1

Simple Harmonic Motion

At first sight the eight physical systems in Figure 1.1 appear to have little in common.

1.1(a) is a simple pendulum, a mass m swinging at the end of a light rigid rod of length l.
1.1(b) is a flat disc supported by a rigid wire through its centre and oscillating through small angles in the plane of its circumference.
1.1(c) is a mass fixed to a wall via a spring of stiffness s sliding to and fro in the x direction on a frictionless plane.
1.1(d) is a mass m at the centre of a light string of length $2l$ fixed at both ends under a constant tension T. The mass vibrates in the plane of the paper.
1.1(e) is a frictionless U-tube of constant cross-sectional area containing a length l of liquid, density ρ, oscillating about its equilibrium position of equal levels in each limb.
1.1(f) is an open flask of volume V and a neck of length l and constant cross-sectional area A in which the air of density ρ vibrates as sound passes across the neck.
1.1(g) is a hydrometer, a body of mass m floating in a liquid of density ρ with a neck of constant cross-sectional area cutting the liquid surface. When depressed slightly from its equilibrium position it performs small vertical oscillations.
1.1(h) is an electrical circuit, an inductance L connected across a capacitance C carrying a charge q.

All of these systems are simple harmonic oscillators which, when slightly disturbed from their equilibrium or rest postion, will oscillate with simple harmonic motion. This is the most fundamental vibration of a single particle or one-dimensional system. A small displacement x from its equilibrium position sets up a restoring force which is proportional to x acting in a direction towards the equilibrium position.

Thus, this restoring force F may be written

$$F = -sx$$

where s, the constant of proportionality, is called the stiffness and the negative sign shows that the force is acting against the direction of increasing displacement and back towards

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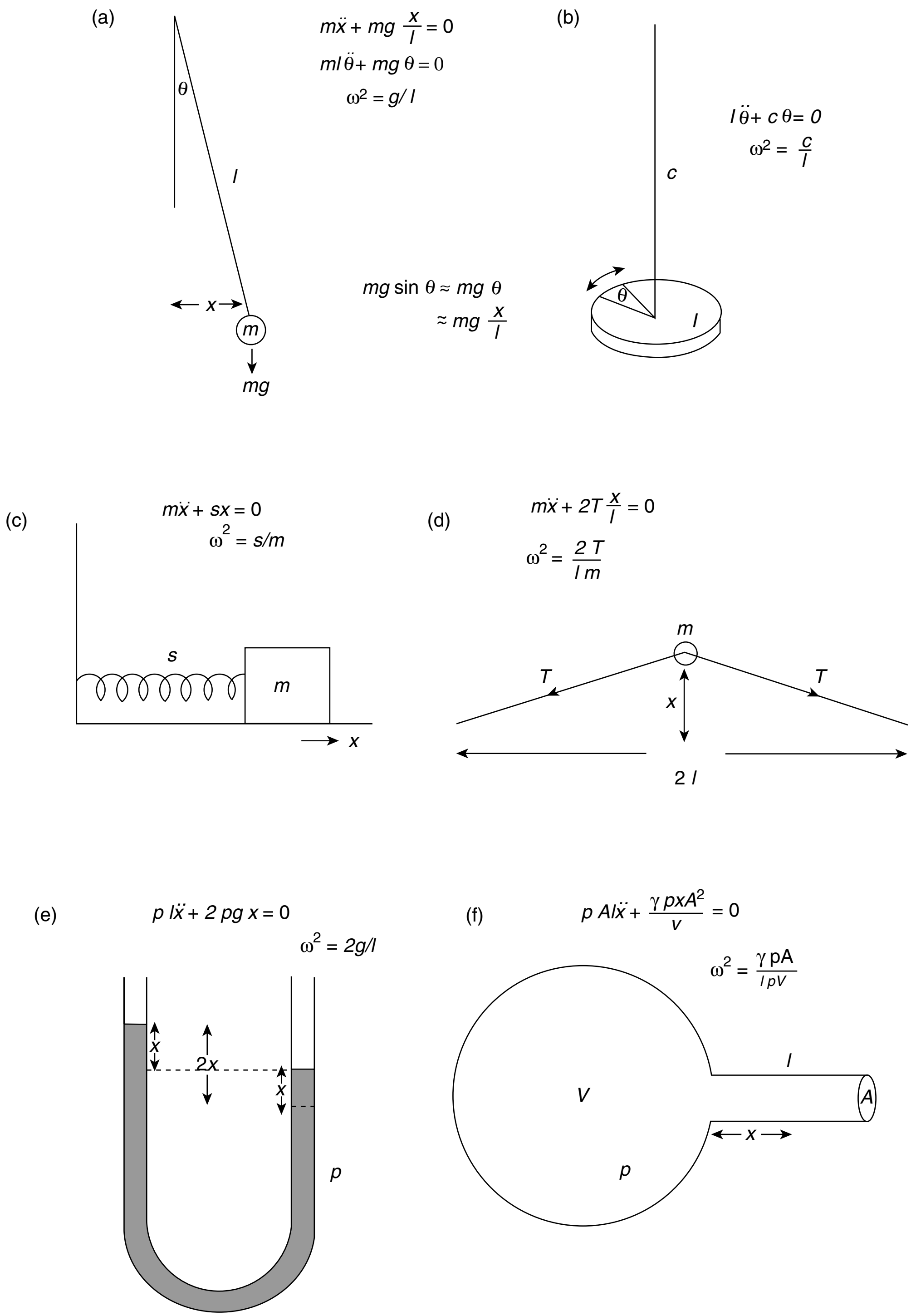
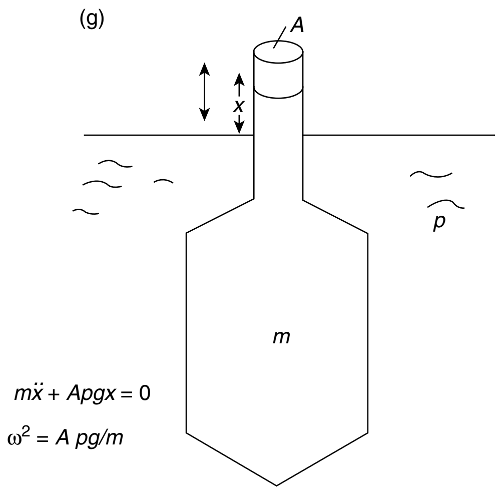

(a)
$m\ddot{x} + mg\frac{x}{l} = 0$
$ml\ddot{\theta} + mg\theta = 0$
$\omega^2 = g/l$
θ
l
x
m
mg
$mg\sin\theta \approx mg\theta \approx mg\frac{x}{l}$
(b)
$I\ddot{\theta} + c\theta = 0$
$\omega^2 = \frac{c}{I}$
c
θ
I
(c)
$m\ddot{x} + sx = 0$
$\omega^2 = s/m$
s
m
x
(d)
$m\ddot{x} + 2T\frac{x}{l} = 0$
$\omega^2 = \frac{2T}{lm}$
m
T
T
x
$2l$
(e)
$\rho l\ddot{x} + 2\rho gx = 0$
$\omega^2 = 2g/l$
x
$2x$
x
ρ
(f)
$\rho Al\ddot{x} + \frac{\gamma pxA^2}{V} = 0$
$\omega^2 = \frac{\gamma pA}{l\rho V}$
l
V
A
x
p



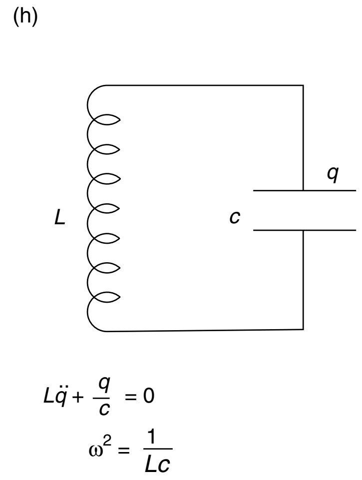


Figure 1.1 Simple harmonic oscillators with their equations of motion and angular frequencies ω of oscillation. (a) A simple pendulum. (b) A torsional pendulum. (c) A mass on a frictionless plane connected by a spring to a wall. (d) A mass at the centre of a string under constant tension T. (e) A fixed length of non-viscous liquid in a U-tube of constant cross-section. (f) An acoustic Helmholtz resonator. (g) A hydrometer mass m in a liquid of density ρ. (h) An electrical $L\,C$ resonant circuit

the equilibrium position. A constant value of the stiffness restricts the displacement x to small values (this is Hooke's Law of Elasticity). The stiffness s is obviously the restoring force per unit distance (or displacement) and has the dimensions

$$\frac{\text{force}}{\text{distance}} \equiv \frac{MLT^{-2}}{L}$$

The equation of motion of such a disturbed system is given by the dynamic balance between the forces acting on the system, which by Newton's Law is

$$\text{mass times acceleration} = \text{restoring force}$$

or

$$m\ddot{x} = -sx$$

where the acceleration

$$\ddot{x} = \frac{\mathrm{d}^2x}{\mathrm{d}t^2}$$

This gives

$$m\ddot{x} + sx = 0$$

or

$$\ddot{x} + \frac{s}{m}x = 0$$

where the dimensions of

$$\frac{s}{m} \text{ are } \frac{MLT^{-2}}{ML} = T^{-2} = \nu^2$$

Here T is a time, or period of oscillation, the reciprocal of ν which is the frequency with which the system oscillates.

