Cubes and Cube Roots

Pattern Formed By Perfect Cubes

A perfect cube is a number that can be obtained by multiplying a number with itself three times.

For example, 1 × 1 × 1 = 1, 2 × 2 × 2 = 8, 3 × 3 × 3 = 27, etc. are all examples of perfect cubes.

There are some very interesting patterns exhibited by perfect cubes. One of them deals with the difference between two consecutive perfect cubes.

Similarly, another pattern followed by the sum of consecutive perfect cubes is

$$1^3 = \left(\frac{1\times 2}{2}\right)^2$$

$$1^3 + 2^3 = \left(\frac{2\times 3}{2}\right)^2$$

$$1^3 + 2^3 + 3^3 = \left(\frac{3\times 4}{2}\right)^2$$

$$1^3 + 2^3 + 3^3 + 4^3 = \left(\frac{4\times 5}{2}\right)^2$$

$$1^3 + 2^3 + 3^3 + 4^3 + 5^3 = \left(\frac{5\times 6}{2}\right)^2$$

Using this pattern, we can find the value of large expressions like $1^3 + 2^3 + 3^3 + 4^3 + \ldots + 20^3$ as

$$1^3 + 2^3 + 3^3 + 4^3 + \ldots + 20^3 = \left(\frac{20\times 21}{2}\right)^2 = (210)^2 = 44100$$

Let us discuss one more example based on the properties exhibited by perfect cubes.

Example 1:

Look at the following pattern.

$1^3 = 1$

$11^3 = 1331$

$101^3 = 1030301$

$1001^3 = 1003003001$

Using this pattern, find the value of 10001^3 and 100001^3.

Solution:

Using the given pattern, we can write

$10001^3 = 1000300030001$

$100001^3 = 1000030000300001$

Identification of Perfect Cubes by Prime Factorisation

The cube of an integer is the number obtained on multiplying that integer with itself three times.

Also, the cube of an integer is written with the index 3 taking the integer as the base.

For example, cube of 2 is written as 2^3.

Thus, $2^3 = 2 \times 2 \times 2 = = 4 \times 2 = 8$

Similarly, cubes of few more integers are as follows:

$(-3)^3 = (-3) \times (-3) \times (-3) = 9 \times (-3) = -27$

$11^3 = 11 \times 11 \times 11 = 121 \times 11 = 1331$

$(-15)^3 = (-15) \times (-15) \times (-15) = 225 \times (-15) = -3375$

$20^3 = 20 \times 20 \times 20 = 400 \times 20 = 8000$

From the above examples, it can be observed that:

- **The cube of a positive number is always positive.**
- **The cube of a negative number is always negative.**

The cube of an integer is known as perfect cube.

In the above examples, 8, –27, 1331, –3375 and 8000 all are perfect cubes or cube numbers.

The following table lists the cubes of natural numbers from 1 to 20.

Number	**Cube of the Number**	**Number**	**Cube of the Number**
1	1	11	1331
2	8	12	1728
3	27	13	2197
4	64	14	2744
5	125	15	3375
6	216	16	4096
7	343	17	4913
8	512	18	5832
9	729	19	6859
10	1000	20	8000

We can thus define a perfect cube as follows:

A number is said to be a perfect cube or a cube number, if it is obtained by multiplying a natural number with itself three times.

From the above calculations, we can see that 64 and 125 are the cubes of two consecutive natural numbers (4 and 5 respectively). We know that there is no natural number between 4 and 5. Therefore, any number between 64 and 125, say 100, 73, etc., are not perfect cubes.

This method can also be used to identify whether a given number is a perfect cube or not. Let us take the example of the number 1224. Now, we know that $10^3 = 1000$ and $11^3 = 1331$. Now, the given number 1224 lies between two consecutive perfect cubes 1000 and 1331. Thus, it cannot be a perfect cube. However, this is not a very convenient method to use, especially in the case of large numbers.

Therefore, let us go through the following video to understand the method used for identifying a perfect cube.

Let us discuss one more example based on this property.

Example 1:

Check whether the numbers 5832 and 6400 are perfect cubes or not.

Solution:

The number 5832 can be expressed as a product of its prime factors as

$5832 = \underline{2 \times 2 \times 2} \times \underline{3 \times 3 \times 3} \times \underline{3 \times 3 \times 3}$

Here, each of the prime factors occurs in groups of three.

Hence, 5832 is a perfect cube.

