Chapter 3

Partial Differential Equations

CHAPTER HIGHLIGHTS

- *Fourier series*
- *Heat equation*

Fourier Series

Periodic Function A function $f(x)$ is said to be periodic if $f(x + a) = f(x)$ for all x. The least value of a is called the period of $f(x)$.

Example: $\sin x$, $\cos x$ are periodic functions with period 2π.

NOTES

1. $f(x)$ and $g(x)$ are periodic functions with period k then $af(x) + bg(x)$ is also a periodic function with period k.
2. If $f(x)$ is a periodic function with period k, then the period of $f(ax)$ is $\frac{k}{a}$.
3. If the periods of functions $f(x)$, $g(x)$ and $h(x)$ are a, b, c, respectively, then the period of $f(x) + g(x) + h(x)$ is the lcm of a, b and c.

Euler's Formula for the Fourier Coefficients

Let $f(x)$ is a periodic function whose period is 2π and is integrable over a period. Then $f(x)$ can be represented by trigonometric series.

$$f(x) = \frac{a_0}{2} + \int_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

where a_0 a_n, b_n are called Fourier coefficients and these are obtained by

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx,$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx \text{ for } n = 1, 2, 3, \ldots$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx \text{ for } n = 1, 2, 3, \ldots$$

SOLVED EXAMPLES

Example 1

Obtain the Fourier series expansion of $f(x) = e^x$ in $(0, 2\pi)$.

Solution

$$a_0 = \frac{1}{\pi}\int_{0}^{2\pi} f(x)dx$$

$$= \frac{1}{\pi}\int_{0}^{2\pi} e^x dx$$

$$= \frac{1}{\pi} e^x \Big]_0^{2\pi} = \frac{1}{\pi}(e^{2\pi} - 1) \qquad (1)$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx dx$$
$$= \frac{1}{\pi}\int_0^{2\pi} e^x \cos nx dx$$

we know that $\int e^{ax}\cos bx dx$

$$= \frac{e^{ax}}{a^2+b^2}[a\cos bx + b\sin bx]$$

$$\therefore a_n = \frac{1}{\pi}\left[\frac{e^x}{1+n^2}(\cos nx + n\sin nx)\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[\frac{e^{2\pi}}{1+n^2}(\cos 2\pi n) - \frac{1}{1+n^2}\right]$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx dx = \frac{1}{\pi}\int_0^{2\pi} e^x \sin nx dx$$

$$= \frac{1}{\pi}\frac{e^x(\sin nx - n\cos nx]_0^{2\pi}}{1+n^2}$$

$$\left(\because \int e^{ax}\sin bx dx = \frac{e^{ax}}{a^2+b^2}\right)(a\sin bx - b\cos bx))$$

$$= \frac{1}{\pi}\frac{1}{1+n^2}(n - e^{2\pi} n\cos 2\pi n)$$

$$= \frac{n}{\pi(1+n^2)}(1 - e^{2\pi}\cos 2\pi n)$$

$$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

$$= \frac{1}{2\pi}(e^{2\pi} - 1) + \sum_{n=1}^{\infty}$$

$$\left[\frac{1}{\pi}\frac{1}{1+n^2}(e^{2\pi}\cos 2\pi n - 1) + \frac{n}{\pi(1+n^2)}(1 - e^{2\pi}\cos 2\pi n)\right]$$

Even and Odd Functions

Even function: A function $f(x)$ is said to be even if $f(-x) = f(x)$ for all x.

Example: x^2, $\cos x$

Odd function: A function $f(x)$ is said to be odd if $f(-x) = -f(x)$ for all x

Example: x^3, $\sin x$

NOTES

1. The sum of two odd functions is odd.
2. The product of an odd function and an even function is odd.
3. Product of two odd functions is even.

Fourier Series for Odd Function and Even Function

Case 1: Let $f(x)$ is an even function in $(-\pi, \pi)$. Then the Fourier series of the even function contains only cosine terms and is known as Fourier cosine series and it is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx, \quad \text{where}$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)dx, a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx dx$$

Case 2: If $f(x)$ is an odd function, then the Fourier series of an odd function contains only sine terms, and is known as Fourier sine series.

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx,$$

where $$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx dx$$

Example 2

Expand the function $f(x) = \frac{\pi^2}{24} - \frac{x^2}{8}$ in Fourier series in the interval $(-\pi, \pi)$.

Solution

$$f(x) = \frac{\pi^2}{24} - \frac{x^2}{8}$$

$$f(-x) = \frac{\pi^2}{24} - \frac{(-x)^2}{8} = \frac{\pi^2}{24} - \frac{x^2}{8} = f(x)$$

$\therefore f(x)$ is an even function.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)dx = \frac{2}{2\pi}\int_0^{\pi}\frac{\pi^2}{24} - \frac{x^2}{8} dx$$

$$= \frac{1}{\pi}\left(\frac{\pi^2 x}{24} - \frac{x^3}{24}\right)\Bigg]_0^{\pi} = 0$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx dx\, s$$

$$= \frac{2}{\pi}\int_0^{\pi}\left(\frac{\pi^2}{24} - \frac{x^2}{8}\right)\cos nx dx$$

$$= \frac{2}{\pi}\left\{\begin{array}{l}\left(\frac{\pi^2}{24} - \frac{x^2}{8}\right) - \left(\frac{\sin nx}{n}\right)\Bigg]_0^{\pi} \\ -\frac{1}{8}\int_0^{\pi}\left(\frac{-(-2x)\sin nx}{n}\right)dx\end{array}\right\}$$

$$= -\frac{2}{\pi}\int_0^{\pi}\frac{2x}{8}\frac{\sin nx}{n}dx$$

$$= \frac{2}{\pi}\left[\frac{2}{8n}\left\{\left[\frac{-x\cos nx}{n}\right]_0^{\pi} - \int_0^{\pi}\frac{\cos nx}{n}dx\right\}\right]$$

$$= \frac{-4}{8\pi n^2}(\pi\cos n\pi)$$

$$= \frac{-1}{2n^2}(\text{con}\, n\pi), n = 1, 2, 3, \ldots$$

$$= \frac{(-1)^{n+1}}{2n^2}$$

$$\therefore f(x) = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{2n^2}\cos nx$$

$$= \frac{1}{2}\left[\cos x - \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} - \cdots\right].$$

Function of Any Period (P = 2L)

