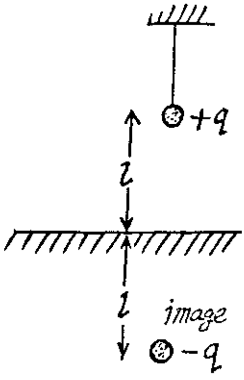
3.2 CONDUCTORS AND DIELECTRICS IN AN ELECTRIC FIELD

3.54 When the ball is charged, for the equilibrium of ball, electric force on it must counter balance the excess spring force, exerted, on the ball due to the extension in the spring.



Thus $F_{el} = F_{spr}$

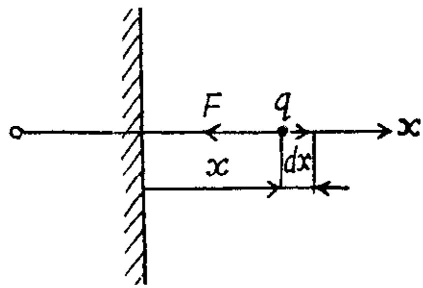
or, $\frac{q^2}{4\pi\varepsilon_0 (2l)^2} = \kappa x$, (The force on the charge q might be considered as arised from attraction by the electrical image)

or, $q = 4l\sqrt{\pi\varepsilon_0 \kappa x}$,

sought charge on the sphere.

3.55 By definition, the work of this force done upon an elementry displacement dx (Fig.) is given by

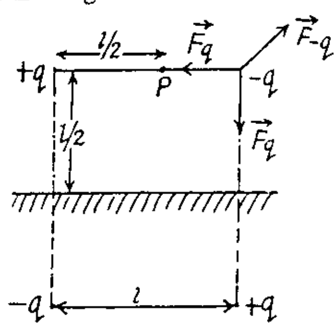
$$dA = F_x\, dx = -\frac{q^2}{4\pi\varepsilon_0 (2x)^2}\, dx,$$



where the expression for the force is obtained with the help of the image method. Integrating this equation over x between l and ∞, we find

$$A = -\frac{q^2}{16\pi\varepsilon_0}\int_l^\infty \frac{dx}{x^2} = -\frac{q^2}{16\pi\varepsilon_0 l}.$$

3.56 (a) Using the concept of electrical image, it is clear that the magnitude of the force acting on each charge,



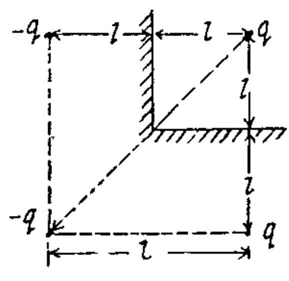
$$|\vec{F}| = \sqrt{2}\,\frac{q^2}{4\pi\varepsilon_0 l^2} - \frac{q^2}{4\pi\varepsilon_0 (\sqrt{2}l)^2}$$

$$= \frac{q^2}{8\pi\varepsilon_0 l^2}(2\sqrt{2} - 1)$$

(b) Also, from the figure, magnitude of electrical field strength at P

$$E = 2\left(1 - \frac{1}{5\sqrt{5}}\right)\frac{q}{\pi\varepsilon_0 l^2}$$

3.57 Using the concept of electrical image, it is easily seen that the force on the charge q is,

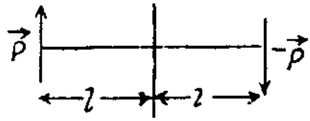


$$F = \frac{\sqrt{2}\,q^2}{4\pi\varepsilon_0 (2l)^2} + \frac{(-q)^2}{4\pi\varepsilon_0 (2\sqrt{2}\,l)^2}$$

$$= \frac{(2\sqrt{2} - 1)\,q^2}{32\pi\varepsilon_0 l^2} \quad \text{(It is attractive)}$$

3.58 Using the concept of electrical image, force on the dipole $\vec{p}$,

$\vec{F} = p\,\frac{\partial \vec{E}}{\partial l}$, where $\vec{E}$ is field at the location of $\vec{p}$ due to $(-\vec{p})$



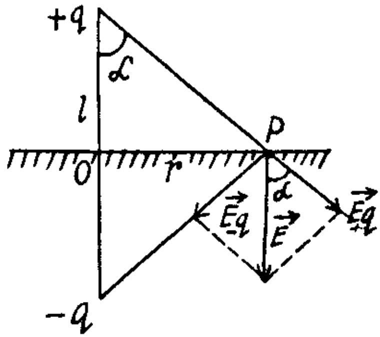
$$\text{or,}\quad |\vec{F}| = \left|\frac{\partial \vec{E}}{\partial l}\right| p = \frac{3\,p^2}{32\,\pi\,\varepsilon_0\,l^4}$$

$$\text{as,}\quad |\vec{E}| = \frac{p}{4\,\pi\,\varepsilon_0\,(2l)^3}$$

3.59 To find the surface charge density, we must know the electric field at the point P (Fig.) which is at a distance r from the point O.

Using the image mirror method, the field at P,

$$E = 2\,E\cos\alpha = 2\,\frac{q}{4\,\pi\,\varepsilon_0\,x^2}\,\frac{l}{x} = \frac{q\,l}{2\,\pi\,\varepsilon_0\,(l^2 + r^2)^{3/2}}$$



Now from Gauss' theorem the surface charge density on conductor is connected with the electric field near its surface (in vaccum) through the relation $\sigma = \varepsilon_0\,E_n$, where E_n is the projection of $\vec{E}$ onto the outward normal $\vec{n}$ (with respect to the conductor).

As our field strength $\vec{E} \uparrow\downarrow \vec{n}$, so

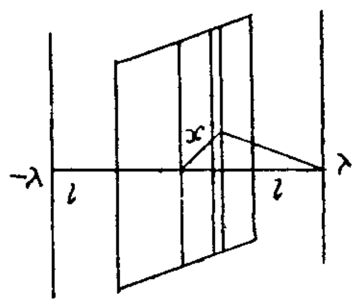
$$\sigma = -\,\varepsilon_0\,E = -\,\frac{ql}{2\,\pi\,(l^2 + r^2)^{3/2}}$$

3.60 (a) The force F_1 on unit length of the thread is given by

$$F_1 = \lambda\,E_1$$

where E_1 is the field at the thread due to image charge :

$$E_1 = \frac{-\lambda}{2\pi\varepsilon_0\,(2l)}$$



$$\text{Thus}\qquad F_1 = \frac{-\,\lambda^2}{4\pi\varepsilon_0\,l}$$

minus singn means that the force is one of attraction.

(b) There is an image thread with charge density $-\lambda$ behind the conducting plane. We calculate the electric field on the conductor. It is

$$E\,(x) = E_n\,(x) = \frac{\lambda l}{\pi\,\varepsilon_0\,(x^2 + l^2)}$$

on considering the thread and its image.

