Exercise 16.5

Chapter 16 Vector Calculus Exercise 16.5 1E

(a)If $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ is a vector field on $\mathbb{R}^3$ and the partial derivatives of P, Q, and R all exist, then the curl of $\mathbf{F}$ is the vector field on $\mathbb{R}^3$ defined by

$$\text{curl } \mathbf{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\mathbf{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\mathbf{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathbf{k}.$$

We have $\mathbf{F}(x, y, z) = (x + yz)\mathbf{i} + (y + xz)\mathbf{j} + (z + xy)\mathbf{k}$.
Then, $P = x + yz$, $Q = y + xz$, and $R = z + xy$.

Find $\frac{\partial P}{\partial y}, \frac{\partial P}{\partial z}, \frac{\partial Q}{\partial x}, \frac{\partial Q}{\partial z}, \frac{\partial R}{\partial x}$, and $\frac{\partial R}{\partial y}$.

$$\frac{\partial P}{\partial y} = \frac{\partial}{\partial y}(x + yz) = z$$

$$\frac{\partial P}{\partial z} = \frac{\partial}{\partial z}(x + yz) = y$$

$$\frac{\partial Q}{\partial x} = \frac{\partial}{\partial x}(y + xz) = z$$

$$\frac{\partial Q}{\partial z} = \frac{\partial}{\partial z}(y + xz) = x$$

$$\frac{\partial R}{\partial x} = \frac{\partial}{\partial x}(z + xy) = y$$

$$\frac{\partial R}{\partial y} = \frac{\partial}{\partial y}(z + xy) = x$$

Substitute the known values in the expression for curl $\mathbf{F}$.

$$\begin{aligned}\text{curl } \mathbf{F} &= (x - x)\mathbf{i} + (y - y)\mathbf{j} + (z - z)\mathbf{k} \\ &= 0\mathbf{i} + 0\mathbf{j} + 0\mathbf{k} \\ &= 0\end{aligned}$$

Thus, we get $\boxed{\text{curl } \mathbf{F} = 0}$.

(b)If $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ is a vector field on $\mathbb{R}^3$ and the partial derivatives of P, Q, and R exist, then the divergence of $\mathbf{F}$ is defined as $\text{div } \mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$.

Find $\frac{\partial P}{\partial x}$, $\frac{\partial Q}{\partial y}$, and $\frac{\partial R}{\partial z}$

$$\frac{\partial P}{\partial x} = \frac{\partial}{\partial x}(x + yz)$$
$$= 1$$
$$\frac{\partial Q}{\partial y} = \frac{\partial}{\partial y}(y + xz)$$
$$= 1$$
$$\frac{\partial R}{\partial z} = \frac{\partial}{\partial z}(z + xy)$$
$$= 1$$

Then, $\text{div } \mathbf{F} = 1 + \ + 1$ or $\text{div } \mathbf{F} = 3$

Thus, we get $\boxed{\text{div } \mathbf{F} = 3}$.

Chapter 16 Vector Calculus Exercise 16.5 2E

(a)If $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ is a vector field on $\mathbb{R}^3$ and the partial derivatives of P, Q, and R all exist, then the curl of $\mathbf{F}$ is the vector field on $\mathbb{R}^3$ defined by

$$\text{curl } \mathbf{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\mathbf{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\mathbf{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathbf{k}.$$

We have $\mathbf{F}(x, y, z) = xy^2z^3\mathbf{i} + x^3yz^2\mathbf{j} + x^2y^3z\mathbf{k}$.
Then, $P = xy^2z^3$, $Q = x^3yz^2$, and $R = x^2y^3z$.

Find $\frac{\partial P}{\partial y}$, $\frac{\partial P}{\partial z}$, $\frac{\partial Q}{\partial x}$, $\frac{\partial Q}{\partial z}$, $\frac{\partial R}{\partial x}$, and $\frac{\partial R}{\partial y}$.

$$\frac{\partial P}{\partial y} = \frac{\partial}{\partial y}\left(xy^2z^3\right)$$
$$= 2xyz^3$$
$$\frac{\partial P}{\partial z} = \frac{\partial}{\partial z}\left(xy^2z^3\right)$$
$$= 3xy^2z^2$$
$$\frac{\partial Q}{\partial x} = \frac{\partial}{\partial x}\left(x^3yz^2\right)$$
$$= 3x^2yz^2$$
$$\frac{\partial Q}{\partial z} = \frac{\partial}{\partial z}\left(x^3yz^2\right)$$
$$= 2x^3yz$$
$$\frac{\partial R}{\partial x} = \frac{\partial}{\partial x}\left(x^2y^3z\right)$$
$$= 2xy^3z$$
$$\frac{\partial R}{\partial y} = \frac{\partial}{\partial y}\left(x^2y^3z\right)$$
$$= 3x^2y^2z$$

Substitute the known values in $\text{curl } \mathbf{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\mathbf{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\mathbf{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathbf{k}$.

$$\text{curl } \mathbf{F} = \left(3x^2y^2z - 2x^3yz\right)\mathbf{i} + \left(3xy^2z^2 - 2xy^3z\right)\mathbf{j} + \left(3x^2yz^2 - 2xyz^3\right)\mathbf{k}$$
$$= x^2yz(3y - 2x)\mathbf{i} + xy^2z(3z - 2y)\mathbf{j} + xyz^2(3x - 2z)\mathbf{k}$$

Thus, we get $\boxed{\text{curl } \mathbf{F} = x^2yz(3y - 2x)\mathbf{i} + xy^2z(3z - 2y)\mathbf{j} + xyz^2(3x - 2z)\mathbf{k}}$.

(b)If $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ is a vector field on $\mathbb{R}^3$ and the partial derivatives of P, Q, and R exist, then the divergence of $\mathbf{F}$ is defined as $\text{div } \mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$.

Find $\frac{\partial P}{\partial x}$, $\frac{\partial Q}{\partial y}$, and $\frac{\partial R}{\partial z}$

$$\begin{aligned}\frac{\partial P}{\partial x} &= \frac{\partial}{\partial x}\left(xy^2z^3\right) \\ &= y^2z^3 \\ \frac{\partial Q}{\partial y} &= \frac{\partial}{\partial y}\left(x^3yz^2\right) \\ &= x^3z^2 \\ \frac{\partial R}{\partial z} &= \frac{\partial}{\partial z}\left(x^2y^3z\right) \\ &= x^2y^3\end{aligned}$$

Then, $\text{div } \mathbf{F} = y^2z^3 + x^3z^2 + x^2y^3$.