If the function $f(x)$ is of period $P = 2L$ has a Fourier series, then $f(x)$ can be expressed as,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\cos\frac{n\pi}{L}x + b_n\sin\frac{n\pi}{L}x)$$

where the Fourier coefficients are as follows:

$$a_0 = \frac{1}{L}\int_{-L}^{L}f(x)dx$$

$$a_n = \frac{1}{L}\int_{-L}^{L}f(x)\cos\frac{n\pi}{L}xdx$$

$$b_n = \frac{1}{L}\int_{-L}^{L}f(x)\sin\frac{n\pi}{L}xdx$$

Fourier Series of Even and Odd Functions Let $f(x)$ be an even function in $(-L, L)$, then the Fourier series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}a_n\cos\frac{n\pi x}{L}$$

Where $$a_0 = \frac{1}{L}\int_{-L}^{L}f(x)dx = \frac{2}{L}\int_0^{L}f(x)dx$$

$$a_n = \frac{2}{L}\int_{-L}^{L}f(x)\cos\frac{n\pi x}{L}dx$$

Let $f(x)$ be an odd function in $(-L, L)$ then Fourier series is

$$f(x) = \sum_{n=1}^{\infty}b_n\sin\frac{n\pi x}{L}$$

where $b_n = \frac{2}{L}\int_0^{L}f(x)\sin\frac{n\pi}{L}dx.$

Half Range Expansion

In the pervious examples we define the function $f(x)$ with the period $2L$.

Suppose $f(x)$ is not periodic function and defined in half the interval say $(0, L)$ of lengths L. such expansions are known as half range expansions or half range Fourier series. In particular a half range expansion containing only cosine series of $f(x)$ in the interval $(0, L)$ in a similar way half range Fourier sine series contains only sine terms. To find the Fourier series of $f(x)$ which is neither periodic nor even nor odd we obtain Fourier cosine series and Fourier sine series of $f(x)$ as follows. We define a function $g(x)$ such that $g(x) = f(x)$ in the interval from $(0, L)$ and $g(x)$ is an even function in $(-L, L)$ and is periodic with period $2L$ and g(x) is obtained by previous methods which are discussed earlier. Similarly we can obtain a fourier sine series as follows. Assume $f(x) = h(x)$ in $(0, L)$ and $h(x)$ is an odd function in the interval $(-L, L)$ with period $2L$ and evaluate $h(x)$ by pervious methods which are discussed earlier.

Example 3

If $f(x) = 1 - x$ in $0 < x < 1$ find Fourier cosine series and Fourier sine series.

Solution

Given $f(x) = 1 - x$ in $0 < x < 1$ since $f(x)$ is neither periodic nor even nor odd function.

Let us assume $g(x) = f(x) = 1 - x$ in $0 < x < 1$

$$= 1 + x \text{ in } -1 < x < 0$$

$\therefore$ $g(x)$ is even function in $(-1, 1)$

$$\therefore g(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}a_n\cos\frac{n\pi x}{L}$$

$$a_0 = \frac{2}{L}\int_0^{L}f(x)dx = 2\int_0^{1}f(x)dx \text{ (here } L = 1)$$

$$= 2\int_0^{1}(1-x)dx = 2\left[x - \frac{x^2}{2}\right]_0^1 = \frac{1}{2}\times 2 = 1$$

$$a_n = \frac{2}{L}\int_0^{L}f(x)\cos\frac{n\pi x}{L}dx$$

$$= 2\int_0^{1}(1-x)\cos n\pi x dx$$

$$= 2\left[(1-x)\frac{\sin n\pi x}{n\pi}\Big|_0^1 - \int_0^{1}(-1)\frac{\sin n\pi xx}{n\pi}\right]$$

$$= 2\left[(1-x)\frac{\sin n\pi x}{n\pi}\Big|_0^1 - \int_0^1 (-1)\frac{\sin n\pi x}{n\pi}dx\right]$$

$$= -2\frac{\cos n\pi x}{n^2\pi^2}\Big]_0^1 = 2\left(\frac{1}{n^2\pi^2} - \frac{\cos n\pi}{n^2\pi^2}\right)$$

$$= \frac{2}{n^2\pi^2}(1-(-1)^n).$$

∴ Fourier cosine series is

$$g(x) = \frac{1}{2} + \frac{2}{\pi^2}\sum_{n=1}^{\infty}\frac{1-(-1)^2}{n^2}\cos n\pi x$$

Fourier sine series in (0, 1)

$$\therefore\ h(x) = f(x) = 1 - x;\ 0 < x < 1$$
$$= -(1+x);\ -1 < x < 0$$

$h(x)$ is an odd function

$$\therefore h(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}$$

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx$$

$$= 2\int_0^1 (1-x)\sin n\pi x dx$$

$$= 2(1-x)\frac{\cos n\pi x}{-n\pi}\Big]_0^1 - 2\int_0^1 \frac{\cos n\pi x}{n\pi}dx = 2/n\pi$$

$$\therefore h(x) = 2/\pi \int_{n=1}^{\infty}\frac{1}{n}\sin(n\pi x)$$

Partial Differential Equations (PDE)

An equation involving two or more independent variables and a dependent variable and its partial derivatives is called a partial differential equation.

$$\therefore\quad f\left(x, y, z, \frac{\partial z}{\partial z}, \frac{\partial z}{\partial y}\ldots\right) = 0.$$

Standard Notation

$$\frac{\partial z}{\partial x} = p = z_x,\quad \frac{\partial z}{\partial y} = q = z_y$$

$$\frac{\partial^2 z}{\partial x^2} = r = z_{xx},\quad \frac{\partial^2 z}{\partial y^2} = t = z_{yy}$$

$$\frac{\partial^2 z}{\partial x \partial y} = z_{xy} = s$$

Formation of Partial Differential Equations

Partial differential equation can be formed by two ways.

1. By eliminating arbitrary constants.
2. By eliminating arbitrary functions.

Formation of PDE by Eliminating Arbitrary Constants

Consider a function $f(x, y, z, a, b) = 0$ where a, b, are arbitrary constants.

Differentiating this partially wrt, x and y eliminate a, b from these equations we get an equation $f(x, y, z, p, q) = 0$, which is partial differential equation of first order.

Example 4

$z = ax^2 - by^2$, a, b are arbitrary constants.

