# Fundamental Concepts

- A **variable** is something that does not have a fixed value. The value of a variable varies.
- Variables are represented by English letters such as *x*, *y*, *z*, *a*, *b*, *c* etc.
- A combination of variables, numbers and operators (+, –, × and ÷) is known as **expression**.

- Using different operations on variables and numbers, expressions such as $\frac{1}{7} - 4y$, $9x - 5$, can be formed.

**Example:**

Meena's age is 4 years less than 7 times the age of Ravi. Express it using variables.

**Solution:**

Let the age of Ravi be *x* years.

7 times the age of Ravi can be expressed as 7*x*.

4 years less than 7 times the age of Ravi can be written as 7*x* – 4.

∴ Age of Meena = (7*x* – 4)

- **Polynomial**

An algebraic expression in which the exponents of the variables are non-negative integers are called polynomials. For example, $3x^4 + 2x^3 + x + 9$, $3x^4$ etc are polynomials.

- **Constant polynomial:** A constant polynomial is of the form $p(x) = k$, where *k* is a real number. For example, –9, 10, 0 are constant polynomials.
- **Zero polynomial:** A constant polynomial '0' is called zero polynomial.

**General form of a polynomial:**

A polynomial of the form $p(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$, where $a_0, a_1 \ldots a_r$ are constants and $a_n \neq 0$.

Here, $a_0, a_1 \ldots a_n$ are the respective coefficients of $x^0, x^1, x^2, \ldots x^n$ and $n$ is the power of the variable $x$.

$a_n x^n, a_{n-1} x^{n-1} - a_0$ and $a_0 \neq 0$ are called the terms of $p(x)$.

- **Classification of polynomials on the basis of number of terms**
    - A polynomial having one term is called a monomial e.g. $3x$, $25t^3$ etc.
    - A polynomial having two terms is called a binomial e.g. $2t - 6$, $3x^4 + 2x$ etc.
    - A polynomial having three terms is called a trinomial. e.g. $3x^4 + 8x + 7$ etc.

- **Degree**

The degree of a polynomial is the highest exponent of the variable of the polynomial. For example, the degree of polynomial $3x^4 + 2x^3 + x + 9$ is 4.

The degree of a term of a polynomial is the value of the exponent of the term.

- **Classification of polynomial according to their degrees**

- A polynomial of degree one is called a linear polynomial e.g. $3x + 2$, $4x$, $x + 9$.
- A polynomial of degree two is called a quadratic polynomial. e.g. $x^2 + 9$, $3x^2 + 4x + 6$ ·
- A polynomial of degree three is called a cubic polynomial e.g. $10x^3 + 3$, $9x^3$ .

**Note:** The degree of a non-zero constant polynomial is zero and the degree of a zero polynomial is not defined.

- Algebraic expressions are formed by combining variables with constants using operations of addition, subtraction, multiplication and division.

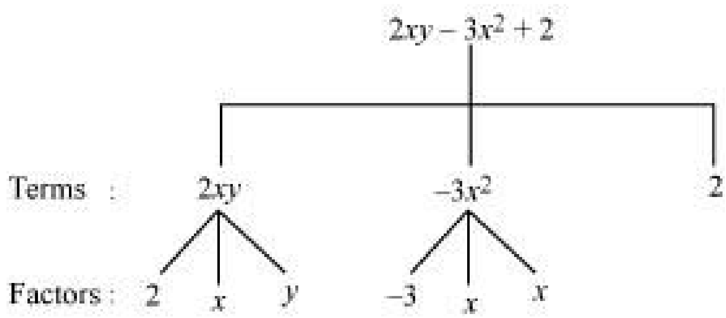
  For example: 4xy, 2x2 – 3, 7xy + 2x, etc.

In an algebraic expression, say 2xy – 3x2 + 2; 2xy, (–3x2), 2 are known as the terms of the expression.

The expression 2xy – 3x2 +2 is formed by adding the terms 2xy, (–3x2) and 2 where 2, x, y are factors of the term 2xy; (–3), x, x are factors of the term (–3x2); 2 is the factor of the term 2.

For an expression, the terms and its factors can be represented easily and elegantly by a tree diagram.

Tree diagram for the expression 2xy – 3x2 +2:




Note: In an expression, 1 is not taken as separate factor.

- The numerical factor of a term is known as its coefficient. For example, for the term –3x2y, the coefficient is (–3).
- The terms having the same algebraic factors are called like terms, while the terms having different algebraic factors are called unlike terms.