Thus, we get $\boxed{\text{div } \mathbf{F} = y^2z^3 + x^3z^2 + x^2y^3}$.

Chapter 16 Vector Calculus Exercise 16.5 3E

img src="https://c4.staticflickr.com/1/560/31125237483_8d46d0cef3_o.jpg" width="607" height="265" alt="Stewart-Calculus-7e-Solutions-Chapter-16.5-Vector-Calculus-3E">

From the vector field,

$P = xye^z, Q = 0$ and $R = yze^x$

Find the partial derivatives of $P, Q,$ and R as follows:

Determine the value of $\frac{\partial P}{\partial y}, \frac{\partial P}{\partial z}, \frac{\partial Q}{\partial x}, \frac{\partial Q}{\partial z}, \frac{\partial R}{\partial y},$ and $\frac{\partial R}{\partial y}$ as follows:

$$\begin{aligned}\frac{\partial P}{\partial y} &= \frac{\partial}{\partial y}\left(xye^z\right) \\ &= xe^z \\ \frac{\partial P}{\partial z} &= \frac{\partial}{\partial z}\left(xye^z\right) \\ &= xy\frac{\partial}{\partial z}\left(e^z\right) \\ &= xye^z\end{aligned}$$

Continue the above step,

$$\begin{aligned}\frac{\partial R}{\partial y} &= \frac{\partial}{\partial y}\left(yze^x\right) \\ &= ze^x\frac{\partial}{\partial y}(y) \\ &= ze^x \\ \frac{\partial R}{\partial x} &= \frac{\partial}{\partial x}\left(yze^x\right) \\ &= yz\frac{\partial}{\partial x}\left(e^x\right) \\ &= yze^x\end{aligned}$$

$$\frac{\partial Q}{\partial x} = 0$$

$$\frac{\partial Q}{\partial z} = 0$$

Substitute these values in equation (1).

$$\begin{aligned}\text{curl } \mathbf{F} &= \left(\frac{\partial R}{\partial y}-\frac{\partial Q}{\partial z}\right)\mathbf{i}+\left(\frac{\partial P}{\partial z}-\frac{\partial R}{\partial x}\right)\mathbf{j}+\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)\mathbf{k}\\ &= \left(ze^x-0\right)\mathbf{i}+\left(xye^z-yze^x\right)\mathbf{j}+\left(0-xe^z\right)\mathbf{k}\\ &= ze^x\mathbf{i}+\left(xye^z-yze^x\right)\mathbf{j}-xe^z\mathbf{k}\end{aligned}$$

Thus, the required value is,

$$\boxed{\text{curl } \mathbf{F} = ze^x\mathbf{i}+\left(xye^z-yze^x\right)\mathbf{j}-xe^z\mathbf{k}}$$

Find the divergence of $F(x,y,z)=xye^z\mathbf{i}+0\mathbf{j}+yze^x\mathbf{k}$ as follows:

Here,

$P=xye^z, Q=0 \text{ and } R=yze^x$

If $\mathbf{F}(x,y,z)=P\mathbf{i}+Q\mathbf{j}+R\mathbf{k}$ is a vector field on $\mathbb{R}^3$ and $\frac{\partial P}{\partial x},\frac{\partial Q}{\partial y}$ and $\frac{\partial R}{\partial z}$ exist, then the $\textbf{div}\mathbf{F}$ is the function of three variables defined by, $\text{div } \mathbf{F}=\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}+\frac{\partial R}{\partial z}$

Find the partial derivatives as follows:

Determine the value of $\frac{\partial P}{\partial x},\frac{\partial Q}{\partial y},$ and $\frac{\partial R}{\partial z}$ as follows:

$$\begin{aligned}\frac{\partial P}{\partial x} &= \frac{\partial}{\partial x}\left(xye^z\right)\\ &= ye^z\end{aligned}$$

$$\begin{aligned}\frac{\partial Q}{\partial y} &= \frac{\partial}{\partial y}(0)\\ &= 0\end{aligned}$$

$$\begin{aligned}\frac{\partial R}{\partial z} &= \frac{\partial}{\partial z}\left(yze^x\right)\\ &= ye^x\end{aligned}$$

Substitute the values of $\frac{\partial P}{\partial x},\frac{\partial Q}{\partial y}$ and $\frac{\partial R}{\partial z}$ in $\text{div } \mathbf{F}=\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}+\frac{\partial R}{\partial z}$.

$$\begin{aligned}\text{div } \mathbf{F} &= \frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}+\frac{\partial R}{\partial z}\\ &= ye^z+0+ye^x\\ &= y\left(e^z+e^x\right)\end{aligned}$$

Therefore, the required value is,

$$\boxed{\text{div}\mathbf{F}=y\left(e^z+e^x\right)}$$

Chapter 16 Vector Calculus Exercise 16.5 4E

(a)If $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ is a vector field on $\mathbb{R}^3$ and the partial derivatives of P, Q, and R all exist, then the curl of $\mathbf{F}$ is the vector field on $\mathbb{R}^3$ defined by

$$\text{curl } \mathbf{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\mathbf{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\mathbf{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathbf{k}.$$

We have $\mathbf{F}(x, y, z) = \sin yz\mathbf{i} + \sin zx\mathbf{j} + \sin xy\mathbf{k}$.
Then, $P = \sin yz$, $Q = \sin zx$, and $R = \sin xy$.

