CLASS : XII, Chapter - 6 APPLICATION OF DERIVATIVES

EXERCISE : 6.1 (Rate of Change of quantities)

Q No 1 : Find the rate of change of the area of a circle with respect to its radius r when (a) $r = 3$ cm (b) $r = 4$ cm.

Sol : Let A denote the area of circle and 'r' its radius.

Then $A = \pi r^2$

$$\Rightarrow \frac{dA}{dr} = \pi(2r) = 2\pi r$$

(a) $$\left(\frac{dA}{dr}\right)_{r=3cm} = 2\pi(3cm) = 6\pi \; cm^2/cm.$$

(b) $$\left(\frac{dA}{dr}\right)_{r=4cm} = 2\pi(4cm) = 8\pi \; cm^2/cm.$$

Q No 2 : The volume of a cube is increasing at rate of $8 cm^3/s$. How fast is the surface area increasing when the length of an edge is 12 cm ?

Sol : Let edge of cube at any instant of time be x.
Let V be the volume and S be the surface area.

The $V = x^3$ and $S = 6x^2$.

$$\Rightarrow \frac{dV}{dt} = 3x^2 \cdot \frac{dx}{dt} \quad \text{and} \quad \frac{dS}{dt} = 12x \cdot \frac{dx}{dt}$$

Now $\frac{dV}{dt} = 8 \; cm^3/sec.$ and when $x = 12$ cm.

$$\Rightarrow 8 = 3(12)^2 \cdot \frac{dx}{dt} \Rightarrow \frac{dx}{dt} = \frac{8}{3 \times 12 \times 12} = \frac{1}{54} \; cm/sec$$

$$\therefore \left(\frac{dS}{dt}\right)_{x=12} = 12 \times 12 \times \frac{1}{54} = \frac{8}{3} \; cm^2/sec.$$

Q No 3 : The radius of a circle is increasing uniformly at the rate of $3 cm/s$. Find the rate at which the area of circle is increasing when radius is 10 cm.

Sol: Let at any instant of time 't', the radius of circle be r, Area be A then.

$$A = \pi r^2$$

$$\Rightarrow \frac{dA}{dt} = \pi(2r)\cdot\frac{dr}{dt}$$

Now $r = 10$ cm. and $\frac{dr}{dt} = 3$ cm/sec (given)

$$\Rightarrow \frac{dA}{dt} = 2\pi \times 10 \times 3 = 60\pi \text{ cm}^2/\text{sec}.$$

$\therefore$ Rate of increase of Area $= 60\pi$ cm^2/sec.

Q No 4: An edge of a variable cube is increasing at the rate of 3 cm/sec. How fast is the volume of the cube increasing when edge is 10cm long.?

Sol: Let at any instant of time t, the edge of cube be x. Then Volume $V = x^3$.

$$\Rightarrow \frac{dV}{dt} = 3x^2\frac{dx}{dt}$$

$$= 3(10)^2 \times (3 \text{ cm/sec})$$ $\left\{\because x = 10\text{ cm and Rate of change of } x, \frac{dx}{dt} = 3 \text{ (given)}\right.$

$$= 900 \text{ cm}^3/\text{sec}.$$

$\therefore$ Rate of increase of volume $= 900$ cm^2/sec.

Q No. 5: A stone is dropped into a quiet lake and waves move in circles at the speed of 5 cm/sec. At the instant when the radius of of circular wave is 8 cm, how fast is the enclosed area increasing?

Sol: Let at any time t radius of circular wave be r and enclosed area A then $A = \pi r^2$.

$$\Rightarrow \frac{dA}{dt} = \pi(2r)\frac{dr}{dt}$$

$$= \pi(2 \times 8\text{cm})(5\text{ cm/sec})$$ $\left[\because r = 8\text{cm and } \frac{dr}{dt} = 5\text{ cm/sec.}\right.$

$$= 80 \text{ cm}^2/\text{sec}$$

$\Rightarrow$ Rate of increase of enclosed Area $= 80$ cm^2/sec.