Thus

$$\sigma\,(x) = \varepsilon_0\,E_n = \frac{\lambda l}{\pi\,(x^2 + l^2)}$$

3.61 (a) At O,

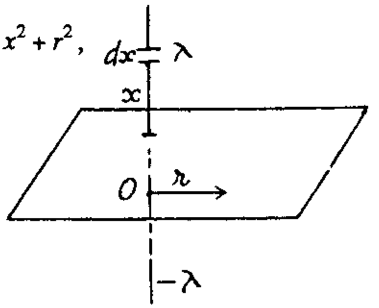
$$E_n(O) = 2\int_l^\infty \frac{\lambda\,dx}{4\pi\varepsilon_0 x^2} = \frac{\lambda}{2\pi\varepsilon_0 l}$$

So $\sigma(O) = \varepsilon_0 E_n = \dfrac{\lambda}{2\pi l}$

(b) $$E_n(r) = 2\int_l^\infty \frac{\lambda\,dx}{4\pi\varepsilon_0(x^2+r^2)}\frac{x}{(x^2+r^2)^{1/2}} = \frac{\lambda}{2\pi\varepsilon_0}\int_l^\infty \frac{x\,dx}{(x^2+r^2)^{3/2}}$$

$$= \frac{\lambda}{4\pi\varepsilon_0}\int_{l^2+r^2}^\infty \frac{dy}{y^{3/2}},\ \text{on putting } y = x^2+r^2,$$

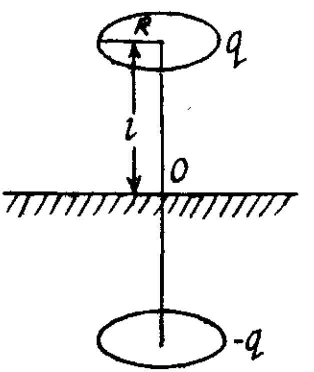
$$= \frac{\lambda}{2\pi\varepsilon_0\sqrt{l^2+r^2}}$$



Hence $\sigma(r) = \varepsilon_0 E_n = \dfrac{\lambda}{2\pi\sqrt{l^2+r^2}}$

3.62 It can be easily seen that in accordance with the image method, a charge $-q$ must be located on a similar ring but on the other side of the conducting plane. (Fig.) at the same perpendicular distance. From the solution of 3.9 net electric field at O,

$$\vec{E} = 2\frac{ql}{4\pi\varepsilon_0(R^2+l^2)^{3/2}}(-\vec{n})$$ where $\vec{n}$ is outward normal with respect to the conducting plane.



Now $E_n = \dfrac{\sigma}{\varepsilon_0}$

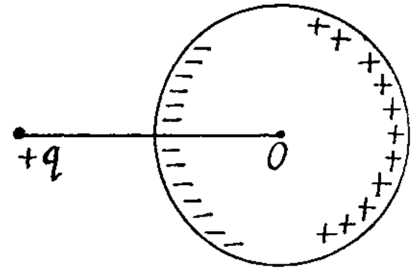
Hence $\sigma = \dfrac{-ql}{2\pi(R^2+l^2)^{3/2}}$

where minus sign indicates that the induced carge is opposite in sign to that of charge $q > 0$.

3.63 Potential φ is the same for all the points of the sphere. Thus we calculate its value at the centre O of the sphere. Thus we can calculate its value at the centre O of the sphere, because only for this point, it can be calculated in the most simple way.

$$\varphi = \frac{1}{4\pi\varepsilon_0}\frac{q}{l} + \varphi' \qquad (1)$$

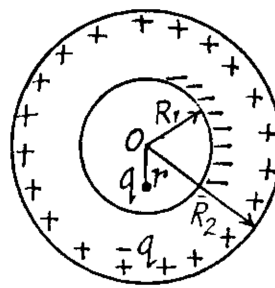
where the first term is the potential of the charge q, while the second is the potential due to the charges induced on the surface of the sphere. But since all induced charges are at the same distance equal to the radius of the circle from the point C and the total induced charge is equal to zero, $\varphi' = 0$, as well. Thus equation (1) is reduced to the form,



$$\varphi = \frac{1}{4\pi\varepsilon_0}\frac{q}{l}$$

3.64 As the sphere has conducting layers, charge $-q$ is induced on the inner surface of the sphere q and consequently charge $+q$ is induced on the outer layer as the sphere as a whole is uncharged.

Hence, the potential at O is given by,



$$\varphi_0 = \frac{q}{4\pi\varepsilon_0 r} + \frac{(-q)}{4\pi\varepsilon_0 R_1} + \frac{q}{4\pi\varepsilon_0 R_2}$$

It should be noticed that the potential can be found in such a simple way only at O, since all the induced charges are at the same distance from this point, and their distribution, (which is unknown to us), does not play any role.

3.65 Potential at the inside sphere,

$$\varphi_a = \frac{q_1}{4\pi\varepsilon_0 a} + \frac{q_2}{4\pi\varepsilon_0 b}$$

Obviously $\qquad \varphi_a = 0$ for $q_2 = -\frac{b}{a}q_1 \qquad (1)$

When $\qquad r \geq b,$

$$\varphi_r = \frac{q_1}{4\pi\varepsilon_0 r} + \frac{q_2}{4\pi\varepsilon_0 r} = \frac{q_1}{4\pi\varepsilon_0}\left(1 - \frac{b}{a}\right) \Big/ r, \text{ using Eq. (1).}$$

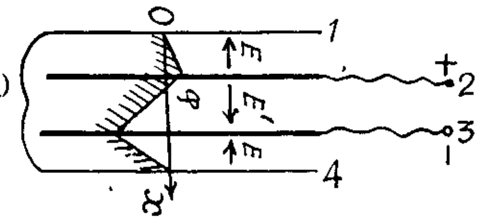
And when $r \leq b$

$$\varphi_r = \frac{q_1}{4\pi\varepsilon_0 r} + \frac{q_2}{4\pi\varepsilon_0 b} = \frac{q_1}{4\pi\varepsilon_0}\left(\frac{1}{r} - \frac{1}{a}\right)$$

3.66 (a) As the metallic plates 1 and 4 are isolated and conncted by means of a conductor, $\varphi_1 = \varphi_4$. Plates 2 and 3 have the same amount of positive and negative charges and due to induction, plates 1 and 4 are respectively negatively and positively charged and in addition to it all the four plates are located a small but at equal distance d relative to each other, the magnitude of electric field strength between 1 - 2 and 3 - 4 are both equal in magnitude and direction (say $\vec{E}$). Let $\vec{E'}$ be the field strength between the plates 2 and 3, which is directed form 2 to 3. Hence $\vec{E'} \uparrow\downarrow \vec{E}$ (Fig.).

