4/11/2013

Sets - Theory

Topics →

* Sets
* Relations.
* Functions.
* Partial orders.
* Lattices.
* Boolean Algebras.
* Groups.

I) Set →

A set can be defined as a well-defined unordered collection of distinct elements.

ex → $A = \{1, 2, 3, 4, \ldots, \infty\}$. (Repitition not allowed and order not important)

$B = \{x \mid (x \text{ is a positive integer}) \text{ and } 1 \leq x \leq 10\}$.

↓

$B = \{1, 2, 3, \ldots, 10\}$.

Null set (empty set) →
A set with no elements is called a "Null/empty set". Denoted as 'ϕ'

ex → $A = \{x \mid x$ is a prime no. and $8 \le x \le 10\}$.
↑ null set

Subset →

If every element of A is also an element of B, then A is subset of B.

ex → $A = \{a, b, c\}$ So, $A \subseteq B$

$B = \{a, b, c, \dots\}$.

* Note → For any set A, A and ϕ are called trivial subsets of A.

Proper Subset →

Any subset of A which is not a trivial subset of A is called proper subset of A.
It is denoted by '$\subset$'

ex → $A = \{1, 2, 3, 4\}$, $B = \{2, 4\}$.

$\therefore$ $B \subset A$ ← proper subset

* Note → If ($A \subseteq B$ and $B \subseteq A$), then $A = B$.

Power set of a set →

If A is a finite set then set of all subsets of A is called "power set of A".
It is denoted by $P(A)$

trivial subsets

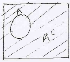
ex → If $A = \{a, b, c\}$, then $P(A) = \{\phi, \{a\}, \{b\}, \{c\}, \{a,b\}, \{b,c\}, \{a,c\}, \{a,b,c\}\}$

A

* Note → If $|A| = n$, then $|P(A)| = 2^n$.

Universal set →

Set of all objects under discussion. It is denoted as 'U'



Complement of a set →

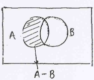
If A is any set then complement of A, denoted by $\bar{A}$ or A^c is defined as

$$A^c = \{x \mid x \notin A \text{ and } x \in U\}.$$

Set Difference →

If A and B are two sets, then

$$A - B = \{x \mid x \in A \text{ and } x \notin B\}.$$



ex : If $A = \{1, 2, 3, 4, 5\}$ and $B = \{1, 3, 5, 7, 9\}$.

then $A - B = \{2, 4\}$.

Set Intersection →

If A and B are two sets, then
$A \cap B = \{x \mid x \in A \text{ and } x \in B\}$.

ex: $A \cap B = \{1, 3, 5\}$. (prev. ex.)

Set Union →

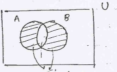
If A and B are two sets, then
$A \cup B = \{x \mid x \in A \text{ or } x \in B \text{ or } x \in A \cap B\}$.

ex: $A \cup B = \{1, 3, 5, 2, 4, 7, 9\}$.

Note → If $A \cap B$ is empty set, then A and B are called disjoint sets.

Symmetric Difference / Boolean sum →

$A \Delta B / A \oplus B = \{x \mid x \in A \text{ or } x \in B \text{ but } x \notin A \cap B\}$.



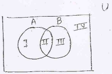
$A \oplus B / A \Delta B$

ex →
$A \oplus B = \{2, 4, 7, 9\}$.

Note → a) The symmetric difference of A and B →

$A \oplus B = (A - B) \cup (B - A)$
$= (A \cup B) - (A \cap B)$

b)



I. $(A-B) = (A \cap B^c)$

II. $(A \cap B)$

III. $(B-A) = (B \cap A^c)$

IV. $(A \cup B)^c = (A^c \cap B^c)$

c) For any 3 sets A, B, C, the following properties hold good:

1) If $A \subseteq B$, then $A \cup B = B$ and $A \cap B = A$

2) $(A^c)^c = A$

3) <u>Commutative laws:</u>

i) $(A \cup B) = (B \cup A)$

ii) $(A \cap B) = (B \cap A)$

iii) $(A \oplus B) = (B \oplus A)$

4) <u>Associative laws:</u>

i) $(A \cup B) \cup C = A \cup (B \cup C)$.

ii) $(A \cap B) \cap C = A \cap (B \cap C)$.

iii) $(A \oplus B) \oplus C = A \oplus (B \oplus C)$.

5) Distributive laws :

i) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

ii) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

6) De Morgan's laws :

i) $(A \cup B)^c = (A^c \cap B^c)$

ii) $(A \cap B)^c = (A^c \cup B^c)$

* iii) $A - (B \cup C) = (A - B) \cap (A - C)$

* iv) $A - (B \cap C) = (A - B) \cup (A - C)$

7) Idempotent laws :

i) $A \cup A = A$

ii) $B \cap B = B$

8) Absorption laws :

i) $A \cup (A \cap B) = A$

ii) $A \cap (A \cup B) = A$

9) Modular laws :

i) $(A \cup B) \cap C = A \cup (B \cap C)$ iff $A \subseteq C$

ii) $(A \cap B) \cup C = A \cap (B \cup C)$ iff $C \subseteq A$

10) $A \cup \phi = A$, $A \cap \phi = \phi$

$A \cup U = U$, $A \cap U = A$

$A \cup A^c = U$, $A \cap A^c = \phi$

$\phi, \{\phi\}, \{\phi, \phi\}$ (A)

$A = \{a, b, c, d\}$ $P(A) = \{\phi, \ldots$

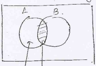
Q.1. which of the following is not true ?

✓ a) $A - (A - B) = B$ not true

b) $A - (A - B) = (A \cap B)$ true

c) $(A \cap B) \cup (A \cap B^c) = A$ true

d) $B \cap (A \cup B) = B$ true.



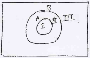
Q.2. which of the following is not true ?

a) If $(A \subset B)$, then $(B^c \subset A^c)$. true

b) $A \cap P(A) = \phi$ true

✓c) $A \cap P(A) = A$ not true.

d) $P(A) \cap P(P(A)) = \{\phi\}$ not true.



$A^c = II \,\&\, III$, $B^c = III$

$\therefore B^c \subset A^c$.

$A = \{a, b\}$

$P(A) = \{\phi, \{a\}, \{b\}, \{a, b\}\}$

$\therefore$ no common element.

Q.3. If $A = \phi$ then $|P(P(A))| = ?$

→ $A = \{\ \}$

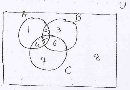
$P(A) = \{A\}$.

$P(P(A)) = \{A, P(A)\}$. $\therefore |P(P(A))| = \boxed{2}$

$A \oplus B$.

Q.4. Which of the following is not true ?

a) $(A-B)-C = (A-C)-B$. true



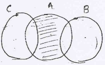
$\{1,4\} - \{4,5,6,7\}$

$= \{1\} =$ LHS.

$\{1,2\} - \{2,3,5,6\} = \{1\} =$ RHS.

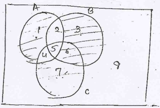
$\therefore$ LHS = RHS.

b) $(A-B)-C = (A-C)-(B-C)$. true



$(A-C)-(B-C)$.

c) $A \oplus (B \cup C) = (A \oplus B) \cup (A \oplus C)$ not true.



LHS $= \{1,2,4,5\} \oplus \{2,3,4,5,6,7\}$

$= \{1,3,6,7\}$.

R.H.S = $\{1,4,3,6\} \cup \{1,2,6,7\}$

$= \{1,2,3,4,6,7\}$.