However, when we solve the equation of motion we shall find that the behaviour of x with time has a sinusoidal or cosinusoidal dependence, and it will prove more appropriate to consider, not ν, but the angular frequency $\omega = 2\pi\nu$ so that the period

$$T = \frac{1}{\nu} = 2\pi\sqrt{\frac{m}{s}}$$

where s/m is now written as ω^2. Thus the equation of simple harmonic motion

$$\ddot{x} + \frac{s}{m}x = 0$$

becomes

$$\boxed{\ddot{x} + \omega^2 x = 0} \tag{1.1}$$

(Problem 1.1)

Displacement in Simple Harmonic Motion

The behaviour of a simple harmonic oscillator is expressed in terms of its displacement x from equilibrium, its velocity $\dot{x}$, and its acceleration $\ddot{x}$ at any given time. If we try the solution

$$x = A\cos\omega t$$

where A is a constant with the same dimensions as x, we shall find that it satisfies the equation of motion

$$\ddot{x} + \omega^2 x = 0$$

for

$$\dot{x} = -A\omega\sin\omega t$$

and

$$\ddot{x} = -A\omega^2\cos\omega t = -\omega^2 x$$

Another solution

$$x = B \sin \omega t$$

is equally valid, where B has the same dimensions as A, for then

$$\dot{x} = B\omega \cos \omega t$$

and

$$\ddot{x} = -B\omega^2 \sin \omega t = -\omega^2 \mathrm{x}$$

The complete or general solution of equation (1.1) is given by the addition or superposition of both values for x so we have

$$x = A \cos \omega t + B \sin \omega t \tag{1.2}$$

with

$$\ddot{x} = -\omega^2 (A \cos \omega t + B \sin \omega t) = -\omega^2 x$$

where A and B are determined by the values of x and $\dot{x}$ at a specified time. If we rewrite the constants as

$$A = a \sin \phi \quad \text{and} \quad B = a \cos \phi$$

where ϕ is a constant angle, then

$$A^2 + B^2 = a^2 (\sin^2\phi + \cos^2\phi) = a^2$$

so that

$$a = \sqrt{A^2 + B^2}$$

and

$$\begin{aligned} x &= a \sin \phi \cos \omega t + a \cos \phi \sin \omega t \\ &= a \sin (\omega t + \phi) \end{aligned}$$

The maximum value of sin $(\omega t + \phi)$ is unity so the constant a is the maximum value of x, known as the amplitude of displacement. The limiting values of $\sin(\omega t + \phi)$ are ± 1 so the system will oscillate between the values of $x = \pm a$ and we shall see that the magnitude of a is determined by the total energy of the oscillator.

The angle ϕ is called the 'phase constant' for the following reason. Simple harmonic motion is often introduced by reference to 'circular motion' because each possible value of the displacement x can be represented by the projection of a radius vector of constant length a on the diameter of the circle traced by the tip of the vector as it rotates in a positive

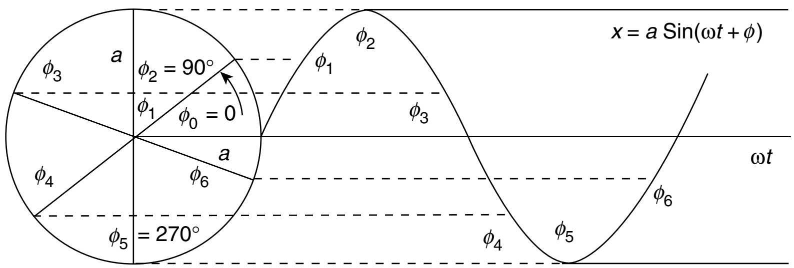


Figure 1.2 Sinusoidal displacement of simple harmonic oscillator with time, showing variation of starting point in cycle in terms of phase angle ϕ

anticlockwise direction with a constant angular velocity ω. Each rotation, as the radius vector sweeps through a phase angle of 2π rad, therefore corresponds to a complete vibration of the oscillator. In the solution

$$x = a \sin(\omega t + \phi)$$

the phase constant ϕ, measured in radians, defines the position in the cycle of oscillation at the time $t = 0$, so that the position in the cycle from which the oscillator started to move is

$$x = a \sin \phi$$

The solution

$$x = a \sin \omega t$$

defines the displacement only of that system which starts from the origin $x = 0$ at time $t = 0$ but the inclusion of ϕ in the solution

$$x = a \sin(\omega t + \phi)$$

where ϕ may take all values between zero and 2π allows the motion to be defined from any starting point in the cycle. This is illustrated in Figure 1.2 for various values of ϕ.

(Problems 1.2, 1.3, 1.4, 1.5)

Velocity and Acceleration in Simple Harmonic Motion

The values of the velocity and acceleration in simple harmonic motion for

$$x = a \sin(\omega t + \phi)$$

are given by

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \dot{x} = a\omega \cos(\omega t + \phi)$$

and

$$\frac{d^2x}{dt^2} = \ddot{x} = -a\omega^2 \sin(\omega t + \phi)$$

The maximum value of the velocity $a\omega$ is called the velocity *amplitude* and the *acceleration amplitude* is given by $a\omega^2$.

From Figure 1.2 we see that a positive phase angle of $\pi/2$ rad converts a sine into a cosine curve. Thus the velocity

$$\dot{x} = a\omega \cos(\omega t + \phi)$$

leads the displacement

$$x = a \sin(\omega t + \phi)$$

by a phase angle of $\pi/2$ rad and its maxima and minima are always a quarter of a cycle ahead of those of the displacement; the velocity is a maximum when the displacement is zero and is zero at maximum displacement. The acceleration is 'anti-phase' (π rad) with respect to the displacement, being maximum positive when the displacement is maximum negative and vice versa. These features are shown in Figure 1.3.

Often, the relative displacement or motion between two oscillators having the same frequency and amplitude may be considered in terms of their phase difference $\phi_1 - \phi_2$ which can have any value because one system may have started several cycles before the other and each complete cycle of vibration represents a change in the phase angle of $\phi = 2\pi$. When the motions of the two systems are diametrically opposed; that is, one has

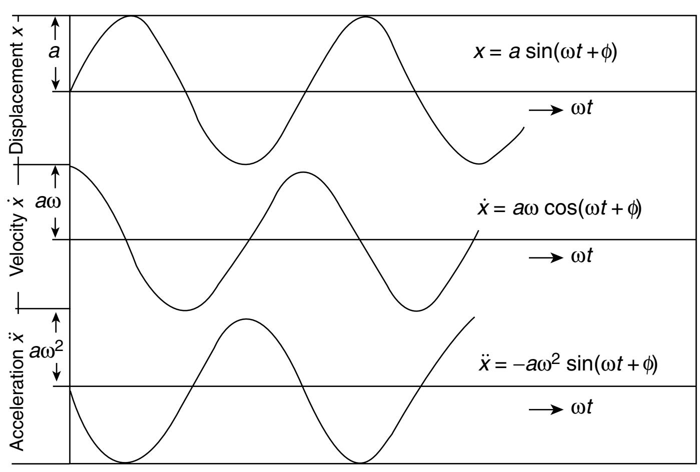


Figure 1.3 Variation with time of displacement, velocity and acceleration in simple harmonic motion. Displacement lags velocity by $\pi/2$ rad and is π rad out of phase with the acceleration. The initial phase constant ϕ is taken as zero

$x = +a$ whilst the other is at $x = -a$, the systems are 'anti-phase' and the total phase difference

$$\phi_1 - \phi_2 = n\pi \,\text{rad}$$

where n is an *odd* integer. Identical systems 'in phase' have

$$\phi_1 - \phi_2 = 2n\pi \,\text{rad}$$

where n is any integer. They have exactly equal values of displacement, velocity and acceleration at any instant.

(Problems 1.6, 1.7, 1.8, 1.9)

Non-linearity

If the stiffness s is constant, then the restoring force $F = -sx$, when plotted versus x, will produce a straight line and the system is said to be linear. The displacement of a linear simple harmonic motion system follows a sine or cosine behaviour. Non-linearity results when the stiffness s is not constant but varies with displacement x (see the beginning of Chapter 14).

Energy of a Simple Harmonic Oscillator

The fact that the velocity is zero at maximum displacement in simple harmonic motion and is a maximum at zero displacement illustrates the important concept of an exchange between kinetic and potential energy. In an ideal case the total energy remains constant but this is never realized in practice. If no energy is dissipated then all the potential energy becomes kinetic energy and vice versa, so that the values of (a) the total energy at any time, (b) the maximum potential energy and (c) the maximum kinetic energy will all be equal; that is

$$E_{\text{total}} = \text{KE} + \text{PE} = \text{KE}_{\text{max}} = \text{PE}_{\text{max}}$$

The solution $x = a \sin(\omega t + \phi)$ implies that the total energy remains constant because the amplitude of displacement $x = \pm a$ is regained every half cycle at the position of maximum potential energy; when energy is lost the amplitude gradually decays as we shall see later in Chapter 2. The potential energy is found by summing all the small elements of work $sx.\,dx$ (force sx times distance dx) *done by the system against the restoring force* over the range zero to x where $x = 0$ gives zero potential energy.

Thus the potential energy =

$$\int_0^x sx \cdot \mathrm{d}x = \tfrac{1}{2}sx^2$$

The kinetic energy is given by $\frac{1}{2}m\dot{x}^2$ so that the total energy

$$\boxed{E = \tfrac{1}{2}m\dot{x}^2 + \tfrac{1}{2}sx^2}$$

Since E is constant we have

$$\frac{\mathrm{d}E}{\mathrm{d}t} = (m\ddot{x} + sx)\dot{x} = 0$$

giving again the equation of motion

$$m\ddot{x} + sx = 0$$

The maximum potential energy occurs at $x = \pm a$ and is therefore

$$\mathrm{PE}_{\max} = \tfrac{1}{2}sa^2$$

The maximum kinetic energy is

$$\begin{aligned}\mathrm{KE}_{\max} = (\tfrac{1}{2}m\dot{x}^2)_{\max} &= \tfrac{1}{2}ma^2\omega^2[\cos^2(\omega t + \phi)]_{\max} \\ &= \tfrac{1}{2}ma^2\omega^2\end{aligned}$$

when the cosine factor is unity.

But $m\omega^2 = s$ so the maximum values of the potential and kinetic energies are equal, showing that the energy exchange is complete.