Q.No. 6: The radius of a circle is increasing at the rate of 0.7 cm/sec. What is the rate of increase of circumference?

Sol: Let at any time 't' r be the radius and C be the circumference of circle, then.

$$C = 2\pi r$$

$$\Rightarrow \frac{dC}{dt} = 2\pi \cdot \frac{dr}{dt} = 2\pi(0.7 \text{ cm/sec}) \quad \left[\because \frac{dr}{dt} = 0.7 \text{ cm/sec given}\right]$$

$$= 1.4\pi \text{ cm/sec}$$

$\therefore$ Rate of increase of circumference $= 1.4\pi$ cm/sec.

Q.No 7. The length x of a rectangle is decreasing at rate of 5 cm/minute and the width y is increasing at rate of 4 cm/min. When $x = 8$ cm and $y = 6$ cm, find the rates of change of (a) the perimeter and (b) the area of rect.

Sol: According to question $\frac{dx}{dt} = -5$ cm/min $\quad$ [Decrease is taken as $-$ve]

$$\frac{dy}{dt} = 4 \text{ cm/min.}$$

(a) Now Perimeter $P = 2(\text{length} + \text{Width})$

$$= 2(x+y)$$

$$\therefore \frac{dP}{dt} = 2\left(\frac{dx}{dt} + \frac{dy}{dt}\right)$$

$$= 2(-5+4)$$

$$= -2 \text{ cm/min}$$

$\therefore$ Perimeter of Rectangle is decreasing at rate of 2 cm/min.

(b) Area(A) = length × Width $= xy$.

$$\Rightarrow \frac{dA}{dt} = x \cdot \frac{dy}{dt} + y \cdot \frac{dx}{dt} \quad \left\{\because x = 8\text{ cm}, y = 6\text{ cm}, \frac{dx}{dt} = -5, \frac{dy}{dt} = 4 \text{ cm/min (given)}\right.$$

$$= 8(4) + 6(-5)$$

$$= 32 - 30 = 2 \text{ cm}^2/\text{min}$$

$\Rightarrow$ Area is increasing at rate of 2 cm²/min

Q No 8: A balloon which always remains spherical on inflation, is being inflated by pumping in 900 cubic centimetres of gas per second. Find the rate at which radius of balloon increases when radius is 15 cm.

Sol. Let at any time 't' the radius of balloon be 'r' and its volume be V, then

$$V = \frac{4}{3}\pi r^3$$

$$\frac{dV}{dt} = \frac{4}{3}\pi\left(3r^2 \cdot \frac{dr}{dt}\right)$$

$$\Rightarrow \quad 900 = \frac{4}{3}\pi\left(3(15)^2 \cdot \frac{dr}{dt}\right) \qquad \left[\because \frac{dV}{dt} = 900\,cm^3/sec,\ r = 15\text{ cm given}\right]$$

$$\Rightarrow \quad \frac{dr}{dt} = \frac{900 \times 3}{4\pi \times 3 \times 15 \times 15} = \frac{1}{\pi}\text{ cm/sec.}$$

$\therefore$ Rate of increase of radius of balloon $= \frac{1}{\pi}$ cm/sec.

Q No 9: A balloon which always remains sperical has variable radius. Find the rate at which its volume is increasing with radius when later is 10 cm.

Sol: Let at any time t, radius of spherical balloon be r. and its volume $V = \frac{4}{3}\pi r^3$.

$$\Rightarrow \quad \frac{dV}{dr} = \frac{4}{3}\pi(3r^2) = 4\pi r^2\ cm^3/cm.$$

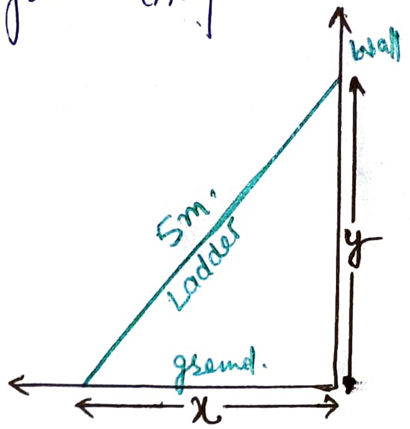
When $r = 10$, $\frac{dV}{dr} = 4 \times \pi \times 10 \times 10 = 400\pi\ cm^2/cm.$

$\therefore$ Rate of change of volume w.r.t change in radius $= 400\pi\ cm^2/cm$

Q No. 10: A ladder 5 m long is leaning against a wall. The bottom of the ladder is pulled along the ground, away from the wall at rate of 2 cm/s. How fast is its height on wall decreasing when foot of ladder is 4 m. away from wall?