  For example: 13x2y, – 23x2y are like terms; 12xy, 3x2 are unlike terms

- Addition and subtraction of algebraic expressions:
  - The sum or difference of two like terms is a like term, with its numerical coefficient equal to the sum or difference of the numerical coefficients of the two like terms.
  - When algebraic expressions are added, the like terms are added and unlike terms are left as they were.

    **Example :** Subtract ($x^2$–$2y^2$+$y$) from the sum of (–$2x^2$+$3x$+2) and

    $\left(-2y+3x^2+5x\right)$
    **Solution:**

$$\left(-2x^2+3x+2\right)+\left(-2y+3x^2+5x\right)$$
$$=\left(-2x^2+3x^2\right)+(3x+5x)-2y+2 \qquad \text{[Rearranging terms]}$$
$$=x^2+8x-2y+2$$
$$\therefore \left(x^2+8x-2y+2\right)-\left(x^2-2y^2+y\right)$$
$$=x^2+8x-2y+2-x^2+2y^2-y$$
$$=\left(x^2-x^2\right)+2y^2+8x+(-2y-y)+2 \qquad \text{[Rearranging terms]}$$
$$=2y^2+8x-3y+2$$

• **Addition and subtraction of algebraic expressions** are done by adding and subtracting like terms.

• We change the sign of each term of subtrahend in case of subtraction of algebraic expressions.

• There are two methods for addition and subtraction of algebraic expressions namely, **vertical** and **horizontal method**.

For example, $-2x^2+5xy-z$ and $3xy+x^2-2z$ can be added using horizontal and vertical method as follows:

$$\begin{array}{r} -2x^2+5xy-1z \\ +\quad 1x^2+3xy-2z \\ \hline -1x^2+8xy-3z \\ \hline \end{array}$$

$$\left(-2x^2+5xy-z\right)+\left(3xy+x^2-2z\right)$$

$$=\left(-2x^2+x^2\right)+(5xy+3xy)+(-z-2z)$$

$$=-x^2+8xy-3z$$

$-2x^2+5xy-z$ can be subtracted from $3xy+x^2-2z$ using horizontal and vertical method as follows:

$$\begin{array}{r} 3xy + 1x^2 - 2z \\ 5xy - 2x^2 - 1z \\ (-) \quad (+) \quad (+) \\ \hline -2xy + 3x^2 - z \\ \hline \end{array}$$

$$\left(3xy + x^2 - 2z\right) - \left(-2x^2 + 5xy - z\right)$$

$$= 3xy + x^2 - 2z + 2x^2 - 5xy + z$$

$$= (3xy - 5xy) + \left(x^2 + 2x^2\right) + (-2z + z)$$

$$= -2xy + 3x^2 - z$$

- The multiplication of a monomial by a monomial gives a monomial. While performing multiplication, the coefficients of the two monomials are multiplied and the powers of different variables in the two monomials are multiplied by using the rules of exponents and powers.

  $(-2ab^2c) \times (3abc^2) = (-2 \times 3) \times (a \times a \times b^2 \times b \times c \times c^2) = -6a^2b^3c^3$

  The multiplication of three or more monomials is also performed similarly.

  $(xy) \times (3yz) \times (3x^2z^2)$

  $= (3 \times 3) \times (x \times x^2) \times (y \times y) \times (z \times z^2)$

  $= 9x^3y^2z^3$

- There are two ways of arrangement of multiplication while multiplying a monomial by a binomial or trinomial or polynomial. These are horizontal arrangement and vertical arrangement.

  Multiplication in **horizontal arrangement** can be performed as follows:

  Here, we arrange monomial and polynomial both horizontally and multiply every term in the polynomial by the monomial by making use of distributive law.

  $5a \times (2b + a - 3b + c)$

  $= (5a \times 2b) + (5a \times a) + (5a \times (-3b)) + (5a \times c)$

$= 10ab + 5a^2 - 15ab + 5ac$

$= 5a^2 - 5ab + 5ac$

Multiplication in **vertical arrangement** can be performed as follows:

$$\begin{array}{r} 4x^2 + 2x \\ \times \quad 3x \\ \hline 12x^3 + 6x \\ \hline \end{array}$$

Here, we have first multiplied $3x$ with $2x$ and wrote the product with sign at the bottom. After doing this, we have multiplied $3x$ with $4x^2$ and wrote the product with sign at the bottom.