According to the problem

$$E' d = \Delta\varphi = \varphi_2 - \varphi_3 \qquad (1)$$



In addition to

$$\varphi_1 - \varphi_4 = 0 = (\varphi_1 - \varphi_2) + (\varphi_2 - \varphi_3) + (\varphi_3 - \varphi_4)$$

or, $\quad 0 = -Ed + \Delta\varphi - Ed$

or, $\quad \Delta\varphi = 2Ed$ or $E = \dfrac{\Delta\varphi}{2d}$

Hence $\quad E = \dfrac{E'}{2} = \dfrac{\Delta\varphi}{2d} \qquad (2)$

(b) Since $E \alpha \sigma$, we can state that according to equation (2) for part (a) the charge on the plate 2 is divided into two parts; such that 1 / 3 rd of it lies on the upper side and 2 / 3 rd on its lower face.

Thus charge density of upper face of plate 2 or of plate 1 or plate 4 and lower face of

$3\sigma = \varepsilon_0 E = \dfrac{\varepsilon_0 \Delta\varphi}{2d}$ and charge density of lower face of 2 or upper face of 3

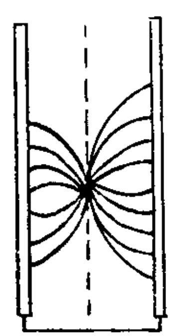
$$\sigma' = \varepsilon_0 E' = \varepsilon_0 \frac{\Delta\varphi}{d}$$

Hence the net charge density of plate 2 or 3 becomes $\sigma + \sigma' = \dfrac{3\,\varepsilon_0\,\Delta\,\varphi}{2\,d}$, which is obvious from the argument.

3.67 The problem of point charge between two conducting planes is more easily tackled (if we want only the total charge induced on the planes) if we replace the point charge by a uniformly charged plane sheet.

Let σ be the charge density on this sheet and E_1, E_2 outward electric field on the two sides of this sheet.

Then $\quad E_1 + E_2 = \dfrac{\sigma}{\varepsilon_0}$



The conducting planes will be assumed to be grounded. Then $E_1 x = E_2 (l - x)$.

Hence $\quad E_1 = \dfrac{\sigma}{l\,\varepsilon_0}(l - x)$, $E_2 = \dfrac{\sigma}{l\,\varepsilon_0} x$

This means that the induced charge density on the plane conductors are

$$\sigma_1 = -\frac{\sigma}{l}(l - x),\ \sigma_2 = -\frac{\sigma}{l} x$$

Hence $q_1 = -\dfrac{q}{l}(l - x)$, $q_2 = -\dfrac{q}{l}\, x$

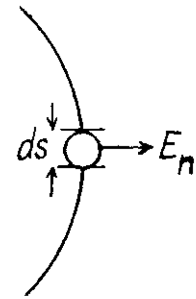
3.68 Near the conductor $E = E_n = \frac{\sigma}{\varepsilon_0}$

This field can be written as the sum of two parts E_1 and E_2. E_1 is the electric field due to an infinitesimal area dS.

Very near it $E_1 = \pm \frac{\sigma}{2\,\varepsilon_0}$

The remaining part contributes $E_2 = \frac{\sigma}{2\,\varepsilon_0}$ on both sides. In calculating the force on the element dS we drop E_1 (because it is a self-force.) Thus

$$\frac{dF}{dS} = \sigma \cdot \frac{\sigma}{2\,\varepsilon_0} = \frac{\sigma^2}{2\,\varepsilon_0}$$

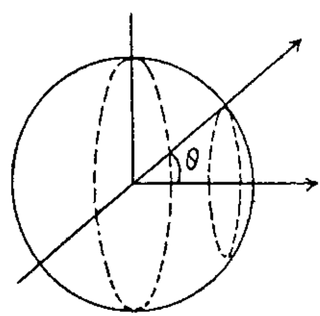


3.69 The total force on the hemisphere is

$$F = \int_0^{\pi/2} \frac{\sigma^2}{2\,\varepsilon_0} \cdot \cos\theta \cdot 2\,\pi\,R \sin\theta\, R\, d\theta$$

$$= \frac{2\,\pi\,R^2\,\sigma^2}{2\,\varepsilon_0} \int_0^{\pi/2} \cos\theta \sin\theta\, d\theta$$

$$= \frac{2\,\pi\,R^2}{2\,\varepsilon_0} \times \frac{1}{2} \times \left(\frac{q}{4\,\pi\,R^2}\right)^2 = \frac{q^2}{32\,\pi\,\varepsilon_0\,R}.$$



3.70 We know that the force acting on the area element dS of a conductor is,

$$d\vec{F} = \frac{1}{2}\sigma\vec{E}\,dS \qquad (1)$$

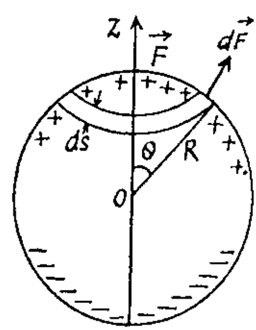
It follows from symmetry considerations that the resultant force $\vec{F}$ is directed along the z-axis, and hence it can be represented as the sum (integral) of the projection of elementary forces (1) onto the z-axis :

$$dF_z = dF\cos\theta \qquad (2)$$

For simplicity let us consider an element area $dS = 2\,\pi\,R\sin\theta\,R\,d\theta$(Fig.). Now considering that $E = \sigma/\varepsilon_0$. Equation (2) takes the from

$$dF_z = \frac{\pi\,\sigma^2\,R^2}{\varepsilon_0}\sin\theta\cos\theta\,d\theta$$

$$= -\left(\frac{\pi\,\sigma_0^2\,R^2}{\varepsilon_0}\right)\cos^3\theta\,d\cos\theta$$



Integrating this expression over the half sphere (i.e. with respect to cos θ between 1 and 0),

we obtain $$F = F_z = \frac{\pi \sigma_0^2 R^2}{4 \varepsilon_0}$$

3.71 The total polarization is $P = (\varepsilon - 1)\, \varepsilon_0 E$. This must equals $\frac{n_0 p}{N}$ where n_0 is the concerntation of water molecules. Thus

$$N = \frac{n_0 p}{(\varepsilon - 1)\, \varepsilon_0 E} = 2{\cdot}93 \times 10^3 \text{ on putting the values}$$

3.72 From the general formula

$$\vec{E} = \frac{1}{4 \pi \varepsilon_0} \frac{3 \vec{p} \cdot \vec{r} \vec{r} - \vec{p} r^2}{r^5}$$

$$\vec{E} = \frac{1}{4 \pi \varepsilon_0} \frac{2 \vec{p}}{l^3}, \text{ where } r = l \text{ and } \vec{r} \uparrow\uparrow \vec{p}$$

This will cause the induction of a dipole moment.

$$\vec{p}_{ind} = \beta \frac{1}{4 \pi \varepsilon_0} \frac{2 \vec{p}}{l^3} \times \varepsilon_0$$

