Chapter
Relations and Functions

Topic-1: Types of Relations, Inverse of a Relation, Mappings, Mapping of Functions, Kinds of Mapping of Functions

1 MCQs with One Correct Answer

1. If the function $f:\mathbb{R}\to\mathbb{R}$ is defined by $f(x)=|x|(x-\sin x)$, then which of the following statements is **TRUE**? **[Adv. 2020]**
   (a) f is one-one, but **NOT** onto
   (b) f is onto, but **NOT** one-one
   (c) f is **BOTH** one-one and onto
   (d) f is **NEITHER** one-one **NOR** onto
2. The function $f:[0,3]\to[1,29]$, defined by $f(x)=2x^3-15x^2+36x+1$, is **[2012]**
   (a) one–one and onto (b) onto but not one–one
   (c) one–one but not onto (d) neither one–one nor onto
3. Let f, g and h be real-valued functions defined on the interval [0, 1] by $f(x)=e^{x^2}+e^{-x^2}, g(x)=xe^{x^2}+e^{-x^2}$ and $h(x)=x^2e^{x^2}+e^{-x^2}$. If a, b and c denote, respectively, the absolute maximum of f, g and h on [0, 1], then **[2010]**
   (a) $a=b$ and $c\neq b$ (b) $a=c$ and $a\neq b$
   (c) $a\neq b$ and $c\neq b$ (d) $a=b=c$
4. If the functions f(x) and g(x) are defined on R → R such that
   $$f(x)=\begin{cases}0, & x\in\text{rational}\\ x, & x\in\text{irrational}\end{cases};\ g(x)=\begin{cases}0, & x\in\text{irrational}\\ x, & x\in\text{rational}\end{cases}$$
   then $(f-g)(x)$ is **[2005S]**
   (a) one-one & onto
   (b) neither one-one nor onto
   (c) one-one but not onto
   (d) onto but not one-one
5. If $f:[0,\infty)\to[0,\infty)$ and $f(x)=\dfrac{x}{1+x}$, then f is **[2003S]**
   (a) one-one and onto (b) one-one but not onto
   (c) onto but not one-one (d) neither one-one nor onto
6. Let function $f:R\to R$ be defined by $f(x)=2x+\sin x$ for $x\in R$, then f is **[2002S]**
   (a) one-to-one and onto
   (b) one-to-one but NOT onto
   (c) onto but NOT one-to-one
   (d) neither one-to-one nor onto
7. Suppose $f(x)=(x+1)^2$ for $x\geq-1$. If $g(x)$ is the function whose graph is the reflection of the graph of $f(x)$ with respect to the line $y=x$, then $g(x)$ equals **[2002S]**
   (a) $-\sqrt{x}-1, x\geq 0$ (b) $\dfrac{1}{(x+1)^2}, x>-1$
   (c) $\sqrt{x+1}, x\geq-1$ (d) $\sqrt{x}-1, x\geq 0$
8. Let $E=\{1,2,3,4\}$ and $F=\{1,2\}$. Then the number of onto functions from E to F is **[2001S]**
   (a) 14 (b) 16 (c) 12 (d) 8
9. The domain of definition of the function $f(x)$ is given by the equation $2^x+2^y=2$ is **[2000S]**
   (a) $0<x\leq 1$ (b) $0\leq x\leq 1$
   (c) $-\infty<x\leq 0$ (d) $-\infty<x<1$
10. Let $f:R\to R$ be any function. Define $g:R\to R$ by $g(x)=|f(x)|$ for all x. Then g is **[2000S]**
    (a) onto if f is onto
    (b) one-one if f is one-one
    (c) continuous if f is continuous
    (d) differentiable if f is differentiable.
11. Let $f(x)$ be defined for all $x>0$ and be continuous. Let $f(x)$ satisfy $f\left(\dfrac{x}{y}\right)=f(x)-f(y)$ for all x, y and $f(e)=1$. Then **[1995S]**
    (a) $f(x)$ is bounded (b) $f\left(\dfrac{1}{x}\right)\to 0$ as $x\to 0$
    (c) $xf(x)\to 1$ as $x\to 0$ (d) $f(x)=\ln x$

12. The function $f(x) = |px - q| + r|x|,\ x \in (-\infty, \infty)$ where $p > 0, q > 0, r > 0$ assumes its minimum value only on one point if **[1995]**
(a) $p \neq q$ (b) $r \neq q$
(c) $r \neq p$ (d) $p = q = r$

13. Which of the following functions is periodic? **[1983 - 1 Mark]**
(a) $f(x) = x - [x]$ where [x] denotes the largest integer less than or equal to the real number x
(b) $f(x) = \sin\frac{1}{x}$ for $x \neq 0$, $f(0) = 0$
(c) $f(x) = x \cos x$
(d) none of these

14. If x satisfies $|x-1| + |x-2| + |x-3| \geq 6$, then **[1983 - 1 Mark]**
(a) $0 \leq x \leq 4$ (b) $x \leq -2$ or $x \geq 4$
(c) $x \leq 0$ or $x \geq 4$ (d) None of these

15. Let $f(x) = |x - 1|$. Then **[1983 - 1 Mark]**
(a) $f(x^2) = (f(x))^2$ (b) $f(x + y) = f(x) + f(y)$
(c) $f(|x|) = |f(x)|$ (d) None of these

16. The entire graphs of the equation $y = x^2 + kx - x + 9$ is strictly above the x-axis if and only if **[1979]**
(a) $k < 7$ (b) $-5 < k < 7$
(c) $k > -5$ (d) None of these.

17. Let R be the set of real numbers. If $f: R \to R$ is a function defined by $f(x) = x^2$, then f is : **[1979]**
(a) Injective but not surjective
(b) Surjective but not injective
(c) Bijective
(d) None of these.

3 Numeric/ New Stem Based Questions

18. Let X be a set with exactly 5 elements and Y be a set with exactly 7 elements. If α is the number of one-one functions from X to Y and β is the number of onto functions from Y to X, then the value of $\frac{1}{5!}(\beta - \alpha)$ is ______. **[Adv. 2018]**

4 Fill in the Blanks

19. If f is an even function defined on the interval (-5, 5), then four real values of x satisfying the equation $f(x) = f\left(\frac{x+1}{x+2}\right)$ are ...., ..., ..., and .... **[1996 - 1 Mark]**

20. There are exactly two distinct linear functions, ..., and .... which map [– 1, 1] onto [0, 2]. **[1989 - 2 Marks]**

21. Let A be a set of n distinct elements. Then the total number of distinct functions from A to A is .... and out of these .... are onto functions. **[1985 - 2 Marks]**

5 True / False

22. If $f_1(x)$ and $f_2(x)$ are defined on domains D_1 and D_2 respectively, then $f_1(x) + f_2(x)$ is defined on $D_1 \cup D_2$. **[1988 - 1 Mark]**

23. The function $f(x) = \frac{x^2 + 4x + 30}{x^2 - 8x + 18}$ is not one -to -one. **[1983 - 1 Mark]**

24. For real numbers x and y, we write $x * y$ if $x - y + \sqrt{2}$ is an irrational number. Then, the relation* is an equivalence relation. **[1981 - 2 Marks]**

6 MCQs with One or More than One Correct Answer

25. Let $a \in R$ and let $f: R \to R$ be given by $f(x) = x^5 - 5x + a$. Then **[Adv. 2014]**
(a) $f(x)$ has three real roots if $a > 4$
(b) $f(x)$ has only real root if $a > 4$
(c) $f(x)$ has three real roots if $a < -4$
(d) $f(x)$ has three real roots if $-4 < a < 4$

26. The function $f(x) = 2|x| + |x + 2| - ||x + 2| - 2|x||$ has a local minimum or a local maximum at $x =$ **[Adv. 2013]**
(a) -2 (b) $\frac{-2}{3}$ (c) 2 (d) $\frac{2}{3}$

27. Let $f: (-1, 1) \to IR$ be such that $f(\cos 4\theta) = \frac{2}{2 - \sec^2\theta}$ for $\theta \in \left(0, \frac{\pi}{4}\right) \cup \left(\frac{\pi}{4}, \frac{\pi}{2}\right)$. Then the value (s) of $f\left(\frac{1}{3}\right)$ is (are) **[Adv. 2012]**
(a) $1 - \sqrt{\frac{3}{2}}$ (b) $1 + \sqrt{\frac{3}{2}}$ (c) $1 - \sqrt{\frac{2}{3}}$ (d) $1 + \sqrt{\frac{2}{3}}$

28. If $f(x) = \cos[\pi^2]x + \cos[-\pi^2]x$, where [x] stands for the greatest integer function, then **[1991 - 2 Marks]**
(a) $f\left(\frac{\pi}{2}\right) = -1$ (b) $f(\pi) = 1$
(c) $f(-\pi) = 0$ (d) $f\left(\frac{\pi}{4}\right) = 1$

29. Let $g(x)$ be a function defined on [– 1, 1]. If the area of the equilateral triangle with two of its vertices at (0,0) and $[x, g(x)]$ is $\frac{\sqrt{3}}{4}$, then the function $g(x)$ is **[1989 - 2 Marks]**
(a) $g(x) = \pm\sqrt{1 - x^2}$ (b) $g(x) = \sqrt{1 - x^2}$
(c) $g(x) = -\sqrt{1 - x^2}$ (d) $g(x) = \sqrt{1 + x^2}$

30. If $y = f(x) = \frac{x+2}{x-1}$ then **[1984 - 3 Marks]**
(a) $x = f(y)$ (b) $f(1) = 3$
(c) y increases with x for $x < 1$
(d) f is a rational function of x

7 Match the Following

31. Match the statements given in Column-I with the intervals/union of intervals given in Column-II. [2011]

Column-I	Column-II
(A) The set $\{\text{Re}\left(\frac{2iz}{1-z^2}\right)$: z is a complex number, $|z|=1\, z \neq \pm 1\}$ is	(p) $(-\infty,-1)\cup(1,\infty)$
(B) The domain of the function $f(x)=\sin^{-1}\left(\frac{8(3)^{x-2}}{1-3^{2(x-1)}}\right)$ is	(q) $(-\infty,-0)\cup(0,\infty)$
(C) If $f(\theta)=\begin{vmatrix} 1 & \tan\theta & 1 \\ -\tan\theta & 1 & \tan\theta \\ -1 & -\tan\theta & 1 \end{vmatrix}$, then the set $\left\{f(\theta): 0\le\theta<\frac{\pi}{2}\right\}$ is	(r) $[2,\infty)$
	(s) $(-\infty,-1]\cup[1,\infty)$
(D) If $f(x)=x^{3/2}$ $(3x-10), x\ge 0$ then $f(x)$ is increasing in	(t) $(-\infty,0]\cup[2,\infty)$ 1

32. Let $f(x)=\frac{x^2-6x+5}{x^2-5x+6}$

Match of expressions/statements in Column I with expressions/statements in Column II and indicate your answer by darkening the appropriate bubbles in the 4 × 4 matrix given in the ORS. [2007 -6 marks]

Column I	Column II
(A) If $-1<x<1$, then $f(x)$ satisfies	(p) $0<f(x)<1$
(B) If $1<x<2$, then $f(x)$ satisfies	(q) $f(x)<0$
(C) If $3<x<5$, then $f(x)$ satisfies	(r) $f(x)>0$
(D) If $x>5$, then $f(x)$ satisfies	(s) $f(x)<1$

33. Let the function defined in column 1 have domain $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$ and range $(-\infty,\infty)$ [1992 - 2 Marks]

Column I	Column II
(A) $1+2x$	(p) onto but not one-one
(B) $\tan x$	(q) one- one but not onto
	(r) one- one and onto
	(s) neither one-one nor onto

8 Comprehension/Passage Based Questions

PARAGRAPH "I"

Let S = {1, 2, 3, 4, 5, 6} and X be the set of all relations R from S to S that satisfy both the following properties:

(i) R has exactly 6 elements.

(ii) For each (a, b) ∈ R, we have $|a-b|\ge 2$.

Let Y = {R ∈ X : The range of R has exactly one element} and

Z = {R ∈ X : R is a function from S to S}.

Let $n(A)$ denote the number of elements in a set A.