Similarly, we can multiply a trinomial with monomial as follows:

$$\begin{array}{r} 2y^3 - 5y + 1 \\ \times \qquad 2y \\ \hline 4y^4 - 10y^2 + 2y \\ \hline \end{array}$$

- While multiplying a polynomial by a binomial (or trinomial) in horizontal arrangement, we multiply it term by term. That is, every term of the polynomial is multiplied by every term of the binomial (or trinomial).

**Example:**

Simplify $(x + 2y)(x + 3) - (2x + 1)(y + x + 1)$.

**Solution:**

$(x + 2y)(x + 3) = x(x + 3) + 2y(x + 3)$

$= x^2 + 3x + 2xy + 6y$

$(2x + 1)(y + x + 1) = 2x(y + x + 1) + 1(y + x + 1)$

$= 2xy + 2x^2 + 2x + y + x + 1$

$= 2xy + 2x^2 + 3x + y + 1$

$\therefore (x + 2y)(x + 3) – (2x + 1)(y + x + 1) = x^2 + 3x + 2xy + 6y – 2xy – 2x^2 – 3x – y – 1$

$= – x^2 + 5y – 1$

- We can also perform multiplication of two polynomials using vertical arrangement.

  For example,

$$\begin{array}{r} l + 6m + 7n \\ \times \quad l + 3m \\ \hline 3lm + 18m^2 + 21mn \\ +l^2 + 6lm + 7nl \\ \hline l^2 + 9lm + 18m^2 + 21mn + 7nl \\ \hline \end{array}$$

- **Division of a polynomial by a monomial using long division method**

  **Example:**
  Divide $x^4 – 2x^3 – 2x^2 + 7x – 15$ by $x – 2$.
  **Solution:**

$$\begin{array}{r|l} & x^3 - 2x + 3 \\ \hline x - 2 & x^4 - 2x^3 - 2x \ + 7x - 15 \\ & x^4 - 2x^3 \\ & - \quad + \\ \hline & - 2x \ + 7x - 15 \\ & -2x^2 + 4x \\ & + \quad - \\ \hline & 3x - 15 \\ & 3x - 6 \\ & - \quad + \\ \hline & -9 \end{array}$$

Divison of polynomials by monomials also satisfy Division algorithm i.e., **Dividend = Divisor × Quotient + Remainder**

It can be easily verified that here

$$(x^4 - 2x^3 - 2x^2 + 7x - 15) = (x - 2)(x^3 - 2x + 3) + (-9).$$

- **Factorization by using the identity, $x^2 + (a + b)x + ab = (x + a)(x + b)$.**

  To apply this identity in an expression of the type $x^2 + px + q$, we observe the coefficient of $x$ and the constant term.

  Two numbers, $a$ and $b$, are chosen such that their product is $q$ and their sum is $p$.

  i.e., $a + b = p$ and $ab = q$

  Then, the expression, $x^2 + px + q$, becomes $(x + a)(x + b)$.

  **Example:**

  Factorize $a^2 - 2a - 8$.

  **Solution:**

  Observe that, $-8 = (-4) \times 2$ and $(-4) + 2 = -2$

  Therefore, $a^2 - 2a - 8 = a^2 - 4a + 2a - 8$

  $= a(a - 4) + 2(a - 4)$

  $= (a - 4)(a + 2)$

- An identity is an equality which is true for all values of the variables in it. It helps us in shortening our calculations.
- Identities for "Square of Sum or Difference of Two Terms" are:
  - $(a + b)^2 = a^2 + 2ab + b^2$
  - $(a - b)^2 = a^2 - 2ab + b^2$

  **Example:**

Evaluate $(5x + 2y)^2 - (3x - y)^2$.

**Solution:**

Using identities (i) and (ii), we obtain

$(5x + 2y)^2 = (5x)^2 + 2\ (5x)\ (2y) + (2y)^2$

$= 25x^2 + 20xy + 4y^2$

$(3x - y)^2 = (3x)^2 - 2\ (3x)\ (y) + (y)^2$

$= 9x^2 - 6xy + y^2$

$\therefore (5x + 2y)^2 - (3x - y)^2 = 25x^2 + 20xy + 4y^2 - 9x^2 + 6xy - y^2 = 16x^2 + 26xy + 3y^2$

- $(a + b)\ (a - b) = a^2 - b^2$

**Example:**

Evaluate $95 \times 105$.