Thus the force,

$$\vec{F} = \frac{\beta}{4 \pi} \frac{2p}{l^3} \frac{\partial}{\partial l} \frac{1}{4 \pi \varepsilon_0} \frac{2p}{l^3} = \frac{3 \beta p^2}{4 \pi^2 \varepsilon_0 l^7}$$

3.73 The electric field E at distance x from the centre of the ring is,

$$E(x) = \frac{qx}{4 \pi \varepsilon_0 (R^2 + x^2)^{3/2}}$$

The induced dipole moment is $p = \beta \varepsilon_0 E = \frac{q \beta x}{4 \pi (R^2 + x^2)^{3/2}}$

The force on this molecule is

$$F = p \frac{\partial}{\partial x} E = \frac{q \beta x}{4 \pi (R^2 + x^2)^{3/2}} \frac{q}{4 \pi \varepsilon_0} \frac{\partial}{\partial x} \frac{x}{(R^2 + x^2)^{3/2}} = \frac{q^2 \beta}{16 \pi^2 \varepsilon_0} \frac{x (R^2 - 2x^2)}{(R^2 + x^2)^4}$$

This vanishes for $x = \frac{\pm R}{\sqrt{2}}$ (apart from $x = 0$, $x = \infty$)

It is maximum when

$$\frac{\partial}{\partial x} \frac{x (R^2 - x^2 \times 2)}{(R^2 + x^2)^4} = 0$$

or, $$(R^2 - 2x^2)(R^2 + x^2) - 4x^2 (R^2 + x^2) - 8x^2 (R^2 - 2x^2) = 0$$

or, $$R^4 - 13 x^2 R^2 + 10 x^4 = 0 \text{ or, } x^2 = \frac{R^2}{20}\left(13 \pm \sqrt{129}\right)$$

or, $x = \frac{R}{\sqrt{20}} \sqrt{13 \pm \sqrt{129}}$ (on either side), Plot of $F_x(x)$ is as shown in the answersheet.

3.74 Inside the ball

$$\vec{D}(\vec{r}\,) = \frac{q}{4\pi}\frac{\vec{r}}{r^3} = \varepsilon\,\varepsilon_0\,\vec{E}.$$

Also $\qquad \varepsilon_0\vec{E} + \vec{P} = \vec{D}$ or $\vec{P} = \dfrac{\varepsilon - 1}{\varepsilon}\vec{D} = \dfrac{\varepsilon - 1}{\varepsilon}\dfrac{q}{4\pi}\dfrac{\vec{r}}{r^3}$

Also, $\qquad q' = -\oint \vec{P}\cdot d\vec{S} = -\dfrac{\varepsilon - 1}{\varepsilon}\dfrac{q}{4\pi}\int d\Omega = -\dfrac{\varepsilon - 1}{\varepsilon}q$

3.75 $D_{diel} = \varepsilon\,\varepsilon_0\,E_{diel} = D_{conductor} = \sigma$ or, $E_{diel} = \dfrac{\sigma}{\varepsilon\,\varepsilon_0}$

$$P_n = (\varepsilon - 1)\,\varepsilon_0\,E_{diel} = \frac{\varepsilon - 1}{\varepsilon}\sigma$$

$$\sigma' = -P_n = -\frac{\varepsilon - 1}{\varepsilon}\sigma$$

This is the surface density of bound charges.

3.76 From the solution of the previous problem q'_{in} = charge on the interior surface of the conductor

$$= -(\varepsilon - 1)/\varepsilon \int \sigma\, dS = -\frac{\varepsilon - 1}{\varepsilon}q$$

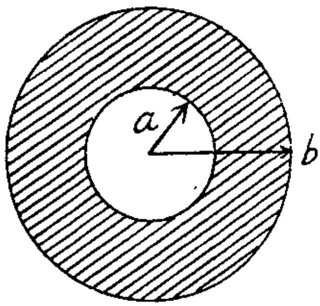
Since the dielectric as a whole is neutral there must be a total charge equal to $q'_{outer} = +\dfrac{\varepsilon - 1}{\varepsilon}q$ on the outer surface of the dielectric.

3.77 (a) Positive extraneous charge is distributed uniformly over the internal surface layer. Let σ_0 be the surface density of the charge.

Clearly, $\qquad E = 0$, for $r < a$

For $a < r$

$\varepsilon_0 E \times 4\pi r^2 = 4\pi a^2 \sigma_0$ by Gauss theorem.



or, $\qquad E = \dfrac{\sigma_0}{\varepsilon_0\,\varepsilon}\left(\dfrac{a}{r}\right)^2, \; a < r < b$

For $r > b$, similarly

$$E = \frac{\sigma_0}{\varepsilon_0}\left(\frac{a}{r}\right)^2, \; r > b$$

Now, $\qquad E = -\dfrac{\partial\varphi}{\partial r}.$

So by integration from infinity where $\varphi(\infty) = 0$,

$$\varphi = \frac{\sigma_0\, a^2}{\varepsilon_0\, r}\; r > b$$

$a < r < b \quad \varphi = \dfrac{\sigma_0 a^2}{\varepsilon \varepsilon r} + B$, B is a constant

or by continuity, $\varphi = \dfrac{\sigma_0 a^2}{\varepsilon_0 \varepsilon}\left(\dfrac{1}{r} - \dfrac{1}{b}\right) + \dfrac{\sigma_0 a^2}{\varepsilon_0 b}$, $a < r < b$

For $\quad r < a$. $\varphi = A =$ Constant

By continuity, $\varphi = \dfrac{\sigma_0 a^2}{\varepsilon_0 \varepsilon}\left(\dfrac{1}{a} - \dfrac{1}{b}\right) + \dfrac{\sigma_0 a^2}{\varepsilon_0 b}$

(b) Positive extraneous charge is distributed uniformly over the internal volume of the dielectric

Let $\rho_0 =$ Volume density of the charge in the dielectric, for $a < r < b$.

$$E = 0, \quad r < a$$

$$\varepsilon_0 \varepsilon \, 4 \pi r^2 E = \frac{4\pi}{3}(r^3 - a^3)\,\rho_0\,, \ (a < r < b)$$

or,
$$E = \frac{\rho_0}{3\,\varepsilon_0\,\varepsilon}\left(r - \frac{a^3}{r^2}\right)$$

$$E = \frac{4\pi}{3}(b^3 - a^3)\,\rho_0 \Big/ \varepsilon_0\, 4\pi r^2, \ r > b$$

or,
$$E = \frac{(b^3 - a^3)\,\rho_0}{3\,\varepsilon_0\, r^2} \quad \text{for} \quad r > b$$

By integration,

$$\varphi = \frac{(b^3 - a^3)\,\rho_0}{3\,\varepsilon_0\, r} \quad \text{for} \quad r > b$$

or,
$$\varphi = B - \frac{\rho_0}{3\,\varepsilon_0\,\varepsilon}\left(\frac{r^2}{2} + \frac{a^3}{r}\right), \ a < r < b$$