Find $\frac{\partial P}{\partial y}$, $\frac{\partial P}{\partial z}$, $\frac{\partial Q}{\partial x}$, $\frac{\partial Q}{\partial z}$, $\frac{\partial R}{\partial x}$, and $\frac{\partial R}{\partial y}$.

$$\frac{\partial P}{\partial y} = \frac{\partial}{\partial y}(\sin yz) = z\cos yz$$

$$\frac{\partial P}{\partial z} = \frac{\partial}{\partial z}(\sin yz) = y\cos yz$$

$$\frac{\partial Q}{\partial x} = \frac{\partial}{\partial x}(\sin zx) = z\cos zx$$

$$\frac{\partial Q}{\partial z} = \frac{\partial}{\partial z}(\sin zx) = x\cos zx$$

$$\frac{\partial R}{\partial x} = \frac{\partial}{\partial x}(\sin xy) = y\cos xy$$

$$\frac{\partial R}{\partial y} = \frac{\partial}{\partial y}(\sin xy) = x\cos xy$$

Substitute the known values in $\text{curl } \mathbf{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\mathbf{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\mathbf{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathbf{k}$.

$$\begin{aligned}\text{curl } \mathbf{F} &= (x\cos xy - x\cos zx)\mathbf{i} + (y\cos yz - y\cos xy)\mathbf{j} + (z\cos zx - z\cos yz)\mathbf{k} \\ &= x(\cos xy - \cos zx)\mathbf{i} + y(\cos yz - \cos xy)\mathbf{j} + z(\cos zx - \cos yz)\mathbf{k}\end{aligned}$$

Thus, we get $\text{curl } \mathbf{F} = x(\cos xy - \cos zx)\mathbf{i} + y(\cos yz - \cos xy)\mathbf{j} + z(\cos zx - \cos yz)\mathbf{k}$.

(b)If $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ is a vector field on $\mathbb{R}^3$ and the partial derivatives of P, Q, and R exist, then the divergence of $\mathbf{F}$ is defined as $\text{div } \mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$.

Find $\frac{\partial P}{\partial x}$, $\frac{\partial Q}{\partial y}$, and $\frac{\partial R}{\partial z}$

$$\frac{\partial P}{\partial x} = \frac{\partial}{\partial x}(\sin yz) = 0$$

$$\frac{\partial Q}{\partial y} = \frac{\partial}{\partial y}(\sin zx) = 0$$

$$\frac{\partial R}{\partial z} = \frac{\partial}{\partial z}(\sin xy) = 0$$

Then, $\text{div } \mathbf{F} = 0 + 0 + 0$.
Thus, we get $\text{div } \mathbf{F} = 0$.

Chapter 16 Vector Calculus Exercise 16.5 5E

Given vector field is $F(x,y,z)=\frac{1}{\sqrt{x^2+y^2+z^2}}(xi+yj+zk)$

(a) Curl $F=\nabla\times F$

$$=\begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{x}{\sqrt{x^2+y^2+z^2}} & \frac{y}{\sqrt{x^2+y^2+z^2}} & \frac{z}{\sqrt{x^2+y^2+z^2}} \end{vmatrix}$$

$$=\left[\frac{\partial}{\partial y}\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right)-\frac{\partial}{\partial z}\left(\frac{y}{\sqrt{x^2+y^2+z^2}}\right)\right]i-\left[\frac{\partial}{\partial x}\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right)-\frac{\partial}{\partial z}\left(\frac{x}{\sqrt{x^2+y}}\right.\right.$$

$$\left[\frac{\partial}{\partial x}\left(\frac{y}{\sqrt{x^2+y^2+z^2}}\right)-\frac{\partial}{\partial y}\left(\frac{x}{\sqrt{x^2+y^2+z^2}}\right)\right]k$$

$$=\left[z\frac{\partial}{\partial y}\left(x^2+y^2+z^2\right)^{\frac{-1}{2}}-y\frac{\partial}{\partial z}\left(x^2+y^2+z^2\right)^{\frac{-1}{2}}\right]i-\left[z\frac{\partial}{\partial x}\left(x^2+y^2+z^2\right)^{\frac{-1}{2}}-x\frac{\partial}{\partial z}\left(x^2+\right.\right.$$

$$\left[y\frac{\partial}{\partial x}\left(x^2+y^2+z^2\right)^{\frac{-1}{2}}-x\frac{\partial}{\partial y}\left(x^2+y^2+z^2\right)^{\frac{-1}{2}}\right]k$$

$$=\left[z\left(\frac{-1}{2}\right)\left(x^2+y^2+z^2\right)^{-\frac{1}{2}-1}\frac{\partial}{\partial y}\left(x^2+y^2+z^2\right)-y\left(\frac{-1}{2}\right)\left(x^2+y^2+z^2\right)^{-\frac{1}{2}-1}\frac{\partial}{\partial z}\left(x^2+y^2\right.\right.$$

$$\left[z\left(\frac{-1}{2}\right)\left(x^2+y^2+z^2\right)^{-\frac{1}{2}-1}\frac{\partial}{\partial x}\left(x^2+y^2+z^2\right)-x\left(\frac{-1}{2}\right)\left(x^2+y^2+z^2\right)^{-\frac{1}{2}-1}\frac{\partial}{\partial z}\left(x^2+y^2\right.\right.$$

$$\left[y\left(\frac{-1}{2}\right)\left(x^2+y^2+z^2\right)^{-\frac{1}{2}-1}\frac{\partial}{\partial x}\left(x^2+y^2+z^2\right)-x\left(\frac{-1}{2}\right)\left(x^2+y^2+z^2\right)^{-\frac{1}{2}-1}\frac{\partial}{\partial y}\left(x^2+y^2\right.\right.$$

$$=\left[z\left(\frac{-1}{2}\right)\left(x^2+y^2+z^2\right)^{-\frac{3}{2}}(2y)-y\left(\frac{-1}{2}\right)\left(x^2+y^2+z^2\right)^{-\frac{3}{2}}(2z)\right]i-$$

$$\left[z\left(\frac{-1}{2}\right)\left(x^2+y^2+z^2\right)^{-\frac{3}{2}}(2x)-x\left(\frac{-1}{2}\right)\left(x^2+y^2+z^2\right)^{-\frac{3}{2}}(2z)\right]j+$$

$$\left[y\left(\frac{-1}{2}\right)\left(x^2+y^2+z^2\right)^{-\frac{3}{2}}(2x)-x\left(\frac{-1}{2}\right)\left(x^2+y^2+z^2\right)^{-\frac{3}{2}}(2y)\right]k$$

$$=\left[-yz\left(x^2+y^2+z^2\right)^{-\frac{3}{2}}+yz\left(x^2+y^2+z^2\right)^{-\frac{3}{2}}\right]i-\left[-xz\left(x^2+y^2+z^2\right)^{-\frac{3}{2}}+xz\left(x^2+y^2-\right.\right.$$