**Solution:**

We have, $95 \times 105 = (100 - 5) \times (100 + 5)$

$= (100)^2 - (5)^2$ [Using identity $(a + b)\ (a - b) = a^2 - b^2$]

$= 10000 - 25$

$= 9975$

- **Some of the expressions can also be factorized by making use of the following identities.**
    1. $a^2 + 2ab + b^2 = (a + b)^2$
    2. $a^2 - 2ab + b^2 = (a - b)^2$
    3. $a^2 - b^2 = (a + b)\ (a - b)$

For example, the expression $4x^2 + 12xy + 9y^2 - 4$ can be factorized as follows:
$4x^2 + 12xy + 9y^2 - 4$
$= (2x^2) + 2(2x)(3y) + (3y)^2 - 4$
$= (2x + 3y)^2 - 4$ [Using the identity, $a^2 + 2ab + b^2 = (a + b)^2$]
$= (2x + 3y)^2 - (2)^2$
$= (2x + 3y + 2)(2x + 3y - 2)$ [Using the identity, $a^2 - b^2 = (a + b)(a - b)$]

- Fractions involving polynomial either in numerator or denominator (or both) are called **algebraic fractions.**

  For example $\frac{12x}{9}$, $\frac{12y}{18}$ etc.

- An algebraic fraction is said to be in its **simplest form or lowest form**, if the numerator and denominator have no common factor, except 1.

  For example, the algebraic fraction $\frac{12x}{9}$ in simplest form is equal to $\frac{4x}{3}$.

- Simplifying an algebraic expression having integral denominator is same as adding or subtracting two unlike fractions.

  **Example:**

  Simplify the algebraic expression $\left(\frac{4p}{3} - \frac{4p}{5}\right) \div \left(\frac{1}{3p} + \frac{1}{p}\right)$.

  **Solution:**

  We have $\left(\frac{4p}{3} - \frac{4p}{5}\right) = \left(\frac{5 \times 4p - 3 \times 4p}{15}\right)$ [LCM of 3 and 5 is 15]

  $= \left(\frac{20p - 12p}{15}\right) = \frac{8p}{15}$

Similarly, $\left(\frac{1}{3p}+\frac{1}{p}\right)=\left(\frac{1+3\times1}{3p}\right)$ [LCM of 3*p* and *p* is 3*p*]

$$=\frac{4}{3p}$$

$$\therefore\left(\frac{4p}{3}-\frac{4p}{5}\right)\div\left(\frac{1}{3p}+\frac{1}{p}\right)=\frac{8p}{15}\div\frac{4}{3p}$$

$$=\frac{8p}{15}\times\frac{3p}{4}$$

$$=\frac{2p^2}{5}$$

- An **algebraic expression** may contain some brackets, namely line bracket, common bracket, curly bracket, or rectangular brackets, and some mathematical operations. An expression enclosed within a bracket is considered as a single quantity even though it may consist of many terms.
- For simplifying an expression, we remove the bracket by the following rules:

  (i) If '+' sign occurs before a bracket, then the signs of all the terms inside the bracket do not change.

  (ii) If '−' sign occurs before a bracket, then the signs of all the terms inside the bracket change.

  Brackets are removed in the order of

  (a) line brackets

  (b) common brackets

  (c) curly brackets

  (d) rectangular brackets

  **Example:**

  Simplify $3e^2-\left[d^2-4\left\{f^2-\left(2e^2-\overline{f^2+d^2}\right)\right\}\right]$

  **Solution:**

$$3e^2 - \left[d^2 - 4\left\{f^2 - \left(2e^2 - \overline{f^2 + d^2}\right)\right\}\right]$$

$= 3e^2 - [d^2 - 4\ \{f^2 - (2e^2 - f^2 - d^2)\}]$ [Line bracket is removed]

$= 3e^2 - [d^2 - 4\ \{f^2 - 2e^2 + f^2 + d^2\}]$ [Common bracket is removed]

$= 3e^2 - [d^2 - 4\ \{2f^2 - 2e^2 + d^2\}]$

$= 3e^2 - [d^2 - 8f^2 + 8e^2 - 4d^2]$ [Curly bracket is removed]

$= 3e^2 - [-3d^2 - 8f^2 + 8e^2]$

$= 3e^2 + 3d^2 + 8f^2 - 8e^2$ [Rectangular bracket is removed]

$= 3d^2 - 5e^2 + 8f^2$