Solution

Given

$$z = ax^2 - by^2 \tag{1}$$

Differentiating z partially wrt x,

$$\frac{\partial z}{\partial x} = 2ax \ \Rightarrow\ p = 2ax \ \Rightarrow\ a = \frac{p}{2x}$$

Differentiate z partially wrt y,

$$\frac{\partial z}{\partial y} = 2by,\ \text{i.e.,}\ q = -2by$$

$$\Rightarrow\ b = \frac{-q}{2y}$$

Substituting the values of a and b in Eq. (1), we get

$$z = \frac{p}{2x}x^2 + \frac{q}{2y},\ y^2$$

$2z = px + qy$ which is a partial differential equation of order 1.

Formation of PDE by Eliminating Arbitrary Function

Consider $z = f(u)$ (1)

f is an arbitrary function in u and u is function in x, y, z.

Now differentiate Eq. (1) wrt x, y partially by chain rule we get

$$\frac{\partial z}{\partial x} = \frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial z}\cdot\frac{\partial z}{\partial x} \tag{2}$$

$$\frac{\partial z}{\partial y} = \frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial z}\cdot\frac{\partial z}{\partial y} \tag{3}$$

by eliminating the arbitrary functions from Eqs. (1), (2), (3) we get a PDE of first order.

Formation of PDE when Two Arbitrary Functions are Involved

When two arbitrary functions are involved, we differentiate the given equation two times and eliminate the two arbitrary functions from the equation obtained.

Example 5

Form the partial differential equation of

$$z = \frac{f(x)}{g(y)}$$

Solution

Given $z = \dfrac{f(x)}{g(y)}$

$$p = z_x = \frac{f'(x)}{g(y)} \quad (1)$$

$$q = z_y = \frac{-f(x)}{[g(y)]^2} \cdot g'(y) \quad (2)$$

$$s = \frac{\partial^2 z}{\partial x \partial y} = \frac{-f'(x)}{[g(y)]^2} \cdot g'(y) \quad (3)$$

$$\text{Eq.(1)} \times \text{Eq.(2)} = pq = \frac{f'(x)}{g(y)} \cdot \left(\frac{-f(x) \cdot g'(y)}{[g(y)]^2} \right) = -s \cdot z$$

$\because \quad pq + sz = 0$

Forming PDE by the Elimination of Arbitrary Function of Specific Functions

Consider $f(u, v) = 0$

Where u, v are the functions in x, y, z.

Differentiate the above equation wrt x and y by chain rule and eliminate the $\dfrac{\partial F}{\partial u}$, $\dfrac{\partial F}{\partial v}$ and convert them in the form $Pp + Qq = R$, which is a first order linear PDE where P, Q, R are functions of x, y, z.

Linear Equation of First Order

Linear equation of first order is $Pp + Qq = R$. This is also called Lagrange's equation, where P, Q, R are the functions in x, y, and z.

Procedure For solving Lagrange's Equations

Take the auxiliary equation as

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}.$$

Solve any two equations and take the solutions as u and v. The complete solution is $\phi(u, v) = 0$ or $u = f(v)$.

Example 6

Solve $(z - y)p + (x - z)q = y - x$.

Solution

Auxiliary equation is

$$\frac{dx}{z - y} = \frac{dy}{x - z} = \frac{dz}{y - x}.$$

Using the multipliers as x, y, z we get

$$\frac{x\,dx + y\,dy + z\,dz}{x(z - y) + y(x - z) + z(y - x)}$$

$$= xdx + ydy + zdz = 0$$

$$\therefore \quad x^2 + y^2 + z^2 = 0$$

and also $\dfrac{dx + dy + dz}{z - y + x - z + y - z} = 0$

$$dx + dy + dz = 0, x + y + z = 0.$$

$\therefore$ The required solution is $x^2 + y^2 + z^2 = f(x + y + z)$.

Non-linear Equations of First Order

There are four types of non linear equations of first order.

Type 1:

$$f(p, q) = 0.$$

If the given equation contains only p and q then the solution is taken as $z = ax + by + c$. Where a, b and c are arbitrary, such that $f(a, b) = 0$.

Example 7

Solve $2p + 3q = 5$

Solution

Given $2p + 3q = 5$

$z = ax + by + c$.

Where $2a + 3b = 5$,

$$b = \frac{5 - 2a}{3}$$

$\therefore$ The solution is $z = ax + \left(\dfrac{5 - 2a}{3}\right) y + c$.

Type 2:

$$f(z, p, q) = 0$$

When the equation is not containing x and y then to solve the equation assume $u = x + ay$ and substitute $p = \dfrac{dz}{du}$, $q = a\dfrac{dz}{du}$.

Solve the resulting equation and replace u by $x + ay$.

Type 3:

$$f(x, p) = g(y, q).$$

The equation is not containing z.

Assume $f(x, p) = a$ and $g(y, q) = a$.

Solve the equations for p and q and then write the solution.

Example 8

Solve $p^2 - q^2 = x^2 - y^2$.

Solution

$$p^2 - q^2 = x^2 - y^2$$

$$p^2 - x^2 = -y^2 + q^2$$

Let $\quad p^2 - x^2 = a^2 = -y^2 + q^2$

$$p^2 = a^2 + x^2 \qquad q^2 = y^2 + a^2$$

$$p = \sqrt{a^2 + x^2} \qquad q = \sqrt{a^2 + y^2}$$

$\therefore$ Take $dz = pdx + qdy$

Integrating on both sides, $\int dz = \int pdx + \int qdy$

$$z = \int \sqrt{a^2 + x^2 dx} + \int \sqrt{a^2 + y^2 dy}$$

$$= \frac{x}{2}\sqrt{a^2 + x^2} + \frac{a^2}{2}\sinh^{-1}\frac{x}{a} + \frac{y}{2}\sqrt{a^2 + y^2}$$

$$+ \frac{a^2}{2}\sinh^{-1}\frac{y}{a} + b.$$

Type 4:

$$z = px + qy + f(p, q)$$

The equation in the above form is Clairaut's equation. The solution is $z = ax + by + f(a, b)$.