$\therefore$ L.H.S $\neq$ R.H.S.

d) $A-(B \cup C) = (A-B) \cap (A-C)$. true

L.H.S = $\{1,2,4,5\} - \{2,3,4,5,6,7\}$

$= \{1\}$.

R.H.S = $\{1,4\} \cap \{1,2\}$.

$= \{1\}$.

L.H.S = R.H.S.

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Reflexive Relation →

A relation R on a set A is said to be reflexive,
if $(xRx)\ \forall x \in A$

i.e. $(x,x) \in R \quad \forall x \in A$.

<u>Note</u> → The diagonal relation on set A is reflexive and any superset of diagonal relation is also reflexive.

Ex. → Let A be $A = \{a, b, c\}$.

✓ $R_1 = \{(a,a), (b,b), (c,c)\}$ ← (The smallest reflexive relation on A is diagonal relation.)

✓ $R_2 = \{(a,a), (b,b), (c,c), (a,b), (b,a), (a,c)\}$.

✓ $R_3 = A \times A$ ← (The largest reflexive relation on set A).

Ex. → Let $A = \{1, 2, 3, \dots n\}$, then no of reflexive relations possible on A is ?

→ No. of non diagonal elements in $|A \times A| = n^2 - n$

$= n \cdot (n-1)$

∴ No. of subsets possible with $n(n-1)$ elements is $2^{n \cdot (n-1)}$.

∴ No. of reflexive relations $= \boxed{2^{n(n-1)}}$.

Q. 2 No. of relations on A which are not reflexive is?

→ Total no. of relations on A $= 2^{(n^2)}$

No. of reflexive relations on A $= 2^{n(n-1)}$

∴ No. of nonreflexive relations on A $= 2^{n^2} - 2^{n(n-1)}$

Note →

3\. 1) The relation '$\leq$' is reflexive on any set of real nos.

2) The relation 'is a divisor of' denoted by '$|$' is reflexive on any set of non-zero real nos.
(Every number is a divisor of itself.)

3) The relation 'is a subset of' denoted by '$\subseteq$' is reflexive on any collection of sets.
(Every set is a subset of itself).

Irreflexive Relation →

The relation 'R' on a set A is called irreflexive if x is not related to x i.e. $x \not R x \ \forall x \in A$ i.e. the orderedered pair $(x,x) \notin R \ \forall x \in A$

ex → Let $A = \{a,b,c\}$

✓ $R_1 = \{ \ \}$ ← (The smallest irreflexive relation on A is empty relation).

✓ $R_2 = \{(a,b),(b,c),(c,b)\}$

✓ $R_3 = A \times A - \underset{\substack{\uparrow \\ \text{diagonal relation.}}}{\Delta_A}$ ← (The largest irreflexive relat^n on A) i.e. $R_3 = \{(a,b), (b,a), (b,c), (c,b), (a,c), (c,a)\}$.

Note →

If A is a set with 'n' elements, then no. of irreflexive relations possible on A = $\boxed{2^{n\cdot(n-1)}}$

Q.1. Let A be a set with n elements, then no. of relations on A which are reflexive or irreflexive ?

→ $= 2^{n\cdot(n-1)} + 2^{n(n-1)}$

$= 2 \cdot 2^{n\cdot(n-1)} = \boxed{2^{n\cdot(n-1)+1}}$

Q.2. " – " no. of relations on A which are neither reflexive nor irreflexive ?

→ $= \boxed{2^{n^2} - 2^{n\cdot(n-1)+1}}$

(OR)

The diagonal elements pairs can be selected as $2^n - 2$ ways.
$\{(1,1), (2,2), \ldots (n,n)\}$.

∴ no. of relations which are neither reflexive nor irreflexive $= \boxed{(2^n - 2) \cdot 2^{n(n-1)}}$

Note → 1) The relation '<' is irreflexive on any set of real nos.

2) The relation '∈' is irreflexive on any collection of sets.

Symmetric Relation :-

A relation R on a set A is said to be symmetric if (xRy) then (yRx) $\forall x, y \in A$ i.e. if the ordered pair $(x, y) \in R$ then $(y, x) \in R$. $\forall x, y \in A$

ex :- Let $A = \{a, b, c\}$, then

✓ $R_1 = \{\ \}$ ← (The smallest symmetric relation on A is an empty relation).

✓ $R_2 = \{(a,a), (c,c)\}$.

✓ $R_3 = \{(a,b), (b,a)\}$

✓ $R_4 = A \times A$ ← (The largest symmetric relation on A).

te-1) Let A be a set with 'n' elements, no. of symmetric relations possible on A is, :-

No. of symmetric relations with diagonal pairs $= 2^n$.

No. of symmetric relations with nondiagonal pairs $= 2^{\frac{n(n-1)}{2}}$

∴ total no. of symmetric relations on A

$$= 2^n \cdot 2^{\frac{n(n-1)}{2}}$$

$$= \boxed{2^{\frac{n(n+1)}{2}}}$$

2) Let A be a set with 'n' elements then no. of relations on A which are reflexive and symmetric are →

$$= 2^{{}^nC_2} = \boxed{2^{\frac{n \cdot (n-1)}{2}}}$$

3) No. of relations which are reflexive but not symmetric →

$$= \underset{\uparrow}{2^{n \cdot (n-1)}} - 2^{\frac{n(n-1)}{2}}$$ ← no. of both refl. and symm. relations.

no. of refl. relations.

4) No. of relations which are symmetric but not reflexive →

$$\underset{\uparrow}{2^{\frac{n \cdot (n+1)}{2}}} - 2^{\frac{n \cdot (n-1)}{2}}$$

no. of symm. relations.

5) No. of relations which are neither reflexive nor symmetric →

$$2^{(n^2)} - \left(2^{\frac{n \cdot (n+1)}{2}} + 2^{n(n-1)} - 2^{\frac{n \cdot (n-1)}{2}}\right)$$

Q.1. The relation 'x' is a brother of 'y' is (symmetric or not?) symmetric on any set of men.

The relation is not symmetric on set of all people, because if x is a brother of y, y can be a sister of x.

2. The relation '' is a complement of' is symmetric on a boolean algebra.

Anti-Symmetric Relation :-

A relation 'R' on a set 'A' is said to be antisymmetric if (xRy and yRx) then ($x=y$) $\forall\ x, y \in A$

ex :- 1) The relation '$\leq$' is antisymmetric on any set of real nos.
(if $a \leq b$ and $b \leq a$, then $a=b$)

2) The relation '<' is antisymmetric on any set of real nos.
($\underbrace{\text{if } (2<3) \text{ and } (3>2)}_{F}$, then $a=b$). (always true)
whatever

3) The relation 'is a divisor of' denoted as '|' is antisymmetric on any set of +ve real nos.
(same logic as above).

4) The relation $\subseteq$ (set inclusion) is antisymmetric on any collection of sets.
(also, i) proper subset of '$\subset$'
ii) super set of '$\supseteq$').

Let $A = \{a, b, c\}$.

$R_1 = \{\ \}$ ← (The smallest antisymmetric relation on A)

$R_2 = \{(b,b), (a,a), (c,c)\}$.

$R_3 = \{(a,b), (c,b)\}$.

$R_4 = \{\underbrace{(a,a), (b,b), (c,c)}_{\text{All diagonal pairs}}, \underbrace{(a,b), (b,c), (c,a)}_{\text{half of nondiagonal pairs.}}\}$. ← (A largest antisymmetric relation on A)

Note: If A is a set with 'n' elements, then

1) No. of elements in a largest antisymmetric relation on A

$= n + \frac{n \cdot (n-1)}{2} = \boxed{\frac{n \cdot (n+1)}{2}}$

2) Let A be a set with 'n' elements, then the no. of antisymmetric relations possible on A →

$= \underbrace{2^n}_{\uparrow} \cdot \underbrace{3^{\frac{n \cdot (n-1)}{2}}}_{\uparrow}$ } ← No. of antisymmetric relations with nondiagonal pairs. $= 2^n \cdot 3^{{}^nC_2}$.