By continuity

$$\frac{b^3 - a^3}{3\,\varepsilon_0\, b}\,\rho_0 = B - \frac{\rho_0}{3\,\varepsilon_0\,\varepsilon}\left(\frac{b^2}{2} + \frac{a^3}{b}\right)$$

or,
$$B = \frac{\rho_0}{3\,\varepsilon_0\,\varepsilon}\left\{\frac{\varepsilon\,(b^3 - a^3)}{b} + \left(\frac{b^2}{2} + \frac{a^3}{b}\right)\right\}$$

Finally
$$\varphi = B - \frac{\rho_0}{3\,\varepsilon_0\,\varepsilon}\left(\frac{a^2}{2} + a^2\right) = B - \frac{\rho_0\, a^2}{2\,\varepsilon_0\,\varepsilon}, \ r < a$$

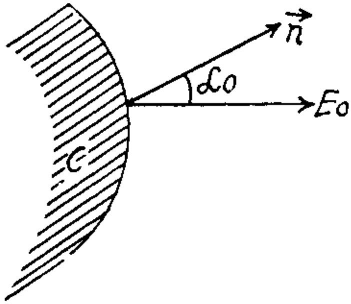
On the basis of obtained expressions $E(r)$ and $(\varphi)(r)$ can be plotted as shown in the answer-sheet.

3.78 Let the field in the dielectric be $\vec{E}$ making an angle α with $\vec{n}$. Then we have the boundary conditions,

$E_0 \cos \alpha_0 = \varepsilon E \cos \alpha$ and $E_0 \sin \alpha_0 = E \sin \alpha$

So $E = E_0 \sqrt{\sin^2 \alpha_0 + \frac{1}{\varepsilon^2} \cos^2 \alpha_0}$ and $\tan \alpha = \varepsilon \tan \alpha_0$

In the dielectric the normal component of the induction vector is

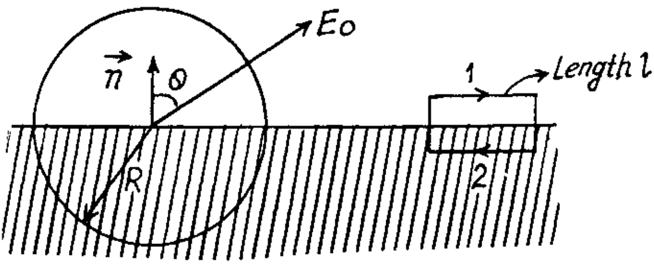


$D_n = \varepsilon_0 \varepsilon E_n = \varepsilon_0 \varepsilon E \cos \alpha = \varepsilon_0 E_0 \cos \alpha_0$

$\sigma' = P_n = D_n - \varepsilon_0 E_n = \left(1 - \frac{1}{\varepsilon}\right) \varepsilon_0 E_0 \cos \alpha_0$

or, $\sigma' = \frac{\varepsilon - 1}{\varepsilon} \varepsilon_0 E_0 \cos \alpha_0$

3.79 From the previous problem, $\sigma' = \varepsilon_0 \frac{\varepsilon - 1}{\varepsilon} E_0 \cos \theta$



(a) Then $\oint \vec{E} \cdot d\vec{S} = \frac{1}{\varepsilon_0} Q = \pi R^2 E_0 \cos \theta \frac{\varepsilon - 1}{\varepsilon}$

(b) $\oint \vec{D} \cdot d\vec{l} = (D_{1t} - D_{2t})\, l = (\varepsilon_0 E_0 \sin \theta - \varepsilon \varepsilon_0 E_0 \sin \theta) = -(\varepsilon - 1) \varepsilon_0 E_0\, l \sin \theta$

3.80 (a) $\operatorname{div} \vec{D} = \frac{\partial D_x}{\partial x} = \rho$ and $D = \rho\, l$

$$E_x = \frac{\rho\, l}{\varepsilon \varepsilon_0},\ l < d \text{ and } E_x = \frac{\rho\, d}{\varepsilon_0} \text{ constant for } l > d$$

$\varphi(x) = -\frac{\rho\, l^2}{2\varepsilon \varepsilon_0}$, $l < d$ and $\varphi(x) = A - \frac{\rho\, ld}{\varepsilon_0}$, $l > d$ then $\varphi(x) = \frac{\rho\, d}{\varepsilon_0}\left(d - \frac{d}{2\varepsilon} - l\right)$,

by continuity.

On the basis of obtained expressions $E_x(x)$ and $\varphi(x)$ can be plotted as shown in the figure of answersheet.

(b) $\rho' = -\text{div}\,\vec{P} = -\text{div}\,(\varepsilon - 1)\,\varepsilon_0 \vec{E} = -\rho\,\frac{(\varepsilon - 1)}{\varepsilon}$

$\sigma' = P_{1n} - P_{2n}$, where n is the normal from 1 to 2.

$= P_{1n}$, ($\vec{P}_2 = 0$ as 2 is vacuum.)

$= (\rho d - \rho\, d/\varepsilon) = \rho\, d\,\frac{\varepsilon - 1}{\varepsilon}$

3.81 $\text{div}\vec{D} = \frac{l}{r^2}\frac{\partial}{\partial r} r^2 D_r = \rho$

$r^2 D_r = \rho\,\frac{r^3}{3} + A \quad D_r = \frac{1}{3}\,\rho\, r + \frac{A}{r^2},\ r < R$

$A = 0$ as $D_r \neq \infty$ at $r = 0$,Thus, $E_r = \frac{\rho r}{3\,\varepsilon\,\varepsilon_0}$

For $\quad r > R,\ D_r = \frac{B}{r^2}$

By continuity of D_r at $r = R$; $B = \frac{\rho R^3}{3}$

so, $$E_r = \frac{\rho R^3}{3\,\varepsilon_0 r^2},\ r > R$$

$$\varphi = \frac{\rho R^3}{3\,\varepsilon_0\, r},\ r > R \text{ and } \varphi = -\frac{\rho r^2}{6\,\varepsilon\,\varepsilon_0} + C,\ r < R$$

$$C = +\frac{\rho\, R^2}{3\,\varepsilon_0} + \frac{\rho\, R^2}{6\,\varepsilon\,\varepsilon_0}, \text{ by continuity of } \varphi.$$

See answer sheet for graphs of $E\,(r)$ and $\varphi\,(r)$

(b) $\rho' = \text{div}\,\vec{P} = -\frac{1}{r^2}\frac{\partial}{\partial r}\left\{\frac{r^3}{3}\rho\left(1 - \frac{1}{\varepsilon}\right)\right\} = -\frac{\rho\,(\varepsilon - 1)}{\varepsilon}$

$\sigma' = P_{1r} - P_{2r} = P_{1r} = \frac{1}{3}\,\rho\, R\left(1 - \frac{1}{\varepsilon}\right)$

3.82 Because there is a discontinuity in polarization at the boundary of the dielectric disc, a bound surface charge appears, which is the source of the electric field inside and outside the disc.