The total energy at any instant of time or value of x is

$$\begin{aligned}E &= \tfrac{1}{2}m\dot{x}^2 + \tfrac{1}{2}sx^2 \\ &= \tfrac{1}{2}ma^2\omega^2[\cos^2(\omega t + \phi) + \sin^2(\omega t + \phi)] \\ &= \tfrac{1}{2}ma^2\omega^2 \\ &= \tfrac{1}{2}sa^2\end{aligned}$$

as we should expect.

Figure 1.4 shows the distribution of energy versus displacement for simple harmonic motion. Note that the potential energy curve

$$\mathrm{PE} = \tfrac{1}{2}\,sx^2 = \tfrac{1}{2}\,ma^2\omega^2\sin^2(\omega t + \phi)$$

is parabolic with respect to x and is symmetric about $x = 0$, so that energy is stored in the oscillator both when x is positive and when it is negative, e.g. a spring stores energy whether compressed or extended, as does a gas in compression or rarefaction. The kinetic energy curve

$$\mathrm{KE} = \tfrac{1}{2}m\dot{x}^2 = \tfrac{1}{2}ma^2\omega^2\cos^2(\omega t + \phi)$$

is parabolic with respect to both x and $\dot{x}$. The inversion of one curve with respect to the other displays the $\pi/2$ phase difference between the displacement (related to the potential energy) and the velocity (related to the kinetic energy).

For any value of the displacement x the sum of the ordinates of both curves equals the total constant energy E.

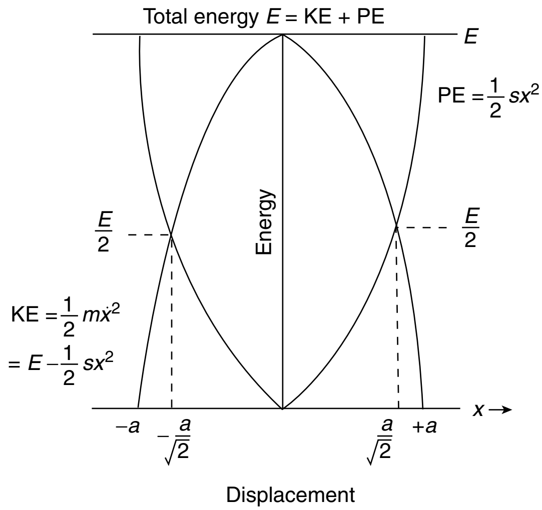


Figure 1.4 Parabolic representation of potential energy and kinetic energy of simple harmonic motion versus displacement. Inversion of one curve with respect to the other shows a 90° phase difference. At any displacement value the sum of the ordinates of the curves equals the total constant energy E

(Problems 1.10, 1.11, 1.12)

Simple Harmonic Oscillations in an Electrical System

So far we have discussed the simple harmonic motion of the mechanical and fluid systems of Figure 1.1, chiefly in terms of the inertial mass stretching the weightless spring of stiffness s. The stiffness s of a spring defines the difficulty of stretching; the reciprocal of the stiffness, the compliance C (where $s = 1/C$) defines the ease with which the spring is stretched and potential energy stored. This notation of compliance C is useful when discussing the simple harmonic oscillations of the electrical circuit of Figure 1.1(h) and Figure 1.5, where an inductance L is connected across the plates of a capacitance C. The force equation of the mechanical and fluid examples now becomes the voltage equation

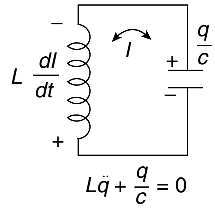


Figure 1.5 Electrical system which oscillates simple harmonically. The sum of the voltages around the circuit is given by Kirchhoff's law as $L\,dI/dt + q/C = 0$

(balance of voltages) of the electrical circuit, but the form and solution of the equations and the oscillatory behaviour of the systems are identical.

In the absence of resistance the energy of the electrical system remains constant and is exchanged between the *magnetic* field energy stored in the inductance and the *electric* field energy stored between the plates of the capacitance. At any instant, the voltage across the inductance is

$$V = -L\frac{\mathrm{d}I}{\mathrm{d}t} = -L\frac{\mathrm{d}^2 q}{\mathrm{d}t^2}$$

where I is the current flowing and q is the charge on the capacitor, the negative sign showing that the voltage opposes the increase of current. This equals the voltage q/C across the capacitance so that

$$L\ddot{q} + q/C = 0 \qquad \text{(Kirchhoff's Law)}$$

or

$$\ddot{q} + \omega^2 q = 0$$

where

$$\omega^2 = \frac{1}{LC}$$

The energy stored in the magnetic field or inductive part of the circuit throughout the cycle, as the current increases from 0 to I, is formed by integrating the power at any instant with respect to time; that is

$$E_{\mathrm{L}} = \int VI \cdot \mathrm{d}t$$

(where V is the magnitude of the voltage across the inductance).

So

$$\begin{aligned} E_{\mathrm{L}} = \int VI\,\mathrm{d}t = \int L\frac{\mathrm{d}I}{\mathrm{d}t}I\,\mathrm{d}t = \int_0^I LI\,\mathrm{d}I \\ = \tfrac{1}{2}LI^2 = \tfrac{1}{2}L\dot{q}^2 \end{aligned}$$

The potential energy stored mechanically by the spring is now stored electrostatically by the capacitance and equals

$$\tfrac{1}{2}CV^2 = \frac{q^2}{2C}$$

Comparison between the equations for the mechanical and electrical oscillators

$$\text{mechanical (force)} \rightarrow m\ddot{x} + sx = 0$$
$$\text{electrical (voltage)} \rightarrow L\ddot{q} + \frac{q}{C} = 0$$
$$\text{mechanical (energy)} \rightarrow \tfrac{1}{2}m\dot{x}^2 + \tfrac{1}{2}sx^2 = E$$
$$\text{electrical (energy)} \rightarrow \frac{1}{2}L\dot{q}^2 + \frac{1}{2}\frac{q^2}{C} = E$$

shows that magnetic field inertia (defined by the inductance L) controls the rate of change of current for a given voltage in a circuit in exactly the same way as the inertial mass controls the change of velocity for a given force. Magnetic inertial or inductive behaviour arises from the tendency of the magnetic flux threading a circuit to remain constant and reaction to any change in its value generates a voltage and hence a current which flows to oppose the change of flux. This is the physical basis of Fleming's right-hand rule.

Superposition of Two Simple Harmonic Vibrations in One Dimension

(1) Vibrations Having Equal Frequencies

In the following chapters we shall meet physical situations which involve the superposition of two or more simple harmonic vibrations on the same system.

We have already seen how the displacement in simple harmonic motion may be represented in magnitude and phase by a constant length vector rotating in the positive (anticlockwise) sense with a constant angular velocity ω. To find the resulting motion of a system which moves in the x direction under the simultaneous effect of two simple harmonic oscillations of equal angular frequencies but of different amplitudes and phases, we can represent each simple harmonic motion by its appropriate vector and carry out a vector addition.

If the displacement of the first motion is given by

$$x_1 = a_1 \cos(\omega t + \phi_1)$$

and that of the second by

$$x_2 = a_2 \cos(\omega t + \phi_2)$$

then Figure 1.6 shows that the resulting displacement amplitude R is given by

$$\begin{aligned} R^2 &= (a_1 + a_2 \cos\delta)^2 + (a_2 \sin\delta)^2 \\ &= a_1^2 + a_2^2 + 2a_1 a_2 \cos\delta \end{aligned}$$

where $\delta = \phi_2 - \phi_1$ is constant.

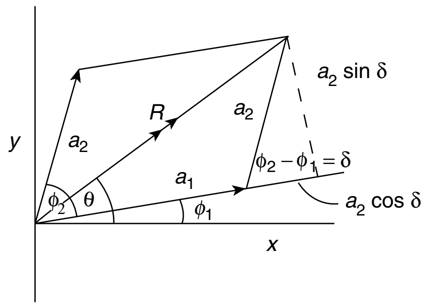


Figure 1.6 Addition of vectors, each representing simple harmonic motion along the x axis at angular frequency ω to give a resulting simple harmonic motion displacement $x = R\cos(\omega t + \theta)$ --- here shown for $t = 0$

The phase constant θ of R is given by

$$\tan\theta = \frac{a_1 \sin\phi_1 + a_2 \sin\phi_2}{a_1 \cos\phi_1 + a_2 \cos\phi_2}$$

so the resulting simple harmonic motion has a displacement

$$x = R\cos(\omega t + \theta)$$

an oscillation of the same frequency ω but having an amplitude R and a phase constant θ.

(Problem 1.13)

(2) Vibrations Having Different Frequencies

Suppose we now consider what happens when two vibrations of equal amplitudes but different frequencies are superposed. If we express them as

$$x_1 = a\sin\omega_1 t$$

and

$$x_2 = a\sin\omega_2 t$$

where

$$\omega_2 > \omega_1$$

then the resulting displacement is given by

$$\begin{aligned} x = x_1 + x_2 &= a(\sin \omega_1 t + \sin \omega_2 t) \\ &= 2a \sin \frac{(\omega_1 + \omega_2)t}{2} \cos \frac{(\omega_2 - \omega_1)t}{2} \end{aligned}$$

This expression is illustrated in Figure 1.7. It represents a sinusoidal oscillation at the average frequency $(\omega_1 + \omega_2)/2$ having a displacement amplitude of $2a$ which modulates; that is, varies between $2a$ and zero under the influence of the cosine term of a much slower frequency equal to half the difference $(\omega_2 - \omega_1)/2$ between the original frequencies.

When ω_1 and ω_2 are almost equal the sine term has a frequency very close to both ω_1 and ω_2 whilst the cosine envelope modulates the amplitude $2a$ at a frequency $(\omega_2 - \omega_1)/2$ which is very slow.