(Since, each nondiagonal combination can apper in 3 ways).

no. of antisymmetric relations with diagonal pairs.

For

$(1,2) \to (1,2)$ only

$\to (2,1)$ only.

$\to$ none.

$= \boxed{2^n \cdot 3^{\frac{n \cdot (n-1)}{2}}}$

3) Let $A = \{1, 2, 3, \ldots, n\}$. No. of relations on A which are symmetric and antisymmetric are $= 2^n$.

$$\{(1,1), (2,2), (3,3), \ldots, (n,n)\} \rightarrow (2^n)$$

4) No. of relations on A which are symmetric or antisymmetric

$$= \boxed{\underbrace{2^{\frac{n(n+1)}{2}}}_{\text{symm.}} + \underbrace{2^n \cdot 3^{\frac{n(n-1)}{2}}}_{\text{antisymm}} - 2^n} \leftarrow \text{symm. and antisymm.}$$

5) No. of relations on A which are symmetric but not antisymmetric

$$= \boxed{2^{\frac{n(n+1)}{2}} - 2^n}$$

6) No. of relations on A which are irreflexive and and antisymmetric

$$= 3^{^nC_2} = \boxed{3^{\frac{n(n-1)}{2}}}$$

antisymm. relatn with nondiagonal pairs.

7). Any subset of antisymmetric relation is also antisymm

Asymmetric Relation →

A relation 'R' on a set 'A' is said to be asymmetric if (xRy) then $(y \not R x)$ $\forall\, x, y \in A$

Note→1) Every asymmetric relation is antisymmetric.

2) In a asymmetric relation, diagonal pairs are not allowed. whereas, in antisymmetric relation, diagonal pairs can be present.

3) Every assymmetric relation is also irreflexive.

ex. → Let $A = \{a, b, c\}$

✓ $R_1 = \{\ \}$ ← (The smallest assymmetric relation on A is empty relation).

✓ $R_2 = \{(a,b), (a,c)\}$

✓ $R_3 = \{(a,b), (a,c), (b,c)\}$ ← (A largest asymmetric relation on A).

Note ⇒

1) If A is a set with 'n' elements, then no. of asymmetric relations possible on A

$= 3^{nC_2} = \boxed{(3)^{\frac{n(n-1)}{2}}}$ ← (Diagonal element pairs are not allowed).

Here also, 3 possibilities $1, 2 \rightarrow$ 1, 2 or 2, 1 or none

2) The only relation on A which is symmetric and asymmetric $= 1$ i.e. empty relation.

3) The no. of relations which are asymmetric and reflexive $= 0$. So, there is no relation which is asymmetric and reflective.

The no. of relations on A which are irreflexive but not asymmetric is

$$= \boxed{2^{n(n-1)} - 3^{\frac{n(n-1)}{2}}}$$

total irreflexive — irreflexive relations that are asymmetric.

4) The relations '$<$' is asymmetric on any set of real nos. (if $a<b$, then $b \not< a$)

($>$, $\subset$, $\supset$)

Transitive Relation →

A relation 'R' on a set 'A' is said to be transitive if (xRy) and (yRz), then (xRz) $\forall x,y,z \in A$

ex → If $A = \{a, b, c\}$, then

✓ $R_1 = \{\ \}$ ← (The smallest transitive relation on A is empty relation).

✓ $R_2 = \{(a,a), (c,c), (b,b)\}$

✓ $R_3 = \{(a,b), (c,c)\}$.

✓ $R_4 = \{(a,b), (a,c)\}$.

✓ $R_5 = \{(a,b), (b,c), (a,c)\}$.

(a,b) (b,a) $(a,b) | (b,a)$ ①
$a'b, (b,a), (a,a), (b,b)$.
(a,a) (b,b) ③
2^{n^2}

✓ $R_6 = \{(a,b), (b,a), (a,a), (b,b)\}$.

$R_7 = A \times A$ ← (The largest transitive relation on A)

Note →

1) If $A = \{a,b\}$, no. of transitive relations possible on A

→ The following relations on A are not transitive →

$R_1 = \{(a,b), (b,a)\}$

$R_2 = \{(a,b), (b,a), (a,a)\}$.

$R_3 = \{(a,b), (b,a), (b,b)\}$

Required no. of relations are $(2^{n^2} - 3)$ where $n = 2$

$= 2^4 - 3 = 16 - 3 = \boxed{13}$.

2) The relation '$\leq$' is transitive on any set of real nos.
($a \leq b$, $b \leq c$) then $a \leq c$).

$(\geq, <, >, |, \subset, \supset, \subseteq, \supseteq)$

3). The relation '$|$' is a divisor of '$|$' is transitive on any set of real nos. (If $2|4$, $4|8$; then $2|8$).

(set inclusion relation)

4) The relation 'is a subset of $\subset$' is transitive on any collection of sets. (if $A \subset B$, $B \subset C$; then $A \subset C$).

Equivalence Relations →

A relation 'R' on a set 'A' is said to be an 'equivalence relation on 'A' if 'R' is :

i) Reflexive ,
ii) Symmetric and
iii) Transitive.

ex → Let $A = \{a, b, c\}$, then how many equivalence relations are possible on A ?

→ ✓ $R_1 = \{(a,a), (b,b), (c,c)\}$ ← (The smallest equivalence relation on A i.e. diagonal relation)

✓ $R_2 = \{(a,a), (b,b), (c,c), (a,b), (b,a)\}$

✓ $R_3 = \{(a,a), (b,b), (c,c), (b,c), (c,b)\}$.

✓ $R_4 = \{(a,a), (b,b), (c,c), (a,c), (c,a)\}$

✓ $R_5 = \{(a,a), (b,b), (c,c), (a,b), (b,c), (a,c), (c,a), (c,b), (b,a)\}$ ← (The largest equivalence relation).

$A \times A$

Note →

1) ∴ total no. of equivalence relations for $(n=3) = \boxed{5}$.

2) total no. of equivalence relations for $(n=4) = \boxed{15}$

Q.1. which of the following is not an equivalence relation on set of all real nos A?

a) $R_1 = \{(a,b) \mid `a-b'$ is an integer$\}$.   equivalence relatn ✓
i.e. $aR_1b \Leftrightarrow (a-b)$ is an integer.

→ 1) we have, '$a-a$' is an integer. i.e. $\underline{0}$.
∴ aR_1a $\forall a \in A$
∴ R_1 is reflexive relation.

2) Also, if $a-b$ is an integer, $b-a$ is also an integer.
∴ if aR_1b, then bR_1a.
∴ R_1 is symmetric.

3) if aR_1b, so, $a-b$ is an integer.
also, bR_1c, so, $b-c$ is an integer.
∴ $a-c$ is also an integer.

∴ aR_1c.

∴ R_1 is transitive.

∴ R_1 is an equivalence relation.

b) $R_2 = \{(a,b) \mid `a-b'$ is divisible by $5\}$.   equivalence relatn ✓

→ 1) 0 is divisible by 5.   ∴ $a-a=0$ div. by 5.
∴ aR_2a ✓

∴ R_2 is reflexive.

2) $a-b$ div. by 5, then $b-a$ also div. by 5
(10 - 5)   (5 - 10)
∴ R_2 is symmetric. (∵ if aR_2b exists, bR_2a exists).