Sol. Let foot of ladder is at a distance of x from wall and top is at height y at any time t.

Then $x^2+y^2=(5)^2$ [Using Pythagoras thm]



$\Rightarrow \quad 2x\frac{dx}{dt}+2y\frac{dy}{dt}=\frac{d}{dt}(25)$ ---(1)

Given that $\frac{dx}{dt} = 2cm/sec. = \frac{2}{100} m/sec$

When $x=4$, $(4)^2+y^2=(5)^2$

$\Rightarrow \quad y=\sqrt{25-16}=\sqrt{9}=3m$

$\therefore$ from (1) $2x\frac{dx}{dt}+2y\frac{dy}{dt}=\frac{d}{dt}(25)$

$\Rightarrow \quad x\frac{dx}{dt}+y\cdot\frac{dy}{dt}=0 \qquad [\because 25 \text{ is constant}]$

$\Rightarrow \quad 4\times\frac{2}{100}+3\frac{dy}{dt}=0 \Rightarrow 3\frac{dy}{dt}=-\frac{8}{100}$

$\Rightarrow \quad \frac{dy}{dt}=-\frac{8}{100}\times\frac{1}{3} m/sec = -\frac{8}{3} cm/sec.$

$\therefore$ Rate of decrease of height on wall $=\frac{8}{3} cm/sec.$

QNo 11: A particle moves along the curve $6y=x^3+2$. find the points on the curve at which the y-coordinate is changing 8 times as fast as the x-coordinate.

Sol.

Given $6y=x^3+2$

$\Rightarrow \quad 6\frac{dy}{dx}=3x^2$

$\Rightarrow \quad 6\times 8=3x^2 \qquad [\because \frac{dy}{dx}=8 \text{ (given)}]$

$\Rightarrow \quad x^2=16 \Rightarrow x=\pm 4$

Now When $x=4$ $\quad 6y=x^3+2 \Rightarrow y=\frac{64+2}{6}=\frac{66}{6}=11$

When $x=-4$ $\quad 6y=(-4)^3+2 \Rightarrow y=\frac{-64+2}{6}=\frac{-62}{6}=-\frac{31}{3}$

$\therefore$ Required points are $(4,11)$ and $\left(-4,-\frac{31}{3}\right)$.

Q No 12: The radius of airbubble is increasing at rate of $\frac{1}{2}$ cm/s. At what rate is the volume of bubble increasing when the radius is 1 cm?

Sol. Let r be the radius and V be the volume at any time t.

Then $V = \frac{4}{3}\pi r^3$

$$\Rightarrow \frac{dV}{dt} = \left(\frac{4}{3}\pi\right)\left(3r^2\frac{dr}{dt}\right)$$

$$= 4\pi r^2\frac{dr}{dt} = 4\pi(1)^2\left(\frac{1}{2}\,cm/sec\right) = 2\pi\,cm^3/sec.$$

$\therefore$ Rate of Increase of the volume of bubble $= 2\pi\,cm^3/sec.$

Q No. 13 A balloon which always remains spherical has a variable diameter $\frac{3}{2}(2x+1)$. Find the rate of change of its volume w.r.t x.