34. If $n(X)={}^{m}C_6$, then the value of m is ________. [Adv. 2024]

35. If the value of $n(Y)+n(Z)$ is k^2, then $|k|$ is ______. [Adv. 2024]

10 Subjective Problems

36. Let $f(x)=Ax^2+Bx+C$ where A, B, C are real numbers. Prove that if $f(x)$ is an integer whenever x is an integer, then the numbers $2A$, $A+B$ and C are all integers. Conversely, prove that if the numbers $2A$, $A+B$ and C are all integers then $f(x)$ is an integer whenever x is an integer. [1998 - 8 Marks]

37. A function $f: IR \to IR$, where IR is the set of real numbers, is defined by $f(x)=\frac{\alpha x^2+6x-8}{\alpha+6x-8x^2}$. Find the interval of values of α for which f is onto. Is the function one-to-one for $\alpha=3$? Justify your answer. [1996 - 5 Marks]

38. Let $\{x\}$ and $[x]$ denotes the fractional and integral part of a real number x respectively. Solve $4\{x\}=x+[x]$. [1994 - 4 Marks]

39. Find the natural number 'a' for which $\sum_{k=1}^{n} f(a+k)=16(2^n-1)$, where the function '$f$' satisfies the relation $f(x+y)=f(x)f(y)$ for all natural numbers x, y and further $f(1)=2$. [1992 - 6 Marks]

40. Let R be the set of real numbers and $f: R\to R$ be such that for all x and y in R $|f(x)-f(y)|\le|x-y|^3$. Prove that $f(x)$ is a constant. [1988 - 2 Marks]

41. A relation R on the set of complex numbers is defined by $z_1 R z_2$ if and only if $\frac{z_1-z_2}{z_1+z_2}$ is real. Show that R is an equivalence relation. [1982 - 2 Marks]

42. Let A and B be two sets each with a finite number of elements. Assume that there is an injective mapping from A to B and that there is an injective mapping from B to A. Prove that there is a bijective mapping from A to B. [1981 - 2 Marks]

43. Consider the following relations in the set of real numbers R.

$R = \{(x, y); x \in R, y \in R, x^2 + y^2 \le 25\}$

$R' = \left\{(x, y): x \in R, y \in R, y \ge \frac{4}{9}x^2\right\}$

Find the domain and range of $R \cap R'$. Is the relation $R \cap R'$ a function? **[1979]**

44. If $f(x) = x^9 - 6x^8 - 2x^7 + 12x^6 + x^4 - 7x^3 + 6x^2 + x - 3$, find $f(6)$. **[1979]**

45. Draw the graph of $y = |x|^{1/2}$ for $-1 \le x \le 1$. **[1978]**

46. Find the domain and range of the function $f(x) = \frac{x^2}{1+x^2}$. Is the function one-to-one? **[1978]**

Topic-2: Composite Functions & Relations, Inverse of a Function, Binary Operations

1 MCQs with One Correct Answer

1. Let $f(x) = x^2$ and $g(x) = \sin x$ for all $x \in R$. Then the set of all x satisfying $(f \circ g \circ g \circ f)(x) = (g \circ g \circ f)(x)$, where $(f \circ g)(x) = f(g(x))$, is **[Adv. 2011]**
   (a) $\pm\sqrt{n\pi}, n \in \{0, 1, 2, ...\}$
   (b) $\pm\sqrt{n\pi}, n \in \{1, 2, ...\}$
   (c) $\frac{\pi}{2} + 2n\pi, n \in \{.... -2, -1, 0, 1, 2....\}$
   (d) $2n\pi, n \in \{... -2, -1, 0, 1, 2, ...\}$

2. X and Y are two sets and $f: X \to Y$. If $\{f(c) = y; c \subset X, y \subset Y\}$ and $\{f^{-1}(d) = x; d \subset Y, x \subset X\}$, then the true statement is **[2005S]**
   (a) $f(f^{-1}(b)) = b$ (b) $f^{-1}(f(a)) = a$
   (c) $f(f^{-1}(b)) = b, b \subset y$ (d) $f^{-1}(f(a)) = a, a \subset x$

3. If $f(x) = \sin x + \cos x$, $g(x) = x^2 - 1$, then $g(f(x))$ is invertible in the domain **[2004S]**
   (a) $\left[0, \frac{\pi}{2}\right]$ (b) $\left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$
   (c) $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ (d) $[0, \pi]$

4. If $f(x) = x^2 + 2bx + 2c^2$ *and* $g(x) = -x^2 - 2cx + b^2$ such that $\min f(x) > \max g(x)$, then the relation between b and c, is **[2003S]**
   (a) no real value of b & c (b) $0 < c < b\sqrt{2}$
   (c) $|c| < |b|\sqrt{2}$ (d) $|c| > |b|\sqrt{2}$

5. Domain of definition of the function $f(x) = \sqrt{\sin^{-1}(2x) + \frac{\pi}{6}}$ for real valued x, is **[2003S]**
   (a) $\left[-\frac{1}{4}, \frac{1}{2}\right]$ (b) $\left[-\frac{1}{2}, \frac{1}{2}\right]$
   (c) $\left(-\frac{1}{2}, \frac{1}{9}\right)$ (d) $\left[-\frac{1}{4}, \frac{1}{4}\right]$

6. Let $f(x) = \frac{\alpha x}{x+1}, x \ne -1$. Then, for what value of α is $f(f(x)) = x$? **[2001S]**
   (a) $\sqrt{2}$ (b) $-\sqrt{2}$
   (c) 1 (d) -1

7. The domain of definition of $f(x) = \frac{\log_2(x+3)}{x^2 + 3x + 2}$ is **[2001S]**
   (a) $R \setminus \{-1, -2\}$ (b) $(-2, \infty)$
   (c) $R \setminus \{-1, -2, -3\}$ (d) $(-3, \infty) - \{-1, -2\}$

8. If $f: [1, \infty) \to [2, \infty)$ is given by $f(x) = x + \frac{1}{x}$ then $f^{-1}(x)$ equals **[2001S]**
   (a) $(x + \sqrt{x^2 - 4})/2$ (b) $x/(1 + x^2)$
   (c) $\left(x - \sqrt{x^2 - 4}\right)/2$ (d) $1 + \sqrt{x^2 - 4}$

9. Let $g(x) = 1 + x - [x]$ and $f(x) = \begin{cases} -1, & x < 0 \\ 0, & x = 0 \\ 1, & x > 0 \end{cases}$. Then for all x, $f(g(x))$ is equal to **[2001S]**
   (a) x (b) 1
   (c) $f(x)$ (d) $g(x)$

10. If the function $f: [1, \infty) \to [1, \infty)$ is defined by $f(x) = 2^{x(x-1)}$, then $f^{-1}(x)$ is **[1999 - 2 Marks]**

(a) $\left(\frac{1}{2}\right)^{x(x-1)}$ (b) $\frac{1}{2}(1+\sqrt{1+4\log_2 x})$

(c) $\frac{1}{2}(1-\sqrt{1+4\log_2 x})$ (d) not defined

11. Let $f(x)=(x+1)^2-1, x\geq -1$. Then the set $\{x: f(x)=f^{-1}(x)\}$ is **[1995]**

(a) $\left\{0,-1,\frac{-3+i\sqrt{3}}{2},\frac{-3-i\sqrt{3}}{2}\right\}$

(b) $\{0,1,-1\}$

(c) $\{0,-1\}$

(d) empty

12. Let $f(x)=\sin x$ and $g(x)=\ln|x|$. If the ranges of the composition functions fog and gof are R_1 and R_2 respectively, then **[1994 - 2 Marks]**

(a) $R_1=\{u:-1\leq u<1\}, R_2=\{v:-\infty<v<0\}$

(b) $R_1=\{u:-\infty<u<0\}, R_2=\{v:-1\leq v\leq 0\}$

(c) $R_1=\{u:-1<u<1\}, R_2=\{v:-\infty<v<0\}$

(d) $R_1=\{u:-1\leq u\leq 1\}, R_2=\{v:-\infty<v\leq 0\}$

13. The domain of definition of the function $y=\frac{1}{\log_{10}(1-x)}+\sqrt{x+2}$ is **[1983 - 1 Mark]**

(a) $(-3,-2)$ excluding -2.5 (b) $[0,1]$ excluding 0.5

(c) $[-2,1)$ excluding 0 (d) none of these

14. If $f(x)=\cos(\ln x)$, then $f(x)f(y)-\frac{1}{2}\left[f\left(\frac{x}{y}\right)+f(xy)\right]$ has the value **[1983 - 1 Mark]**

(a) -1 (b) $1/2$

(c) -2 (d) none of these

2 Integer Value Answer/ Non-Negative Integer

15. The value of $\left((\log_2 9)^2\right)^{\frac{1}{\log_2(\log_2 9)}}\times(\sqrt{7})^{\frac{1}{\log_4 7}}$ is ________. **[Adv. 2018]**

16. Let $f:[0,4\pi]\to[0,\pi]$ be defined by $f(x)=\cos^{-1}(\cos x)$. The number of points $x\in[0,4\pi]$ satisfying the equation $g(x)=\frac{10-x}{10}$ is **[Adv. 2014]**

4 Fill in the Blanks

17. If $f(x)=\sin^2 x+\sin^2\left(x+\frac{\pi}{3}\right)+\cos x\cos\left(x+\frac{\pi}{3}\right)$ and $g\left(\frac{5}{4}\right)=1$, then $(gof)(x)=$ .... **[1996 - 2 Marks]**

18. If $f(x)=\sin\ln\left(\frac{\sqrt{4-x^2}}{1-x}\right)$, then domain of $f(x)$ is .... and its range is .... **[1985 - 2 Marks]**

19. The domain of the function $f(x)=\sin^{-1}(\log_2\frac{x^2}{2})$ is given by .... **[1984 - 2 Marks]**

20. The values of $f(x)=3\sin\left(\sqrt{\frac{\pi^2}{16}-x^2}\right)$ lie in the interval .... **[1983 - 1 Mark]**

5 True / False

21. If $f(x)=(a-x^n)^{1/n}$ where $a>0$ and n is a positive integer, then $f[f(x)]=x$. **[1983 - 1 Mark]**

6 MCQs with One or More than One Correct Answer

22. Let $f(x)=\sin\left(\frac{\pi}{6}\sin\left(\frac{\pi}{2}\sin x\right)\right)$ for all $x\in R$ and $g(x)=\frac{\pi}{2}\sin x$ for all $x\in R$. Let $(fog)(x)$ denote $f(g(x))$ and $(gof)(x)$ denote $g(f(x))$. Then which of the following is (are) true? **[Adv. 2015]**

(a) Range of f is $\left[-\frac{1}{2},\frac{1}{2}\right]$

(b) Range of fog is $\left[-\frac{1}{2},\frac{1}{2}\right]$

(c) $\lim\limits_{x\to 0}\frac{f(x)}{g(x)}=\frac{\pi}{6}$

(d) There is an $x\in R$ such that $(gof)(x)=1$

23. Let $f:\left(-\frac{\pi}{2},\frac{\pi}{2}\right)\to R$ be given by $f(x)=(\log(\sec x+\tan x))^3$. Then **[Adv. 2014]**

(a) $f(x)$ is an odd function

(b) $f(x)$ is one-one function

(c) $f(x)$ is an onto function

(d) $f(x)$ is an even function

24. Let $f:(0,1)\to\mathbf{R}$ be defined by $f(x)=\frac{b-x}{1-bx}$, where b is a constant such that $0<b<1$. Then **[2011]**

(a) f is not invertible on $(0,1)$

(b) $f\neq f^{-1}$ on $(0,1)$ and $f'(b)=\frac{1}{f'(0)}$

(c) $f=f^{-1}$ on $(0,1)$ and $f'(b)=\frac{1}{f'(0)}$

(d) f^{-1} is differentiable $(0,1)$

25. If $g(f(x)) = |\sin x|$ and $f(g(x)) = (\sin\sqrt{x})^2$, then [1998 - 2 Marks]

(a) $f(x) = \sin^2 x$, $g(x) = \sqrt{x}$

(b) $f(x) = \sin x$, $g(x) = |x|$

(c) $f(x) = x^2$, $g(x) = \sin\sqrt{x}$

(d) f and g cannot be determined.

26. If $f(x) = 3x - 5$, then $f^{-1}(x)$ [1998 - 2 Marks]

(a) is given by $\dfrac{1}{3x-5}$

(b) is given by $\dfrac{x+5}{3}$

(c) does not exist because f is not one-one

(d) does not exist because f is not onto.