Acoustically this growth and decay of the amplitude is registered as 'beats' of strong reinforcement when two sounds of almost equal frequency are heard. The frequency of the 'beats' is $(\omega_2 - \omega_1)$, the difference between the separate frequencies (not half the difference) because the maximum amplitude of $2a$ occurs twice in every period associated with the frequency $(\omega_2 - \omega_1)/2$. We shall meet this situation again when we consider the coupling of two oscillators in Chapter 4 and the wave group of two components in Chapter 5.

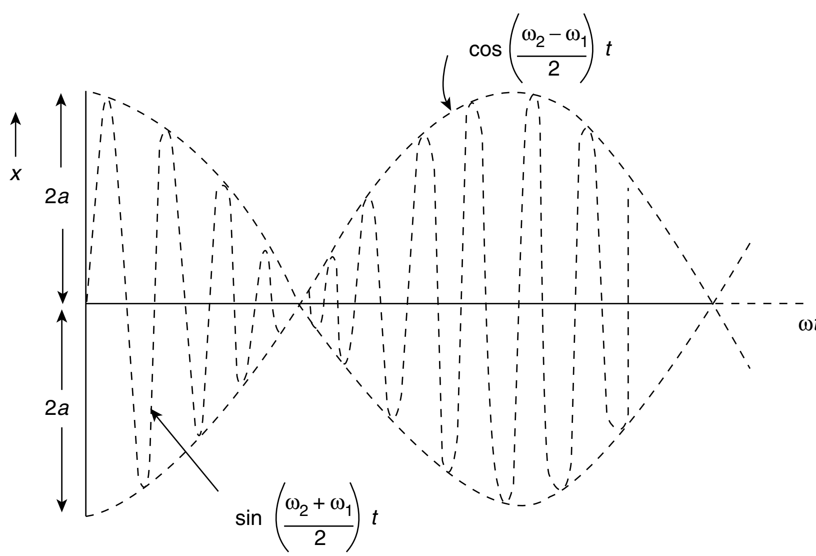


Figure 1.7 Superposition of two simple harmonic displacements $x_1 = a \sin \omega_1 t$ and $x_2 = a \sin \omega_2 t$ when $\omega_2 > \omega_1$. The slow cos $[(\omega_2 - \omega_1)/2]t$ envelope modulates the sin $[(\omega_2 + \omega_1)/2]t$ curve between the values $x = \pm 2a$

Superposition of Two Perpendicular Simple Harmonic Vibrations

(1) Vibrations Having Equal Frequencies

Suppose that a particle moves under the simultaneous influence of two simple harmonic vibrations of equal frequency, one along the x axis, the other along the perpendicular y axis. What is its subsequent motion?

This displacements may be written

$$x = a_1 \sin(\omega t + \phi_1)$$
$$y = a_2 \sin(\omega t + \phi_2)$$

and the path followed by the particle is formed by eliminating the time t from these equations to leave an expression involving only x and y and the constants ϕ_1 and ϕ_2.

Expanding the arguments of the sines we have

$$\frac{x}{a_1} = \sin\omega t \cos\phi_1 + \cos\omega t \sin\phi_1$$

and

$$\frac{y}{a_2} = \sin\omega t \cos\phi_2 + \cos\omega t \sin\phi_2$$

If we carry out the process

$$\left(\frac{x}{a_1}\sin\phi_2 - \frac{y}{a_2}\sin\phi_1\right)^2 + \left(\frac{y}{a_2}\cos\phi_1 - \frac{x}{a_1}\cos\phi_2\right)^2$$

this will yield

$$\frac{x^2}{a_1^2} + \frac{y^2}{a_2^2} - \frac{2xy}{a_1 a_2}\cos(\phi_2 - \phi_1) = \sin^2(\phi_2 - \phi_1) \tag{1.3}$$

which is the general equation for an ellipse.

In the most general case the axes of the ellipse are inclined to the x and y axes, but these become the principal axes when the phase difference

$$\phi_2 - \phi_1 = \frac{\pi}{2}$$

Equation (1.3) then takes the familiar form

$$\frac{x^2}{a_1^2} + \frac{y^2}{a_2^2} = 1$$

that is, an ellipse with semi-axes a_1 and a_2.

If $a_1 = a_2 = a$ this becomes the circle

$$x^2 + y^2 = a^2$$

When

$$\phi_2 - \phi_1 = 0, 2\pi, 4\pi, \text{ etc.}$$

the equation simplifies to

$$y = \frac{a_2}{a_1}x$$

which is a straight line through the origin of slope a_2/a_1.

Again for $\phi_2 - \phi_1 = \pi$, 3π, 5π, etc., we obtain

$$y = -\frac{a_2}{a_1}x$$

a straight line through the origin of equal but opposite slope.

The paths traced out by the particle for various values of $\delta = \phi_2 - \phi_1$ are shown in Figure 1.8 and are most easily demonstrated on a cathode ray oscilloscope.

When

$$\phi_2 - \phi_1 = 0,\ \pi,\ 2\pi, \text{ etc.}$$

and the ellipse degenerates into a straight line, the resulting vibration lies wholly in one plane and the oscillations are said to be *plane polarized*.

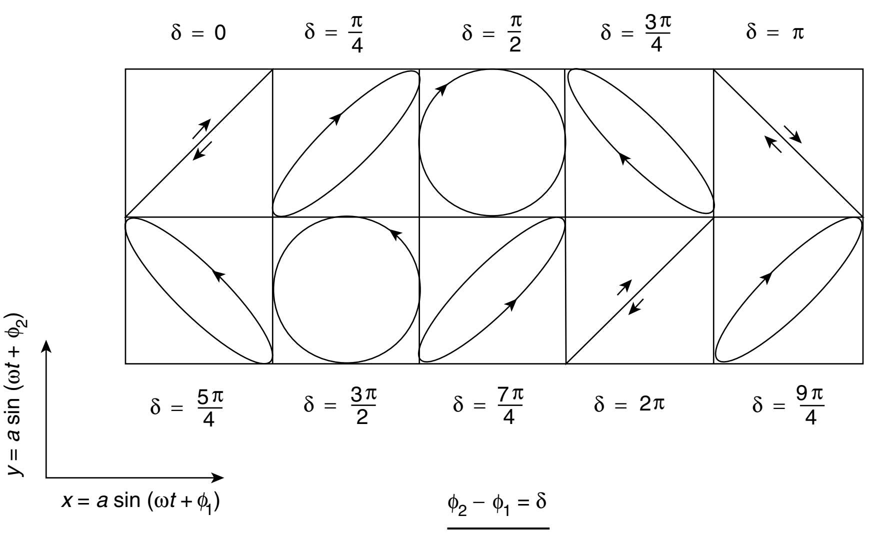


Figure 1.8 Paths traced by a system vibrating simultaneously in two perpendicular directions with simple harmonic motions of equal frequency. The phase angle δ is the angle by which the y motion leads the x motion

Convention defines the plane of polarization as that plane perpendicular to the plane containing the vibrations. Similarly the other values of

$$\phi_2 - \phi_1$$

yield *circular* or *elliptic* polarization where the tip of the vector resultant traces out the appropriate conic section.

(Problems 1.14, 1.15, 1.16)

*Polarization

Polarization is a fundamental topic in optics and arises from the superposition of two perpendicular simple harmonic optical vibrations. We shall see in Chapter 8 that when a light wave is plane polarized its electrical field oscillation lies within a single plane and traces a sinusoidal curve along the direction of wave motion. Substances such as quartz and calcite are capable of splitting light into two waves whose planes of polarization are perpendicular to each other. Except in a specified direction, known as the optic axis, these waves have different velocities. One wave, the ordinary or O wave, travels at the same velocity in all directions and its electric field vibrations are always perpendicular to the optic axis. The extraordinary or E wave has a velocity which is direction-dependent. Both ordinary and extraordinary light have their own refractive indices, and thus quartz and calcite are known as doubly refracting materials. When the ordinary light is faster, as in quartz, a crystal of the substance is defined as positive, but in calcite the extraordinary light is faster and its crystal is negative. The surfaces, spheres and ellipsoids, which are the loci of the values of the wave velocities in any direction are shown in Figure 1.9(a), and for a

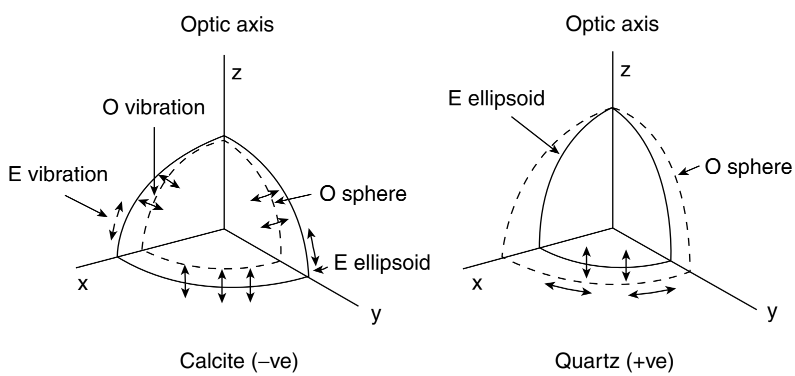


Figure 1.9a Ordinary (spherical) and extraordinary (elliposoidal) wave surfaces in doubly refracting calcite and quartz. In calcite the E wave is faster than the O wave, except along the optic axis. In quartz the O wave is faster. The O vibrations are always perpendicular to the optic axis, and the O and E vibrations are always tangential to their wave surfaces

*This section may be omitted at a first reading.