3) $a-b$ div. by 5 ; $b-c$ div. by 5
$(10-5)$ $(5-15)$

$\therefore$ $a-c$ also div. by 5 $(10-15)$

(∵ if aR_2b, bR_2c exists, aR_2c also exists).

$\therefore$ R_2 is transitive.

c) $R_3 = \{(a,b) \mid 'a-b' \text{ is an odd no.}\}$. not equivalence relatⁿ.

→ 1) if $a-a=0$, 0 is not an odd no.
$\therefore$ R_3 is not reflexive.

2) if $a-b =$ odd no, then $b-a =$ odd no.
$\therefore$ R_3 is symmetric.

3) $a-b =$ odd no. , $b-c =$ odd no.
$\therefore$ $a-c$ = need not be odd.
$\therefore$ R_3 is not transitive.

$\therefore$ Given relation is not an equivalence relation.

d) $R_4 = \{(a,b) \mid 'a-b' \text{ is an even no.}\}$. equivalence relatⁿ

Reflexive → $a-a=0$.

Symmetric → $a-b =$ even ; $b-a =$ even.

transitive → $a-b =$ even, $b-c =$ even ; $a-c =$ even.

Partial Ordering Relation (Partial order) →

A relation 'R' on a set 'A' is said to be a partial ordering relation (partial order) if R is reflexive, antisymmetric and transitive.

Partially ordered set (poset) →

A set 'A' with a partial order 'R' defined on 'A' is called Partially ordered set (poset) and it is denoted by $[A; R]$

ex. → 1) The relation '$\leq$' is a partial order on any set of real nos. A i.e. the set A w.r.t. '$\leq$' is a 'poset.'

$$[A; \leq]$$

2) The relation 'is a divisor of' ('|') is a partial order relation on any set of +ve integers.
i.e. $[A; |]$ is a poset.

3) The relation 'is a subset of ($\subseteq$)' is a partial order relatn on any collection of sets 'S'
i.e. $[S; \subseteq]$ is a poset.

4) Let $A = \{a, b, c\}$,

✓ $R_1 = \{(a,a), (b,b), (c,c)\}$ ← (The smallest partial order on A)

Note → and also, the only relation on set A which is both an equivalence relation and a partial order is the diagonal relatn of (A) given above.

$R_2 = \{(a,a), (b,b), (c,c), (a,b), (b,c), (a,c)\}$.
← (Largest partial order on A).

* 19 different partial orders are possible on this set A.

Totally ordered set → (Linearly ordered set / chain)
A poset $[A; R]$ is called a "totally ordered set" if every pair of elements in A are comparable
i.e. aRb or bRa $\forall a, b \in A$

Ex → 1) If A is any set of real nos. then the poset $[A; \leq]$ is a totally-ordered set

2) If $A = \{1, 2, 3, \ldots, 10\}$ then the poset $[A; |]$ is not a totally ordered set.

because 2 is not related to 3 and also 3 is not related to 2. So, 2 and 3 are not comparable.

3) If $A = \{1, 2, 6, 30, 60, 300\}$, then $[A; |]$ is a totally ordered set. because each pair is comparable.

4) If $S = \{\phi, \{a\}, \{b\}, \{a, b\}\}$, then $[S; \subseteq]$ is not a totally ordered set because singleton sets $\{a\}$ and $\{b\}$ are not comparable.

$\therefore \{a\} \nsubseteq \{b\}$ and $\{b\} \nsubseteq \{a\}$.

$1 \le 2 \le 3$.

power

toset

dig. re

5) If $S = \{\phi, \{a\}, \{a,b\}, \{a,b,c\}\}$ is, then $[S; \subseteq]$ is a totally ordered set.

Q.1. Let $A = \{a,b,c\}$, which of the following is not true?

a) $R_1 = \{(a,a),(c,c)\}$ is symmetric, antisymmetric and transitive on A. (correct)

b) $R_2 = \{(a,b),(b,a),(a,c)\}$ is symmetric and antisymmetric (false).

(c,a) should be there.

The relation R_2 is neither symmetric nor antisymm.
The relation R_2 is only irreflexive.

c) $R_3 = \{(a,b),(b,a),(c,c)\}$ is symmetric but not antisymmetric (correct).

d) $R_4 = \{(a,b),(b,c),(c,c)\}$ is antisymmetric but not symmetric (correct).

Q.2. Let $A = \{a,b,c,d\}$ and a relation on set A is defined as $R = \{(a,a),(b,a),(b,b),(b,c),(b,d),(c,a),(c,b),(c,c),(c,d)\}$ is

which of the foll. is true?

a) R is an equivalence relat^n.
→ There is no pair (d,d). ∴ it is not reflexive.
∴ It is not an equivalence relat^n.

b) R is an irreflexive or antisymm. relat^n.
→ Not irreflexive: (a,a), (b,b), (c,c) are present.
Not antisymm: (b,c) and (c,b) are present.

(a,b) (b,c) (a,c).

c) R is symmetric or asymmetric relatn.

→ Not symmetric → $(b,a) \in R$ but $(a,b) \notin R$

Not asymmetric → diagonal elements are present.

d) R is transitive

3. Let A = set of all real nos.

$R = \{(a,b) \mid b = a^k$ for some integer $k\}$.

i.e. $aRb \Leftrightarrow b = a^k$

e.g. $2R8 \Leftrightarrow 8 = 2^3$.

a) R is an equivalence relation.

b) R is partial order.

c) R is reflexive and symmetric but not transitive.

d) R is a total order.

→ $a = a^1$ $\therefore aRa$. $\therefore$ R is reflexive.

R is not symmetric →

$2 = 4$ i.e. $2 = 2^2$ $\therefore 2R4$.

but $4 \neq 2^?$ $\therefore 4 \not R 2$.

$(a,b),(b,c),(c,c)$ $a,b,\ b,a$ (a,a) 804 a4 $2 R_? \ 8^{?}$ 64 $a,b \quad b,a\ (a,a)$

R is not a total order →
$2 \not R 3$ and also $3 \not R 2$. ∴ Not every pair comparable.

R is antisymmetric →
if aRb and bRa then $a=b$.

R is transitive → then
if aRb and bRc, aRc
$(2R8)$ $(8R64)$ $(2R64)$

∴ Given relation is Reflexive, antisymmetric and transitive.

Q.4. which of the foll. statements is not true ?

a) If a relation R on a set A is symmetric and transitive then R is reflexive. (false)
→ Let $A = \{a, b, c\}$
$R_1 = \{\}$ ← symmetric and transitive but not reflexive.
$R_2 = \{(a,b),(b,a),(a,a)$ $(b,b)\}$ ← symmetric and transitive but not reflexive.

b) If a relation R on a set A is irreflexive and transitive then R is antisymmetric. (true)
→ Suppose, the given statement is false.
Let R be a relation on A which is irreflexive and transitive but not antisymmetric.

Now, let $(a,b) \in R$ and $(b,a) \in R$ ($\because R$ is not antisymmetric).

$(a,a) \in R$ ($\because R$ is transitive)

$\Rightarrow$ R is not irreflexive because a diagonal pair exists. which is a contradiction to our hypothesis. i.e. R is irreflexive.

$\therefore$ the given statement is true.

c) If R and S are antisymmetric relations on a set A then $(R \cup S)$ and $(R \cap S)$ are also antisymmetric. (false)

$\rightarrow$ Let $A = \{a,b,c\}$. $R = \{(a,b)\}$, $S = \{(b,a)\}$ $\Big\}$ antisymmetric.