7 Match the Following

27. Let $E_1 = \left\{x \in \mathbb{R} : x \neq 1 \text{ and } \dfrac{x}{x-1} > 0\right\}$ and

$E_2 = \left\{x \in E_1 : \sin^{-1}\left(\log_e\left(\dfrac{x}{x-1}\right)\right) \text{ is a real number}\right\}$.

(Here, the inverse trigonometric function $\sin^{-1} x$ assumes values in $\left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$).

Let $f : E_1 \to \mathbb{R}$ be the function defined by $f(x) = \log_e\left(\dfrac{x}{x-1}\right)$ and $g : E_2 \to \mathbb{R}$ be the function defined by

$g(x) = \sin^{-1}\left(\log_e\left(\dfrac{x}{x-1}\right)\right)$. [Adv. 2018]

LIST - I	LIST - II
P. The range of f is	1. $\left(-\infty, \dfrac{1}{1-e}\right] \cup \left[\dfrac{e}{e-1}, \infty\right)$
Q. The range of g contains	2. $(0, 1)$
R. The domain of f contains	3. $\left[-\dfrac{1}{2}, \dfrac{1}{2}\right]$
S. The domain of g is	4. $(-\infty, 0) \cup (0, \infty)$
	5. $\left(-\infty, \dfrac{e}{e-1}\right]$
	6. $(-\infty, 0) \cup \left(\dfrac{1}{2}, \dfrac{e}{e-1}\right]$

The correct option is:

(a) P → 4; Q → 2; R → 1; S → 1

(b) P → 3; Q → 3; R → 6; S → 5

(c) P → 4; Q → 2; R → 1; S → 6

(d) P → 4; Q → 3; R → 6; S → 5

10 Subjective Problem

28. Let f be a one-one function with domain $\{x, y, z\}$ and range $\{1, 2, 3\}$. It is given that exactly one of the following statements is true and the remaining two are false $f(x) = 1, f(y) \neq 1, f(z) \neq 2$ determine $f^{-1}(1)$. [1982 - 3 Marks]

Answer Key

Topic-1 : Types of Relations, Inverse of a Relation, Mappings, Mapping of Functions, Kinds of Mapping of Functions

1. (c) **2.** (b) **3.** (d) **4.** (a) **5.** (b) **6.** (a) **7.** (d) **8.** (a) **9.** (d) **10.** (c)
11. (d) **12.** (c) **13.** (a) **14.** (c) **15.** (d) **16.** (b) **17.** (d) **18.** (119) **19.** $\dfrac{3\pm\sqrt{5}}{2}, \dfrac{-3\pm\sqrt{5}}{2}$
20. $x+1, -x+1$ **21.** $n^n, n!$ **22.** (False) **23.** (True) **24.** (False) **25.** (b, d) **26.** (a, b) **27.** (a, b) **28.** (a, c)
29. (b, c) **30.** (a, d) **31.** (A) → (s), (B) → (t), (C) → (r), (D) → (r)
32. (A) → (r), (s), (p); (B) → (q), (s); (C) → (q), (s); (D) → (r), (s), (p) **33.** (A) → (q); (B) → (r)
34. (20) **35.** (36)

Topic-2 : Composite Functions & Relations, Inverse of a Function, Binary Operations

1. (a) **2.** (d) **3.** (b) **4.** (d) **5.** (a) **6.** (d) **7.** (d) **8.** (a) **9.** (b) **10.** (b)
11. (c) **12.** (d) **13.** (c) **14.** (d) **15.** (8)0 **16.** (3) **17.** (1) **18.** (–2, 1) [–1, 1]
19. $[-2,-1] \cup [1,2]$ **20.** $[0, 3/\sqrt{2}]$ **21.** (True) **22.** (a, b, c) **23.** (a, b, c)
24. (a, b) **25.** (a) **26.** (b) **27.** (a)

Hints & Solutions

Topic-1: Types of Relations, Inverse of a Relation, Mappings, Mapping of Functions, Kinds of Mapping of Functions

1. **(c)** $f(x)$ is a non-periodic, continous and odd function

$$f(x)=\begin{cases}-x^2+x\sin x, x<0\\ x^2-x\sin x, x\geq 0\end{cases}$$

$$f'(x)=\begin{cases}-2x+\sin x+x\cos x,\ x<0\\ 2x-\sin x-x\cos x,\ x\geq 0\end{cases}$$

$$f'(x)=\begin{cases}-(x-\sin x)-x(1-\cos x), x<0\\ (x-\sin x)+x(1-\cos x), x>0\end{cases}$$

$\because$ $x-\sin x<0$ if $x<0$ and

$1-\cos x>0,\ \forall\ x\in R$

$\therefore$ $-(x-\sin x)-x(1-\cos x)>0$ if $x<0$

and $(x-\sin x)+x(1-\cos x)>0$ if $x>0$

$\Rightarrow$ $f'(x)>0\ \forall\ x\in R \Rightarrow$ $f(x)$ is increasing in R

$\Rightarrow$ $f(x)$ is one-one

$$\therefore \underset{x\to-\infty}{Lt}\left(-x^2\right)\left(1-\frac{\sin x}{x}\right)=-\infty \therefore \underset{x\to\infty}{Lt}\ x^2\left(1-\frac{\sin x}{x}\right)=\infty$$

$\Rightarrow$ Range of $f(x)$ = R $\Rightarrow$ $f(x)$ is an onto function

2. **(b)** Given : $f(x)=2x^3-15x^2+36x+1$

$\Rightarrow$ $f'(x)=6x^2-30x+36$

$=6(x^2-5x+6)=6(x-2)(x-3)$

$\because$ $f'(x)>0\ \ \forall\ x\in[0,2)$ and $f'(x)<0\ \ \forall\ x\in(2,3)$

$\therefore$ $f(x)$ is increasing on $[0, 2)$ and decreasing on $(2, 3)$

$\therefore$ $f(x)$ is many one on $[0, 3]$

Also $f(0)=1, f(2)=29, f(3)=28$

$\therefore$ Absolute min = 1 and Absolute max = 29

$\therefore$ Range of $f=[1, 29]$ = codomain

Hence f is onto.

3. **(d)** $f(x)=e^{x^2}+e^{-x^2}\Rightarrow f'(x)=2x\left(e^{x^2}-e^{-x^2}\right)\geq 0,$

$\forall$ x $\in[0,1]$

$\therefore$ f (x) is an increasing function on [0, 1]

$$\therefore\ f_{max}=f(1)=e+\frac{1}{e}=a\ ;\ g(x)=xe^{x^2}+e^{-x^2}$$

$$\Rightarrow\ g'(x)=(2x^2+1)e^{x^2}-2xe^{-x^2}\geq 0,\forall x\in[0,1]$$

$\therefore$ $g(x)$ is an increasing function on [0, 1]

$$\therefore\ g_{\max}=g(1)=e+\frac{1}{e}=b$$

$$h(x)=x^2e^{x^2}+e^{-x^2}$$

$$\Rightarrow\ h'(x)=2x\left[e^{x^2}(1+x^2)-e^{-x^2}\right]\geq 0,\forall x\in[0,1]$$

$\therefore$ $h(x)$ is an increasing function on $[0,1]$

$$\therefore\ h_{\max}=h(1)=e+\frac{1}{e}=c\quad \therefore a=b=c.$$

4. **(a)** Given $f(x)$ and $g(x)$ defined on $R\to R$

and $f(x)=\begin{cases}0, & x\in\text{rational}\\ x, & x\in\text{irrational}\end{cases}$

$g(x)=\begin{cases}0, & x\in\text{irrational}\\ x, & x\in\text{rational}\end{cases}$

$\therefore (f-g):R\to R$ such that

$(f-g)(x)=\begin{cases}-x, & x\in\text{rational}\\ x, & x\in\text{irrational}\end{cases}$

Since $(f-g):R\to R$ for any x, then there is only one value of $(f(x)-g(x))$ whether x is rational or irrational. Moreover as $x\in R, f(x)-g(x)$ also belongs to R. Therefore, $(f-g)$ is one-one onto.

5. **(b)** Given : $f:[0,\infty)\to[0,\infty)$ and $f(x)=\dfrac{x}{x+1}$

$$\therefore\ f'(x)=\frac{1+x-x}{(1+x)^2}=\frac{1}{(1+x)^2}>0\ \forall\ x$$

$\therefore$ f is an increasing function $\Rightarrow f$ is one-one.

Now, $D_f=[0,\infty)$

For range let $\dfrac{x}{1+x}=y\Rightarrow x=\dfrac{y}{1-y}$

Now, $x\geq 0\Rightarrow 0\leq y<1$

$\therefore R_f=[0,1)\neq$ Co-domain, $\therefore$ f is not onto.

6. **(a)** Given : $f(x)=2x+\sin x,\ x\in R$

$\Rightarrow f'(x)=2+\cos x$. Now $-1\leq\cos x\leq 1$

$\Rightarrow 1\leq 2+\cos x\leq 3\Rightarrow 1\leq 2+\cos x\leq 3$

$\therefore f'(x)>0,\ \forall\ x\in R$

$\Rightarrow f(x)$ is strictly increasing and therefore one-one

Also as $x\to\infty$, $f(x)\to\infty$ and $x\to-\infty$, $f(x)\to-\infty$

$\therefore$ Range of $f(x)=R$ = domain of $f(x)\Rightarrow f(x)$ is onto.

Hence, $f(x)$ is one-one and onto.

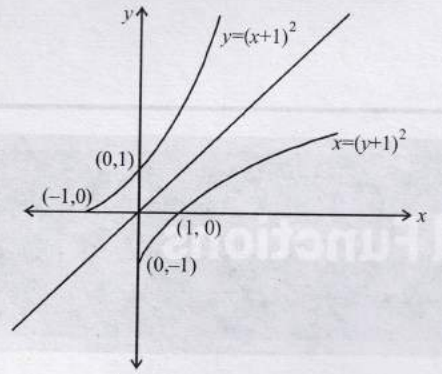
7. **(d)** Given : $f(x)=(x+1)^2, x\geq-1$

If $g(x)$ is the reflection of $f(x)$ in the line $y=x$, then it can be obtained by interchanging x and y in $f(x)$

i.e., $y=(x+1)^2$ changes to $x=(y+1)^2$

$$\Rightarrow\ y+1=\sqrt{x}\ \ [y+1\neq-\sqrt{x},\text{ since } y\geq-1]$$

$$\Rightarrow\ y=\sqrt{x}-1\quad\text{defined }\forall\ x\geq 0$$



$\therefore g(x) = \sqrt{x} - 1 \quad \forall\ x \geq 0$

8. **(a)** $E = \{1, 2, 3, 4\}$ and $F = \{1, 2\}$

From E to F we can define, in all, $2 \times 2 \times 2 \times 2 = 16$ functions (2 options for each element of E) out of which 2 are into, when all the elements of E either map to 1 or to 2.

$\therefore$ Number of onto functions = $16 - 2 = 14$

9. **(d)** Given : $2^x + 2^y = 2\ \forall\ x, y \in R$

but $2^x, 2^y > 0\ \forall\ x, y \in R$

$\therefore \quad 2^x = 2 - 2^y < 2 \Rightarrow 0 < 2^x < 2 \Rightarrow x < 1$

Hence domain = $(-\infty, 1)$

10. **(c)** Let $h(x) = |x|$

$\therefore \quad g(x) = |f(x)| = h(f(x))$

Since composition of two continuous functions is continuous, therefore g is continuous if f is continuous.

11. **(d)** $f(x)$ is continuous and defined for all $x > 0$.