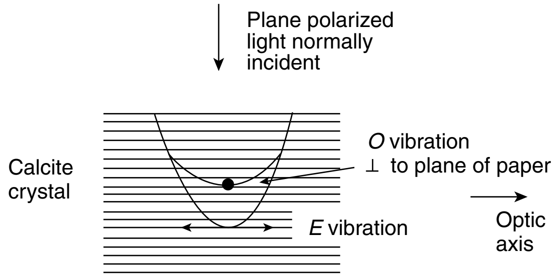


Figure 1.9b Plane polarized light normally incident on a calcite crystal face cut parallel to its optic axis. The advance of the *E* wave over the *O* wave is equivalent to a gain in phase

given direction the electric field vibrations of the separate waves are tangential to the surface of the sphere or ellipsoid as shown. Figure 1.9(b) shows plane polarized light normally incident on a calcite crystal cut parallel to its optic axis. Within the crystal the faster *E* wave has vibrations parallel to the optic axis, while the *O* wave vibrations are perpendicular to the plane of the paper. The velocity difference results in a phase gain of the *E* vibration over the *O* vibration which increases with the thickness of the crystal. Figure 1.9(c) shows plane polarized light normally incident on the crystal of Figure 1.9(b) with its vibration at an angle of 45° of the optic axis. The crystal splits the vibration into

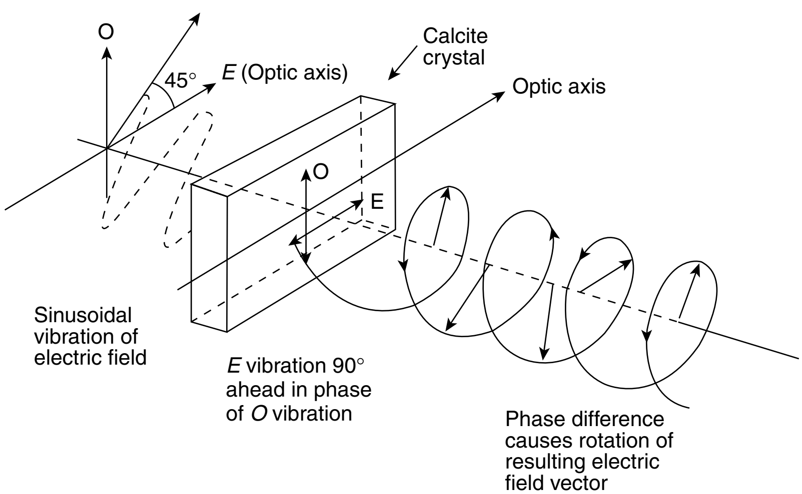


Figure 1.9c The crystal of Fig. 1.9c is thick enough to produce a phase gain of $\pi/2$ rad in the *E* wave over the *O* wave. Wave recombination on leaving the crystal produces circularly polarized light

equal E and O components, and for a given thickness the E wave emerges with a phase gain of 90° over the O component. Recombination of the two vibrations produces circularly polarized light, of which the electric field vector now traces a helix in the anticlockwise direction as shown.

(2) Vibrations Having Different Frequencies (Lissajous Figures)

When the frequencies of the two perpendicular simple harmonic vibrations are not equal the resulting motion becomes more complicated. The patterns which are traced are called Lissajous figures and examples of these are shown in Figure 1.10 where the axial frequencies bear the simple ratios shown and

$$\delta = \phi_2 - \phi_1 = 0 \text{ (on the left)}$$
$$= \frac{\pi}{2} \text{ (on the right)}$$

If the amplitudes of the vibrations are respectively a and b the resulting Lissajous figure will always be contained within the rectangle of sides $2a$ and $2b$. The sides of the rectangle will be tangential to the curve at a number of points and the ratio of the numbers of these tangential points along the x axis to those along the y axis is the inverse of the ratio of the corresponding frequencies (as indicated in Figure 1.10).

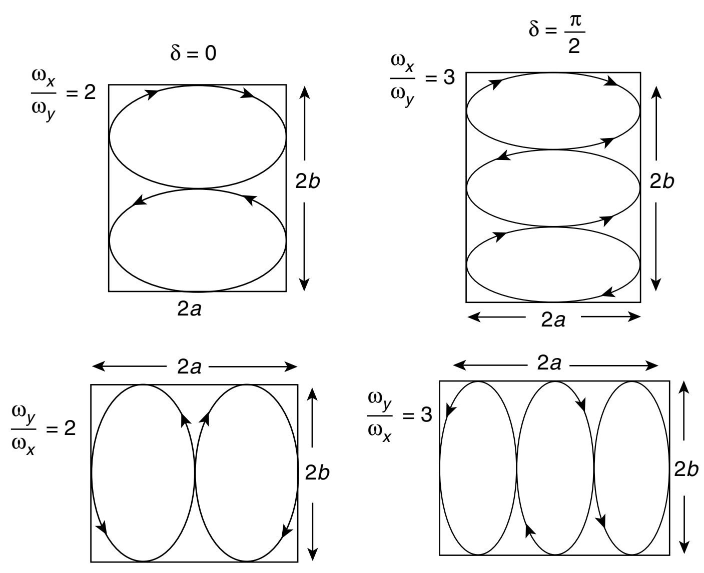


Figure 1.10 Simple Lissajous figures produced by perpendicular simple harmonic motions of different angular frequencies

Superposition of a Large Number n of Simple Harmonic Vibrations of Equal Amplitude a and Equal Successive Phase Difference δ

Figure 1.11 shows the addition of n vectors of equal length a, each representing a simple harmonic vibration with a constant phase difference δ from its neighbour. Two general physical situations are characterized by such a superposition. The first is met in Chapter 5 as a wave group problem where the phase difference δ arises from a small *frequency difference*, $\delta\omega$, between consecutive components. The second appears in Chapter 12 where the intensity of optical interference and diffraction patterns are considered. There, the superposed harmonic vibrations will have the same frequency but each component will have a constant phase difference from its neighbour because of the extra *distance* it has travelled.

The figure displays the mathematical expression

$$R\cos(\omega t+\alpha)=a\cos\omega t+a\cos(\omega t+\delta)+a\cos(\omega t+2\delta)$$
$$+\cdots+a\cos(\omega t+[n-1]\delta)$$

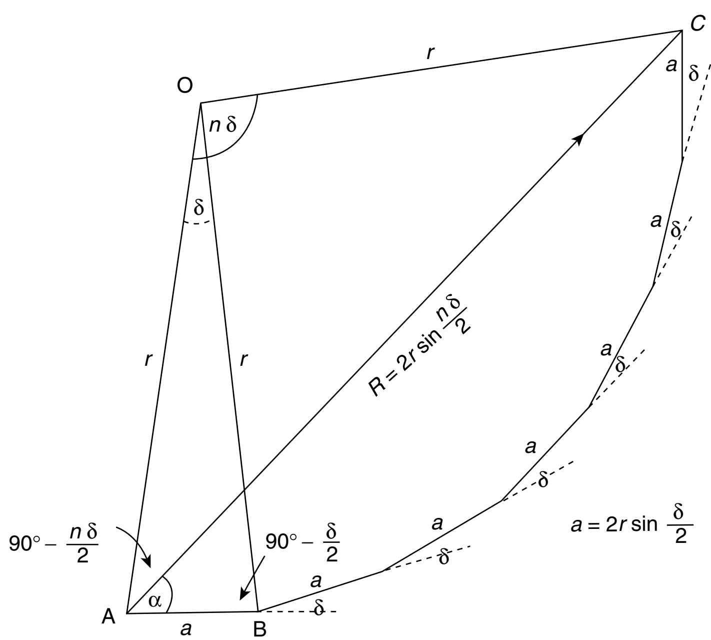


Figure 1.11 Vector superposition of a large number n of simple harmonic vibrations of equal amplitude a and equal successive phase difference δ. The amplitude of the resultant

$$R=2r\sin\frac{n\delta}{2}=a\frac{\sin n\delta/2}{\sin\delta/2}$$

and its phase with respect to the first contribution is given by

$$\alpha=(n-1)\delta/2$$

where R is the magnitude of the resultant and α is its phase difference with respect to the first component $a\cos\omega t$.

Geometrically we see that each length

$$a = 2r\sin\frac{\delta}{2}$$

where r is the radius of the circle enclosing the (incomplete) polygon.

From the isosceles triangle OAC the magnitude of the resultant

$$R = 2r\sin\frac{n\delta}{2} = a\frac{\sin n\delta/2}{\sin\delta/2}$$

and its phase angle is seen to be

$$\alpha = \mathrm{O\hat{A}B} - \mathrm{O\hat{A}C}$$

In the isosceles triangle OAC

$$\mathrm{\hat{O}AC} = 90^\circ - \frac{n\delta}{2}$$

and in the isosceles triangle OAB

$$\mathrm{O\hat{A}B} = 90^\circ - \frac{\delta}{2}$$

so

$$\alpha = \left(90^\circ - \frac{\delta}{2}\right) - \left(90^\circ - \frac{n\delta}{2}\right) = (n-1)\frac{\delta}{2}$$

that is, half the phase difference between the first and the last contributions. Hence the resultant

$$R\cos(\omega t + \alpha) = a\frac{\sin n\delta/2}{\sin\delta/2}\cos\left[\omega t + (n-1)\frac{\delta}{2}\right]$$

We shall obtain the same result later in this chapter as an example on the use of exponential notation.