Classification of Second Order Homogeneous Linear Equations

A second order linear homogeneous PDE of the form

$$A\frac{\partial^2\phi}{\partial x^2} + B\frac{\partial^2\phi}{\partial x\partial y} + C\frac{\partial^2\phi}{\partial y^2} + D\frac{\partial\phi}{\partial x} + E\frac{\partial\phi}{\partial y} + F\phi(x, y) = 0 \quad (1)$$

Where A, B, C, D, E and F are either functions of x and y only or constants, is called

1. a parabolic equation, if $B^2 - 4AC = 0$
2. an elliptic equation, if $B^2 - 4AC < 0$
3. a hyperbolic equation, if $B^2 - 4AC > 0$

Examples:

1. Consider the one-dimensional heat equation:

$$\frac{\partial u}{\partial t} = c^2\frac{\partial^2 u}{\partial x^2}$$

$$\Rightarrow \quad c^2\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial t} = 0$$

Comparing it with Eq. (1), we have

$A = c^2$, $B = 0$ and $C = 0$

$\therefore$ $B^2 - 4AC = 0^2 - 4 \times c^2 \times 0 = 0$

$\therefore$ One dimensional heat equation is parabolic.

Similarly, it can be easily observed that

2. One-dimensional wave equation:

$\frac{\partial^2 y}{\partial t^2} = c^2\frac{\partial^2 y}{\partial x^2}$ is hyperbolic ($B^2 - 4AC > 0$) and

3. The Laplace equation:

$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ is elliptic ($B^2 - 4AC < 0$)

Method of Separation of Variables

Consider a PDE involving a dependent variable u and two independent variables x and y. In the method of separation of variables, we find a solution of the PDE in the form of a product of a function of x and a function of y, i.e., we write

$$u(x, y) = X(x) \cdot Y(y) \quad (1)$$

Then $\quad \frac{\partial u}{\partial x} = \frac{\partial}{\partial x}(XY) = X'Y; \frac{\partial u}{\partial y} = \frac{\partial}{\partial y}(XY) = XY'$

$\frac{\partial^2 u}{\partial x^2} = X''Y, \frac{\partial^2 u}{\partial x\partial y} = X'Y', \frac{\partial^2 u}{\partial y^2} = XY''$ and so on

Here $\quad X' = \frac{dX}{dx}; Y' = \frac{dY}{dy}; X'' = \frac{d^2X}{dx^2}; Y'' = \frac{d^2Y}{dy^2}$.

Substituting these in the given PDE, separating X and its derivatives from Y and its derivatives, finding solutions for x and y and substituting them in Eq. (1), we get the solution of the given PDE

This is best explained through the examples given below:

Example 9

Solve $xp + yq = 0$ by the method of separation of variables.

Solution

For the PDE,

$$xp + yq = 0 \quad (1)$$

Let $\quad z = X(x) \cdot Y(y) \quad (2)$

be the solution

$$\therefore \quad p = \frac{\partial z}{\partial x} = X'Y \text{ and } q = \frac{\partial z}{\partial y} = XY'$$

Substituting these in Eq. (1)

$$x X' Y + y XY' = 0$$

$$\Rightarrow xX'Y = -y XY'$$

$$\Rightarrow \quad x\frac{X'}{X} = -y\frac{Y'}{Y} \quad (3)$$

In Eq. (3), as LHS is a function of x alone and RHS is a function of y alone, they are equal only if each of them is equal to some constant

$$\therefore \quad x\frac{X'}{X} = -y\frac{Y'}{Y} = k\,(\text{say}) \quad (4)$$

Where k is a constant

From Eq. (4), $x\frac{X'}{X} = k \quad \Rightarrow \quad xX' = kX$

$$\Rightarrow \quad x\frac{dX}{dx} = kx$$

$$\Rightarrow \quad \frac{dX}{X} = k \cdot \frac{dx}{x}$$

Integrating on both sides we have,

$$\int \frac{dx}{X} = k \int \frac{dx}{x}$$

$$\Rightarrow \quad \log X = k \log x + \log C_1$$

$$\Rightarrow \quad \log X = \log x^k C$$

$$\Rightarrow \quad X = C_1 x^k \qquad (5)$$

Again from Eq. (4), $-y\dfrac{Y'}{Y} = k$

$$\Rightarrow \quad -yY' = kY$$

$$\Rightarrow \quad y\frac{dY}{dy} = -kY$$

$$\Rightarrow \quad \frac{dY}{Y} = -k\frac{dy}{y}$$

Integrating on both sides,

$$\int \frac{dY}{Y} = -k \int \frac{dy}{y}$$

$$\Rightarrow \quad \log Y = -k \log y + \log C_2$$

$$\Rightarrow \quad \log Y = \log y^{-k} C_2$$

$$\Rightarrow \quad Y = C_2 y^{-k} \qquad (6)$$

Substituting Eqs. (5) and (6) in Eq. (2), we get the solution of Eq. (1) as,

$$z = (C_1 x^k)(C_2 y^{-k})$$

$$= C_1 C_2 x^k y^{-k}$$

$$\therefore \quad z = C\left(\frac{x}{y}\right)^k \text{ where } C = C_1 C_2.$$

Example 10

Solve the PDE $u_x + u_t = 3u$; $u(0, t) = 4e^t$ by the method of separation of variables.

Solution

Let $u = X(x).\ T(t)$ (1)

be the solution of the PDE

$$u_x + u_t = 3u \qquad (2)$$

$$u = XT \quad \Rightarrow \quad u_x = \frac{\partial u}{\partial x} = X'T \text{ and } u_t$$

$$= \frac{\partial u}{\partial t} = XT'$$

Substituting these in Eq. (2), we get

$$X'T + XT' = 3XT$$

Dividing throughout by XT, we have

$$\frac{X'}{X} + \frac{T'}{T} = 3$$

$$\Rightarrow \quad \frac{X'}{X} = \frac{-T'}{T} + 3 = k \qquad (3), \text{(say)}$$

From Eq. (3), $\dfrac{X'}{X} = k \quad \Rightarrow \quad X' = kX$

$$\Rightarrow \quad X' - kX = 0 \qquad (4)$$

Which is a linear equation with its auxiliary equation being

$$m - k = 0 \quad \Rightarrow \quad m = k$$

Hence its solution is $X = C_1 e^{kx}$ (5)

Again from Eq. (3), $\dfrac{-T'}{T} + 3 = k$

$$\Rightarrow \quad \frac{T'}{T} = 3 - k$$

$$\Rightarrow \quad T' = (3-k)T$$

$$\Rightarrow \quad T' - (3-k)\,T = 0 \qquad (6)$$

Which is a linear equation with its auxiliary equation being $m - (3 - k) = 0$

$$\Rightarrow \quad m = 3 - k$$

$\therefore$ The solution of Eq. (6) is $T = C_2 e^{(3-k)t}$ (7)