Also $f\left(\dfrac{x}{y}\right) = f(x) - f(y)$ and $f(e) = 1$

$\Rightarrow$ Clearly $f(x) = \ell n\, x$, satisfies all these properties

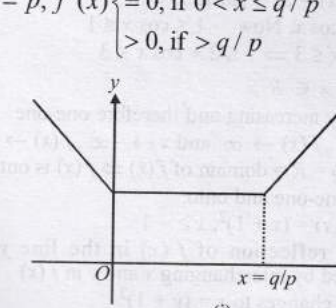
$\therefore f(x) = \ell n\, x$

12. **(c)** $f(x) = |px - q| + r|x|$

$$= \begin{cases} -px + q - rx, & x \leq 0 \\ -px + q + rx, & 0 < x \leq q/p \\ px - q + rx, & q/p < x \end{cases}$$

$$f'(x) = \begin{cases} -p - r, & x \leq 0 \\ -p + r, & 0 < x \leq q/p \\ p + r, & q/p < x \end{cases}$$

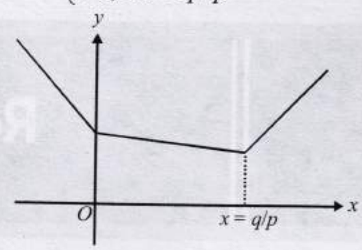
For $r = p, f'(x) \begin{cases} < 0, \text{ if } x < 0 \\ = 0, \text{ if } 0 < x \leq q/p \\ > 0, \text{ if } > q/p \end{cases}$



(i)

From graph (i), infinite many points for minima value of $f(x)$

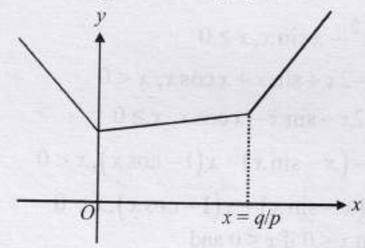
For $r < p, f'(x) \begin{cases} < 0, \text{ if } x \leq 0 \\ < 0, \text{ if } 0 < x \leq q/p \\ > 0, \text{ if } x < q/p \end{cases}$



(ii)

From graph (ii), only point of minima of $f(x)$ at $x = q/p$

For $r > p, f'(x) \begin{cases} < 0, \text{ if } x \leq 0 \\ > 0, \text{ if } 0 < x \leq q/p \\ > 0, \text{ if } x > q/p \end{cases}$

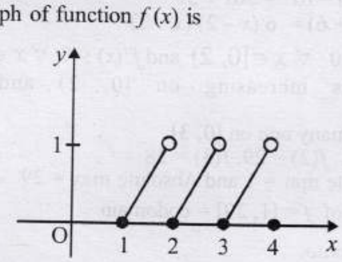


(iii)

From graph (iii), only one point of minima of $f(x)$ at $x = 0$

13. **(a)** $f(x) = x - [x] = \begin{cases} \dots \\ x - 1, & 1 \leq x < 2 \\ x - 2, & 2 \leq x < 3 \\ x - 3, & 3 \leq x < 4 \\ \dots \end{cases}$

$\therefore \quad$ Graph of function $f(x)$ is

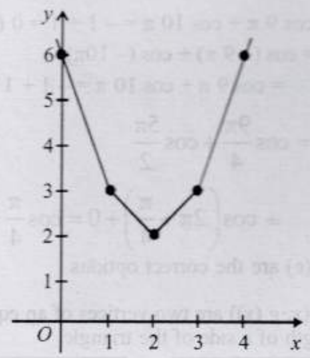


Clearly it is a periodic function with period 1.

14. **(c)** $|x - 1| + |x - 2| + |x - 3| \geq 6$

Consider $f(x) = |x - 1| + |x - 2| + |x - 3|$

$$\therefore \quad f(x) = \begin{cases} 6 - 3x, & x < 1 \\ 4 - x, & 1 \leq x < 2 \\ x, & 2 \leq x < 3 \\ 3x - 6, & x \geq 3 \end{cases}$$



From the graph of $f(x)$, it is clear that $f(x) \geq 6$ for $x \leq 0$ or $x \geq 4$

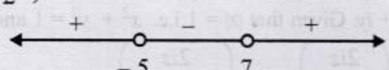
15. **(d)** Given : $f(x) = |x-1| = \begin{cases} -x+1, & x<1 \\ x-1, & x \geq 1 \end{cases}$

Consider $f(x^2) = (f(x))^2$
If it is true, it should be true for all x.
Put $x = 2$, then
LHS $= f(2^2) = |4-1| = 3$ and RHS $= (f(2))^2 = 1$
Since, L.H.S. ≠ R.H.S.
∴ (a) is not correct.
Consider $f(x+y) = f(x) + f(y)$
Put $x = 2, y = 5$, then
L.H.S. $= f(7) = 6$ and R.H.S $= f(2) + f(5) = 1 + 4 = 5$
∴ (b) is not correct.
Consider $f(|x|) = |f(x)|$
Put $x = -5$, then L.H.S. $= f(|-5|) = f(5) = 4$
and R.H.S. $= |f(-5)| = |-5-1| = 6$
∴ (c) is not correct.
∴ (d) is the correct alternative.

16. **(b)** $y = x^2 + (k-1)x + 9 = \left(x + \frac{k+1}{2}\right)^2 + 9 - \left(\frac{k-1}{2}\right)^2$

For entire graph to be above x-axis, we should have

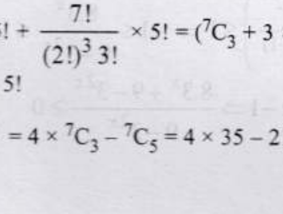
$$9 - \left(\frac{k-1}{2}\right)^2 > 0$$



$\Rightarrow \quad k^2 - 2k - 35 < 0 \Rightarrow (k-7)(k+5) < 0$
$\Rightarrow \quad -5 < k < 7$

17. **(d)** $f(x) = x^2$ is many one as $f(1) = f(-1) = 1$
Also f is into as – ve real number have no pre-image.
∴ F is neither injective nor surjective.

18. **(119)** Here n (X) = 5 and n (Y) = 7
Number of one-one function $= \alpha = {}^7C_5 \times 5!$
and Number of onto function Y to X = β



$$= \frac{7!}{3!\,4!} \times 5! + \frac{7!}{(2!)^3\,3!} \times 5! = ({}^7C_3 + 3 \times {}^7C_3)\,5!$$

$= 4 \times {}^7C_3 \times 5!$

$$\Rightarrow \quad \frac{\beta - \alpha}{5!} = 4 \times {}^7C_3 - {}^7C_5 = 4 \times 35 - 21 = 119$$

19. Given an even function $f(x) = f\left(\frac{x+1}{x+2}\right)$

$$\therefore \quad f(x) = f(-x) = f\left(\frac{-x+1}{-x+2}\right)$$

$$\Rightarrow \quad x = \frac{-x+1}{-x+2} \Rightarrow x^2 - 3x + 1 = 0 \quad \Rightarrow x = \frac{3 \pm \sqrt{5}}{2}$$

Also $f(x) = f\left(\frac{x+1}{x+2}\right) = f(-x)$

$$\Rightarrow \quad \frac{x+1}{x+2} = -x \Rightarrow x^2 + 3x + 1 = 0 \Rightarrow x = \frac{-3 \pm \sqrt{5}}{2}$$

∴ Four values of x are

$$\frac{3+\sqrt{5}}{2}, \frac{3-\sqrt{5}}{2}, \frac{-3+\sqrt{5}}{2} \text{ and } \frac{-3-\sqrt{5}}{2}$$

20. Every linear function is either strictly increasing or strictly decreasing. If $f(x) = ax + b$, $D_f = [p, q]$, $R_f = [m, n]$.
Then $f(p) = m$ and $f(q) = n$, if $f(x)$ is strictly increasing and $f(p) = n$, $f(q) = m$, if $f(x)$ is strictly decreasing function.
Let the linear function $f(x) = ax + b$, maps $[-1, 1]$ onto $[0, 2]$.
Then $f(-1) = 0$ and $f(1) = 2$ or $f(-1) = 2$ and $f(1) = 0$, depending upon $f(x)$ is increasing or decreasing respectively.
$\Rightarrow \quad -a + b = 0$ and $a + b = 2$ ....(i)
or $\quad -a + b = 2$ and $a + b = 0$ ....(ii)
On solving (i), we get $a = 1, b = 1$.
On solving (ii), we get $a = -1, b = 1$
Hence, there are only two functions $f(x) = x + 1$ and $f(x) = -x + 1$.

21. Set A has n distinct elements.
Then to define a function from A to A, we need to associate each element of set A to any one the n elements of set A.
∴ Total number of functions from A to $A = n^n$
Now for an onto function from A to A, we need to associate each element of A to one and only one element of A.
∴ Total number of functions from A to $A = n!$.

22. **(False)** We know that sum of any two functions is defined only on the points where both f_1 as well as f_2 are defined that is $f_1 + f_2$ is defined on $D_1 \cap D_2$.
∴ The given statement is false.

23. **(True)** A function is one-one if it is strictly increasing or strictly decreasing, other wise it is many one.

$$f(x) = \frac{x^2 + 4x + 30}{x^2 - 8x + 18} \Rightarrow f'(x) = \frac{-12[x^2 + 2x - 26]}{(x^2 - 8x + 18)^2}$$

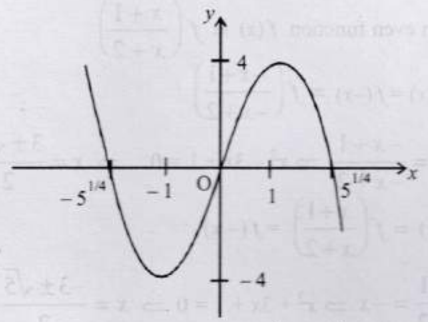
$$\Rightarrow \quad f'(x) = \frac{-12(x - 3\sqrt{3} + 1)(x + 3\sqrt{3} + 1)}{(x^2 - 8x + 18)^2}$$

$\Rightarrow \quad f(x)$ increases on $\left(-3\sqrt{3} - 1, 3\sqrt{3} - 1\right)$ and decreases otherwise.
∴ $\quad f(x)$ is many one.

24. **(False)** Given: $x * y = x - y + \sqrt{2}$

Let $x = 2\sqrt{2}$, $y = \sqrt{2}$
$\Rightarrow x * y = 2\sqrt{2} - \sqrt{2} + \sqrt{2} = 3\sqrt{2}$ (irrational)
and $y * x = \sqrt{2} - 2\sqrt{2} + \sqrt{2} = 0$ (rational)
∴ $x * y \neq y * x$ (Not symmetric)
Hence $*$ is not an equivalence relation.

25. **(b, d)** $f(x) = x^5 - 5x + a$
$f(x) = 0 \Rightarrow x^5 - 5x + a = 0 \Rightarrow a = 5x - x^5 = g(x)$
$\Rightarrow g(x) = 0$ when $x = 0, 5^{1/4}, -5^{1/4}$
and $g'(x) = 0 \Rightarrow x = 1, -1$
Also $g(-1) = -4$ and $g(1) = 4$
Thus graph of $g(x)$ will be as shown below.



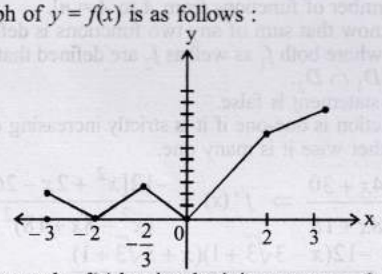
From graph, it is clear that if $a \in (-4, 4)$
then $g(x) = a$ or $f(x) = 0$ has 3 real roots
If $a > 4$ or $a < -4$
then $f(x) = 0$ has only one real root.
$\therefore$ option (b) and (d) are the correct options.