For the moment let us examine the behaviour of the magnitude of the resultant

$$R = a\frac{\sin n\delta/2}{\sin\delta/2}$$

which is not constant but depends on the value of δ. When n is very large δ is very small and the polygon becomes an arc of the circle centre O, of length $na = A$, with R as the chord. Then

$$\alpha = (n-1)\frac{\delta}{2} \approx \frac{n\delta}{2}$$

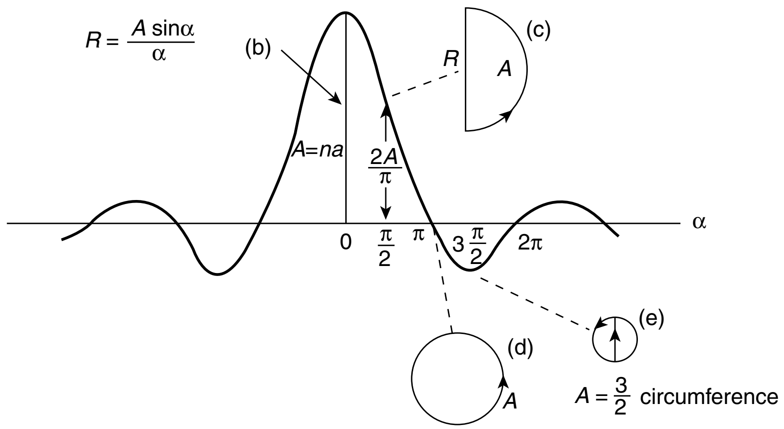


Figure 1.12 (a) Graph of A sin α/α versus α, showing the magnitude of the resultants for (b) $\alpha = 0$; (c) $\alpha = \pi/2$; (d) $\alpha = \pi$ and (e) $\alpha = 3\pi/2$

and

$$\sin\frac{\delta}{2} \rightarrow \frac{\delta}{2} \approx \frac{\alpha}{n}$$

Hence, in this limit,

$$R = a\frac{\sin n\delta/2}{\sin\delta/2} = a\frac{\sin\alpha}{\alpha/n} = na\frac{\sin\alpha}{\alpha} = \frac{A\sin\alpha}{\alpha}$$

The behaviour of $A\ \sin\alpha/\alpha$ versus α is shown in Figure 1.12. The pattern is symmetric about the value $\alpha = 0$ and is zero whenever sin $\alpha = 0$ except at $\alpha \rightarrow 0$ that is, when sin $\alpha/\alpha \rightarrow 1$. When $\alpha = 0$, $\delta = 0$ and the resultant of the n vectors is the straight line of length A, Figure 1.12(b). As δ increases A becomes the arc of a circle until at $\alpha = \pi/2$ the first and last contributions are out of phase $(2\alpha = \pi)$ and the arc A has become a semicircle of which the diameter is the resultant R Figure 1.12(c). A further increase in δ increases α and curls the constant length A into the circumference of a circle $(\alpha = \pi)$ with a zero resultant, Figure 1.12(d). At $\alpha = 3\pi/2$, Figure 1.12(e) the length A is now 3/2 times the circumference of a circle whose diameter is the amplitude of the first minimum.

*Superposition of *n* Equal SHM Vectors of Length *a* with Random Phase

When the phase difference between the successive vectors of the last section may take random values ϕ between zero and 2π (measured from the x axis) the vector superposition and resultant R may be represented by Figure 1.13.

*This section may be omitted at a first reading.

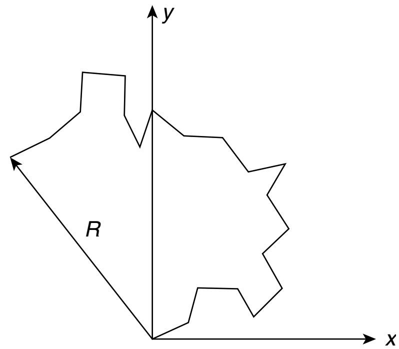


Figure 1.13 The resultant $R = \sqrt{n}a$ of n vectors, each of length a, having random phase. This result is important in optical incoherence and in energy loss from waves from random dissipation processes

The components of R on the x and y axes are given by

$$\begin{aligned} R_x &= a\cos\phi_1 + a\cos\phi_2 + a\cos\phi_3 \ldots a\cos\phi_n \\ &= a\sum_{i=1}^{n}\cos\phi_i \end{aligned}$$

and

$$R_y = a\sum_{i=1}^{n}\sin\phi_i$$

where

$$R^2 = R_x^2 + R_y^2$$

Now

$$R_x^2 = a^2\left(\sum_{i=1}^{n}\cos\phi_i\right)^2 = a^2\left[\sum_{i=1}^{n}\cos^2\phi_i + \sum_{\substack{i=1\\ i\neq j}}^{n}\cos\phi_i\sum_{j=1}^{n}\cos\phi_j\right]$$

In the typical term $2\cos\phi_i\cos\phi_j$ of the double summation, $\cos\phi_i$ and $\cos\phi_j$ have random values between ± 1 and the averaged sum of sets of these products is effectively zero.

The summation

$$\sum_{i=1}^{n}\cos^2\phi_i = n\overline{\cos^2\phi}$$

that is, the number of terms n times the average value $\overline{\cos^2\phi}$ which is the integrated value of $\cos^2\phi$ over the interval zero to 2π divided by the total interval 2π, or

$$\overline{\cos^2\phi} = \frac{1}{2\pi}\int_0^{2\pi} \cos^2\phi \,\mathrm{d}\phi = \frac{1}{2} = \overline{\sin^2\phi}$$

So

$$R_x^2 = a^2\sum_{i=1}^{n} \cos^2\phi_i = na^2\overline{\cos^2\phi_i} = \frac{na^2}{2}$$

and

$$R_y^2 = a^2\sum_{i=1}^{n} \sin^2\phi_i = na^2\overline{\sin^2\phi_i} = \frac{na^2}{2}$$

giving

$$R^2 = R_x^2 + R_y^2 = na^2$$

or

$$R = \sqrt{n}a$$

Thus, the amplitude R of a system subjected to n equal simple harmonic motions of amplitude a with random phases in only $\sqrt{n}a$ whereas, if the motions were all in phase R would equal na.

Such a result illustrates a very important principle of random behaviour.

(Problem 1.17)

Applications

Incoherent Sources in Optics The result above is directly applicable to the problem of coherence in optics. Light sources which are in phase are said to be coherent and this condition is essential for producing optical interference effects experimentally. If the amplitude of a light source is given by the quantity a its intensity is proportional to a^2, n coherent sources have a resulting amplitude na and a total intensity n^2a^2. Incoherent sources have random phases, n such sources each of amplitude a have a resulting amplitude $\sqrt{n}a$ and a total intensity of na^2.

Random Processes and Energy Absorption From our present point of view the importance of random behaviour is the contribution it makes to energy loss or absorption from waves moving through a medium. We shall meet this in all the waves we discuss.

Random processes, for example collisions between particles, in Brownian motion, are of great significance in physics. Diffusion, viscosity or frictional resistance and thermal conductivity are all the result of random collision processes. These energy dissipating phenomena represent the transport of mass, momentum and energy, and change only in the direction of increasing disorder. They are known as 'thermodynamically irreversible' processes and are associated with the increase of entropy. Heat, for example, can flow only from a body at a higher temperature to one at a lower temperature. Using the earlier analysis where the length a is no longer a simple harmonic amplitude but is now the average distance a particle travels between random collisions (its mean free path), we see that after n such collisions (with, on average, equal time intervals between collisions) the particle will, on average, have travelled only a distance $\sqrt{n}a$ from its position at time $t = 0$, so that the distance travelled varies only with the square root of the time elapsed instead of being directly proportional to it. This is a feature of all random processes.

Not all the particles of the system will have travelled a distance $\sqrt{n}a$ but this distance is the most probable and represents a statistical average.

Random behaviour is described by the diffusion equation (see the last section of Chapter 7) and a constant coefficient called the diffusivity of the process will always arise. The dimensions of a diffusivity are always length^2/time and must be interpreted in terms of a characteristic distance of the process which varies only with the square root of time.

Some Useful Mathematics

The Exponential Series

By a 'natural process' of growth or decay we mean a process in which a quantity changes by a constant fraction of itself in a given interval of space or time. A 5% per annum compound interest represents a natural growth law; attenuation processes in physics usually describe natural decay.

The law is expressed differentially as

$$\frac{\mathrm{d}N}{N} = \pm\alpha\,\mathrm{d}x \quad \text{or} \quad \frac{\mathrm{d}N}{N} = \pm\alpha\,\mathrm{d}t$$

where N is the changing quantity, α is a constant and the positive and negative signs represent growth and decay respectively. The derivatives $\mathrm{d}N/\mathrm{d}x$ or $\mathrm{d}N/\mathrm{d}t$ are therefore proportional to the value of N at which the derivative is measured.

Integration yields $N = N_0 e^{\pm\alpha x}$ or $N = N_0 e^{\pm\alpha t}$ where N_0 is the value at x or $t = 0$ and e is the exponential or the base of natural logarithms. The exponential series is defined as

$$\mathrm{e}^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots$$

and is shown graphically for positive and negative x in Figure 1.14. It is important to note that whatever the form of the index of the logarithmic base e, it is the power to which the

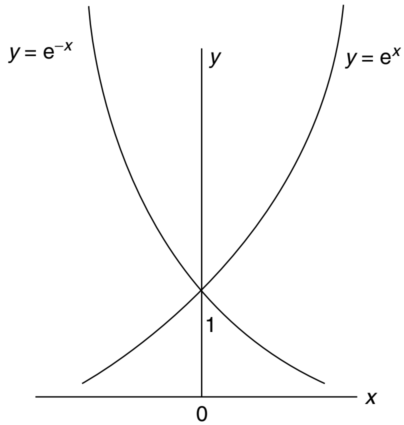


Figure 1.14 The behaviour of the exponential series $y = \mathrm{e}^{x}$ and $y = e^{-x}$

base is raised, and is therefore always non-dimensional. Thus $e^{\alpha x}$ is non-dimensional and α must have the dimensions of x^{-1}. Writing

$$\mathrm{e}^{\alpha x} = 1 + \alpha x + \frac{(\alpha x)^2}{2!} + \frac{(\alpha x)^3}{3!} + \cdots$$

it follows immediately that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}(\mathrm{e}^{\alpha x}) &= \alpha + \frac{2\alpha^2}{2!}x + \frac{3\alpha^3}{3!}x^2 + \cdots \\ &= \alpha\left[1 + \alpha x + \frac{(\alpha x)^2}{2!} + \frac{(\alpha x)^3}{3!}\right) + \cdots\Bigg] \\ &= \alpha \mathrm{e}^{\alpha x}\end{aligned}$$

Similarly

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}(\mathrm{e}^{\alpha x}) = \alpha^2 \mathrm{e}^{\alpha x}$$

In Chapter 2 we shall use $\mathrm{d}(\mathrm{e}^{\alpha t})/\mathrm{d}t = \alpha\,\mathrm{e}^{\alpha t}$ and $\mathrm{d}^2\,(\mathrm{e}^{\alpha t})/\mathrm{d}t^2 = \alpha^2\,\mathrm{e}^{\alpha t}$ on a number of occasions.