$$\left[-yx\left(x^2+y^2+z^2\right)^{-\frac{3}{2}}+yx\left(x^2+y^2+z^2\right)^{-\frac{3}{2}}\right]k$$

$$=0$$

Therefore Curl $F=0$

(b) div $F=\nabla\cdot F$

$$=\frac{\partial}{\partial x}\left[x\left(x^2+y^2+z^2\right)^{\frac{-1}{2}}\right]+\frac{\partial}{\partial y}\left[y\left(x^2+y^2+z^2\right)^{\frac{-1}{2}}\right]+\frac{\partial}{\partial z}\left[z\left(x^2+y^2+z^2\right)^{\frac{-1}{2}}\right]$$

$$=x\left(\frac{-1}{2}\right)\left(x^2+y^2+z^2\right)^{-\frac{1}{2}-1}\frac{\partial}{\partial x}\left(x^2+y^2+z^2\right)+\left(x^2+y^2+z^2\right)^{-\frac{1}{2}}\frac{\partial}{\partial x}(x)+$$

$$y\left(\frac{-1}{2}\right)\left(x^2+y^2+z^2\right)^{-\frac{1}{2}-1}\frac{\partial}{\partial y}\left(x^2+y^2+z^2\right)+\left(x^2+y^2+z^2\right)^{-\frac{1}{2}}\frac{\partial}{\partial y}(y)+$$

$$z\left(\frac{-1}{2}\right)\left(x^2+y^2+z^2\right)^{-\frac{1}{2}-1}\frac{\partial}{\partial z}\left(x^2+y^2+z^2\right)+\left(x^2+y^2+z^2\right)^{-\frac{1}{2}}\frac{\partial}{\partial z}(z)$$

$$= x\left(\frac{-1}{2}\right)\left(x^2+y^2+z^2\right)^{-\frac{3}{2}}(2x)+\left(x^2+y^2+z^2\right)^{-\frac{1}{2}}(1)+$$
$$y\left(\frac{-1}{2}\right)\left(x^2+y^2+z^2\right)^{-\frac{3}{2}}(2y)+\left(x^2+y^2+z^2\right)^{-\frac{1}{2}}(1)+$$
$$z\left(\frac{-1}{2}\right)\left(x^2+y^2+z^2\right)^{-\frac{3}{2}}(2z)+\left(x^2+y^2+z^2\right)^{-\frac{1}{2}}(1)$$
$$= -x^2\left(x^2+y^2+z^2\right)^{-\frac{3}{2}}+\left(x^2+y^2+z^2\right)^{-\frac{1}{2}}-y^2\left(x^2+y^2+z^2\right)^{-\frac{3}{2}}+\left(x^2+y^2+z^2\right)^{-\frac{1}{2}}-$$
$$z^2\left(x^2+y^2+z^2\right)^{-\frac{3}{2}}+\left(x^2+y^2+z^2\right)^{-\frac{1}{2}}$$

$$= -\left(x^2+y^2+z^2\right)\left(x^2+y^2+z^2\right)^{-\frac{3}{2}}+3\left(x^2+y^2+z^2\right)^{-\frac{1}{2}}$$
$$= -\left(x^2+y^2+z^2\right)^{-\frac{3}{2}+1}+3\left(x^2+y^2+z^2\right)^{-\frac{1}{2}}$$
$$= -\left(x^2+y^2+z^2\right)^{-\frac{1}{2}}+3\left(x^2+y^2+z^2\right)^{-\frac{1}{2}}$$
$$= 2\left(x^2+y^2+z^2\right)^{-\frac{1}{2}}$$

Therefore $\text{div } F = \dfrac{2}{\sqrt{x^2+y^2+z^2}}$

Chapter 16 Vector Calculus Exercise 16.5 6E

Given vector field is $F(x,y,z) = e^{xy}\sin z\,j + y\tan^{-1}\left(\frac{x}{z}\right)k$

(a) $\text{Curl } F = \nabla\times F$

$$= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & e^{xy}\sin z & y\tan^{-1}\left(\frac{x}{z}\right) \end{vmatrix}$$

$$= \left[\frac{\partial}{\partial y}\left(y\tan^{-1}\left(\frac{x}{z}\right)\right)-\frac{\partial}{\partial z}\left(e^{xy}\sin z\right)\right]i-\left[\frac{\partial}{\partial x}\left(y\tan^{-1}\left(\frac{x}{z}\right)\right)\right]j+\left[\frac{\partial}{\partial x}\left(e^{xy}\sin z\right)\right]k$$

$$= \left[\tan^{-1}\left(\frac{x}{z}\right)-e^{xy}\frac{\partial}{\partial z}(\sin z)\right]i-\left[y\frac{\partial}{\partial x}\left(\tan^{-1}\left(\frac{x}{z}\right)\right)\right]j+\left[\sin z\frac{\partial}{\partial x}\left(e^{xy}\right)\right]k$$

$$= \left[\tan^{-1}\left(\frac{x}{z}\right)-e^{xy}\cos z\right]i-\left[y\frac{1}{1+\frac{x^2}{z^2}}\frac{\partial}{\partial x}\left(\frac{x}{z}\right)\right]j+\left[(\sin z)e^{xy}\frac{\partial}{\partial x}(xy)\right]k$$

$$= \left[\tan^{-1}\left(\frac{x}{z}\right)-e^{xy}\cos z\right]i-\left[\frac{yz^2}{z^2+x^2}\left(\frac{1}{z}\right)\right]j+\left[(\sin z)e^{xy}(y)\right]k$$

$$= \left[\tan^{-1}\left(\frac{x}{z}\right)-e^{xy}\cos z\right]i-\left[\frac{yz}{z^2+x^2}\right]j+\left[ye^{xy}\sin z\right]k$$

Therefore $\text{Curl } F = \left[\tan^{-1}\left(\frac{x}{z}\right)-e^{xy}\cos z\right]i-\left[\frac{yz}{z^2+x^2}\right]j+\left[ye^{xy}\sin z\right]k$