26. **(a, b)** Given : $f(x) = 2|x| + |x+2| - \|x+2| - 2|x\|$

Critical points of the $f(x)$ can be obtained by solving $|x| = 0$, $|x+2| = 0$ and $\|x+2| - 2|x\| = 0$, which give $x = 0, -2, 2, -\dfrac{2}{3}$

$$\therefore \quad f(x)\begin{cases} -2x-4, & x \le -2 \\ 2x+4, & -2 < x \le -\dfrac{2}{3} \\ -4x, & -\dfrac{2}{3} < x \le 0 \\ 4x, & 0 < x \le 2 \\ 2x+4, & x > 2 \end{cases}$$

Graph of $y = f(x)$ is as follows :



From graph, $f(x)$ has local minimum at $x = -2$ and $x = 0$ and local maximum at $x = -\dfrac{2}{3}$

27. **(a, b)** Given : $f(\cos 4\theta) = \dfrac{2}{2 - \sec^2\theta} = \dfrac{2\cos^2\theta}{2\cos^2\theta - 1}$

$$= \frac{1 + \cos 2\theta}{\cos 2\theta} = 1 + \frac{1}{\cos 2\theta}$$

Let $\cos 4\theta = \dfrac{1}{3} \Rightarrow 2\cos^2 2\theta - 1 = \dfrac{1}{3} \Rightarrow \cos 2\theta = \pm\sqrt{\dfrac{2}{3}}$

$\therefore f(\cos 4\theta) = 1 + \dfrac{1}{\cos 2\theta} = 1 \pm \sqrt{\dfrac{3}{2}}$ or $f\left(\dfrac{1}{3}\right) = 1 \pm \sqrt{\dfrac{3}{2}}$

28. **(a, c)** $f(x) = \cos[\pi^2]x + \cos[-\pi^2]x$
We know that $9 < \pi^2 < 10$ and $-10 < -\pi^2 < -9$
$\Rightarrow [\pi^2] = 9$ and $[-\pi^2] = -10$
$\therefore$ $f(x) = \cos 9x + \cos(-10x)$
$f(x) = \cos 9x + \cos 10x$

(a) $f\left(\dfrac{\pi}{2}\right) = \cos\dfrac{9\pi}{2} + \cos 5\pi = -1$ **(true)**

(b) $f(\pi) = \cos 9\pi + \cos 10\pi = -1 + 1 = 0$ **(false)**

(c) $f(-\pi) = \cos(-9\pi) + \cos(-10\pi)$
$= \cos 9\pi + \cos 10\pi = -1 + 1 = 0$ **(true)**

(d) $f\left(\dfrac{\pi}{4}\right) = \cos\dfrac{9\pi}{4} + \cos\dfrac{5\pi}{2}$

$$= \cos\left(2\pi + \frac{\pi}{4}\right) + 0 = \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}} \quad \textbf{(false)}$$

$\therefore$ (a) and (c) are the correct options.

29. **(b, c)**
As (0, 0) and $(x, g(x))$ are two vertices of an equilateral triangle; therefore, length of a side of the triangle

$$= \sqrt{(x-0)^2 + (g(x)-0)^2} = \sqrt{x^2 + (g(x))^2}$$

$\therefore$ The area of equilateral triangle $= \dfrac{\sqrt{3}}{4}(x^2 + (g(x))^2)$

But given that area of the equilateral triangle $= \dfrac{\sqrt{3}}{4}$

$\therefore$ $(g(x))^2 = 1 - x^2 \Rightarrow g(x) = \pm\sqrt{1 - x^2}$
$\therefore$ (b), (c) are the correct options as (a) is not a function.
($\because$ image of x is not unique)

30. **(a, d)** Given : $f(x) = y = \dfrac{x+2}{x-1}$

(a) $f(x) = \dfrac{x+2}{x-1} = y \Rightarrow x = f(y)$
$\therefore$ (a) is correct
(b) $f(1) \neq 3$ as function is not defined for $x = 1$
$\therefore$ (b) is not correct.

(c) $f'(x) = \dfrac{x-1-x-2}{(x-1)^2} = \dfrac{-3}{(x-1)^2}$

$\therefore$ $f'(x) < 0$, if $x \neq 1 \Rightarrow f(x)$ is decreasing if $x \neq 1$
$\therefore$ (c) is not correct.

(d) $f(x) = \dfrac{x+2}{x-1}$, which is a rational function of x.
$\therefore$ (d) is correct.

31. **(A) → (s), (B) → (t), (C) → (r), (D) → (r)**
Let $z = x + iy$. Given that $|z| = 1$ i.e. $x^2 + y^2 = 1$ and $x \neq \pm 1$

Then $\text{Re}\left(\dfrac{2iz}{1 - z^2}\right) = \text{Re}\left(\dfrac{2iz}{z.\bar{z} - z^2}\right)$

$= \text{Re}\left(\dfrac{2i}{\bar{z} - z}\right) = \text{Re}\left(\dfrac{2i}{-2iy}\right) = \text{Re}\left(\dfrac{-1}{y}\right) = \dfrac{-1}{y}$

where, $x = \sqrt{1 - y^2}$

$-1 \le y \le | \Rightarrow \dfrac{-1}{y} \ge 1$ of $\dfrac{-1}{y} \le -1$

$\therefore \text{Re}\left(\dfrac{2iz}{1 - z^2}\right) \in (-\infty, -1] \cup [1, \infty) \quad \therefore A \to s$

(B) For the domain of $f(x) = \sin^{-1}\left(\dfrac{8(3)^{x-2}}{1 - 3^{2(x-1)}}\right)$

We should have

$$-1 \le \left(\frac{8(3)^{x-2}}{1 - 3^{2(x-1)}}\right) \le 1 \Rightarrow -1 \le \frac{8.3^x}{9 - 3^{2x}} \le 1$$

$$\Rightarrow \quad \frac{8.3^x}{9 - 3^{2x}} \ge -1 \Rightarrow \frac{8.3^x + 9 - 3^{2x}}{9 - 3^{2x}} \ge 0$$

$\Rightarrow \dfrac{(3^x-9)(3^x+1)}{(3^x-3)(3^x+3)} \geq 0$

We know that $3^x > 0$

$+ \quad - \quad +$

$-\infty \quad 1 \quad 2 \quad \infty$

$\therefore \ x \in (-\infty, 1) \cup (2, \infty)$ ...(i)

And $\dfrac{8.3^x}{9-3^{2x}} \leq 1 \Rightarrow \dfrac{8.3^x - 9 + 3^{2x}}{9-3^{2x}} \leq 0$

$\dfrac{(3^x+9)(3^x-1)}{(3^x-3)(3^x+3)} \geq 0$

$+ \quad - \quad +$

$-\infty \quad 0 \quad 1 \quad \infty$

$\therefore x \in (-\infty, 0] \cup (1, \infty)$ ...(ii)

From (i) and (ii), we get $x \in (-\infty, 0] \cup [2, \infty) \ \therefore B \to t$

(C) $f(\theta) = \begin{vmatrix} 1 & \tan\theta & 1 \\ -\tan\theta & 1 & \tan\theta \\ -1 & -\tan\theta & 1 \end{vmatrix}$

Applying $R_1 = R_1 + R_3$

$= \begin{vmatrix} 0 & 0 & 2 \\ -\tan\theta & 1 & \tan\theta \\ -1 & -\tan\theta & 1 \end{vmatrix}$

$= 2\,(1 + \tan^2\theta) = 2\sec^2\theta \geq 2$ for $0 \leq \theta < \dfrac{\pi}{2} \ \therefore C \to r$

(D) $f(x) = x^{3/2}(3x-10), x \geq 0$

$\therefore f'(x) = \dfrac{3}{2}x^{1/2}(3x-10) + x^{3/2}$

For f(x) to be increasing $f'(x) \geq 0$

$\Rightarrow 3x^{3/2}[3x-10+2x] \geq 0 \Rightarrow x^{3/2}(5x-10) \geq 0$

$- \quad - \quad +$

$-\infty \quad 0 \quad 2 \quad \infty$, But $x > 0$

$\therefore f(x)$ is incresing on $[2, \infty)$

$\therefore$ D $\to$ r.

32. **(A) $\to$ (r), (s), (p); (B) $\to$ (q), (s); (C) $\to$ (q), (s);** (D) $\to$ (r), (s), (p)

$f(x) = \dfrac{x^2-6x+5}{x^2-5x+6} = \dfrac{(x-5)(x-1)}{(x-2)(x-3)}$

(A) If $-1 < x < 1$ then $f(x) = \dfrac{(-ve)(-ve)}{(-ve)(-ve)} = +ve$

$\therefore \quad f(x) > 0$ (r)

Also $f(x) - 1 = \dfrac{-x-1}{x^2-5x+6} = -\dfrac{(x+1)}{(x-2)(x-3)}$

For $-1 < x < 1$, $f(x) - 1 = \dfrac{-(+ve)}{(-ve)(-ve)} = -ve$

$\Rightarrow f(x) - 1 < 0 \Rightarrow f(x) < 1$ (s)

$\therefore \ 0 < f(x) < 1$ (p)

(B) If $1 < x < 2$ then $f(x) = \dfrac{(-ve)(+ve)}{(-ve)(-ve)} = -ve$

$\therefore f(x) < 0$ (q) and so $f(x) < 1$ (s)

(C) If $3 < x < 5$ then $f(x) = \dfrac{(-ve)(+ve)}{(+ve)(+ve)} = -ve$

$\therefore f(x) < 0$ (q) and so $f(x) < 1$ (s)

(D) For $x > 5$, $f(x) > 0$ (r)

Also $f(x) - 1 = \dfrac{-(x+1)}{(x-2)(x-3)} < 0$

For $x > 5$, $f(x) < 1$ (s)

$\therefore 0 < f(x) < 1$ (p)

33. **(A) $\to$ (q); (B) $\to$ (r)**

(A) $f(x) = 1 + 2x$, $D_f = (-\pi/2, \pi/2)$

The given function represents a straight line so it is one one.

But $f_{min} = 1 - \pi = f\left(-\dfrac{\pi}{2}\right)$, $f_{max} = 1 + \pi = f\left(\dfrac{\pi}{2}\right)$

$\therefore$ Range of $f = (1-\pi, 1+\pi) \in (-\infty, \infty)$

$\therefore f$ is not onto. Hence (A) $\to$ (q).

(B) $f(x) = \tan x$

It is an increasing function on $(-\pi/2, \pi/2)$ and its range $= (-\infty, \infty)$ = co-domain of f.

$\therefore f$ is one one onto. Hence (B) $\to$ r

34. **(20)** Given $S = \{1, 2, 3, 4, 5, 6\}$ $R : S \to S$

Number of elements in $R = 6$

and for each $(a, b) \in R : |a-b| \geq 2$

$X \to$ set of all relation $R : S \to S$

$a = 1, b = 3, 4, 5, 6 \to$ (4)

$a = 2, b = 4, 5, 6 \to$ (3)

$a = 3, b = 1, 5, 6 \to$ (3)

$a = 4, b = 1, 2, 6 \to$ (3)

$a = 5, b = 1, 2, 3 \to$ (3)

$a = 6, b = 1, 2, 3, 4 \to$ (4)

Total number of ordered pairs (a, b) such that $|a-b| \geq 2 = 20$

$\therefore n(X)$ = number of elements in $X = {}^{20}C_6$

$\therefore m = 20$

35. **(36)** Given set $S = \{1, 2, 3, 4, 5, 6\}$; $R : S \to S$

Number of elements in $R = 6$

and for each $(a, b) \in R$; $|a-b| \geq 2$

$X \to$ set of all relation $R : S \to S$

If

$a=1$	$b=3,4,5,6$	$\to$	4
$a=2$	$b=4,5,6$	$\to$	3
$a=3$	$b=1,5,6$	$\to$	3
$a=4$	$b=1,2,6$	$\to$	3
$a=5$	$b=1,2,3$	$\to$	3
$a=6$	$b=1,2,3,4$	$\to$	4

Total number of ordered pairs (a, b) such that $|a-b| \geq 2 = 20$

$\therefore n(X)$ = number of elements in $X = {}^{20}C_6$

$\therefore m = 20$

$Y = \{R \in X$: The range of R has exactly one element)

From above, if range of R has exactly one element, then maximum number of elements in R will be 4.

$\therefore n(Y) = 0$

$Z = \{R \in X : R \text{ is a function from S to S}\}$

$n(Z) = {}^4C_1 \times {}^3C_1 \times {}^3C_1 \times {}^3C_1 \times {}^3C_1 \times {}^4C_1 = (36)^2$

$n(y) + n(z) = 0 + (36)^2 = k^2$

$\Rightarrow |k| = 36$

36. Let $f(x) = Ax^2 + Bx + C$ is an integer whenever x is an integer.

$\therefore$ $f(0), f(1), f(-1)$ are integers

$\Rightarrow$ $C, A + B + C, A - B + C$ are integers.

$\Rightarrow$ $C, A + B, A - B$ are integers

$\Rightarrow$ $C, A + B, (A + B) + (A - B) = 2A$ are integers.

Conversely suppose $2A$, $A + B$ and C are integers.

Let x be any integer.

Now, $f(x) = Ax^2 + Bx + C = 2A\left[\dfrac{x(x-1)}{2}\right] + (A+B)\ x + C$

Since x is an integer, therefore $x(x-1)/2$ will be also an integer. Also $2A$, $A + B$ and C are integers.

$\therefore$ $f(x)$ is an integer for all integer x.