By taking logarithms it is easily shown that $\mathrm{e}^x\,\mathrm{e}^y = \mathrm{e}^{x+y}$ since $\log_\mathrm{e}\,(\mathrm{e}^x\,\mathrm{e}^y) = \log_\mathrm{e}\mathrm{e}^x + \log_\mathrm{e}\mathrm{e}^y = x + y$.

The Notation $\mathrm{i} = \sqrt{-1}$

The combination of the exponential series with the complex number notation $\mathrm{i} = \sqrt{-1}$ is particularly convenient in physics. Here we shall show the mathematical convenience in expressing sine or cosine (oscillatory) behaviour in the form $\mathrm{e}^{\mathrm{i}x} = \cos x + \mathrm{i}\sin x$.

In Chapter 3 we shall see the additional merit of i in its role of vector operator.

The series representation of sin x is written

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} \cdots$$

and that of cos x is

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} \cdots$$

Since

$$\mathrm{i} = \sqrt{-1}, \mathrm{i}^2 = -1, \mathrm{i}^3 = -\mathrm{i}$$

etc. we have

$$\begin{aligned} \mathrm{e}^{\mathrm{i}x} &= 1 + \mathrm{i}x + \frac{(\mathrm{i}x)^2}{2!} + \frac{(\mathrm{i}x)^3}{3!} + \frac{(\mathrm{i}x)^4}{4!} + \cdots \\ &= 1 + \mathrm{i}x - \frac{x^2}{2!} - \frac{\mathrm{i}x^3}{3!} + \frac{x^4}{4!} + \cdots \\ &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \mathrm{i}\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots\right) \\ &= \cos x + \mathrm{i}\sin x \end{aligned}$$

We also see that

$$\frac{\mathrm{d}}{\mathrm{d}x}(\mathrm{e}^{\mathrm{i}x}) = \mathrm{i}\,\mathrm{e}^{\mathrm{i}x} = \mathrm{i}\cos x - \sin x$$

Often we shall represent a sine or cosine oscillation by the form $\mathrm{e}^{\mathrm{i}x}$ and recover the original form by taking that part of the solution preceded by i in the case of the sine, and the real part of the solution in the case of the cosine.

Examples

(1) In simple harmonic motion ($\ddot{x} + \omega^2 x = 0$) let us try the solution $x = a\,\mathrm{e}^{\mathrm{i}\omega t}\,\mathrm{e}^{\mathrm{i}\phi}$, where a is a constant length, and ϕ (and therefore $\mathrm{e}^{\mathrm{i}\phi}$) is a constant.

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \dot{x} = \mathrm{i}\omega a\,\mathrm{e}^{\mathrm{i}\omega t}\,\mathrm{e}^{\mathrm{i}\phi} = \mathrm{i}\omega x$$

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = \ddot{x} = \mathrm{i}^2\omega^2 a\,\mathrm{e}^{\mathrm{i}\omega t}\,\mathrm{e}^{\mathrm{i}\phi} = -\omega^2 x$$

Therefore

$$\begin{aligned} x &= a\,\mathrm{e}^{\mathrm{i}\omega t}\,\mathrm{e}^{\mathrm{i}\phi} = a\,\mathrm{e}^{\mathrm{i}(\omega t + \phi)} \\ &= a\cos(\omega t + \phi) + \mathrm{i}\,a\sin(\omega t + \phi) \end{aligned}$$

is a complete solution of $\ddot{x} + \omega^2 x = 0$.

On p. 6 we used the sine form of the solution; the cosine form is equally valid and merely involves an advance of $\pi/2$ in the phase ϕ.

(2)

$$\mathrm{e}^{\mathrm{i}x} + \mathrm{e}^{-\mathrm{i}x} = 2\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right) = 2\cos x$$
$$\mathrm{e}^{\mathrm{i}x} - \mathrm{e}^{-\mathrm{i}x} = 2\mathrm{i}\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right) = 2\mathrm{i}\sin x$$

(3) On p. 21 we used a geometrical method to show that the resultant of the superposed harmonic vibrations

$$a\cos\omega t + a\cos(\omega t + \delta) + a\cos(\omega t + 2\delta) + \cdots + a\cos(\omega t + [n-1]\delta)$$
$$= a\frac{\sin n\delta/2}{\sin\delta/2}\cos\left\{\omega t + \left(\frac{n-1}{2}\right)\delta\right\}$$

We can derive the same result using the complex exponential notation and *taking the real part* of the series expressed as the geometrical progression

$$a\,\mathrm{e}^{\mathrm{i}\omega t} + a\,\mathrm{e}^{\mathrm{i}(\omega t+\delta)} + a\,\mathrm{e}^{\mathrm{i}(\omega t+2\delta)} + \cdots + a\,\mathrm{e}^{\mathrm{i}[\omega t+(n-1)\delta]}$$
$$= a\,\mathrm{e}^{\mathrm{i}\omega t}(1 + z + z^2 + \cdots + z^{(n-1)})$$

where $z = \mathrm{e}^{\mathrm{i}\delta}$.

Writing

$$S(z) = 1 + z + z^2 + \cdots + z^{n-1}$$

and

$$z[S(z)] = z + z^2 + \cdots + z^n$$

we have

$$S(z) = \frac{1 - z^n}{1 - z} = \frac{1 - \mathrm{e}^{\mathrm{i}n\delta}}{1 - \mathrm{e}^{\mathrm{i}\delta}}$$

So

$$a\,\mathrm{e}^{\mathrm{i}\omega t}S(z) = a\,\mathrm{e}^{\mathrm{i}\omega t}\frac{1 - \mathrm{e}^{\mathrm{i}n\delta}}{1 - \mathrm{e}^{\mathrm{i}\delta}}$$
$$= a\,\mathrm{e}^{\mathrm{i}\omega t}\frac{\mathrm{e}^{\mathrm{i}n\delta/2}(\mathrm{e}^{-\mathrm{i}n\delta/2} - \mathrm{e}^{\mathrm{i}n\delta/2})}{\mathrm{e}^{\mathrm{i}\delta/2}(\mathrm{e}^{-\mathrm{i}\delta/2} - \mathrm{e}^{\mathrm{i}\delta/2})}$$
$$= a\,\mathrm{e}^{\mathrm{i}[\omega t + (\frac{n-1}{2})\delta]}\frac{\sin n\delta/2}{\sin\delta/2}$$

with the real part

$$= a\cos\left[\omega t + \left(\frac{n-1}{2}\right)\delta\right]\frac{\sin n\delta/2}{\sin\delta/2}$$

which recovers the original cosine term from the complex exponential notation.

(Problem 1.18)

(4) Suppose we represent a harmonic oscillation by the complex exponential form

$$z = a\,\mathrm{e}^{\mathrm{i}\omega t}$$

where a is the amplitude. Replacing i by $-\mathrm{i}$ defines the *complex conjugate*

$$z^* = a\,\mathrm{e}^{-\mathrm{i}\omega t}$$

The use of this conjugate is discussed more fully in Chapter 3 but here we can note that the product of a complex quantity and its conjugate is always equal to the square of the amplitude for

$$\begin{aligned} zz^* &= a^2\,\mathrm{e}^{\mathrm{i}\omega t}\,\mathrm{e}^{-\mathrm{i}\omega t} = a^2\,\mathrm{e}^{(\mathrm{i}-\mathrm{i})\omega t} = a^2\,\mathrm{e}^0 \\ &= a^2 \end{aligned}$$

(Problem 1.19)

Problem 1.1
The equation of motion

$$m\ddot{x} = -sx \quad \text{with} \quad \omega^2 = \frac{s}{m}$$

applies directly to the system in Figure 1.1(c).

If the pendulum bob of Figure 1.1(a) is displaced a small distance x show that the stiffness (restoring force per unit distance) is mg/l and that $\omega^2 = g/l$ where g is the acceleration due to gravity. Now use the small angular displacement θ instead of x and show that ω is the same.

In Figure 1.1(b) the angular oscillations are rotational so the mass is replaced by the moment of inertia I of the disc and the stiffness by the restoring couple of the wire which is C rad^{-1} of angular displacement. Show that $\omega^2 = C/I$.

In Figure 1.1(d) show that the stiffness is $2T/l$ and that $\omega^2 = 2T/lm$.

In Figure 1.1(e) show that the stiffness of the system in $2\rho Ag$, where A is the area of cross section and that $\omega^2 = 2g/l$ where g is the acceleration due to gravity.

In Figure 1.1(f) only the gas in the flask neck oscillates, behaving as a piston of mass ρAl. If the pressure changes are calculated from the equation of state use the adiabatic relation $pV^{\gamma} =$ constant and take logarithms to show that the pressure change in the flask is

$$\mathrm{d}p = -\gamma p \frac{\mathrm{d}V}{V} = -\gamma p \frac{Ax}{V},$$

where x is the gas displacement in the neck. Hence show that $\omega^2 = \gamma pA/l\rho V$. Note that γp is the stiffness of a gas (see Chapter 6).

In Figure 1.1(g), if the cross-sectional area of the neck is A and the hydrometer is a distance x above its normal floating level, the restoring force depends on the volume of liquid displaced (Archimedes' principle). Show that $\omega^2 = g\rho A/m$.

Check the dimensions of ω^2 for each case.