We have for the electric field at the origin.

$$\vec{E} = -\int \frac{\sigma'\, dS}{4\,\pi\,\varepsilon_0\, r^3}\,\vec{r},$$

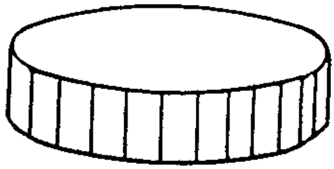
where $\vec{r} =$ radius vector to the origin from the element dS.

$\sigma' = P_n = P\cos\theta$ on the curved surface

($P_n = 0$ on the flat surface.)

Here θ = angle between $\vec{r}$ and $\vec{P}$

By symmetry, $\vec{E}$ will be parallel to $\vec{P}$. Thus



$$E = -\int_0^{2\pi} \frac{P\cos\theta\, R\, d\theta \cdot \cos\theta}{4\pi\varepsilon_0 R^2} \cdot d$$

where, $r = R$ if $d << R$.

So, $E = -\dfrac{Pd}{4\varepsilon_0 R}$ and $\vec{E} = -\dfrac{\vec{P}d}{4\varepsilon_0 R}$

3.83. Since there are no free extraneous charges anywhere

$$\text{div}\,\vec{D} = \frac{\partial D_x}{\partial x} = 0 \quad \text{or,} \quad D_x = \text{Constant}$$

But $D_x = 0$ at ∞, so, $D_x = 0$, every where.

Thus, $\vec{E} = -\dfrac{\vec{P_0}}{\varepsilon_0}\left(1 - \dfrac{x^2}{d^2}\right)$ or, $E_x = -\dfrac{P_0}{\varepsilon_0}\left(1 - \dfrac{x^2}{d^2}\right)$

So, $\varphi = \dfrac{P_0 x}{\varepsilon_0} - \dfrac{P_0 x^3}{3\varepsilon_0 d^2} + \text{constant}$

Hence,

$$\varphi(+d) - \varphi(-d) = \frac{2P_0 d}{\varepsilon_0} - \frac{2P_0 d^3}{3d^2\varepsilon_0} = \frac{4P_0 d}{3\varepsilon_0}$$

3.84 (a) We have $D_1 = D_2$, or, $\varepsilon E_2 = E_1$

Also, $E_1\dfrac{d}{2} + E_2\dfrac{d}{2} = E_0 d$ or, $E_1 + E_2 = 2E_0$

Hence, $E_2 = \dfrac{2E_0}{\varepsilon + 1}$ and $E_1 = \dfrac{2\varepsilon E_0}{\varepsilon + 1}$ and $D_1 = D_2 = \dfrac{2\varepsilon\varepsilon_0 E_0}{\varepsilon + 1}$

(b) $D_1 = D_2$, or, $\varepsilon E_2 = E_1 = \dfrac{\sigma}{\varepsilon_0} = E_0$

Thus, $E_1 = E_0$, $E_2 = \dfrac{E_0}{\varepsilon}$ and $D_1 = D_2 = \varepsilon_0 E_0$

3.85 (a) Constant voltage acros the plates;

$$E_1 = E_2 = E_0\,,\ D_1 = \varepsilon_0 E_0,\ D_2 = \varepsilon_0\,\varepsilon\, E_0$$

(b) Constant charge across the plates;

$$E_1 = E_2\,,\ D_1 = \varepsilon_0 E_1,\ D_2 = \varepsilon\,\varepsilon_0 E_2 = \varepsilon D_1$$

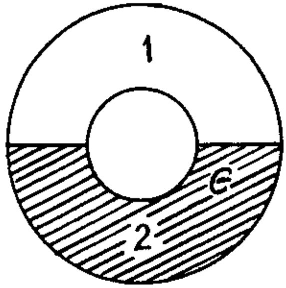
$$E_1(1+\varepsilon) = 2E_0 \quad \text{or} \quad E_1 = E_2 = \frac{2E_0}{\varepsilon+1}$$

3.86 At the interface of the dielectric and vacuum,

$$E_{1t} = E_{2t}$$

The electric field must be radial and

$$E_1 = E_2 = \frac{A}{\varepsilon_0\,\varepsilon\, r^2},\ a < r < b$$



Now, $q = \dfrac{A}{R^2}(2\pi R^2) + \dfrac{A}{\varepsilon R^2}(2\pi R^2)$

$$= A\left(1+\frac{1}{\varepsilon}\right)2\pi$$

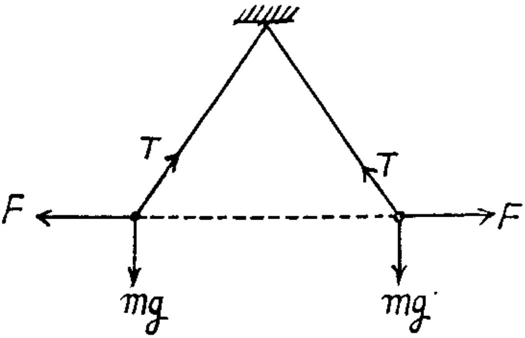
or, $E_1 = E_2 = \dfrac{q}{2\pi\varepsilon_0 r^2(1+\varepsilon)}$

3.87 In air the forces are as shown. In K–oil,

$F \to F' = F/\varepsilon$ and $mg \to mg\left(1 - \dfrac{\rho_0}{\rho}\right)$.

Since the inclinations do not change

$$\frac{1}{\varepsilon} = 1 - \frac{\rho_0}{\rho}$$



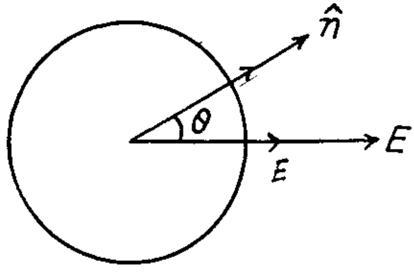
or, $\dfrac{\rho_0}{\rho} = 1 - \dfrac{1}{\varepsilon} = \dfrac{\varepsilon - 1}{\varepsilon}$

or, $\rho = \rho_0\dfrac{\varepsilon}{\varepsilon - 1}$

where ρ_0 is the density of K–oil and ρ that of the material of which the balls are made.

3.88 Within the ball the electric field can be resolved into normal and tangential components.

$$E_n = E\cos\theta,\ E_t = E\sin\theta$$



Then, $D_n = \varepsilon\,\varepsilon_0 E\cos\theta$

and $P_n = (\varepsilon - 1)\ \varepsilon_0 E\cos\theta$

or, $\sigma' = (\varepsilon - 1)\,\varepsilon_0 E\cos\theta$

so, $\sigma_{max} = (\varepsilon - 1)\,\varepsilon_0 E$,

and total charge of one sign,

$$q' = \int_0^1 (\varepsilon - 1)\,\varepsilon_0 E \cos\theta\, 2\pi R^2\, d(\cos\theta) = \pi R^2 \varepsilon_0 (\varepsilon - 1) E$$

(Since we are interested in the total charge of one sign we must intergrate cos θ from 0 to 1 only).