Here, $R \cup S = \{(a,b),(b,a)\} \leftarrow$ not antisymmetric.

$R \cap S = \{\ \} \leftarrow$ antisymmetric. (always).

Any subset of antisymmetric relat^n is antisymm. and $R \cap S$ is a subset of R. Hence, it is always antisymm.

Note: If R is antisymmetric relation, then $R \cap S$ is antisymm. for any relation S on A.

d) If R and S are transitive relations, then $(R \cup S)$ need not be transitive but $(R \cap S)$ is always transitive. (true).

$\rightarrow$ Let $A = \{a,b,c\}$ $R = \{(a,b)\}$, $S = \{(b,c)\}$ $\Big\}$ transitive.

$R \cup S = \{(a,b), (b,c)\} \leftarrow$ not transitive

$\{(a,b)\ (b,c)\}$ a,b
$\{(a,b)\ (b,c)\}$ b,a a,b
a,b

$R \cap S \rightarrow$ always transitive.

Transitive Closure of A Relation →

Let 'R' be any relation on a set 'A' then transitive closure of R denoted as 'R^*' is defined as the smallest transitive relation on A which contains 'R'.

Note → If R is transitive, then $R^* = R$ (OR) R is transitive iff $R = R^*$

ex. Let $A = \{a, b, c\}$
and $R = \{(a,b), (b,c)\}$.

$\therefore R^* = \{(a,b), (b,c), (a,c)\}$.

Reflexive closure of A Relation →

Smallest reflexive relation on A which contains R is called 'Reflexive closure of R' and denoted as '$R^\#$'.

ex → Let $A = \{a, b, c\}$. $R = \{(a,b), (b,c)\}$.

then $R^\# = \{(a,b), (b,c), (a,a), (b,b), (c,c)\}$.

A.A. diagonal relatn.

$\boxed{R^\# = R \cup \delta(A)}$

Symmetric closure of a relation →

Smallest symmetric relation on A which contains R is called symmetric closure of R and denoted as 'R^+'.

$\boxed{R^{+} = (R \cup R^{-1})}$

ex → Let $A = \{a, b, c\}$ and $R = \{(a,b), (b,c)\}$.

$R^{+} = \{(a,b), (b,a), (b,c), (c,b)\}$.

Reflexive Symmetric closure of R →

Reflexive symmetric closure of R
= Symmetric Reflexive closure of R.

i.e. $(R^{+})^{\#} = (R^{\#})^{+}$

ex → $A = \{a, b, c\}$ $R = \{(a,b), (b,c)\}$.

$R^{+} = \{(a,b), (b,c), (b,a), (c,b)\}$.

LHS $= (R^{+})^{\#} = \{(a,b), (b,c), (b,a), (c,b), (a,a), (b,b), (c,c)\}$.

RHS $= (R^{\#})^{+} = \{(a,b), (b,c), (a,a), (b,b), (c,c), (b,a), (c,b)\}$.

Note)

i) Reflexive trans. closure of R

= trans. reflexive closure of R

i.e. $(R^{*})^{\#} = (R^{\#})^{*}$

ii) Symmetric transitive closure of R
$\neq$ (need not be) transitive symmetric closure of R

Let $A = \{a, b, c\}$ $\quad R = \{(a,b), (b,c)\}$.

$(R^*)^+ = \{(a,b), (b,c), (a,c), (b,a), (c,b), (c,a)\}$.

$(R^+)^* = \{(a,b), (b,c), (b,a), (c,b), (a,a), (b,b), (c,c), (a,c), (c,a)\}$.

$= A \times A$

$\therefore (R^*)^+ \neq (R^+)^*$

Q.1) Let $A = \{a, b, c\}$, $R = \{(a,a), (a,c), (b,b), (c,a), (c,b)\}$.
Find transitive closure of R. (R^*).

→ The matrix corresp. to given relatⁿ is

$$\begin{array}{c|ccc|} & a & b & c \\ \hline a & 1 & \cancel{0}\,1 & 1 \\ b & 0 & 1 & 0 \\ c & 1 & 1 & \cancel{0}\,1 \end{array}$$

To find R^*, applying column by row cartesian produ

	I	II	III
column	$\{a,c\}$	$\{b,c\}$	$\{a,c\}$
Row	$\{a,c\}$	$\{b\}$	$\{a,b,c\}$
	(c,c)	(b,b) (c,b)	(a,b)

$\therefore R^* = \{(a,a), (a,c), (b,b), (c,a), (c,b), (a,b), (c,c)\}$.

$= (A \times A) - \{(b,a), (b,c)\}$.

(warshall's algorithm) →

Q.2 let $A = \{a, b, c, d\}$

$R = \{(a,d), (b,a), (b,c), (c,a), (c,d), (d,c)\}$

Find the transitive closure of R i.e. R^*.

→

$$\begin{array}{c|cccc} & a & b & c & d \\ \hline a & \\ b & \\ c & \\ d & \end{array}
\left[\begin{array}{cccc} \cancel{0}1 & 0 & \cancel{0}1 & 1 \\ 1 & 0 & 1 & \cancel{0}1 \\ 1 & 0 & \cancel{0}1 & 1 \\ \cancel{0}1 & 0 & 1 & \cancel{0}1 \end{array}\right]$$

	I	II	III	IV
Column	{b, c}	ϕ	{b, d}	{a, b, c, d}
Row.	{d}	{a, c, d}	{a, d}	{a, c, d}

{b, d} {c, d}

{b, a}, {b, d} {a, a} {a, c} {c, c}
{d, a} {a, d} {a, d} {b, a} {b, c} {b, d}
{c, a} {c, d} {d, a} {d, c}
{d, d}.

$\therefore R^* = \{(a,d), (b,a), (b,c), (c,a), (c,d), (d,c), (b,d), (d,a), (d,d), (a,a), (a,c), (c,c)\}$.

Q.3 Let $A = \{a, b, c, d\}$. The relation R on the set A is defined by $R = \{(a,a), (b,a), (b,b), (b,c), (b,d), (c,a), (c,b), (c,c), (c,d)\}$.

$R^* = ?$

→

$$\begin{array}{c|cccc} & a & b & c & d \\ \hline a & \\ b & \\ c & \\ d & \end{array}
\left[\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & \cancel{0}1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

	I	II	III	IV
column	(a, b, c)	{b, c}	{b, c}	{b, c}
Row	(a).	{a, b, c, d}	{a, b, c, d}	ϕ.

(a, a), (b, a), (c, a)

{b, a} ...

So, no new pairs are added.

$\therefore$ $\boxed{R^* = R}$

$\therefore$ R is transitive relation.

Equivalence classes →

Let R be an equivalence relation on set A ; then for any element $x \in A$, equivalence class of x denoted by $[x]$ is defined as

$$\boxed{[x] = \{ y \mid y \in A \text{ and } (x, y) \in R \}.}$$

Note : 1) We can have $[x] = [y]$ even though $x \neq y$; $x, y \in A$

2) Set of all distinct equivalence classes of the elements of set A define a partition of A w.r.t relation R.

3.4. If $A = \{a, b, c, d, e\}$ and an equivalence relation R on set A is

$R = \{(a,a), (b,b), (c,c), (d,d), (e,e), (a,e), (e,a), (b,d), (d,b)\}$.

Find the partition of A w.r.t. R.

→ eq. class of $A = [a] = \{a, e\}$

$[b] = \{b, d\}$ $[c] = \{c\}$ $[d] = \{b, d\}$.

$[e] = \{a, e\}$.

So, the partitions are distinct equivalence classes.