The number 6400 can be expressed as a product of its prime factors as

$6400 = \underline{2 \times 2 \times 2} \times \underline{2 \times 2 \times 2} \times 2 \times 2 \times 5 \times 5$

Here, the prime factors 2 and 3 do not occur in groups of three.

Hence, 6400 is **not** a perfect cube.

Finding the Smallest Number by Which a Non-perfect Cube Must be Divided by or Multiplied with to Convert It into a Perfect Cube

We use the method of prime factorization to identify a perfect cube. We can also use the same method to find the smallest number with which the given number must be multiplied with or divided by to obtain a perfect cube.

Let us understand this by taking an example of the number 1800.

We first have to prime factorize the number 1800, which can be done as

$$\begin{array}{r|l} 2 & 1800 \\ \hline 2 & 900 \\ \hline 2 & 450 \\ \hline 3 & 225 \\ \hline 3 & 75 \\ \hline 5 & 25 \\ \hline 5 & 5 \\ \hline & 1 \end{array}$$

Now, the number 1800 can be written as a product of its prime factors as

$$1800 = \underline{2\times 2\times 2}\times \underline{3\times 3}\times \underline{5\times 5}$$

Here, the prime factor 2 appears in a group of three whereas the prime factors 3 and 5 do not appear in a group of three (each of them appears twice).

For a number to be a perfect cube, each of its prime factors should occur in groups of three. Thus, we have to make the prime factors 3 and 5 occur in groups of three. For this, we are required to multiply the number by 3 × 5 = 15. The resulting number is 1800 × 15 = 27000, which is a perfect cube ($27000 = 30^3$).

We could also have divided the number 1800 by 3 × 3 × 5 × 5 = 225. The resulting number is 1800 ÷ 225 = 8, which is a perfect cube ($8 = 2^3$).

Let us discuss one more example based on this aspect.

Example 1:

Find the smallest number that has to be multiplied with 91476 to make a perfect cube. Also find the smallest number that has to be divided by 91476 to make a perfect cube.

Solution:

We first have to prime factorize 91476, which can be done as

$$\begin{array}{r|l} 2 & 91476 \\ 2 & 45738 \\ 3 & 22869 \\ 3 & 7623 \\ 3 & 2541 \\ 7 & 847 \\ 11 & 121 \\ 11 & 11 \\ & 1 \end{array}$$

$91476 = \underline{3 \times 3 \times 3} \times \underline{2 \times 2} \times \underline{11 \times 11} \times \underline{7}$

Here, only prime factor 3 appears in a group of three. The prime factors 2, 11 and 7 appear twice, twice and once respectively.

Thus, we have to multiply 91476 with 2 × 11 × 7 × 7, which gives $\underline{3 \times 3 \times 3} \times \underline{2 \times 2 \times 2} \times \underline{11 \times 11 \times 11} \times \underline{7 \times 7 \times 7}$, which is a perfect cube.

2 × 11 × 7 × 7 = 1078

Therefore, 1078 is the smallest number that should be multiplied with 91476 to obtain a perfect cube.

Since each group of 2, 11, and 7 is incomplete, we can remove these groups and still end up with a perfect cube. Therefore, we have to divide 91476 by 2 × 2 × 11 × 11 × 7, which gives us 3 × 3 × 3, which is a perfect cube.

2 × 2 × 11 × 11 × 7 = 3388

Therefore, 3388 is the smallest number that divides 91476 and gives a perfect cube.

Relation between the Units digit of a Number and that of its Cube

The units digit of a number and that of its cube exhibit a particular relationship. Let us try and look into it.

Let us begin by finding the cubes of some numbers ending with 1.

$1^3 = 1$

$11^3 = 1331$

$21^3 = 9261$

Did you observe something here? Note that the cube of each of the numbers also ends with the digit 1. In fact, you can check this for any number ending with 1.

Similarly, if a number ends with 2, then its cube ends with 8. We can, in fact, note down this property for all the digits from 0 to 9 as

If a number has any of the digits 0, 1, 4, 5, 6, and 9 at its units place, then its cube also ends with the same digit.

If a number ends with 2, then its cube ends with 8, and vice-versa.

If a number ends with 3, then its cube ends with 7, and vice-versa.

This property is true for every number and its cube. It can be verified by considering any number.

Let us discuss one more example based on this.

Example 1:

Without directly multiplying, calculate the unit place digit of the cube of 109.

Solution:

The unit place digit of 109 is 9. But we know that if unit place digit of a number is 9, then the unit place digit of its cube is also 9. Therefore, the unit place digit of the cube of 109 is 9.