(b) div F = ∇·F

$$= \frac{\partial}{\partial x}[0] + \frac{\partial}{\partial y}\left[e^{xy}\sin z\right] + \frac{\partial}{\partial z}\left[y\tan^{-1}\left(\frac{x}{z}\right)\right]$$

$$= (\sin z)\frac{\partial}{\partial y}\left(e^{xy}\right) + y\frac{\partial}{\partial z}\left[\tan^{-1}\left(\frac{x}{z}\right)\right]$$

$$= (\sin z)e^{xy}\frac{\partial}{\partial y}(xy) + y\frac{1}{1+\frac{x^2}{z^2}}\frac{\partial}{\partial z}\left(\frac{x}{z}\right)$$

$$= (\sin z)e^{xy}(x) + \frac{yz^2}{z^2+x^2}\left(-\frac{x}{z^2}\right)$$

$$= xe^{xy}\sin z - \frac{yx}{z^2+x^2}$$

Therefore div F = $xe^{xy}\sin z - \frac{yx}{z^2+x^2}$

Chapter 16 Vector Calculus Exercise 16.5 7E

(a)If $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ is a vector field on $\mathbb{R}^3$ and the partial derivatives of P, Q, and R all exist, then the curl of $\mathbf{F}$ is the vector field on $\mathbb{R}^3$ defined by

$$\text{curl } \mathbf{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\mathbf{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\mathbf{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathbf{k}.$$

Let $P = e^x \sin y$, $Q = e^y \sin z$, and $R = e^z \sin x$.

Find $\frac{\partial P}{\partial y}, \frac{\partial P}{\partial z}, \frac{\partial Q}{\partial x}, \frac{\partial Q}{\partial z}, \frac{\partial R}{\partial x}$, and $\frac{\partial R}{\partial y}$.

$$\frac{\partial P}{\partial y} = \frac{\partial}{\partial y}\left(e^x \sin y\right) = e^x \cos y$$

$$\frac{\partial P}{\partial z} = \frac{\partial}{\partial z}\left(e^x \sin y\right) = 0$$

$$\frac{\partial Q}{\partial x} = \frac{\partial}{\partial x}\left(e^y \sin z\right) = 0$$

$$\frac{\partial Q}{\partial z} = \frac{\partial}{\partial z}\left(e^y \sin z\right) = e^y \cos z$$

$$\frac{\partial R}{\partial x} = \frac{\partial}{\partial x}\left(e^z \sin x\right) = e^z \cos x$$

$$\frac{\partial R}{\partial y} = \frac{\partial}{\partial y}\left(e^z \sin x\right) = 0$$

Substitute the known values in $\text{curl } \mathbf{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\mathbf{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\mathbf{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathbf{k}$.

$$\text{curl } \mathbf{F} = \left(0 - e^y \cos z\right)\mathbf{i} + \left(0 - e^z \cos x\right)\mathbf{j} + \left(0 - e^x \cos y\right)\mathbf{k}$$
$$= -e^y \cos z\mathbf{i} - e^z \cos x\mathbf{j} - e^x \cos y\mathbf{k}$$

Thus, we get $\text{curl } \mathbf{F} = \left\langle -e^y \cos z, -e^z \cos x, -e^x \cos y \right\rangle$.

(b)If $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ is a vector field on $\mathbb{R}^3$ and the partial derivatives of P, Q, and R exist, then the divergence of $\mathbf{F}$ is defined as $\text{div } \mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$.

Find $\frac{\partial P}{\partial x}$, $\frac{\partial Q}{\partial y}$, and $\frac{\partial R}{\partial z}$

$$\frac{\partial P}{\partial x} = \frac{\partial}{\partial x}\left(e^x \sin y\right)$$
$$= e^x \sin y$$
$$\frac{\partial Q}{\partial y} = \frac{\partial}{\partial y}\left(e^y \sin z\right)$$
$$= e^y \sin z$$
$$\frac{\partial R}{\partial z} = \frac{\partial}{\partial z}\left(e^z \sin x\right)$$
$$= e^z \sin x$$

Then, $\text{div } \mathbf{F} = e^x \sin y + e^y \sin z + e^z \sin x$.

Thus, we get $\boxed{\text{div } \mathbf{F} = e^x \sin y + e^y \sin z + e^z \sin x}$.

Chapter 16 Vector Calculus Exercise 16.5 8E

(a)If $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ is a vector field on $\mathbb{R}^3$ and the partial derivatives of P, Q, and R all exist, then the curl of $\mathbf{F}$ is the vector field on $\mathbb{R}^3$ defined by

$$\text{curl } \mathbf{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\mathbf{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\mathbf{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathbf{k}.$$

Let $P = \frac{x}{y}$, $Q = \frac{y}{z}$, and $R = \frac{z}{x}$.

Find $\frac{\partial P}{\partial y}$, $\frac{\partial P}{\partial z}$, $\frac{\partial Q}{\partial x}$, $\frac{\partial Q}{\partial z}$, $\frac{\partial R}{\partial x}$, and $\frac{\partial R}{\partial y}$.

$$\frac{\partial P}{\partial y} = \frac{\partial}{\partial y}\left(\frac{x}{y}\right)$$
$$= -\frac{x}{y^2}$$
$$\frac{\partial P}{\partial z} = \frac{\partial}{\partial z}\left(\frac{x}{y}\right)$$
$$= 0$$
$$\frac{\partial Q}{\partial x} = \frac{\partial}{\partial x}\left(\frac{y}{z}\right)$$
$$= 0$$
$$\frac{\partial Q}{\partial z} = \frac{\partial}{\partial z}\left(\frac{y}{z}\right)$$
$$= -\frac{y}{z^2}$$
$$\frac{\partial R}{\partial x} = \frac{\partial}{\partial x}\left(\frac{z}{x}\right)$$
$$= -\frac{z}{x^2}$$
$$\frac{\partial R}{\partial y} = \frac{\partial}{\partial y}\left(\frac{z}{x}\right)$$
$$= 0$$

Substitute the known values in $\text{curl } \mathbf{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\mathbf{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\mathbf{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathbf{k}$.

$$\text{curl } \mathbf{F} = \left(0 + \frac{y}{z^2}\right)\mathbf{i} + \left(0 + \frac{z}{x^2}\right)\mathbf{j} + \left(0 + \frac{x}{y^2}\right)\mathbf{k}$$
$$= \frac{y}{z^2}\mathbf{i} + \frac{z}{x^2}\mathbf{j} + \frac{x}{y^2}\mathbf{k}$$

Thus, we get $\text{curl } \mathbf{F} = \left\langle \frac{y}{z^2}, \frac{z}{x^2}, \frac{x}{y^2} \right\rangle$.