Sol: Let V be the volume of spherical balloon.

then $V = \frac{4}{3}\pi \times \left[\frac{1}{2}\times\frac{3}{2}(2x+1)\right]^3$ $\quad\left[\because \text{Diameter} = \frac{3}{2}(2x+1) \Rightarrow \text{Radius} = \frac{1}{2}\times\frac{3}{2}(2x+1)\right.$

$$= \frac{4}{3}\pi \times \frac{27}{64}(2x+1)^3 = \frac{9\pi}{16}(2x+1)^3$$

$$\therefore \frac{dV}{dx} = \frac{9\pi}{16}\cdot 3(2x+1)^2(2) = \frac{27}{8}\pi(2x+1)^2$$

$\therefore$ Rate of change of Volume w.r.t $x = \frac{27}{8}\pi(2x+1)^2$

Q No 14 Sand is pouring from a pipe at the rate of $12\,cm^3/sec$. The falling sand forms a cone on the ground in such a way that the height of the cone is always one-sixth of the radius of the base. How fast is the height of sand cone increasing when the height is 4cm.

Sol: Let r be the radius and h be the height of cone formed by the falling sand at any time t.

Then ATQ $\quad h = \frac{1}{6}r \quad$ or $\quad r = 6h.$

Then volume $V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi (6h)^2 h = 12\pi h^3$

$\Rightarrow \frac{dV}{dt} = 12\pi \cdot 3h^2 \frac{dh}{dt}$

But $\frac{dV}{dt} = 12$ (given)

$\Rightarrow 12 = 12\pi \, 3h^2 \frac{dh}{dt}$

$\Rightarrow \frac{dh}{dt} = \frac{1}{3\pi h^2}$

When $h = 4$ cm $\quad \frac{dh}{dt} = \frac{1}{3\pi(4)^2} = \frac{1}{48\pi}$

$\therefore$ Height of Sand cone is increasing at rate of $\frac{1}{48\pi}$ when height $h = 4$ cm.

QNo. 15: The total Cost $C(x)$ in Rupees associated with production of x units of an item is given by

$C(x) = 0.007x^3 - 0.003x^2 + 15x + 4000$.

Find marginal cost when 17 units are produced.

Sol: Given $C(x) = 0.007x^3 - 0.003x^2 + 15x + 4000$

$\therefore$ Marginal Cost $= \frac{dC}{dx} = \frac{d}{dx}\left(0.007x^3 - 0.003x^2 + 15x + 4000\right)$

$= 0.007 \times 3x^2 - 0.003 \times 2x + 15 + 0$.

$\therefore \left(\frac{dC}{dx}\right)_{x=17} = 0.007 \times 3 \times (17)^2 - 0.003 \times 2 \times 17 + 15 \times 17$

$= 20.967$

$\therefore$ Marginal Cost when $x = 17$ is 20.967.

QNo. 16: The total revenue in Rupees received from sale of x units of a product is given by $R(x) = 13x^2 + 26x + 15$. find the marginal revenue when $x = 7$.

Sol Given $R(x) = 13x^2 + 26x + 15$

$\therefore$ Marginal Revenue $= \frac{dR}{dx} = 13(2x) + 26(1) + 0 = 26x + 26$

$\therefore \left(\frac{dR}{dx}\right)_{x=7} = 26 \times 7 + 26 = 208$.

Choose Correct answer in Exercise 17 and 18

Q No 17. The rate of change of the area of a circle w.r.t its radius r at r = 6 cm is

(A) 10π (B) 12π (C) 8π (D) 11π.

Sol: If A is the area of circle.

Then $A = \pi r^2$

$$\Rightarrow \frac{dA}{dr} = 2\pi r$$

$$\therefore \left(\frac{dA}{dr}\right)_{r=6} = 2\pi \times 6 = 12\pi.$$

$\therefore$ Correct answer is (B)

Q No 18. The total Revenue in Rupees received from sale of x units of a product is given by.

$R(x) = 3x^2 + 36x + 5$. The marginal revenue when x = 15 is (A) 116 (B) 96 (C) 90 (D) 126.

Sol: Given $R(x) = 3x^2 + 36x + 5$

$$\therefore \text{Marginal Revenue} = \frac{dR}{dx} = 6x + 36$$

$$\therefore \left(\frac{dR}{dx}\right)_{x=1} = 6 \times 15 + 36$$

$$= 90 + 36$$

$$= 126.$$

$\therefore$ Correct answer is 126. option (D)

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