∴ Required partition of A w.r.t. R

$= \{ [a], [b], [c] \}$

$= \{ \{a, e\}, \{b, d\}, \{c\} \}$

Q.5. If $A = \{a, b, c, d, e\}$ and a partition of A is given by $\{\{a, d\}, \{b, c, e\}\}$, then find the equivalence relatn on A w.r.t. Partition.

→ Required equivalence relation on A

= cartesian product of $\{a, d\}$ with itself, cartesian product of $\{b, c, e\}$ with itself.

$= \{ \{a, d\} \times \{a, d\}, \{b, c, e\} \times \{b, c, e\} \}$.

$= \{\{a, a\}, \{a, d\}, \{d, a\}, \{d, d\}, \{b, b\}, \{b, c\}, \{b, e\}, \{c, b\}, \{c, c\}, \{c, e\}, \{e, b\}, \{e, c\}, \{e, e\}\}$.

Q.6. If A is set of all real nos. and an equivalence relatn R on A is given by aRb iff $(a-b)$ is an integer. then.

i) Find $[1]$

ii) Find $[1/2]$

iii) Find $[\sqrt{2}]$.

→ $[x] = \{y \mid y \in A$ and $(x-y)$ is an integer$\}$.

i) $[1] = \{y \mid y \in A$ and $(1-y)$ is an integer$\}$.

= set of all integers.

ii) $[1/2] = \{y \mid y \in A$ and $(1/2 - y)$ is an integer$\}$

$(1-y) = 1-(n+1/2)$ where n is any integer.

$\therefore [1/2] = \{(n+1/2) \mid n$ is an integer$\}$.

$= \{\pm 1/2, \pm 3/2, \pm 5/2, \ldots\}$

iii) $[\sqrt{2}] = \{y \mid y \in A$ and $(\sqrt{2}-y)$ is an integer$\}$

$\therefore \sqrt{2}-y = \sqrt{2}-(n+\sqrt{2})$ where n is any integer

$\therefore [\sqrt{2}] = \{(n+\sqrt{2}) \mid n$ is any integer$\}$.

Q.7. Let $A = \{$set of all integers$\}$ and an equivalence relation R on A is defined by aRb iff $(a-b)$ is divisible by 3.

then find

i) $[0]$

ii) $[1]$

iii) $[2]$.

iv) How many distinct equivalence classes are possible ?

→ i) $[0] = \{ y \mid y \in A$ and $(0-y)$ is div. by $3 \}$.

$= \{ \ldots, -9, -6, -3, 0, 3, 6, 9, \ldots \}$

ii) $[1] = \{ y \mid y \in A$ and $(1-y)$ is div. by $3 \}$

$\therefore 1-y = 1-(3n+1)$, n is any integer.

$\therefore [1] = \{ \ldots -8, -5, -2, 1, 4, 7, 10, \ldots \}$.

iii) $[2] = \{ y \mid y \in A$ and $(2-y)$ is div. by $3 \}$

$\therefore 2-y = 2-(3n+2)$, n is any integer.

$\therefore [2] = \{ \ldots, -7, -4, -1, 2, 5, 8, 11, \ldots \}$.

iv) If we take $3n+3 \rightarrow$ first set gets repeated.
$3n+4 \rightarrow$ second set gets repeated.
$3n+5 \rightarrow$ third set gets repeated.

The given eg. relation divides the given set into 3 distinct equivalence classes.

Q.8. Let A = set of all people. An equivalence relation R on A is defined as aRb iff a and b were born in the same month. How many distinct equivalence classes are possible?

a) 31 b) 12 c) 53 d) 366 e) cannot be determined.

→ Since, we have 12 months.

$\therefore \boxed{12}$.

Least Upper Bound (LUB or Join or Supremum) →

Let $[A; R]$ be a poset. For $a, b \in A$, if there exists an element $c \in A$ such that

i) aRc and bRc

ii) if there exists any other element d such that (aRd) and (bRd) then (cRd), then c is called least upper bound of a and b.

Greatest Lower Bound (GLB or Meet) →

Let $[A; R]$ be a poset. For $a, b \in A$, if there exists an element $c \in A$ such that

i) (cRa) and (cRb)

and

ii) if there exists any other element d such that (dRa) and (dRb) then (dRc),

then c is called Greatest Lower Bound (GLB) of a and b.

Q.9 If A is any set of real nos., then $[A; \le]$ is a poset. For $a, b \in A$ find i) LUB of a and b. $= a \cup b$.

ii) GLB of a and b $= a \cap b$.

→ i) LUB of a and b $=$ max. of $\{a, b\}$.

ii) GLB of a and b $=$ min. of $\{a, b\}$

$c \mid 4$ $c \mid b$

$c \subseteq a$ $c \subseteq b$

$a \subseteq b$ $c \subseteq a$ $c \subseteq b$ $a \cap b$

- If A is any set of positive integers, then $[A; \mid]$ is a poset. For $a, b \in A$

→ LUB of a and b $= a \cup b =$ L.C.M of (a, b).

GLB of a and $b = a \cap b =$ G.C.D. of (a, b).

If S is any collection of sets, then $[S; \subseteq]$ is a poset. For $A, B, \in S$

→ LUB of A and $B =$ $A \cup B$.

GLB of A and B $= A \cap B$

- In a poset, the least upper bound of any two sets/elements if exists, is unique.

The above statement is true for GLB also.

LUB.
$a \subseteq b \quad b \subseteq c$.

Join Semi Lattice → ([A; R])

A poset 'A' w.r.t relation 'R' is called a join semi lattice if least upper bound exists for every pair of elements in A.

Meet Semi Lattice → ([A; R])

A poset 'A' w.r.t relation 'R' is called a meet semi lattice if GLB exists for every pair of elements in A.

Lattice →

A poset 'A' w.r.t 'R' is ([A; R]) is called a lattice if LUB and GLB exists for every pair of elements in 'A'.

3.
ex → Suppose if $A = \{1, 2, 3, \dots, 10\}$ w.r.t. divisibility, then the poset $[A; |]$ is a meet semilattice but not a join semilattice.

→ the given statement is true because the GLB of any two nos. is GCD of the two nos.
The GCD of any two nos. in the given set exists in the set.

However, the given poset is not a join semilattice
ex → LUB of 3 and 4 is equal to LCM of 3 & 4 = 12 which is not present in set.

If $S = \{\{a\}, \{b\}, \{a, b\}\}$ then the poset $[S; \subseteq]$ is a join semilattice but not a meet semilattice.

→ The LUB for $\subseteq$ operation on sets is $A \cup B$. The union of all pairs exists in set. So, it is join semilattice.

However, the GLB of {a} and {b} is ϕ which is not present in S. ∴ it's not meet semilattice.

If A is any set of real nos, then poset [A; ≤] is a lattice.

→ 3, 4 → LUB = 4 always exists
GLB = 3. for any pair of real nos.

∴ It is a lattice.

The poset given in this example is a totally ordered set and every totally ordered set is a lattice.

If A = {1, 3, 6, 12, 36}. The poset {A; |} is a lattice.

→ The given poset is a totally ordered set so true.

If A = set of all +ve integers, then poset [A; |] is a lattice.

→ Lcm of any two +ve integers is +ve integer ∈ A.

GCD of any two +ve integers is not <1.
∴ ∈ A.

[A; |].
∴ It is a lattice.

(+ve) lattice

Note →

If 'n' is a +ve integer then

D_n = set of all +ve divisors of n.

$D_6 = \{1, 2, 3, 6\}$.

$D_{12} = \{1, 2, 3, 4, 6, 12\}$.

$D_{30} = \{1, 2, 3, 5, 6, 10, 15, 30\}$.