(b)If $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ is a vector field on $\mathbb{R}^3$ and the partial derivatives of P, Q, and R exist, then the divergence of $\mathbf{F}$ is defined as $\text{div } \mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$.

Find $\frac{\partial P}{\partial x}, \frac{\partial Q}{\partial y}$, and $\frac{\partial R}{\partial z}$

$$\frac{\partial P}{\partial x} = \frac{\partial}{\partial x}\left(\frac{x}{y}\right)$$
$$= \frac{1}{y}$$
$$\frac{\partial Q}{\partial y} = \frac{\partial}{\partial y}\left(\frac{y}{z}\right)$$
$$= \frac{1}{z}$$
$$\frac{\partial R}{\partial z} = \frac{\partial}{\partial z}\left(\frac{z}{x}\right)$$
$$= \frac{1}{x}$$

Then, $\text{div } \mathbf{F} = \frac{1}{y} + \frac{1}{z} + \frac{1}{x}$.

Thus, we get $\boxed{\text{div } \mathbf{F} = \frac{1}{y} + \frac{1}{z} + \frac{1}{x}}$.

Chapter 16 Vector Calculus Exercise 16.5 9E

(A)

We know $div\,\vec{F} = \vec{\nabla}.\vec{F}$

$$= \frac{\partial}{\partial x}P(x,y,z) + \frac{\partial}{\partial y}Q(x,y,z) + \frac{\partial}{\partial z}R(x,y,z)$$

The vector field given is such that $R(x,y,z) = 0$ and P and Q do not depend on z.

From the diagram we see that x component of $\vec{F}$ is zero, i.e. P(x, y)=0 and the y – component Q(x, y) is a decreasing function of y.

Therefore $div\,\vec{F} = \frac{\partial}{\partial x}(0) + \frac{\partial}{\partial y}Q(x,y) + \frac{\partial}{\partial z}(0)$

$$= \frac{\partial}{\partial y}Q(x,y) < 0$$

i.e. $div\,\vec{F}$ is negative

(B)

$$curl\,\vec{F} = \left(\frac{\partial}{\partial y}R(x,y,z) - \frac{\partial}{\partial z}Q(x,y,z)\right)\hat{i} + \left(\frac{\partial}{\partial z}P(x,y,z) - \frac{\partial}{\partial x}R(x,y,z)\right)\hat{j}$$
$$+\left(\frac{\partial}{\partial x}Q(x,y,z) - \frac{\partial}{\partial y}P(x,y,z)\right)\hat{k}$$
$$= \left(\frac{\partial}{\partial y}(0) - Q(x,y)\right)\hat{i} + \left(\frac{\partial}{\partial z}(0) - \frac{\partial}{\partial x}(0)\right)\hat{j}$$
$$+\left(\frac{\partial}{\partial x}Q(x,y) - \frac{\partial}{\partial y}(0)\right)\hat{k}$$

$$= \frac{-\partial}{\partial z}Q(x,y)\hat{i} + \frac{\partial}{\partial x}Q(x,y)\hat{k}$$

Since Q(x, y) does not depend on Z. Then

$$curl\,\vec{F} = \frac{\partial}{\partial x}Q(x,y)\hat{k}$$

From the diagram we see that Q does not depend upon x either.

Therefore $\frac{\partial}{\partial x}Q(x,y) = 0$ and thus $curl\,\vec{F} = \vec{0}$

Chapter 16 Vector Calculus Exercise 16.5 10E

(a)

The objective is to verify $div\mathbf{F}$ is positive, negative or zero.

Assume $\mathbf{F} = P(x,y,z)\mathbf{i} + Q(x,y,z)\mathbf{j} + R(x,y,z)\mathbf{k}.$

Given that the vector field $\mathbf{F}$ is independent of z (that is, *P* and *Q* does not depend upon *z*) and its *z*-component is 0, that is, $R(x,y,z) = 0.$

Then, $\mathbf{F} = P(x,y)\mathbf{i} + Q(x,y)\mathbf{j} + (0)\mathbf{k}.$

The divergence of the vector field $\mathbf{F} = P(x,y)\mathbf{i} + Q(x,y)\mathbf{j} + (0)\mathbf{k}$ is,

$$\text{div}\,\mathbf{F} = \nabla\cdot\mathbf{F}$$
$$= \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)\cdot\left(P(x,y)\mathbf{i} + Q(x,y)\mathbf{j} + (0)\mathbf{k}\right)$$
$$= \frac{\partial}{\partial x}P(x,y) + \frac{\partial}{\partial y}Q(x,y) + \frac{\partial}{\partial z}(0)$$
$$= \frac{\partial}{\partial x}P(x,y) + \frac{\partial}{\partial y}Q(x,y)$$

The vector field $\mathbf{F}$ points along the $x-$direction i.e. $\frac{\partial Q}{\partial y} = 0$.

$$\text{div}\,\mathbf{F} = \frac{\partial}{\partial x}P(x,y) + 0$$

Thus, $\text{div}\,\mathbf{F} = \frac{\partial}{\partial x}P(x,y)$

The particles moving from a point and entering to a point in the vector field.

Therefore, $\boxed{\text{div}\,\mathbf{F} = 0}$.