Prime Factorisation Method of Finding Cube Roots

We find the cube of an integer by multiplying the same integer with itself three times. For example, we can find the cube of 6 as $6^3 = 6 \times 6 \times 6 = 216$

We can write this expression, i.e. "216 is the cube of 6", in another way as "cube root of 216 is 6". Mathematically, we can express it as $\sqrt[3]{216} = 6$.

Here, the symbol $\sqrt[3]{\ }$ denotes the cube root.

Similarly,

1. $8^3 = 512 \Rightarrow \sqrt[3]{512} = 8$
2. $15^3 = 3375 \Rightarrow \sqrt[3]{3375} = 15$
3. $(-7)^3 = -343 \Rightarrow \sqrt[3]{-343} = -7$
4. $(-12)^3 = -1728 \Rightarrow \sqrt[3]{-1728} = -12$

Hence, we can say that finding the cube root is the inverse operation of finding the cube.

The cube root of a perfect cube can also be found by the method of prime factorization. Let us find the cube root of 21952 by this method.

Now, let us find the cube root of –74088 by the method of prime factorization.

For the same, we will first find the cube root of 74088.

74088 can be factorized as follows:

2	74088
2	37044
2	18522
3	9261
3	3087
3	1029
7	343
7	49
7	7
	1

$\therefore 74088 = (2 \times 2 \times 2) \times (3 \times 3 \times 3) \times (7 \times 7 \times 7)$

$\Rightarrow 74088 = (2 \times 3 \times 7)^3$

$\Rightarrow \sqrt[3]{74088} = 2 \times 3 \times 7$

$\Rightarrow \sqrt[3]{74088} = 42$

$\therefore \sqrt[3]{-74088} = -42$

Let us discuss more examples based on the cube root of a perfect cube by the method of prime factorization.

Example 1:

Find the cube root of 287496 by prime factorization method.

Solution:

The prime factorization of 287496 can be done as follows:

2	287496
2	143748
2	71874
3	35937
3	11979
3	3993
11	1331
11	121
11	11
	1

Thus, the number 287496 can be expressed as a product of its prime factors as

$287496 = (2 \times 2 \times 2) \times (3 \times 3 \times 3) \times (11 \times 11 \times 11)$

$\Rightarrow 287496 = (2 \times 3 \times 11)^3$

$\Rightarrow \sqrt[3]{287496} = 2 \times 3 \times 11$

$\Rightarrow \sqrt[3]{287496} = 66$

Example 2:

Find the cube root of –1157625 by prime factorization method.

Solution:

To find the cube root of –1157625, we will first find the cube root of 1157625.

The prime factorization of 1157625 can be done as follows:

3	1157625
3	385875
3	128625
5	42875
5	8575
5	1715
7	343
7	49
7	7
	1

$\therefore 1157625 = (3 \times 3 \times 3) \times (5 \times 5 \times 5) \times (7 \times 7 \times 7)$

$\Rightarrow 1157625 = (3 \times 5 \times 7)^3$

$\Rightarrow \sqrt[3]{1157625} = 3 \times 5 \times 7$

$\Rightarrow \sqrt[3]{1157625} = 105$

$\therefore \sqrt[3]{-1157625} = -105$

Finding Cube Root Of A Perfect Cube By Estimation

One way of finding the cube root of a number is the method of prime factorization. However, this method is not always convenient. For example, try to find the cube root of the number 704969 by prime factorizing it. Try dividing it by all prime numbers less than 20. You will see that no number divides 704969. It is actually the perfect cube of the prime number 89. We have already taken a lot of time to check the prime factor of 704969 up till 20 and have not been able to solve it. 89 is a large prime number.

In order to overcome such situations, we follow another method of finding the cube root of a perfect cube. This method is known as estimation method or grouping method.

Example 1:

Find the cube root of 97336 through estimation.

Solution:

To find the cube root of 97336 through estimation, first of all we have to make groups of three digits starting from the rightmost digit. By doing so, we will have two groups of numbers as

97	336
↓	↓
Second Group	First Group

Here, the first group is 336. This number ends with 6. We know that 6 comes at the unit's place of a number only when its cube root ends in 6. Therefore, the unit place digit of the cube root of 97336 is 6.

The second group is 97. We know that $4^3 = 64$ and $5^3 = 125$. Also, $64 < 97 < 125$. To find the tens place of the cube root, we have to take the cube root of the smaller number (64) among 64, 97, and 125. Now, $\sqrt[3]{64} = 4$. Therefore, the tens place digit of the cube root of 97336 is 4.

Therefore, $\sqrt[3]{97336} = 46$.