If n is a +ve integer, then the poset $[D_n; |]$ is a lattice.

Note →

Let A = $\{a, b, c\}$.

$P(A) = \{\phi, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, c\}, \{a, b, c\}\}$.

then $[P(A); \subseteq]$ is always a lattice.

If A is a finite set, then poset $[P(A); \subseteq]$ is a lattice.

A lattice 'A' is denoted by the $[L, \underset{\text{LUB}}{\vee}, \underset{\text{GLB}}{\wedge}]$

The following properties hold good in a lattice.

(For any 3 element $a, b, c \in L$),

1) Commutative laws →

$a \vee b = b \vee a$

$a \wedge b = b \wedge a$

2) Associative laws →

$(a \vee b) \vee c = a \vee (b \vee c)$
$(a \wedge b) \wedge c = a \wedge (b \wedge c)$.

3) Idempotent laws →

$a \vee a = a$
$a \wedge a = a$

4) Absorption laws →

$a \vee (a \wedge b) = a$
$a \wedge (a \vee b) = a$

ote →
In a lattice L, $(a \vee b) = b$ iff $(a \wedge b) = a$, $\forall a, b \in L$

Sublattice →
Let L be lattice $[L, \vee, \wedge]$. A subset M of L is called a sublattice of L, if

i) M is a lattice i.e. $[M, \vee, \wedge]$.

ii) for any pair of elements $a, b \in M$, the LUB (and GLB) are same in M and L.

Distributive Lattice →
A lattice $[L, \vee, \wedge]$ is said to be distributive if the following distributive laws hold good →

i) $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$

ii) $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$

$\forall a, b, c \in L$.

Bounded Lattice →

Let L be a lattice w.r.t. R. If there exists an element $I \in L$ such that $(a R I) \forall a \in L$ then I is called upper bound of the lattice L.

Similarly, if there exists an element $O \in L$, such that $(O R a)$ $\forall a \in L$, then O is called lower bound of the lattice L.

In a lattice, if upper bound and lower bound exists then it is called a bounded lattice.

Note → In a bounded lattice, the upper bound (and also lower bound) is unique.

In a bounded lattice, the foll. properties hold good →

1) LUB of a and I i.e. $a \vee I = I$

2) GLB of a and I, i.e. $a \wedge I = a$.

3) LUB of a and O i.e. $a \vee O = a$

4) GLB of a and O i.e. $a \wedge O = O$.

e →

In a lattice or poset,
if aRb then $(a \vee b) = b$ ← LUB
and $(a \wedge b) = a$. ← GLB.

Complement of an element (in a lattice) →

Let L be a bounded lattice, for any element $a \in L$,
if there exists an element $b \in L$ such that
$(a \vee b) = I$
and $(a \wedge b) = 0$,
then b is called "complement of a" written as $\bar{a} = b$
i.e. a and b are complements of each other.

ote → 1) In a lattice, complement of an element may or may not exist.

If it exists, it need not be unique.

2) In a distributive lattice, complement of an element if exists, is unique. In a distributive lattice, each element has atmost one complement.

Complemented Lattice →

Let L be bounded lattice, if each element of L has a complement in L, then L is called a complemented lattice.

In a complemented lattice, each element has at ~~unique~~ least one complement.

Boolean Algebra →

A lattice 'L' is said to be a Boolean Algebra if 'L' is distributive and complemented.

In a boolean algebra, each element has an unique complement.

Hasse Diagram (Poset diagram) →

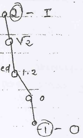
Let $[A; R]$ be a finite poset. On the poset diagram of A,

i) There is a vertex corresponding to each element of A.

ii) An edge between the elements 'a' and 'b' is not present in the diagram if there exists an element $x \in A$ such that (aRx) and (xRb)

iii) An edge between the elements 'a' and 'b' is present iff aRb and there is no element 'x' $\in A$ such that (aRx) and (xRb).

Q.1. If $A = \{-1, 0, 1.2, \sqrt{2}, 2\}$ then no. of edges in the Hasse diagram of the poset $[A; \leq]$.



→ -1 is Lower Bound.

and since every element is related 2, it is upper bound.

∴ No. of edges = $\boxed{4}$.

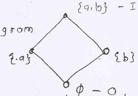
Note: In a totally ordered set, the complement exists only for upper bound (I) and lower bound (O).

∴ the given poset is not a complemented lattice. However, the given poset is a bounded and distributive lattice.

2) Every totally ordered set is a distributive lattice.

Q.2. If $S = \{\phi, \{a\}, \{b\}, \{a,b\}\}$. then number of edges in the poset diagram of the poset $[S; \subseteq]$ is ?

→ ∴ No. of edges in Hasse diagram $= \boxed{4}$



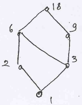
LUB and GLB exists for every pair of elements. Hence, the given poset is distributive.

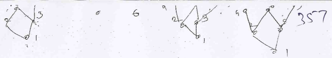
for {a} and {b}, GLB is ϕ and LUB is {a,b}.
Also, given poset is complemented lattice.

The poset given in the example is a <u>boolean</u> <u>algebra.</u>

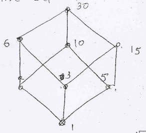
Q.3. If $A = \{1, 2, 3, 6, 9, 18\}$, then no. of edges in the poset diagram of poset $[A; |]$ is ?

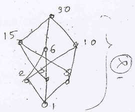
→ ∴ No. of edges $= \boxed{7}$





Q.4. The Let $A = \{1, 2, 3, 5, 6, 10, 15, 30\}$ then





$\therefore$ no. of edges = $\boxed{12}$.

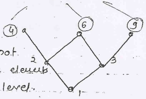
complement of 2 $\rightarrow$ 15 ($\because$ GLB of 2 and 15 is 1 i.e. Lower bound

and LUB of 2 and 15 is 30 i.e. upper bound)

Q.5. Let $A = \{1, 2, 3, 4, 6, 9\}$ then no. of edges in the hasse diagram of poset $[A; /]$ is

$\rightarrow$ No. of edges = $\boxed{6}$

maximal elements.



Here, upper bound does not exist as more than one elements exist at the same top level.

e.g.

Maximal elements $\rightarrow$

If in a poset, an element is not related to any other element then it is called maximal element.

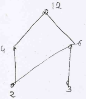
For any two maximal elements, LUB does not exist.

The given poset is not a lattice.

→ However, here, for every pair of elements in the poset, GLB exists. Hence, the given poset is a meet semilattice but not a join semilattice.

Q.6. Let $A = \{2, 3, 4, 6, 12\}$ then no. of edges in the Hasse diagram of the poset $[A; |]$ is ?

→ ∴ No. of edges = $\boxed{5}$.



Here, Lower bound does not exist as there are two minimal elements 2 and 3.

Note →

Minimal elements →

In a poset, an element is called minimal, if no other elements of the poset is related to it.

For any two minimal elements, GLB does not exist.

→ The given poset is not a lattice. However, for every pair of elements in the poset, LUB exists. Hence, the given poset is a join semilattice and not a meet semilattice.

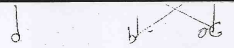
Q.7. The poset $[\{2, 3, 4, 9, 12, 18\}; |]$ is

a) a join semilattice but not a meet semilattice.

b) a meet semilattice but not a join semilattice.

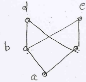
c) a lattice

✓d) neither a join semilattice nor a meet semilattice.



If the hasse diagram (the given poset) has one of the above structures, It is a lattice.