(b)

The objective is to determine whether $\text{curl}\,\mathbf{F} = 0$. If not, in which direction does $\text{curl}\mathbf{F}$ point.

$$\text{curl}\,\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P(x,y) & Q(x,y) & 0 \end{vmatrix}$$

$$= \mathbf{i}\left[\frac{\partial}{\partial y}(0) - \frac{\partial}{\partial z}Q(x,y)\right] - \mathbf{j}\left[\frac{\partial}{\partial x}(0) - \frac{\partial}{\partial z}P(x,y)\right] + \mathbf{k}\left[\frac{\partial}{\partial x}Q(x,y) - \frac{\partial}{\partial y}P(x,y)\right]$$

$$= \mathbf{i}[0-0] - \mathbf{j}[0-0] + \mathbf{k}\left[\frac{\partial}{\partial x}Q(x,y) - \frac{\partial}{\partial y}P(x,y)\right]$$

$$= 0\mathbf{i} - 0\mathbf{j} + \mathbf{k}\left[\frac{\partial}{\partial x}Q(x,y) - 0\right] \qquad \left[\text{Since } \frac{\partial P}{\partial y} = 0\right]$$

$$= \frac{\partial}{\partial x}Q(x,y)$$

$\frac{\partial}{\partial x}Q(x,y) = 0$ when $Q(x,y) = \text{constant}$ so $\text{curl}\,\mathbf{F} = 0.$

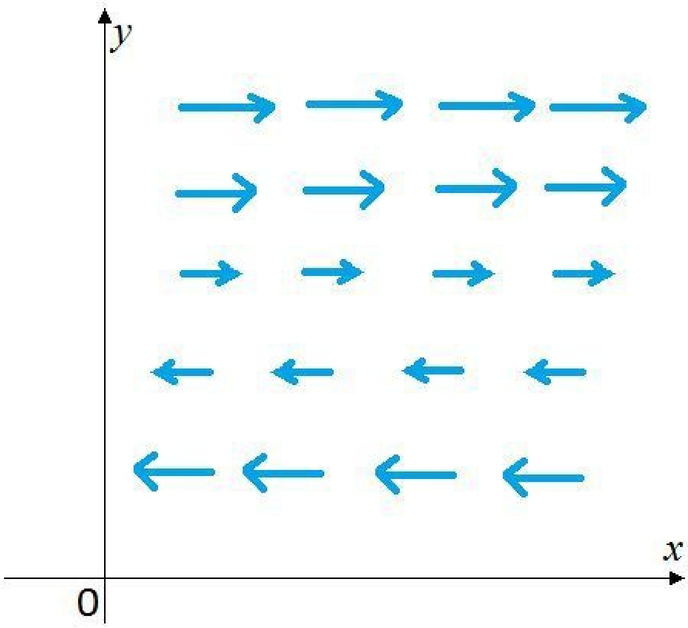
From the graph notice that, the *y*-component $Q(x,y)$ of the field $\mathbf{F}$ is a constant as x increases, so $\frac{\partial Q}{\partial x} \neq 0.$

Thus, $\text{curl}\,\mathbf{F} \neq 0$

Therefore, $\text{curl}\,\mathbf{F}$ points in the negative $z-$direction.

(a)

Consider the vector field $\mathbf{F}$ in the *xy*-plane given below:



Chapter 16 Vector Calculus Exercise 16.5 11E

Let $\mathbf{F} = P(x,y,z)\mathbf{i} + Q(x,y,z)\mathbf{j} + R(x,y,z)\mathbf{k}.$

Given that the vector field $\mathbf{F}$ is independent of z (that is, *P* and *Q* does not depend upon *z*) and its *z*-component is 0, that is, $R(x,y,z) = 0.$

Then, $\mathbf{F} = P(x,y)\mathbf{i} + Q(x,y)\mathbf{j} + (0)\mathbf{k}.$

The divergence of the vector field $\mathbf{F} = P(x,y)\mathbf{i} + Q(x,y)\mathbf{j} + (0)\mathbf{k}$ is,

$$\begin{aligned}
\operatorname{div}\mathbf{F} &= \nabla \cdot \mathbf{F} \\
&= \frac{\partial}{\partial x}P(x,y,z) + \frac{\partial}{\partial y}Q(x,y,z) + \frac{\partial}{\partial z}R(x,y,z) \\
&= \frac{\partial}{\partial x}P(x,y) + \frac{\partial}{\partial y}Q(x,y) + \frac{\partial}{\partial z}(0) \\
&= \frac{\partial}{\partial x}P(x,y) + \frac{\partial}{\partial y}Q(x,y)
\end{aligned}$$

From the diagram notice that *y*-component of $\mathbf{F}$ is zero since all the arrows are parallel to the *x*-axis, that is, $Q(x,y) = 0.$

And *x* component is constant for every *y*, that is, $P(x,y) = c,$ here c is a constant and then $\frac{\partial}{\partial x}P(x,y) = 0.$

Thus,

$$\begin{aligned}
\operatorname{div}\mathbf{F} &= \frac{\partial}{\partial x}P(x,y) + \frac{\partial}{\partial y}Q(x,y) \\
&= 0 + \frac{\partial}{\partial y}(0) \\
&= 0
\end{aligned}$$

Therefore, $\operatorname{div}\mathbf{F} = \boxed{0}.$

(b)

Determine whether $\operatorname{curl}\mathbf{F} = 0.$

The curl of the vector field $\mathbf{F} = P(x,y)\mathbf{i} + Q(x,y)\mathbf{j} + (0)\mathbf{k}$ is,

$$\operatorname{curl}\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P(x,y) & Q(x,y) & 0 \end{vmatrix}$$

$$= \mathbf{i}\left[\frac{\partial}{\partial y}(0) - \frac{\partial}{\partial z}Q(x,y)\right] - \mathbf{j}\left[\frac{\partial}{\partial x}(0) - \frac{\partial}{\partial z}P(x,y)\right] + \mathbf{k}\left[\frac{\partial}{\partial x}Q(x,y) - \frac{\partial}{\partial y}P(x,y)\right]$$

$$= \mathbf{i}[0-0] - \mathbf{j}[0-0] + \mathbf{k}\left[\frac{\partial}{\partial x}Q(x,y) - \frac{\partial}{\partial y}P(x,y)\right]$$

$$= 0\mathbf{i} - 0\mathbf{j} + \mathbf{k}\left[\frac{\partial}{\partial x}(0) - \frac{\partial}{\partial y}P(x,y)\right] \qquad [\text{Since } Q(x,y) = 0]$$

$$= -\left[\frac{\partial}{\partial y}P(x,y)\right]\mathbf{k}$$

From the diagram notice that the length of the arrows increases as *y* increases that is, $P(x,y)$ increases with *y* and thus $\frac{\partial P}{\partial y} > 0.$

And hence curl $\mathbf{F} < 0$ and it points in the negative *z*-direction.

Chapter 16 Vector Calculus Exercise 16.5 12E

(A)

curl f: Meaningless because we can take curl of vector fields only and f is not a vector field.