Substituting Eqs. (5) and (7) in Eq. (1), we get the general solution of given PDE (2) as

$$u = X \cdot T = (C_1 e^{kx})(C_2 e^{(3-k)t})$$

$$= C_1 C_2 e^{kx+(3-k)t}$$

$$\therefore\ u = ce^{kx+(3-k)t};\ \text{where } c = c_1\, c_2$$

$$\therefore \quad u(x, t) = ce^{kx+(3-k)t} \qquad (8)$$

Given $u(0, t) = 4e^t$

$\therefore$ From Eq. (8), $u(0, t) = ce^{(3-k)t} = 4e^t$

Comparing on both sides, we get

$$C = 4,\ 3 - k = 1$$

$$\Rightarrow \quad C = 4;\ k = 2$$

Substituting these in Eq. (8), we get the required solution of Eq. (2) as

$$u(x, t) = 4e^{2x+t}$$

One Dimensional Diffusion Equation The diffusion equation is a partial differential equation that describes density fluctuations in a material undergoing diffusion. The partial differential equation representing the one dimensional diffusion equation is

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2}$$

where $u(x, t)$ is the density of the diffusing material at time t and D is diffusion coefficient

Example 11

Find the solution of the one dimensional diffusion equation $\dfrac{\partial u}{\partial t} = D\dfrac{\partial^2 u}{\partial x^2}$ on the interval $x \in [0, L]$ with initial condition

$$u\,(x, 0) = f(x), \forall x \in [0, L]$$

and Dirichlet's boundary conditions

$$u(0, t) = u(L, t) = 0 \quad \forall t > 0$$

Solution

We will solve the one-dimensional diffusion equation

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2} \qquad (1)$$

by the method of separation of variables.

Let $u(x, t) = X(x)\, T(t)$ (2)

be the solution of Eq. (1)

$$\therefore \frac{\partial u}{\partial x} = X'T \quad \text{and} \quad \frac{\partial^2 u}{\partial x^2} = X''T \quad \text{and} \quad \frac{\partial u}{\partial t} = XT'$$

Substituting these in Eq. (1),

$$XT' = DX''\, T$$

$$\Rightarrow \frac{1}{D}\frac{T'}{T} = \frac{X''}{X} \tag{3}$$

As the left hand side depends only on the variable t and the right hand side depends only on the variable x, both sides are equal to some constant say $-\lambda$

(Negative sign is taken for convenience reason)

From Eq. (3), $\frac{1}{D}\frac{T'}{T} = \frac{X''}{X} = -\lambda$

$$\Rightarrow \frac{1}{D}\frac{T'}{T} = -\lambda \text{ and } \frac{X''}{X} = -\lambda$$

$$\Rightarrow T' + DT = 0 \tag{4}$$

and
$$X'' + \lambda X = 0 \tag{5}$$

Clearly Eqs. (4) and (5) are linear ordinary differential equations involving the variables t and x respectively.

Solving (4), we get

$$T(t) = Ce^{-\lambda Dt} \tag{6}$$

Solving Eq. (5), we get different possible solutions depending on the value of λ as given below.

$$X(x) = \begin{cases} A\sin(\sqrt{\lambda}x) + B\cos(\sqrt{\lambda}x);\ \text{for } \lambda > 0 \\ A'e^{\sqrt{-\lambda}x} + B'e^{\sqrt{-\lambda}x};\ \text{for } \lambda < 0 \\ A''x + B'';\ \text{for } \lambda = 0 \end{cases}$$

Given boundary conditions are

$$u(0, t) = 0 \text{ and } u(L, t) = 0$$

From Eq. (2), $u\,(0, t) = X(0)\, T(t) = 0$

$$\Rightarrow X(0) = 0$$

and
$$u(L, t) = X(L)\, T(t) = 0 \Rightarrow X(L) = 0$$

Taking into account, the boundary conditions $X(0) = 0$ and $X(L) = 0$, the values of $X(x)$ for $\lambda = 0$ and $\lambda < 0$ leads to only the trivial solutions and hence we take the value $X(x)$ given for $\lambda > 0$, which on application of the boundary conditions becomes,

$$X(x) = C_n \sin\left(\frac{n\pi x}{L}\right); \quad n = 1, 2, 3\ldots$$

and Eq. (6) becomes,

$$T(t) = B_n \exp\left(-D\left(\frac{n\pi}{L}\right)^2 t\right), \quad n = 1, 2, 3\ldots$$

where B_n is a constant

$\therefore$ Substituting the values of $X(x)$ and $T(t)$ in Eq. (2),

We get

$$u(x, t)$$

$$= \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{L}x\right)\exp\left(-D\left(\frac{n\pi}{L}\right)^2 t\right) \tag{7}$$

where $A_n = \text{Constant } (= B_n C_n)$

Given initial condition is

$$u(x, 0) = f(x)$$

i.e.,
$$\sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{L}x\right) = f(x) \text{ (From Eq. (7))} \tag{8}$$

By writing $f(x)$ as a half range Fourier sine series in $[0, L]$ we have

$$f(x) = \sum_{n=1}^{\infty} F_n \sin\left(\frac{n\pi}{L}x\right)$$

where $F_n = \frac{2}{L}\int_0^L f(\xi)\sin\left(\frac{n\pi}{L}\xi\right)d\xi$

$\therefore$ Eq. (8) becomes,

$$\sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{L}x\right) = \sum_{n=1}^{\infty} F_n \sin\left(\frac{n\pi}{L}x\right)$$

$$\Rightarrow A_n = F_n = \frac{2}{L}\int_0^L f(\xi)\sin\left(\frac{n\pi}{L}\xi\right)d\xi$$

Substituting the value of A_n in Eq. (7), we get the solution of Eq. (1) as

$$u\,(x, t)$$

$$= \sum_{n=1}^{\infty}\left(\left(\frac{2}{L}\int_0^L f(\xi)\sin\left(\frac{n\pi}{L}\xi\right)d\xi\right)\sin\left(\frac{n\pi x}{L}\right)\right)$$

$$\exp\left(-D\left(\frac{n\pi}{L}\right)^2 t\right)$$

Heat Equation

The heat flow in a body of homogeneous material is governed by the heat equation $\frac{\partial u}{\partial t} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)$ where $c^2 = \frac{k}{\sigma\rho}$ and $u\,(x, y, z, t)$ is the temperature in a body, k is the thermal conductivity, σ is specific heat of the body, ρ is the density of the material and c^2 the constant is called the diffusivity of the body. If the heat flow is in x-direction only then u depends on x and t, then the heat equation becomes $\frac{\partial u}{\partial t} = c^2\left(\frac{\partial^2 u}{\partial x^2}\right)$, which is known as one-dimensional heat equation.