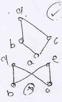
8. The poset diag. of a poset $P = \{a, b, c, d, e\}$ is shown below,



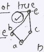
which of the foll. statements is not true?

a) P is not a lattice. true.

b) The subset $\{a, b, c, d\}$ of P is a lattice. true.



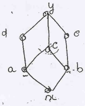
✓ c) The subset $\{b, c, d, e\}$ of P is a lattice. not true



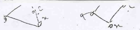
d) The subset $\{a, b, c, e\}$ of P is a lattice. true

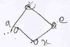
› The given poset is neither join semilattice and meet sublattice because for d and e GLB does not exist.

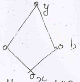
9. The hasse diag. of a Lattice $L = \{x, a, b, c, d, e, y\}$ is shown below.



which of the foll. subsets of L are sublattices of L?





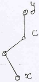


a) $\{x, a, b, y\}$

→ It is a lattice. But the LUB of a and b is not same as in original. In L, LUB of a and b is c, here it is y.

Hence, this is not a sublattice of L.

b) $\{x, a, c, y\}$



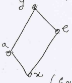
Here, LUB of a and c = c

GLB of a and c = a.

(same as in original).

∴ It is a sublattice of L.

c) $\{x, a, e, y\}$



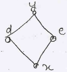
LUB of a and e = y

GLB of a and e = x.

(Same as in given L).

∴ It is a sublattice of L.

d) $\{x, d, e, y\}$

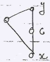


LUB of d and e = y

GLB of d and e = x

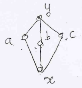
(same as in given L).

e) $\{x, c, d, y\}$

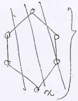


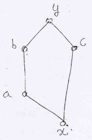
It is not a sublattice of L because GLB of c and d here is x and a in L.

15. which of the following lattices is/are not distributive?



L_1^*





L_2^*

→ i) $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$.

LHS RHS.

$a \vee x = y \wedge y$

$\Rightarrow a \neq y$

$\therefore$ The given L_1^* is not distributive.

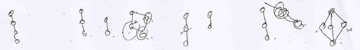
ii) $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$.

$\Rightarrow a \vee x = b \wedge y$

$\Rightarrow a \neq b$

$\therefore$ The given L_2^* is not distributive.

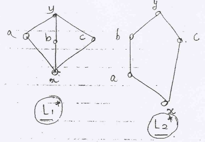
Note> In a distributive lattice, each element can have at most one complement. If it is not true, the lattice is not distributive.



In L_1^*, for the element a, we have two complements b and c. Hence, L_1^* is not a distributive lattice.

In L_2^*, similarly, for the element c, we have two complements a and b. Hence, L_2^* is not a distributive lattice.

Note → A lattice L is not distributive iff L has a sublattice which is isomorphic to L_1^* or L_2^*.



Q 11. Which of the following statements is not true

a) A lattice with 4 or fewer elements is distributive.
→ The statement follows from above thm. true

b) Every totally ordered set is a distributive lattice.
→ The statement follows from thm as a totally ordered set is a chain and a chain cannot have a sublattice isomorphic to L_1^* or L_2^* true

c) Every sublattice of a distributive lattice is also distributive. true
→ This statement follows from above thm.

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d) Every distributive lattice is a bounded lattice. false

→ Let $N = \{1, 2, 3, 4, 5, \ldots \infty\}$

the poset $[N; \leq]$ is a totally ordered set and hence a distributive lattice. But it is not bounded lattice because the upper bound of the lattice does not exist. Hence, it is false.

12. Which of the following is not a distributive lattice ?

a) $[P(A); \subseteq]$ where $A = \{a, b, c, d\}$. ✓

→ The elements of the power set are sets, and for any 3 sets, distributive laws hold good.

∴ $[P(A); \subseteq]$ is a distributive lattice.

b) $[D_{81}; |]$ ✓

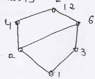
→ $D_{81} = \{1, 3, 9, 27, 81\}$

the given lattice It is a totally ordered set and hence, a distributive lattice.

c) $[D_n; |]$ (∵ n is any positive integer) ✓

→ $[D_n; |]$ is a distributive lattice

e.g. → $[D_{12}; |] = [\{1, 2, 3, 4, 6, 12\}; |]$ is a totally ordered set.



d) $[\{1,2,3,5,30\};|]$ not distributive.

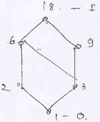
Q.13. For the lattice $[D_{18};|]$, which of the foll. are not true?

a) The complement of $1 = 18$ true

b) " - "of $2 = 9$ true

✓c) " - " of $3 = 6$, false

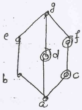
d) " - " of ~~3~~ 6 does not exist. true.
and 3



→ The lattice given in this example is a distributive lattice but not a complemented lattice.

Q.14. For the lattice shown below, how many complements does the element e have?

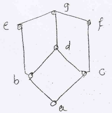
a) 1 b) 2 ✓c) 3 d) 4.



→ f, c and d are complements of e.

The lattice given in this example is a complemented lattice (as each element has at least one complement) but not a distributive lattice (as each element has more than one complement).

15. The lattice shown below is



→ a) dist lattice but not a complemented lattice.

b) complemented but not a dist lattice.

c) Boolean Algebra (both dist. and complemented)

✓d) Neither distributive nor complemented lattice.

→ $\underline{e}$ has $\underline{\text{two complements}}$ f and c. So, not a distr. lattice.

$\underline{d}$ has $\underline{\text{no complement}}$. So, not a complemented lattice.

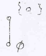
16. If A is a finite set then the poset $[P(A); \subseteq]$ is a Boolean Algebra.

→ Complement of $x = (A - x)$ $\forall\ x \in P(A)$

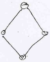
17. If $A = \{a, b, c\}$ then the poset $[P(A); \subseteq]$ is a Boolean algebra.

→ complement of $\{a, c\} = \{b\}$.

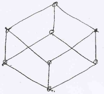
If $A = \{a\}$. → Hasse diagram



If $A = \{a, b\}$ → Hasse diagram



If $A = \{a, b, c\}$ → Hasse diagram



Note: 1) Every Boolean algebra is isomorphic to one of the power sets given in the above ex.

2) So, in the hasse diagram of Boolean algebra, we have 2^n vertices and $n \cdot 2^{n-1}$ edges.

3) A lattice with two elements is a totally ordered set and also a Boolean algebra. A totally ordered set with n elements is a Boolean algebra iff $n = 2$.

Square-Free Integer →

A positive integer 'n' is said to be square free if the set D_n has no perfect squares (except 1).

(OR)

A positive integer n is square free if n is a product of distinct prime nos.

Theorem →

The poset $[D_n ; |]$ is a boolean algebra iff n is a squarefree number/integer.

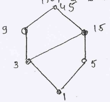
If the poset $[D_n; |]$ is a boolean algebra then complement of $x = \boxed{\frac{n}{x}}$ $\forall x \in D_n$

18. which of the following is not a boolean algebra?

a) $[D_{110}; |]$ → 110 (1, 2, 5, 11) is a squarefree integer. So, given poset is a Boolean algebra.

b) $[D_{91}; |]$ → 91 (1, 7, 13) is a squarefree integer. So, given poset is a Boolean algebra.

c) $[D_{45}; |]$ → 45 (1, 5, 9) is not a squarefree integer. So, it is not a boolean algebra. (it is distributive but not a complemented lattice).



d) $[D_{64}; |]$ → 64 is not a squarefree no. So, not a boolean algebra.

Q.19. In the boolean algebra $[D_{110}; |]$, complement of $22 = ?$

→ complement of $22 = \frac{110}{22} = 5$.

complement of $5 = \frac{110}{5} = 22$.