37. Put $y = \dfrac{\alpha x^2 + 6x - 8}{\alpha + 6x - 8x^2}$

$\Rightarrow$ $(\alpha + 6x - 8x^2)\,y = \alpha x^2 + 6x - 8$

$\Rightarrow$ $(\alpha + 8y)\,x^2 + 6\,(1 - y)\,x - (8 + \alpha y) = 0$

$\because$ x is real, $\therefore$ $D \geq 0$

$\Rightarrow$ $36\,(1 - y)^2 + 4\,(\alpha + 8y)\,(8 + \alpha y) \geq 0$

$\Rightarrow$ $9\,(1 - 2y + y^2) + [8\alpha + (64 + \alpha^2)y + 8\alpha y^2] \geq 0$

$\Rightarrow$ $y^2\,(9 + 8\alpha) + y\,(46 + \alpha^2) + (9 + 8\alpha) \geq 0$ ...(i)

For (i) to hold for each $y \in R$,

$9 + 8\,\alpha > 0$ and $(46 + \alpha^2)^2 - 4\,(9 + 8\alpha)^2 \leq 0$

$\Rightarrow \alpha > -9/8$ and $[46 + \alpha^2 - 2\,(9 + 8\alpha)]\,[46 + \alpha^2 + 2\,(9 + 8\alpha)] \leq 0$

$\Rightarrow$ $\alpha > -9/8$ and $(\alpha^2 - 16\alpha + 28)\,(\alpha^2 + 16\alpha + 64) \leq 0$

$\Rightarrow$ $\alpha > -9/8$ and $(\alpha - 2)\,(\alpha - 14)\,(\alpha + 8)^2 \leq 0$

$\Rightarrow \alpha > -8/9$ and $(\alpha - 2)\,(\alpha - 14) \leq 0$ $[\because (\alpha+8)^2 \geq 0]$

$\Rightarrow \alpha > -8/9$ and $2 \leq \alpha \leq 14$ $\Rightarrow$ $2 \leq \alpha \leq 14$

Thus, $f(x) = \dfrac{\alpha x^2 + 6x - 8}{\alpha + 6x - 8x^2}$ will be onto if $2 \leq \alpha \leq 14$.

When $\alpha = 3$, then $f(x) = \dfrac{3x^2 + 6x - 8}{3 + 6x - 8x^2}$

In this case, $f(x) = 0$ implies, $3x^2 + 6x - 8 = 0$

$\Rightarrow$ $x = \dfrac{-6 \pm \sqrt{36+96}}{6} = \dfrac{-6 \pm 2\sqrt{33}}{6} = \dfrac{1}{3}(-3 \pm \sqrt{33})$

$\therefore$ $f\left[\dfrac{1}{3}(-3+\sqrt{33})\right] = f\left[\dfrac{1}{3}(-3-\sqrt{33})\right] = 0$

Hence, f is not one-to-one at $\alpha = 3$.

38. Given : $4\,\{x\} = x + [x]$,

where $[x]$ = greatest integer $\leq x$

$\{x\}$ = fractional part of x

$\therefore x = [x] + \{x\}$ for any $x \in R$

$\therefore$ Given equation becomes

$4\,\{x\} = [x] + \{x\} + [x]$ $\Rightarrow$ $3\,\{x\} = 2\,[x]$

$\Rightarrow$ $[x] = \dfrac{3}{2}\,\{x\}$ ....(i)

Now $-1 < \{x\} < 1$ $\Rightarrow$ $-\dfrac{3}{2} < \dfrac{3}{2}\{x\} < \dfrac{3}{2}$

$\Rightarrow$ $-\dfrac{3}{2} < [x] < \dfrac{3}{2}$ $\Rightarrow$ $[x] = -1, 0, 1$ (using eqn (i))

If $[x] = -1$

$\Rightarrow -1 = \dfrac{3}{2}\{x\} \Rightarrow \{x\} = -\dfrac{2}{3}$ (using eqn (i))

$\therefore$ $x = [x] + \{x\}$ $\Rightarrow$ $x = -1 + (-2/3) = -5/3$

If $[x] = 0$, then $\dfrac{3}{2}\{x\} = 0$

$\Rightarrow$ $\{x\} = 0$ $\therefore$ $x = 0 + 0 = 0$

If $[x] = 1$, then $\dfrac{3}{2}\{x\} = 1$

$\Rightarrow$ $\{x\} = 2/3$ $\Rightarrow x = 1 + 2/3 = 5/3$

$\therefore$ $x = -5/3, 0, 5/3$

39. Given : $f(x + y) = f(x) f(y)\ \forall\ x, y \in N$ and $f(1) = 2$

To find 'a' such that $\sum_{k=1}^{n} f(a+k) = 16(2^n - 1)$ ...(i)

For this we start with $f(1) = 2$ ...(ii)

$\therefore f(2) = f(1 + 1) = f(1) f(1) \Rightarrow f(2) = 2^2$ [using (ii)]

Similarly we get, $f(3) = 2^3$, $f(4) = 2^4$, ..., $f(n) = 2^n$

Now eq. (i) can be written as

$f(a + 1) + f(a + 2) + f(a + 3) + ... + f(a + n) = 16\,(2^n - 1)$

$\Rightarrow$ $f(a) f(1) + f(a) f(2) + f(a) f(3) + ... + f(a) f(n) = 16\,(2^n - 1)$

$\Rightarrow$ $f(a)\,[f(1) + f(2) + f(3) + ... + f(n)] = 16\,(2^n - 1)$

$\Rightarrow$ $f(a)\,[2 + 2^2 + 2^3 + ... + 2^n] = 16\,(2^n - 1)$

$\Rightarrow$ $f(a)\left[\dfrac{2(2^n-1)}{2-1}\right] = 16(2^n - 1)$

$\therefore$ $f(a) = 8 = 2^3 = f(3) \Rightarrow a = 3$

40. Since $|f(x) - f(y)| \leq |x - y|^3$ is true $\forall\ x, y \in R$

For $x \neq y$, $\dfrac{|f(x)-f(y)|}{|x-y|} \leq |x-y|^2$

$\Rightarrow$ $\lim\limits_{y\to x}\left|\dfrac{f(x)-f(y)}{x-y}\right| \leq \lim\limits_{y\to x}|x-y|^2$

$\Rightarrow$ $\left|\lim\limits_{y\to x}\dfrac{f(x)-f(y)}{x-y}\right| \leq 0$

$\Rightarrow$ $|f'(x)| \leq 0$ $\Rightarrow f'(x) = 0$

$\therefore$ $f(x)$ is a constant function.

41. Given that $z_1 R z_2$ iff $\dfrac{z_1 - z_2}{z_1 + z_2}$ is real.

For reflexive :

$\because$ $\dfrac{z - z}{z + z} = 0$ which is real

$\therefore$ $z R z\ \forall\ z$ $\therefore$ R is reflexive.

For symmetric : Let $z_1 R z_2 \Rightarrow \dfrac{z_1 - z_2}{z_1 + z_2}$ is real

$\Rightarrow -\left(\dfrac{z_1 - z_2}{z_1 + z_2}\right)$ is also real

$\Rightarrow \dfrac{z_2 - z_1}{z_2 + z_1}$ is real $\Rightarrow z_2 R z_1$

$\therefore$ R is symmetric.

For transitive :

Let $z_1 R z_2$

$\Rightarrow \dfrac{z_1 - z_2}{z_1 + z_2}$ is real and $\dfrac{z_2 - z_3}{z_2 + z_3}$ is also real

$\Rightarrow$ $\dfrac{z_1 - z_2}{z_1 + z_2}$ is real $\Rightarrow$ $I_m\left(\dfrac{z_1 - z_2}{z_1 + z_2}\right) = 0$

$\Rightarrow \quad I_m\left(\dfrac{(x_1-x_2)+i(y_1-y_2)}{(x_1+x_2)+i(y_1+y_2)}\right)=0$

$\Rightarrow \quad I_m((x_1-x_2)+i(y_1-y_2))((x_1+x_2)-i(y_1+y_2))=0$

$\Rightarrow \quad (x_1+x_2)(y_1-y_2)-(x_1-x_2)(y_1+y_2)=0$

$\Rightarrow \quad x_2y_1-x_1y_2=0 \Rightarrow \dfrac{x_1}{y_1}=\dfrac{x_2}{y_2}$ ...(i)

and $z_2 \, R \, z_3$

Similarly, $I_m\left(\dfrac{z_2-z_3}{z_2+z_3}\right)=0 \Rightarrow \dfrac{x_2}{y_2}=\dfrac{x_3}{y_3}$ ...(ii)

From (i) and (ii) we get $\dfrac{x_1}{y_1}=\dfrac{x_3}{y_3}$

$\Rightarrow \quad I_m\left(\dfrac{z_1-z_3}{z_1+z_3}\right)=0 \Rightarrow \dfrac{z_1-z_3}{z_1+z_3}$ is real

$\Rightarrow \quad z_1 R \, z_3 \; \therefore R$ is transitive.

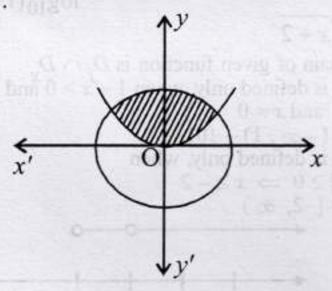
Thus R is reflexive, symmetric and transitive.
Hence R is an equivalence relation.

42. As there is an injective maping from A to B, each element of A has unique image in B. Similarly as there is an injective mapping from B to A, each element of B has unique image in A. So we can conclude that each element of A has unique image in B and each element of B has unique image in A or in other words there is one to one mapping from A to B. Thus there is bijective mapping from A to B.

43. $R = [(x, y); x \in R, y \in R, x^2 + y^2 \le 25]$, which represents all the points inside and on the circle $x^2 + y^2 = 5^2$, with centre (0, 0) and radius = 5,

$$R' = \left\{(x,y): x \in R, y \in R, y \ge \frac{4}{9}x^2\right\},$$

which represents all the points inside and on the upward parabola $x^2 \le \frac{9}{4}y$.



$\therefore \quad R \cap R'$ = The set of all points in shaded region.

Now, $x^2 + y^2 \le 25 \Rightarrow x^2 \le 25 - y^2$ ....(i)

and $y \ge \dfrac{4}{9}x^2 \Rightarrow \dfrac{16x^4}{81} \le y^2$

$\Rightarrow \quad -\dfrac{16x^4}{81} \ge -y^2$

$\Rightarrow \quad 25-\dfrac{16x^4}{81} \ge 25-y^2$ ....(ii)

From (i) and (ii), $x^2 \le 25-\dfrac{16}{81}x^4$

$\Rightarrow \quad 16x^4 + 81x^2 - 2025 \le 0$

$\therefore \quad$ Domain of $R \cap R'$ =

$\{x : x \in R, 16x^4 + 81x^2 - 2025 \le 0\}$ and range of $R \cap R'$

$= \{y : y \in R, y \ge \dfrac{4x^2}{9}, 16x^4 + 81x^2 - 2025 \le 0\}$

$R \cap R'$ is not a function because image of an element is not unique , e..g., (0, 1), (0, 2), (0,3) ..... $\in R \cap R'$.

44. $f(x) = x^9 - 6x^8 - 2x^7 + 12x^6 + x^4 - 7x^3 + 6x^2 + x - 3$

$\therefore \; f(6) = 6^9 - 6 \times 6^8 - 2 \times 6^7 + 12 \times 6^6 + 6^4 - 7 \times 6^3 + 6 \times 6^2 + 6 - 3$

$= 6^9 - 6^9 - 2 \times 6^7 + 2 \times 6^7 + 6^4 - 7 \times 6^3 + 6^3 + 6 - 3 = 3$

45. $y = |x|^{1/2}, -1 \le x \le 1$

$\Rightarrow \quad y = \sqrt{-x}$ if $-1 \le x \le 0 = \sqrt{x}$ if $0 \le x \le 1$

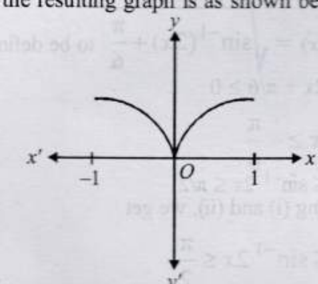
$\Rightarrow \quad y^2 = -x$ if $-1 \le x \le 0$ and $y^2 = x$ if $0 \le x \le 1$

[Here y should be taken always + ve, as by definition y is a + ve square root].

