2 + b_2y + c_2z + d_2 = 0$ is

Given by $\left|\frac{a_1x+b_1y+c_1z+d_1}{\sqrt{a_1^2+b_1^2+c_1^2}}\right| = \pm \left|\frac{a_2x+b_2y+c_2z+d_2}{\sqrt{a_2^2+b_2^2+c_2^2}}\right|$

Of these two bisecting planes , one bisects the acute and the other obtuse angle between the given planes.

(x) Equation of a plane through the intersection of two planes P_1 and P_2 is given by $P_1 + \lambda P_2 = 0$

C Straight Line in Space

(i) Equation of a line through A (x_1, y_1, z_1) and having direction cosines l, m, n are

$\frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n}$ and the lines through (x_1, y_1, z_1) and (x_2, y_2, z_2).

$$\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} = \frac{z-z_1}{z_2-z_1}$$

(ii) Intersection of two planes $a_1x + b_1y + c_1z + d1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$ together represent the unsymmetrical form of the straight line.

(iii) General equation of the plane containing the line $\frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n}$

is $A(x - x_1) + B(y - y_1) + c(z - z_1) = 0$ where $Al + bm + cn = 0$.

Line of greatest slope

AB is the line of intersection of G-plane and H is the horizontal plane. Line of greatest slope on a given plane, drawn through a given point on the plane, is the line through the point 'P' perpendicular to the line of intersetion of the given plane with any horizontal plane.