Wave Equation

The one-dimensional wave equation of a vibrating elastic string is given by,

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2} \text{ where } c^2 = \frac{T}{\rho}$$

Laplace Equation

When the temperature in a homogeneous material is in steady state and the temperature does not vary with time then the heat conduction equation becomes $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=0$ and this is known as

Laplace's equation in cartesian system While solving the boundary value problems the following results may be used

If $u(x, t)$ is a function of x and t

1. $L\left\{\frac{\partial u}{\partial t}\right\}=s\bar{u}(x,s)-u(x,0)$
2. $L\left\{\frac{\partial^2 u}{\partial t^2}\right\}=s^2\bar{u}(x,s)-su(x,0)-u_t(x,0)$
3. $L\left\{\frac{\partial u}{\partial x}\right\}=\frac{d\bar{u}}{dx}$
4. $L\frac{\partial^2 u}{\partial x^2}=\frac{d^2\bar{u}}{dx^2}$ where $L\{u(x,t)\}=\bar{u}(x,s)$

Example 49

Solve the one dimensional heat equation $\frac{\partial u}{\partial t}=2\frac{\partial^2 u}{\partial x^2}$ satisfying the boundary conditions $u(0, t) = 0 = u(4, t)$ and $u(x, 0) = 8\sin 2\pi x$.

Solution

Taking Laplace transform on both sides of the equation $\frac{\partial u}{\partial t}=2\frac{\partial^2 u}{\partial x^2}$

$$L\left\{\frac{\partial u}{\partial t}\right\}=2L\left\{\frac{\partial^2 u}{\partial x^2}\right\}$$

$$s\bar{u}-u(x,0)=2\cdot\frac{d^2\bar{u}}{dx^2}$$

or $\frac{\partial^2\bar{u}}{\partial x^2}-\frac{s}{2}\bar{u}=-4\sin 2\pi x$ as $u(x, 0) = 8\sin 2\pi x$

The general solution of the above equation is $\bar{u}$

$$=Ae^{(\sqrt{S/2})X}+Be^{-(\sqrt{S/2})X}-\frac{4\sin 2\pi x}{-(2\pi)^2-\frac{s}{2}}$$

or

$$\bar{u}=Ae^{\sqrt{S/2}X}+Be^{-\sqrt{S/2}X}+\frac{8\sin 2\pi x}{8\pi^2+s} \quad (1)$$

But $u(0, t) = 0 = u(4, t)$

$\therefore$ $\bar{u}(0,s)=0,\ \bar{u}(4,s)=0$

$\therefore$ From Eq. (1), we have $A + B = 0$

and $0=Ae^{4\sqrt{s/2}}+Be^{-4\sqrt{s/2}}+\frac{8\sin 8\pi}{8\pi^2+s}$

$$\Rightarrow Ae^{4\sqrt{s/2}}+Be^{-4\sqrt{s/2}}=0$$

Solving we get $A = B = 0$

$$\therefore \text{From (1) we have } \bar{u}=\frac{8\sin 2\pi x}{8\pi^2+s}$$

$$\therefore y=L^{-1}\left\{\frac{8}{8\pi^2+s}\sin 2\pi x\right\}$$

i.e., $y=8e^{-8\pi^2 t}\sin 2\pi x.$

Example 50

Solve the wave equation of a stretched string given by $\frac{\partial^2 u}{\partial t^2}=9\frac{\partial^2 u}{\partial x^2}$ satisfying the boundary conditions $u(x, 0) = 0$, $u_t(x,0)=0, x>0$ and $\bar{u}(0,t)=F(t)$, $\lim_{x\to\infty}u(x,t)=0, t\ge 0$.

Solution

Given $\frac{\partial^2 u}{\partial t^2}=9\frac{\partial^2 u}{\partial x^2}$.

Taking Laplace transform on both sides of the equation with the boundary conditions we have

$$L\left\{\frac{\partial^2 u}{\partial t^2}\right\}=9L\left\{\frac{\partial^2 u}{\partial x^2}\right\}$$

or $s^{2\bar{u}}(x,s)-su(x,0)-u_t(x,0)=9\cdot\frac{d^2\bar{u}}{dx^2}$ or $\frac{d^2\bar{u}}{dx^2}-\frac{s^2}{9}\bar{u}=0$ (1)

Also $\bar{u}(0,s)=\int_0^\infty F(t)e^{-st}dt=\bar{F}$ and $u(x, s) = 0$ as $x\to\infty$

$\therefore$ The general solution Eq. of (1) is $\bar{u}(x,s)$

$$=c_1e^{\frac{s}{3}x}+c_2e^{\frac{-s}{3}x}$$

and $\bar{u}(x,s)=0$ as $x\to\infty\Rightarrow c_1=0$

and $\bar{u}(0,s)=\bar{F}(s)=c_2$

Hence, $\bar{u}(x,s)=\bar{F}(s)\,e^{-\frac{sx}{3}}$

$$\therefore u(x,t)=L^{-1}\left\{e^{-\frac{sx}{3}}\bar{F}(s)\right\}$$

$$=\begin{cases}F\left(t-\frac{x}{3}\right), t>\frac{x}{3}\\ 0, t<\frac{x}{3}\end{cases} \text{ as } L^{-}\{\bar{F}(s)=F(t)\},$$

when expressed in terms of Heaviside's unit step function. $u(x, t)$

$$=F\left(t-\frac{x}{3}\right)\cdot H\left(t-\frac{x}{3}\right).$$

Exercises

1. Let $f(x) = \begin{cases} 0 & \text{if} \quad -\pi \le x \le 0 \\ x^3 & \text{if} \quad 0 < x \le \pi \end{cases}$

   Which is a periodic function with period $2p$, then $a_0 =$ ______.