Problem 1.2

Show by the choice of appropriate values for A and B in equation (1.2) that equally valid solutions for x are

$$x = a\cos(\omega t + \phi)$$
$$x = a\sin(\omega t - \phi)$$
$$x = a\cos(\omega t - \phi)$$

and check that these solutions satisfy the equation

$$\ddot{x} + \omega^2 x = 0$$

Problem 1.3

The pendulum in Figure 1.1(a) swings with a displacement amplitude a. If its starting point from rest is

(a) $x = a$

(b) $x = -a$

find the different values of the phase constant ϕ for the solutions

$$x = a\sin(\omega t + \phi)$$
$$x = a\cos(\omega t + \phi)$$
$$x = a\sin(\omega t - \phi)$$
$$x = a\cos(\omega t - \phi)$$

For each of the different values of ϕ, find the values of ωt at which the pendulum swings through the positions

$$x = +a/\sqrt{2}$$
$$x = a/2$$

and

$$x = 0$$

for the first time after release from

$$x = \pm a$$

Problem 1.4

When the electron in a hydrogen atom bound to the nucleus moves a small distance from its equilibrium position, a restoring force per unit distance is given by

$$s = \mathrm{e}^2/4\pi\epsilon_0 r^2$$

where $r = 0.05$ nm may be taken as the radius of the atom. Show that the electron can oscillate with a simple harmonic motion with

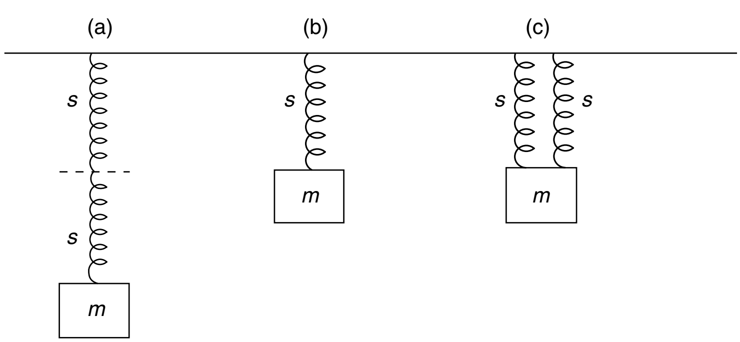
$$\omega_0 \approx 4.5 \times 10^{-16}\,\mathrm{rad\,s^{-1}}$$

If the electron is forced to vibrate at this frequency, in which region of the electromagnetic spectrum would its radiation be found?

$$e = 1.6 \times 10^{-19}\,\mathrm{C},\ \text{electron mass } m_{\mathrm{e}} = 9.1 \times 10^{-31}\,\mathrm{kg}$$
$$\epsilon_0 = 8.85 \times 10^{-12}\,\mathrm{N^{-1}\,m^{-2}\,C^2}$$

Problem 1.5

Show that the values of ω^2 for the three simple harmonic oscillations (a), (b), (c) in the diagram are in the ratio 1 : 2 : 4.



Problem 1.6

The displacement of a simple harmonic oscillator is given by

$$x = a \sin(\omega t + \phi)$$

If the oscillation started at time $t = 0$ from a position x_0 with a velocity $\dot{x} = v_0$ show that

$$\tan\phi = \omega x_0/v_0$$

and

$$a = (x_0^2 + v_0^2/\omega^2)^{1/2}$$

Problem 1.7

A particle oscillates with simple harmonic motion along the x axis with a displacement amplitude a and spends a time $\mathrm{d}t$ in moving from x to $x + \mathrm{d}x$. Show that the probability of finding it between x and $x + \mathrm{d}x$ is given by

$$\frac{\mathrm{d}x}{\pi(a^2 - x^2)^{1/2}}$$

(in wave mechanics such a probability is not zero for $x > a$).

Problem. 1.8

Many identical simple harmonic oscillators are equally spaced along the x axis of a medium and a photograph shows that the locus of their displacements in the y direction is a sine curve. If the distance λ separates oscillators which differ in phase by 2π radians, what is the phase difference between two oscillators a distance x apart?

Problem 1.9

A mass stands on a platform which vibrates simple harmonically in a vertical direction at a frequency of 5 Hz. Show that the mass loses contact with the platform when the displacement exceeds 10^{-2}m.

Problem 1.10

A mass M is suspended at the end of a spring of length l and stiffness s. If the mass of the spring is m and the velocity of an element $\mathrm{d}y$ of its length is proportional to its distance y from the fixed end of the spring, show that the kinetic energy of this element is

$$\frac{1}{2}\left(\frac{m}{l}\mathrm{d}y\right)\left(\frac{y}{l}v\right)^2$$

where v is the velocity of the suspended mass M. Hence, by integrating over the length of the spring, show that its total kinetic energy is $\frac{1}{6}mv^2$ and, from the total energy of the oscillating system, show that the frequency of oscillation is given by

$$\omega^2 = \frac{s}{M + m/3}$$

Problem 1.11

The general form for the energy of a simple harmonic oscillator is

$$E = \tfrac{1}{2}\text{mass (velocity)}^2 + \tfrac{1}{2}\text{stiffness (displacement)}^2$$

Set up the energy equations for the oscillators in Figure 1.1(a), (b), (c), (d), (e), (f) and (g), and use the expression

$$\frac{\mathrm{d}E}{\mathrm{d}t} = 0$$

to derive the equation of motion in each case.

Problem 1.12

The displacement of a simple harmonic oscillator is given by $x = a \sin \omega t$. If the values of the displacement x and the velocity $\dot{x}$ are plotted on perpendicular axes, eliminate t to show that the locus of the points $(x, \dot{x})$ is an ellipse. Show that this ellipse represents a path of constant energy.

Problem 1.13

In Chapter 12 the intensity of the pattern when light from two slits interferes (Young's experiment) will be seen to depend on the superposition of two simple harmonic oscillations of equal amplitude a and phase difference δ. Show that the intensity

$$I = R^2 \propto 4a^2 \cos^2 \delta/2$$

Between what values does the intensity vary?

Problem 1.14

Carry out the process indicated in the text to derive equation (1.3) on p. 15.

Problem 1.15

The co-ordinates of the displacement of a particle of mass m are given by

$$x = a \sin \omega t$$
$$y = b \cos \omega t$$

Eliminate t to show that the particle follows an elliptical path and show by adding its kinetic and potential energy at any position x, y that the ellipse is a path of constant energy equal to the sum of the separate energies of the simple harmonic vibrations.

Prove that the quantity $m(x\dot{y} - y\dot{x})$ is also constant. What does this quantity represent?

Problem 1.16

Two simple harmonic motions of the same frequency vibrate in directions perpendicular to each other along the x and y axes. A phase difference

$$\delta = \phi_2 - \phi_1$$

exists between them such that the principal axes of the resulting elliptical trace are inclined at an angle to the x and y axes. Show that the measurement of two separate values of x (or y) is sufficient to determine the phase difference.
(Hint: use equation (1.3) and measure y(max), and y for $(x = 0.)$

Problem 1.17

Take a random group of $n > 7$ values of ϕ in the range $0 \leq \phi \leq \pi$ and form the product

$$\sum_{\substack{i=1 \\ i \neq j}}^{n} \cos \phi_i \sum_{j=1}^{n} \cos \phi_j$$

Show that the average value obtained for several such groups is negligible with respect to $n/2$.

Problem 1.18

Use the method of example (3) (p. 28) to show that

$$a\sin\omega t + a\sin(\omega t+\delta) + a\sin(\omega t+2\delta) + \cdots + a\sin[\omega t+(n-1)\delta] = a\sin\left[\omega t + \frac{(n-1)}{2}\delta\right]\frac{\sin n\delta/2}{\sin\delta/2}$$

Problem 1.19

If we represent the sum of the series

$$a\cos\omega t + a\cos(\omega t+\delta) + a\cos(\omega t+2\delta) + \cdots + a\cos[\omega t+(n-1)\delta]$$

by the complex exponential form

$$z = a\,\mathrm{e}^{\mathrm{i}\omega t}(1 + \mathrm{e}^{\mathrm{i}\delta} + \mathrm{e}^{\mathrm{i}2\delta} + \cdots + \mathrm{e}^{\mathrm{i}(n-1)\delta})$$

show that

$$zz^* = a^2\frac{\sin^2 n\delta/2}{\sin^2\delta/2}$$

Summary of Important Results

Simple Harmonic Oscillator (mass m, stiffness s, amplitude a)
Equation of motion $\ddot{x} + \omega^2 x = 0$ where $\omega^2 = s/m$
Displacement $x = a\sin(\omega t + \phi)$
Energy $= \frac{1}{2}m\dot{x}^2 + \frac{1}{2}sx^2 = \frac{1}{2}m\omega^2 a^2 = \frac{1}{2}sa^2 =$ constant

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Superposition (Amplitude and Phase) of two SHMs
One-dimensional

Equal ω, different amplitudes, phase difference δ, resultant R where $R^2 = a_1^2 + a_2^2 + 2a_1a_2\cos\delta$
Different ω, equal amplitude,

$$x = x_1 + x_2 = a(\sin\omega_1 t + \sin\omega_2 t) = 2a\sin\frac{(\omega_1+\omega_2)t}{2}\cos\frac{(\omega_2-\omega_1)t}{2}$$

Two-dimensional: perpendicular axes
Equal ω, different amplitude—giving general conic section

$$\frac{x^2}{a_1^2} + \frac{y^2}{a_2^2} - \frac{2xy}{a_1a_2}\cos(\phi_2-\phi_1) = \sin^2(\phi_2-\phi_1)$$

(basis of optical polarization)

Superposition of n SHM Vectors (equal amplitude a , constant successive phase difference δ)

The resultant is $R\cos(\omega t + \alpha)$, where

$$R = a\frac{\sin n\delta/2}{\sin\delta/2}$$

and

$$\alpha = (n-1)\delta/2$$

Important in optical diffraction and wave groups of many components.