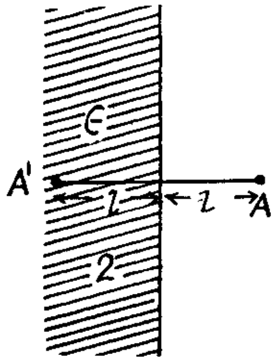
3.89 The charge is at A in the medium 1 and has an image point at A' in the medium 2. The electric field in the medium 1 is due to the actual charge q at A and the image charge q' at A'. The electric field in 2 is due to a corrected charge q' at A. Thus on the boundary between 1 and 2,

$$E_{1n} = \frac{q'}{4\pi\varepsilon_0 r^2}\cos\theta - \frac{q}{4\pi\varepsilon_0 r^2}\cos\theta$$

$$E_{2n} = \frac{-q''}{4\pi\varepsilon_0 r^2}\cos\theta$$

$$E_{1t} = \frac{q'}{4\pi\varepsilon_0 r^2}\sin\theta + \frac{q}{4\pi\varepsilon_0 r^2}\sin\theta$$

$$E_{2t} = \frac{q''}{4\pi\varepsilon_0 r^2}\sin\theta$$



The boundary conditions are

$$D_{1n} = D_{2n} \text{ and } E_{1t} = E_{2t}$$

$$\varepsilon q'' = q - q'$$

$$q'' = q + q'$$

So,
$$q'' = \frac{2q}{\varepsilon + 1}, \quad q' = -\frac{\varepsilon - 1}{\varepsilon + 1} q$$

(a) The surface density of the bound charge on the surface of the dielectric

$$\sigma' = P_{2n} = D_{2n} - \varepsilon_0 E_{2n} = (\varepsilon - 1)\varepsilon_0 E_{2n}$$

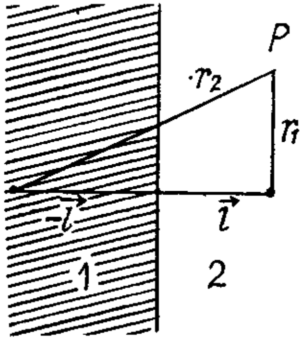
$$= -\frac{\varepsilon - 1}{\varepsilon + 1}\frac{q}{2\pi r^2}\cos\theta = -\frac{\varepsilon - 1}{\varepsilon + 1}\frac{ql}{2\pi r^3}$$

(b) Total bound charge is, $-\frac{\varepsilon - 1}{\varepsilon + 1} q \int_0^\infty \frac{l}{2\pi (l^2 + x^2)^{3/2}} 2\pi x\, dx = -\frac{\varepsilon - 1}{\varepsilon + 1} q$

3.90 The force on the point charge q is due to the bound charges. This can be calculated from the field at this charge after extracting out the self field. This image field is

$$E_{\text{image}} = \frac{\varepsilon - 1}{\varepsilon + 1}\frac{q}{4\pi\varepsilon_0 (2l)^2}$$

Thus,
$$F = \frac{\varepsilon - 1}{\varepsilon + 1}\frac{q^2}{16\pi\varepsilon_0 l^2}$$



3.91 $E_p = \dfrac{q\,\vec{r_1}}{4\pi\varepsilon_0 r_1^3} + \dfrac{q'\vec{r_2}}{4\pi r_2^3\varepsilon_0}$, P in 1

$E_p = \dfrac{q''\,\vec{r_1}}{4\pi\varepsilon_0 r_1^3}$, P in 2

where $q'' = \dfrac{2q}{\varepsilon+1}$, $q' = q'' - q$

In the limit $\vec{l} \to 0$

$\vec{E_p} = \dfrac{(q+q')\,\vec{r}}{4\pi\varepsilon_0 r^3} = \dfrac{q\,\vec{r}}{2\pi\varepsilon_0(1+\varepsilon)r^3}$, in either part.

Thus,
$$E_p = \frac{q}{2\pi\varepsilon_0(1+\varepsilon)r^2}$$

$$\varphi = \frac{q}{2\pi\varepsilon_0(1+\varepsilon)r}$$

$$D = \frac{q}{2\pi\varepsilon_0(1+\varepsilon)r^2} \times \begin{cases} 1 \text{ in vacuum} \\ \varepsilon \text{ in dielectric} \end{cases}$$

3.92 $\vec{E_p} = \dfrac{q\,\vec{r_2}}{4\pi\varepsilon_0\varepsilon r_2^3} + \dfrac{q'\,\vec{r_1}}{4\pi\varepsilon_0 r_1^3}$; P in 2

$\vec{E_p} = \dfrac{q''\,\vec{r_2}}{4\pi\varepsilon_0 r_2^3}$; P in 1

Using the boundary conditions,

$E_{1n} = \varepsilon E_{2n}$, $E_{1t} = E_{2t}$

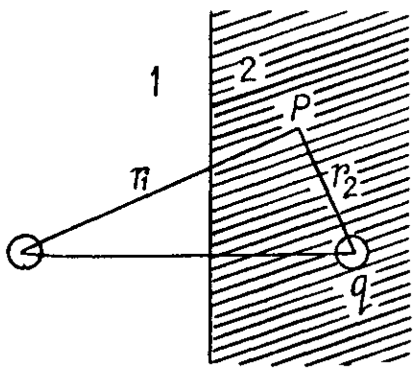
This implies

$q - \varepsilon q' = q''$ and $q + \varepsilon q' = \varepsilon q''$

So, $q'' = \dfrac{2q}{\varepsilon+1}$, $q' = \dfrac{\varepsilon-1}{\varepsilon+1}\dfrac{q}{\varepsilon}$

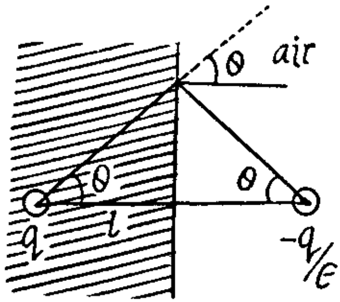
Then, as earlier,

$$\sigma' = \frac{ql}{2\pi r^3}\cdot\left(\frac{\varepsilon-1}{\varepsilon+1}\right)\cdot\frac{1}{\varepsilon}$$



3.93 To calculate the electric field, first we note that an image charge will be needed to ensure that the electric field on the metal boundary is normal to the surface.