Clearly $y^2 = -x$ represents upper half of left handed parabola (upper half as y is + ve)

and $y^2 = x$ represents upper half of right handed parabola.

Therefore the resulting graph is as shown below :



46. Since $f(x)$ is defined and real for all real values of x,

$\therefore \quad$ Domain of f is R.

Clearly $0 \le \dfrac{x^2}{1+x^2} < 1$, for all $x \in R \Rightarrow 0 \le f(x) < 1$

$\Rightarrow$ Range of $f = [0, 1)$

Since $f(1) = f(-1) = 1/2$

$\therefore \quad f$ is not one-to-one.

Topic-2: Composite Functions & Relations, Inverse of a Function, Binary Operations

1. **(a)** Given : $f(x) = x^2$ and $g(x) = \sin x, \forall x \in R$

$\therefore$ (gof) $(x) = \sin x^2$

$\Rightarrow$ (gogof) $(x) = \sin(\sin x^2)$

$\Rightarrow$ (fogogof) $(x) = \sin^2(\sin x^2)$

Since given that (fogogof) (x) = (gogof) (x)

$\therefore \sin^2(\sin x^2) = \sin(\sin x^2)$

$\Rightarrow \sin(\sin x^2) = 0, 1$

$\Rightarrow \sin x^2 = n\pi$ or $\left((4n+1)\dfrac{\pi}{2}\right)$, where $n \in Z$

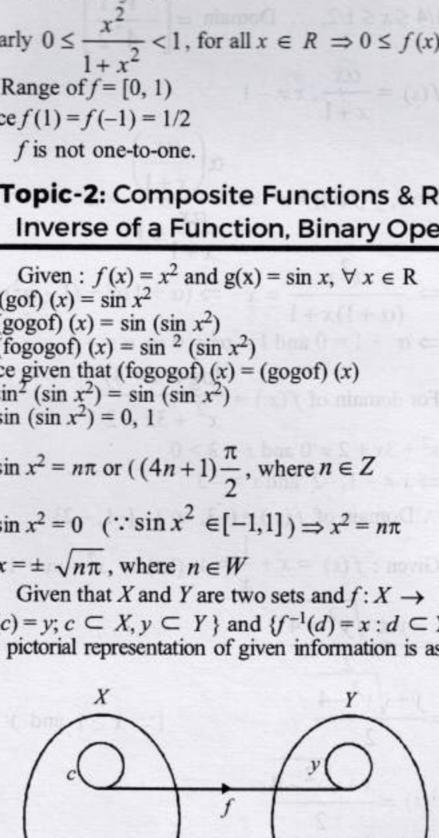
$\Rightarrow \sin x^2 = 0 \quad (\because \sin x^2 \in [-1,1]) \Rightarrow x^2 = n\pi$

$\therefore \; x = \pm\sqrt{n\pi}$, where $n \in W$

2. **(d)** Given that X and Y are two sets and $f : X \to Y$.

$\{f(c) = y; c \subset X, y \subset Y\}$ and $\{f^{-1}(d) = x : d \subset Y, x \subset X\}$

The pictorial representation of given information is as shown:



Since $\quad f^{-1}(d) = x \Rightarrow f(x) = d$.

Now if $a \subset x \Rightarrow f(a) \subset f(x) = d \Rightarrow f^{-1}(f(a)) = a$

Hence, $f^{-1}(f(a)) = a, a \subset x$ is the correct option.

3. **(b)** Given : $f(x) = \sin x + \cos x$ and $g(x) = x^2 - 1$
$\Rightarrow g(f(x)) = (\sin x + \cos x)^2 - 1 = \sin 2x$

Clearly $g(f(x))$ is invertible in $-\frac{\pi}{2} \le 2x \le \frac{\pi}{2}$

[$\because \sin\theta$ is invertible in $-\pi/2 \le \theta \le \pi/2$]

$\Rightarrow -\frac{\pi}{4} \le x \le \frac{\pi}{4}$

4. **(d)** $f(x) = x^2 + 2bx + 2c^2 \Rightarrow f(x) = (x+b)^2 + 2c^2 - b^2$
$\Rightarrow f_{\min} = 2c^2 - b^2$
and $g(x) = -x^2 - 2cx + b^2$
$g(x) = -(x+c)^2 + b^2 + c^2 \Rightarrow g_{\max} = b^2 + c^2$
For $f_{\min} > g_{\max} \Rightarrow 2c^2 - b^2 > b^2 + c^2$
$\Rightarrow c^2 > 2b^2 \Rightarrow |c| > |b|\sqrt{2}$

5. **(a)** For $f(x) = \sqrt{\sin^{-1}(2x) + \frac{\pi}{6}}$ to be defined and real,

$\sin^{-1} 2x + \pi/6 \ge 0$

$\Rightarrow \sin^{-1} 2x \ge -\frac{\pi}{6}$ ...(i)

But $-\pi/2 \le \sin^{-1} 2x \le \pi/2$ ...(ii)

On combining (i) and (ii), we get

$-\frac{\pi}{6} \le \sin^{-1} 2x \le \frac{\pi}{2}$

$\Rightarrow \sin(-\pi/6) \le 2x \le \sin(\pi/2) \Rightarrow -1/2 \le 2x \le 1$

$\Rightarrow -1/4 \le x \le 1/2$, $\therefore$ Domain $= \left[-\frac{1}{4}, \frac{1}{2}\right]$

6. **(d)** $f(x) = \frac{\alpha x}{x+1}, x \ne -1$

Now, $f(f(x)) = x \Rightarrow \frac{\alpha\left(\frac{\alpha x}{x+1}\right)}{\frac{\alpha x}{x+1} + 1} = x$

$\Rightarrow \frac{\alpha^2 x}{(\alpha+1)x + 1} = x \Rightarrow (\alpha+1)x^2 + (1-\alpha^2)x = 0$

$\Rightarrow \alpha + 1 = 0$ and $1 - \alpha^2 = 0 \Rightarrow \alpha = -1$

7. **(d)** For domain of $f(x) = \frac{\log_2(x+3)}{x^2 + 3x + 2}$

$x^2 + 3x + 2 \ne 0$ and $x + 3 > 0$

$\Rightarrow x \ne -1, -2$ and $x > -3$

$\therefore$ Domain of $f(x) = (-3, \infty) - \{-1, -2\}$

8. **(a)** Given : $f(x) = x + \frac{1}{x} = y$ (let) $\Rightarrow x^2 - yx + 1 = 0$

$\Rightarrow x = \frac{y \pm \sqrt{y^2 - 4}}{2}$

$\therefore x = \frac{y + \sqrt{y^2 - 4}}{2}$ [$\because x \ge 1$ and $y \ge 2$]

$\therefore f^{-1}(x) = \frac{x + \sqrt{x^2 - 4}}{2}$

9. **(b)** $g(x) = 1 + x - [x]$

and $f(x) = \begin{cases} -1, & x < 0 \\ 0, & x = 0 \\ 1, & x > 0 \end{cases}$

For integral values of x; $g(x) = 1$

For $x < 0$ (but not integral value); $x - [x] > 0 \Rightarrow g(x) > 1$

For $x > 0$ (but not integral value); $x - [x] > 0 \Rightarrow g(x) > 1$

$\therefore g(x) \ge 1, \forall x \Rightarrow f(g(x)) = 1, \forall x$

10. **(b)** Let $y = 2^{x(x-1)}$
$\Rightarrow x^2 - x - \log_2 y = 0;$

$x = \frac{1}{2}\left(1 \pm \sqrt{1 + 4\log_2 y}\right)$

For $y \ge 1$, $\log_2 y \ge 0 \Rightarrow \sqrt{1 + 4\log_2 y} \ge 0$

$\because x \ge 1$, $\therefore x = \frac{1}{2}\left(1 + \sqrt{1 + 4\log_2 y}\right)$

$\Rightarrow f^{-1}(x) = \frac{1}{2}\left(1 + \sqrt{1 + 4\log_2 x}\right)$

11. **(c)** $f(x) = f^{-1}(x) \Rightarrow fof(x) = x$

$\Rightarrow [(x+1)^2 - 1 + 1]^2 - 1 = x \Rightarrow (x+1)^4 = x + 1$

$\Rightarrow (x+1)[(x+1)^3 - 1] = 0$

$\therefore x = 0$ or -1

$\therefore$ Required set is $\{0, -1\}$

12. **(d)** Given : $f(x) = \sin x$ and $g(x) = \ell n\ |x|$
Now $f \circ g(x) = f(g(x)) = \sin(\ln |x|)$

$\therefore R_1 = \{u : -1 \le u \le 1\}$ ($\because -1 \le \sin\theta \le 1, \forall \theta$)

Also $g \circ f(x) = g(f(x)) = \ln |\sin x|$

$\therefore 0 \le |\sin x| \le 1$

$-\infty < \ln |\sin x| \le 0$

$\therefore R_2 = \{v : -\infty < v \le 0\}$

13. **(c)** $y = \frac{1}{\log_{10}(1-x)} + \sqrt{x+2}$

$\Rightarrow y = f(x) + g(x)$, where $f(x) = \frac{1}{\log_{10}(1-x)}$ and

$g(x) = \sqrt{x+2}$

$\therefore$ Domain of given function is $D_f \cap D_g$

Since $f(x)$ is defined only, when $1 - x > 0$ and $1 - x \ne 1$

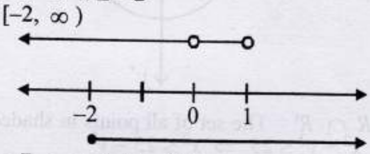
$\Rightarrow x < 1$ and $x \ne 0$

$\therefore D_f = (-\infty, 1) - \{0\}$

Also $g(x)$ is defined only, when

$x + 2 \ge 0 \Rightarrow x \ge -2$

$\therefore D_g = [-2, \infty)$



$\therefore D_f \cap D_g = [-2, 1) - \{0\}$

14. **(d)** Given : $f(x) = \cos(\ell n\ x)$

$\therefore f(x)f(y) - \frac{1}{2}\left[f\left(\frac{x}{y}\right) + f(xy)\right]$

$= \cos(\ell n\ x)\cos(\ell n\ y) - \frac{1}{2}[\cos(\ell n\ x - \ell n\ y) + \cos(\ell n\ x + \ell n\ y)]$

$= \cos(\ell n\ x)\cos(\ell n\ y) - \frac{1}{2}[2\cos(\ell n\ x)\cos(\ell n\ y)] = 0$

15. **(8)** $((\log_2 9)^2)^{\frac{1}{\log_2(\log_2 9)}} \times (\sqrt{7})^{\frac{1}{\log_4 7}}$

$= (\log_2 9)^{2 \times \log_{(\log_2 9)} 2} \times 7^{\frac{1}{2} \times \log_7 4}$

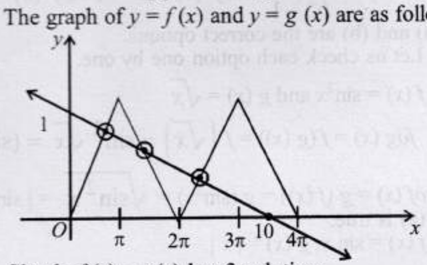
$= (\log_2 9)^{\log_{(\log_2 9)} 4} \times 7^{\log_7 2} = 4 \times 2 = 8$

16. (3) Given : $f:[0,4\pi]\to[0,\pi]$ defined by

$f(x)=\cos^{-1}(\cos x)$

and $g(x)=\dfrac{10-x}{10}=1-\dfrac{x}{10}$

The graph of $y=f(x)$ and $y=g(x)$ are as follows.



Clearly $f(x)=g(x)$ has 3 solutions.