   (A) $\frac{\pi^3}{4}$ (B) $\frac{\pi^3}{8}$

   (C) $\frac{\pi^3}{12}$ (D) $\frac{\pi^3}{16}$

2. The value of the fourier coefficient a_n for $n \ge 2$ for $f(x) = x \sin x$ in $(-\pi, \pi)$ is ______.

   (A) $\cos \frac{(n-1)\pi}{n-1} \cos nx$

   (B) $\cos \frac{2(n-1)\pi}{n-1} \cos nx$

   (C) $\frac{1}{2} + \sum_{n=1}^{\infty} \cos nx$

   (D) $\left[\cos \frac{(n-1)\pi}{n-1} - \cos \frac{(n+1)}{n+1}\right]$

3. If $f(x) = x^3$ in $(-\pi, \pi)$ then the value of b_n is ______.

   (A) $\frac{2}{n^3}[6 + n^2\pi^2]$

   (B) $\frac{2\cos n\pi}{n^3}[6 - n^2\pi^2]$

   (C) $\frac{2}{n^3}[6 - n^2\pi^2]$

   (D) $\frac{\cos n\pi}{n^3}[6 + n^2\pi^2]$

4. Find the Fourier series of $f(x)$ which is defined as follows:

   $f(x) = 2 \qquad 0 < x < 1$

   $\quad = 3 \qquad 1 < x < 2$

   (A) $\sum_{n=1}^{\infty} \sin \frac{\pi x}{2}$

   (B) $\frac{5}{2} - \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{2n-1} \sin \frac{(2n-1)\pi x}{2}$

   (C) $\frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{2n-1} \sin \frac{\pi x}{2}$

   (D) None of these

Direction for questions 5 and 6:

Let $f(x) = \frac{\pi - x}{2}$ in the interval $(0, 2\pi)$

5. The Fourier series of $f(x)$ is ______.

   (A) $\cos x + \frac{\cos 2x}{2} + \frac{\cos 3x}{2} + \cdots$

   (B) $\sin x - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \frac{\sin 4x}{4} +$

   (C) $\sin x + \frac{\sin 2x}{2} + \frac{\sin 3x}{3} + \cdots$

   (D) None of these

6. The value of $\frac{\pi}{4}$ is ______.

   (A) $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \cdots$

   (B) $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{6} + \cdots$

   (C) $1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{9}$

   (D) $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} \cdots$

Direction for questions 7 and 8:

Let $f(x) = x^2$ in the interval $(-\pi, \pi)$.

7. The Fourier series of $f(x)$ is ______.

   (A) $-\frac{\pi^2}{3}$

   (B) $\frac{\pi^2}{3} + \sum_{n=1}^{\infty} \cos nx$

   (C) $\sum_{n=1}^{\infty} \cos nx$

   (D) $\frac{\pi^2}{3} + \sum_{n=1}^{\infty} 4 \frac{(-1)^n}{n^2} \cos nx$

8. The value of $\frac{\pi^2}{12}$ is

   (A) $1 + \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$

   (B) $1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots$

   (C) $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$

   (D) $1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots$

9. The value of $\frac{\pi^2}{6}$ is

(A) $1-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}$

(B) $1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}\cdots$

(C) $1+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}$

(D) None of these

10. The half-range sine series of $f(x) = e^x$ in $0 < x < l$ is ______.

(A) $\sum_{n=1}^{\infty}\frac{1}{1+n^2\pi^2}\,(1 - e(-1)^n)\sin n\pi x$

(B) $2\pi\sum_{n=1}^{\infty}\frac{n}{1+n^2\pi^2}\,(1 - e(-1)^n)\sin n\pi x$

(C) $\sum_{n=1}^{\infty}\frac{1}{1+n^2\pi^2}\,(1 - e(-1)^n)\sin n\pi x$

(D) None of these

11. The order and degree of the $\frac{\partial^2 z}{\partial x^2}+3xy\left(\frac{\partial z}{\partial x}\right)^2+5\frac{\partial z}{\partial y}$ $= 8$ are

(A) 1, 1 (B) 1, 2

(C) 2, 1 (D) 2, 2

12. The differential equation whose solution is $z = (x - a)(y - b)$ is _______.

(A) $pq = 2z$ (B) $pq = z$

(C) $p = 2zq$ (D) $p = zq$

13. Form a PDE of $z = (x - y)\,\phi\,(x^2 - y^2)$

(A) $py - xq = z$ (B) $py + xq = z$

(C) $px + yq = z$ (D) $px - yq = z$

14. The solution of $x^2p + y^2q = (x + y)z$ is ________.

(A) $f(xy, x - y) = 0.$ (B) $f\left(\frac{xy}{z},\frac{x-y}{z}\right)=0$

(C) $f(zx, z - x) = 0$ (D) None of these

15. Solve $(2p + 1)\,q = pz$

(A) $a\log(z - a) = x - ay + b$

(B) $2a\log(z + a) = ay + b$

(C) $2a\log(z - a) = x + ay + b$

(D) $a\log(z + a) = 3x + ay + b$

16. The solution of $q^2x\,(1 + y^2) = py^2$ is _______.

(A) $z = a\,(1 + y^2)$

(B) $z = \frac{ax^2}{2} - a\,(1 + y^2) + b$

(C) $z=\frac{ax^2}{2}+\sqrt{a(1+y^2)}+b$

(D) $z=\frac{ax}{2}+\sqrt{a(1+y^2)}+b$

17. Solve $pqz = q^2(yp + q^2) + p^2(xq + p^2)$.

(A) $z = ax + by + \frac{a^3}{b}+\frac{b^3}{a}$

(B) $z = ax - by$

(C) $z = ax + by + \frac{a}{a^3}+\frac{b}{b^3}$

(D) None of these

18. In the process of solving the partial differential equation $\frac{\partial^2 u}{\partial x^2}+5\frac{\partial^2 u}{dy^2}=0$ by the method of separation of variables, the linear differential equation involving the independent variable 'X' is _____. (Here k is a constant)

(A) $\frac{d^2X}{dx^2}+kX(x)=0$

(B) $\frac{d^2X}{dx^2}-kX(x)=0$

(C) $\frac{d^2X}{dx^2}+k\frac{dx}{dx}+k^2X(x)=0$

(D) $\frac{d^2X}{dx^2}-k\frac{dx}{dx}+2k\,X(x)=0$

19. The second order partial differential equation $3x^2\frac{\partial^2 u}{\partial x^2}$ $-6xy\frac{\partial^2 u}{\partial x\partial y}+3y^2\frac{\partial^2 u}{\partial y^2}-5\frac{\partial u}{\partial x}+7\frac{\partial u}{\partial y}=6x^2y$ is ______.