The image charge must have magnitude $-\frac{q}{\varepsilon}$ so that the tangential component of the electric field may vanish. Now,



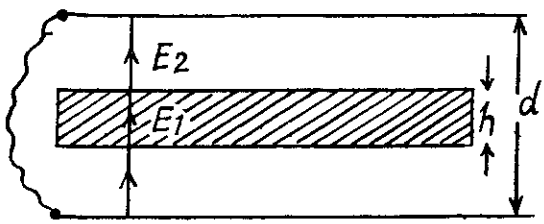
$$E_n = \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{\varepsilon r^2}\right) 2\cos\theta = \frac{ql}{2\pi\varepsilon_0 \varepsilon r^3}$$

Then $P_n = D_n - \varepsilon_0 E_n = \frac{(\varepsilon - 1)\,ql}{2\pi\varepsilon r^3} = \sigma'$

This is the density of bound charge on the surface.

3.94 Since the condenser plates are connected,

$$E_1 h + E_2 (d - h) = 0$$



and $\qquad P + \varepsilon_0 E_1 = \varepsilon_0 E_2$

or, $\qquad E_1 + \frac{P}{\varepsilon_0} = E_2$

Thus, $E_2 d - \frac{Ph}{\varepsilon_0} = 0$, or, $E_2 = \frac{Ph}{\varepsilon_0 d}$

$$E_1 = -\frac{P}{\varepsilon_0}\left(1 - \frac{h}{d}\right)$$

3.95 Given $\vec{P} = \alpha \vec{r}$, where $\vec{r} =$ distance from the axis. The space density of charges is given by, $\rho' = -\operatorname{div} \vec{P} = -2\alpha$

On using. $\qquad \operatorname{div} \vec{r} = \frac{1}{r}\frac{\partial}{\partial r}(\vec{r}\cdot\vec{r}) = 2$

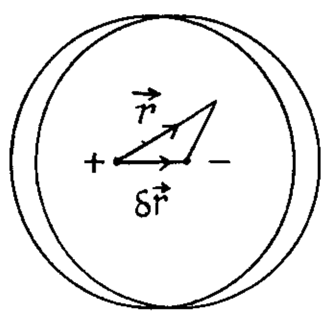
3.96 In a uniformly charged sphere,

$$E_r = \frac{\rho_0 r}{3\varepsilon_0} \quad \text{or,} \quad \vec{E} = \frac{\rho_0}{3\varepsilon_0}\vec{r}$$

The total electric field is

$$\vec{E} = \frac{1}{3\varepsilon_0}\rho_0 \vec{r} - \frac{1}{3\varepsilon_0}(\vec{r} - \delta\vec{r})\,\rho_0$$

$$= \frac{1}{3\varepsilon_0}\rho_0 \delta\vec{r} = -\frac{\vec{P}}{3\varepsilon_0}$$



where $\rho\,\delta\vec{r} = -\vec{P}$ (dipole moment is defined with its direction being from the *-ve* charge to *+ve* charge.)

The potential outside is

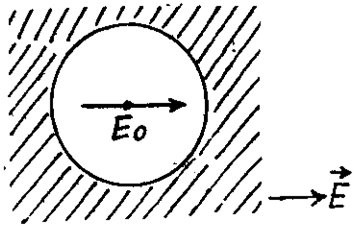
$$\varphi = \frac{1}{4\pi\varepsilon_0}\left(\frac{Q}{r} - \frac{Q}{|\vec{r} - \delta\vec{r}|}\right), = \frac{\vec{p}_0\cdot\vec{r}}{4\pi\varepsilon_0 r^3},\ r > R$$

where $\vec{p}_0 = -\frac{4\pi}{3}R^3\,\rho_0\,\partial\vec{r}$ is the total dipole moment.

3.97 The electric field $\vec{E_0}$ in a spherical cavity in a uniform dielectric of permittivity ε is related to the far away field $\vec{E}$, in the following manner. Imagine the cavity to be filled up with the dielectric. Then there will be a uniform field $\vec{E}$ everywhere and a polarization $\vec{P}$, given by,

$$\vec{P} = (\varepsilon - 1)\,\varepsilon_0 \vec{E}$$

Now take out the sphere making the cavity, the electric field inside the sphere will be $-\dfrac{\vec{P}}{3\varepsilon_0}$



By superposition. $\vec{E_0} - \dfrac{\vec{P}}{3\varepsilon_0} = \vec{E}$

or, $\vec{E_0} = \vec{E} + \dfrac{1}{3}(\varepsilon - 1)\vec{E} = \dfrac{1}{3}(\varepsilon + 2)\vec{E}$

3.98 By superposition the field $\vec{E}$ inside the ball is given by

$$\vec{E} = \vec{E_0} - \frac{\vec{P}}{3\varepsilon_0}$$

On the other hand, if the sphere is not too small, the macroscopic equation $\vec{P} = (\varepsilon - 1)\,\varepsilon_0 \vec{E}$ must hold. Thus,

$$\vec{E}\left(1 + \frac{1}{3}(\varepsilon - 1)\right) = \vec{E_0} \quad \text{or,} \quad \vec{E} = \frac{3\vec{E_0}}{\varepsilon + 2}$$

Also
$$\vec{P} = 3\,\varepsilon_0 \frac{\varepsilon - 1}{\varepsilon + 2}\vec{E_0}$$

3.99 This is to be handled by the same trick as in 3.96. We have effectively a two dimensional situation. For a uniform cylinder full of charge with charge density ρ_0 (charge per unit volume), the electric field E at an inside point is along the (cylindrical) radius vector $\vec{r}$ and equal to,

$$\vec{E} = \frac{1}{2\varepsilon_0}\rho\,\vec{r}$$

$$\left(\operatorname{div}\vec{E} = \frac{l}{r}\frac{\partial}{\partial r}(rE_r) = \frac{\rho}{\varepsilon_0}, \quad \text{hence,} \quad E_r = \frac{\rho}{2\varepsilon_0}r\right)$$

Therefore the polarized cylinder can be thought of as two equal and opposite charge distributions displaced with respect to each other

$$\vec{E} = \frac{1}{2\varepsilon_0}\rho\,\vec{r} - \frac{1}{2\varepsilon_0}\rho\,(\vec{r} - \delta\vec{r}) = \frac{1}{2\varepsilon_0}\rho\,\delta\,\vec{r} = -\frac{\vec{P}}{2\varepsilon_0}$$

Since $\vec{P} = -\rho\,\delta\,\vec{r}$. (direction of electric dipole moment vector being from the negative charge to positive charge .)

3.100 As in 3.98, we write $\vec{E} = \vec{E_0} - \dfrac{\vec{P}}{2\varepsilon_0}$

using here the result of the foregoing problem.

Also
$$\vec{P} = (\varepsilon - 1)\,\varepsilon_0 \vec{E}$$

So, $\vec{E}\left(\dfrac{\varepsilon + 1}{2}\right) = \vec{E_0}$, or, $\vec{E} = \dfrac{2\vec{E_0}}{\varepsilon + 1}$ and $\vec{P} = 2\varepsilon_0 \dfrac{\varepsilon - 1}{\varepsilon + 1}\vec{E_0}$