17. $f(x)=\sin^2 x+\sin^2\left(x+\dfrac{\pi}{3}\right)+\cos x\cos\left(x+\dfrac{\pi}{3}\right)$

$=\sin^2 x+\left[\sin\left(x+\dfrac{\pi}{3}\right)\right]^2+\cos x\cos\left(x+\dfrac{\pi}{3}\right)$

$=\sin^2 x+\dfrac{1}{4}(\sin x+\sqrt{3}\cos x)^2$

$+\dfrac{1}{2}\cos x(\cos x-\sqrt{3}\sin x)$

$=\dfrac{5}{4}(\sin^2 x+\cos^2 x)=\dfrac{5}{4}$

$\therefore\quad (gof)\,x=g[f(x)]=g(5/4)=1$

18. Given function is, $f(x)=\sin\left[\ell n\left(\dfrac{\sqrt{4-x^2}}{1-x}\right)\right]$

For $\ell n\left(\dfrac{\sqrt{4-x^2}}{1-x}\right)$ to be defined $\dfrac{\sqrt{4-x^2}}{1-x}>0$

$\Rightarrow 1-x>0$ and $4-x^2>0\Rightarrow x<1$ and $-2<x<2$

Combining these two inequalities, we get $x\in(-2,1)$

$\therefore$ Domain of $f(x)$ is $(-2,1)$

Since $\sin\theta$ always lies in $[-1,1]$.

$\therefore$ Range of $f(x)$ is $[-1,1]$

19. The function $f(x)=\sin^{-1}\left(\log_2\dfrac{x^2}{2}\right)$ will be defined

if $-1\le\log_2\left(\dfrac{x^2}{2}\right)\le 1$

$\Rightarrow\quad 2^{-1}\le\dfrac{x^2}{2}\le 2^1\quad\Rightarrow\quad 1\le x^2\le 4$

$\Rightarrow -2\le x\le -1$ or $1\le x\le 2\Rightarrow x\in[-2,-1]\cup[1,2]$

20. Given : $f(x)=3\sin\left(\sqrt{\dfrac{\pi^2}{16}-x^2}\right)$

For the given function to be defined

$\dfrac{\pi^2}{16}-x^2\ge 0\Rightarrow -\pi/4\le x\le\pi/4$

$\therefore$ Domain $=[-\pi/4,\pi/4]$

Now, for $x\in[-\pi/4,\pi/4]$, $\sqrt{\pi^2/16-x^2}\in[0,\pi/4]$ and sine function increases on $[0,\pi/4]$.

$\therefore\quad 0\le\sin\sqrt{\dfrac{\pi^2}{16}-x^2}\le 1/\sqrt{2}$

$\Rightarrow\quad 0\le 3\sin\sqrt{\dfrac{\pi^2}{16}-x^2}\le 3/\sqrt{2}$

$\therefore\quad f(x)=[0,3/\sqrt{2}]$

21. **(True)** $f(x)=(a-x^n)^{1/n}$, $a>0$, n is a positive integer

$f(f(x))=f[(a-x^n)^{1/n}]=[a-\{(a-x^n)^{1/n}\}^n]^{1/n}$

$=(a-a+x^n)^{1/n}=x$

22. **(a, b, c)**

$f(x)=\sin\left(\dfrac{\pi}{6}\sin\left(\dfrac{\pi}{2}\sin x\right)\right)$

$-1\le\sin x\le 1\Rightarrow -\dfrac{\pi}{2}\le\dfrac{\pi}{2}\sin x\le\dfrac{\pi}{2}$

$\Rightarrow -1\le\sin\left(\dfrac{\pi}{2}\sin x\right)\le 1\Rightarrow\dfrac{-\pi}{6}\le\dfrac{\pi}{6}\sin\left(\dfrac{\pi}{2}\sin x\right)\le\dfrac{\pi}{6}$

$\Rightarrow\dfrac{-1}{2}\le\sin\left[\dfrac{\pi}{6}\sin\left(\dfrac{\pi}{2}\sin x\right)\right]\le\dfrac{1}{2}$

$\therefore\quad$ Range of $f=\left[\dfrac{-1}{2},\dfrac{1}{2}\right]$

Now, $fog(x)=\sin\left[\dfrac{\pi}{6}\sin\left(\dfrac{\pi}{2}\sin\left(\dfrac{\pi}{2}\sin x\right)\right)\right]$

Range of $fog=\left[\dfrac{-1}{2},\dfrac{1}{2}\right]$

Now, $\lim\limits_{x\to 0}\dfrac{f(x)}{g(x)}=\lim\limits_{x\to 0}\dfrac{\sin\left(\frac{\pi}{6}\sin\left(\frac{\pi}{2}\sin x\right)\right)}{\frac{\pi}{2}\sin x}$

$=\lim\limits_{x\to 0}\dfrac{\sin\left(\frac{\pi}{6}\sin\left(\frac{\pi}{2}\sin x\right)\right)}{\frac{\pi}{6}\sin\left(\frac{\pi}{2}\sin x\right)}\times\dfrac{\frac{\pi}{6}\sin\left(\frac{\pi}{2}\sin x\right)}{\frac{\pi}{2}\sin x}=\pi/6$

$gof(x)=\dfrac{\pi}{2}\sin\left(\sin\left(\dfrac{\pi}{6}\sin\left(\dfrac{\pi}{2}\sin x\right)\right)\right)$

$-\dfrac{\pi}{2}\sin\left(\dfrac{1}{2}\right)\le g(f(x))\le\dfrac{\pi}{2}\sin\left(\dfrac{1}{2}\right)$

Let $\dfrac{\pi}{2}\sin\left(\dfrac{1}{2}\right)=p$

Clearly $0<p<1$

$\therefore\quad -\dfrac{\pi}{2}\sin\left(\dfrac{1}{2}\right)\le g(f(x))\le\dfrac{\pi}{2}\sin\left(\dfrac{1}{2}\right)$

$-p\le g(f(x))\le p\Rightarrow o<p<1$

$\therefore\quad gof(x)\ne 1$ for any $x\in R$.

23. **(a, b, c)** Given : $f:\left(-\dfrac{\pi}{2},\dfrac{\pi}{2}\right)\to R$ is given by

$f(x)=(\log(\sec x+\tan x))^3$

$f(-x) = (\log(\sec x - \tan x))^3$

$= \left[\log\left(\frac{(\sec x - \tan x)(\sec x + \tan x)}{\sec x + \tan x}\right)\right]^3$

$= \left[\log\left(\frac{1}{\sec x + \tan x}\right)\right]^3 = \left[-\log(\sec x + \tan x)\right]^3$

$= -\left[\log(\sec x + \tan x)\right]^3 = -f(x)$

$\therefore$ $f(x)$ is an odd function.

$\therefore$ option (a) is correct and (d) is not correct.

Now, $f'(x) = 3\left[\log(\sec x + \tan x)\right]^2 . \frac{\sec x \tan x + \sec^2 x}{\sec x + \tan x}$

$= 3\sec x\left[\log(\sec x + \tan x)\right]^2 > 0\ \forall x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$\therefore$ $f(x)$ is increasing on $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

We know that strictly increasing function is one one.

$\therefore f$ is one one, hence (b) is the correct option.

Also $\lim_{x \to \frac{\pi}{2}^-} \left[\log(\sec x + \tan x)\right]^3 \to \infty$

and $\lim_{x \to \frac{\pi}{2}^+} \left[\log(\sec x + \tan x)\right]^3 \to -\infty$

$\therefore$ Range of $f = (-\infty, \infty) = R =$ Domain

$\therefore f$ is an onto function.

$\therefore$ option (c) is correct.

24. **(a, b)** Given : $f(x) = \frac{b-x}{1-bx}, 0 < b < 1$

Let $f(x_1) = f(x_2) \Rightarrow \frac{b-x_1}{1-bx_1} = \frac{b-x_2}{1-bx_2}$

$\Rightarrow b - b^2 x_2 - x_1 + bx_1x_2 = b - x_2 - b^2x_1 + bx_1 x_2$

$\Rightarrow x_2 (1 - b^2) = x_1 (1 - b^2) \Rightarrow x_1 = x_2$ as $1 - b^2 \neq 0$

$\therefore f$ is one one.

Also $\frac{b-x}{1-bx} = y \Rightarrow b - x = y - bxy$

$\Rightarrow (by - 1)x = y - b \Rightarrow x = \frac{y-b}{by-1}$

For $y = \frac{1}{b}$, x is not defined

$\therefore f$ is not onto and hence nor invertible.

Also $f'(x) = \frac{-1(1-bx)-(-b)(b-x)}{(1-bx)^2} = \frac{b^2-1}{(1-bx)^2}$

$\therefore \quad f'(b) = \frac{1}{b^2-1}$ and $f'(0) = b^2 - 1 \Rightarrow f'(b) = \frac{1}{f'(0)}$

$\therefore$ (a) and (b) are the correct options.

25. **(a)** Let us check each option one by one.

(a) $f(x) = \sin^2 x$ and $g(x) = \sqrt{x}$

Now, $fog(x) = f(g(x)) = f(\sqrt{x}) = \sin^2\sqrt{x} = (\sin\sqrt{x})^2$

and $gof(x) = g(f(x)) = g(\sin^2 x) = \sqrt{\sin^2 x} = |\sin x|$

$\therefore$ (a) is true.

(b) $f(x) = \sin x, g(x) = |x|$

$fog(x) = f(g(x)) = f(|x|) = \sin|x| \neq (\sin\sqrt{x})^2$

$\therefore$ (b) is not true

(c) $f(x) = x^2, g(x) = \sin\sqrt{x}$

$fog(x) = f(g(x)) = f(\sin\sqrt{x}) = (\sin\sqrt{x})^2$

and $(gof)(x) = g(f(x)) = g(x^2) = \sin\sqrt{x^2}$
$= \sin|x| \neq |\sin x|$

$\therefore$ (c) is not true.

26. **(b)** $f(x) = 3x - 5$ is strictly increasing on R.

$\therefore f^{-1}(x)$ exists.

Let $y = f(x) = 3x - 5$

$\Rightarrow y + 5 = 3x \Rightarrow x = \frac{y+5}{3}$ ...(i)

$\because y = f(x) \Rightarrow x = f^{-1}(y)$ ...(ii)

From (i) and (ii),

$f^{-1}(y) = \frac{y+5}{3} \Rightarrow f^{-1}(x) = \frac{x+5}{3}$

27. **(a)** For $E_1, \frac{x}{x-1} > 0$ and $x \neq 1 \Rightarrow x \in (-\infty, 0) \cup (1, \infty)$

For E_2, $-1 \leq \log_e\left(\frac{x}{x-1}\right) \leq 1 \Rightarrow \frac{1}{e} \leq \frac{x}{x-1} \leq e$

$\Rightarrow \frac{1}{e} \leq 1 + \frac{1}{x-1} \leq e \Rightarrow \frac{1}{e} - 1 \leq \frac{1}{x-1} \leq e - 1$

$\Rightarrow (x-1) \in \left(-\infty, \frac{e}{1-e}\right] \cup \left[\frac{1}{e-1}, \infty\right)$

$\Rightarrow x \in \left(-\infty, \frac{1}{e-1}\right] \cup \left[\frac{e}{e-1}, \infty\right)$

For E_1, $\frac{x}{x-1} \in (0, \infty) - \{1\}$

$\Rightarrow \log_e\left(\frac{x}{x-1}\right) \in (-\infty, \infty) - \{0\}$

$\Rightarrow f(x) \in (-\infty, 0) \cup (0, \infty)$

$g(x) = \sin^{-1}\left(\log_e\left(\frac{x}{x-1}\right)\right) \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$

28. f is one one function,

$D_f = \{x, y, z\}$; $R_f = \{1, 2, 3\}$

Exactly one of the following is true :

$f(x) = 1, f(y) \neq 1, f(z) \neq 2$

To determine f^{-1} (1):

Case I: $f(x) = 1$ is true.

$\Rightarrow$ $f(y) \neq 1, f(z) \neq 2$ are false.

$\Rightarrow$ $f(y) = 1, f(z) = 2$ are true.

But $f(x) = 1, f(y) = 1$ are true, is not possible as f is one to one.

$\therefore$ This case is not possible.

Case II: $f(y) \neq 1$ is true.

$\Rightarrow$ $f(x) = 1$ and $f(z) \neq 2$ are false

$\Rightarrow$ $f(x) \neq 1$ and $f(z) = 2$ are true

Thus, $f(x) \neq 1, f(y) \neq 1, f(z) = 2$

$\Rightarrow$ Either $f(x)$ or $f(y) = 2$. So, f is not one to one

$\therefore$ This case is also not possible.

Case III: $f(z) \neq 2$ is true

$\therefore$ $f(x) = 1$ and $f(y) \neq 1$ are false.

$\Rightarrow$ $f(x) \neq 1$ and $f(y) = 1$ are true.

$\therefore$ $f^{-1}(1) = y$