(A) elliptic equation

(B) parabolic equation

(C) hyperbolic equation

(D) depends on the value of x and y

20. Which of the following partial differential equations represents the one-dimensional diffusion equation?

(A) $\frac{\partial^2 u}{\partial t^2}=c^2\frac{\partial^2 u}{\partial x^2}$

(B) $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$

(C) $\frac{\partial u}{\partial t}=D\frac{\partial^2 u}{\partial x^2}$

(D) $\frac{\partial^2 u}{\partial x^2}=c^2\frac{\partial^2 u}{\partial y^2}+\frac{\partial u}{\partial x}$

21. In the one-dimensional diffusion equation, $\frac{\partial u}{\partial t}=D\frac{\partial^2 u}{\partial x^2}$, $u(x, t)$ and D represent respectively

(A) density of and diffusion coefficient.
(B) diffusion and density coefficient.
(C) viscosity and diffusion coefficient.
(D) diffusion and viscosity coefficient.

22. Which of the following pair can be represented by the same partial differential equation? (Except possibly a change in the constant multiplying the partial derivatives)
(A) The one-dimensional wave equation and the one -dimensional heat equation.
(B) The one-dimensional wave equation and the two -dimensional Laplace equation.
(C) The one-dimensional heat equation and the two-dimensional Laplace equation.
(D) The one-dimensional heat equation and the one-dimensional diffusion equation.

23. Solution of the one dimensional heat equation $\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}$ $x > 0$, $t > 0$ satisfying the boundary condition $u(0, t) = 1$, $u(x, 0) = 0$ is ______.

(A) $erf\left(\frac{x}{2\sqrt{t}}\right)$ (B) $erf\left(\frac{\sqrt{x}}{2\sqrt{t}}\right)$

(C) $erf\left(\frac{1}{2\sqrt{t}}\right)$ (D) $erf\left(\frac{x}{\sqrt{t}}\right)$

24. A string is stretched between two fixed points follows the equation $\frac{\partial^2 y}{\partial t^2}=a^2\frac{\partial^2 y}{\partial x^2}$ $(t > 0, x > 0)$ satisfying the boundary conditions $y(x, 0) = 0$, $x > 0$ and $y(0, t) = t$ $\underset{x\to\infty}{\text{Lt}}\ y(x,t)=0, t\geq 0$, Find $y(x, t)$ in terms of Heaviside's unit step function.
(A) $(t - x)\,H(t - x)$
(B) $\left(t-\frac{x}{a}\right)H\left(t-\frac{x}{a}\right)$
(C) $(t - xa)\,H(t - xa)$
(D) None of these

25. The one dimensional wave equation is ______.
(A) $\frac{\partial u}{\partial t}=c\frac{\partial u}{\partial x}$
(B) $\frac{\partial^2 u}{\partial t^2}=c^2\frac{\partial^2 u}{\partial x^2}$
(C) $\frac{\partial^2 u}{\partial x^2}=c^2\frac{\partial^2 u}{\partial t^2}$
(D) $\frac{\partial^2 u}{\partial t^2}=c^2\frac{\partial u}{\partial x}$

PREVIOUS YEARS' QUESTIONS

1. The equation $K_x\frac{\partial^2 h}{\partial x^2}+K_z\frac{\partial^2 h}{\partial z^2}=0$ can be transformed to $\frac{\partial^2 h}{\partial x_t^2}+\frac{\partial^2 h}{\partial z^2}=0$ by substituting **[GATE, 2008]**

(A) $x_t = x\frac{K_z}{K_x}$ (B) $x_t = x\frac{K_x}{K_z}$

(C) $x_t = x\sqrt{\frac{K_x}{K_z}}$ (D) $x_t = x\sqrt{\frac{K_z}{K_x}}$

2. The partial differential equation that can be formed from $z = ax + by + ab$ has the form (with $p=\frac{\partial z}{\partial x}$ and $q=\frac{\partial z}{\partial y}$) **[GATE, 2010]**
(A) $z = px + qy$
(B) $z = px + pq$
(C) $z = px + qy + pq$
(D) $z = qy + pq$

3. The Fourier series of the function,
$f(x) = 0, \quad -\pi < x \leq 0$
$= \pi - x. \quad 0 < x < \pi$
In the interval $[-\pi, \pi]$ is
$f(x)=\frac{\pi}{4}+\frac{2}{\pi}\left[\frac{\cos x}{1^2}+\frac{\cos 3x}{3^2}+\cdots\right]+$
$\left[\frac{\sin x}{1}+\frac{\sin 2x}{2}+\frac{\sin 3x}{3}+\cdots\right]$
The convergence of the above Fourier series at $x = 0$ gives **[GATE, 2016]**

(A) $\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6}$ (B) $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}=\frac{\pi^2}{12}$

(C) $\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}=\frac{\pi^2}{8}$ (D) $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{2n-1}=\frac{\pi^2}{4}$

4. The type of partial differential equation $\frac{\partial^2 p}{\partial x^2}+\frac{\partial^2 p}{\partial y^2}+3\frac{\partial^2 p}{\partial x\partial y}+2\frac{\partial p}{\partial x}-\frac{\partial p}{\partial y}=0$ is **[GATE, 2016]**

(A) elliptic
(B) parabolic
(C) hyperbolic
(D) None of these

5. The solution of the partial differential equation $\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2}$ is of the form **[GATE, 2016]**

(A) $C\cos(kt)[C_1 e^{(\sqrt{k/\alpha})x} + C_2 e^{-(\sqrt{k/\alpha})x}]$

(B) $Ce^{kt}[C_1 e^{(\sqrt{k/\alpha})x} + C_2 e^{-(\sqrt{k/\alpha})x}]$

(C) $Ce^{kt}\left[C_1\cos\left(\sqrt{\frac{k}{\alpha}}\right)x + C_2\cos\left(-\sqrt{\frac{k}{\alpha}}\right)x\right]$

(D) $C\sin(kt)\left[C_1\cos\left(\sqrt{\frac{k}{\alpha}}\right)x + C_2\cos\left(-\sqrt{\frac{k}{\alpha}}\right)x\right]$

Answer Keys

Exercises

1. A	**2.** D	**3.** B	**4.** B	**5.** C	**6.** D	**7.** D	**8.** D	**9.** B	**10.** B
11. C	**12.** B	**13.** B	**14.** B	**15.** C	**16.** C	**17.** A	**18.** B	**19.** B	**20.** C
21. A	**22.** D	**23.** A	**24.** B	**25.** B					

Previous Years' Questions

1. D	**2.** C	**3.** C	**4.** C	**5.** B