(B)

grad f: Meaningful, This is a vector field

(C)

div $\vec{F}$: Meaningful and scalar field

(D)

curl (grad f): Meaningful and vector field

(E)

grad $\vec{F}$: Meaningless because gradient is applied to scalar functions and $\vec{F}$ is a vector function

(F)

grad (div $\vec{F}$): Meaningful and vector field

(G)

div (grad f): Meaningful and scalar field

(H)

grad (div f): Meaningless because f is a scalar function and divergence of scalar function is not defined.

(I)

curl (curl $\vec{F}$): Meaningful and vector field

(J)

div (div $\vec{F}$): Meaningless because div $\vec{F}$ is a scalar quantity and divergence of scalar function is not defined

(K)

(grad f) × (div $\vec{F}$): Meaningless because div $\vec{F}$ is a scalar quantity and we can not take its cross product

(L)

div (curl (grad f)): Meaningful and scalar field.

Chapter 16 Vector Calculus Exercise 16.5 13E

Given that $F(x, y, z) = y^2 z^3 i + 2xyz^3 j + 3xy^2 z^2 k$

$$\text{Curl } f = \nabla \times f = \begin{bmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 z^3 & 2xyz^3 & 3xy^2 z^2 \end{bmatrix}$$

$$= \left[\frac{\partial}{\partial y}\left(3xy^2 z^2\right) - \frac{\partial}{\partial z}\left(2xyz^3\right)\right]i - \left[\frac{\partial}{\partial x}\left(3xy^2 z^2\right) - \frac{\partial}{\partial z}\left(y^2 z^3\right)\right]j + \left[\frac{\partial}{\partial x}\left(2xy\right.\right.$$

$$= \left[6xyz^2 - 6xyz^2\right]i - \left[3y^2 z^2 - 3y^2 z^2\right]j + \left[2yz^3 - 2yz^3\right]k$$

Curl $f = 0$, therefore vector field is Conservative

Applying technique for finding f , we have

$f_x(x, y, z) = y^2 z^3$ ....(i)

$f_y(x, y, z) = 2xyz^3$ ....(ii)

$f_z(x, y, z) = 3xy^2 z^2$ ....(iii)

Integrating (i) with respect to x, we obtain

$$f(x, y, z) = xy^2z^3 + g(y, z) \text{..............................(iv)}$$

Differentiating (iv) with respect to y, we get
$f_y(x, y, z) = 2xyz^3 + g_y(y, z)$,

so, Comparison with (ii) gives
$g_y(y, z) = 0$

Thus, $g(y, z) = h(z)$ and

$$f_z(x, y, x) = 3xy^2z^2 + h'(z)$$

Then(iii) gives $h'(z) = 0$,

Therefore

$$f(x, y, z) = xy^2z^3 + k$$

This is the required solution

Chapter 16 Vector Calculus Exercise 16.5 14E

$$\text{curl } f = \nabla f = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xyz^2 & x^2yz^2 & x^2y^2z \end{vmatrix}$$

$$= \left[\frac{\partial}{\partial y}\left(x^2y^2z\right) - \frac{\partial}{\partial z}\left(x^2yz^2\right)\right]i - j\left[\frac{\partial}{\partial x}\left(x^2y^2z\right) - \frac{\partial}{\partial z}\left(xyz^2\right)\right]$$
$$+k\left[\frac{\partial}{\partial x}\left(x^2yz^2\right) - \frac{\partial}{\partial y}\left(xyz^2\right)\right]$$
$$= \left(2x^2yz - 2x^2yz\right)i - \left(2xy^2z - 2xyz\right)j + \left(2xyz^2 - xz^2\right)k$$

$$= -\left(2xy^2z - 2xyz\right)j + \left(2xyz^2 - xz^2\right)k$$

$\therefore$ curl $f \neq 0$, the vector fuld is not conservative

Chapter 16 Vector Calculus Exercise 16.5 15E

If $\mathbf{F}$ is a vector field on $\mathbb{R}^3$ and curl $\mathbf{F} = 0$, then $\mathbf{F}$ is a conservative vector field.

We know that $\text{curl } \mathbf{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\mathbf{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\mathbf{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathbf{k}$.

Let $P = 3xy^2z^2$, $Q = 2x^2yz^2$, and $R = 3x^2y^2z^2$.

Find $\frac{\partial P}{\partial y}, \frac{\partial P}{\partial z}, \frac{\partial Q}{\partial x}, \frac{\partial Q}{\partial z}, \frac{\partial R}{\partial x}$, and $\frac{\partial R}{\partial y}$.

$$\begin{aligned}\frac{\partial P}{\partial y} &= \frac{\partial}{\partial y}\left(3xy^2z^2\right)\\ &= 6xyz^2\end{aligned}$$

$$\begin{aligned}\frac{\partial P}{\partial z} &= \frac{\partial}{\partial z}\left(3xy^2z^2\right)\\ &= 6xy^2z\end{aligned}$$

$$\begin{aligned}\frac{\partial Q}{\partial x} &= \frac{\partial}{\partial x}\left(2x^2yz^3\right)\\ &= 4xyz^3\end{aligned}$$

$$\begin{aligned}\frac{\partial Q}{\partial z} &= \frac{\partial}{\partial z}\left(2x^2yz^3\right)\\ &= 6x^2yz^2\end{aligned}$$

$$\begin{aligned}\frac{\partial R}{\partial x} &= \frac{\partial}{\partial x}\left(3x^2y^2z^2\right)\\ &= 6xy^2z^2\end{aligned}$$

$$\begin{aligned}\frac{\partial R}{\partial y} &= \frac{\partial}{\partial y}\left(3x^2y^2z^2\right)\\ &= 6x^2yz^2\end{aligned}$$

Substitute the known values in $\text{curl } \mathbf{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\mathbf{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\mathbf{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathbf{k}$.

$$\begin{aligned}\text{curl } \mathbf{F} &= \left(6x^2yz^2 - 6x^2yz^2\right)\mathbf{i} + \left(6xy^2z - 6xy^2z^2\right)\mathbf{j} + \left(4xyz^3 - 6xyz^2\right)\mathbf{k}\\ &= 0\mathbf{i} + 6xy^2z(1-z)\mathbf{j} + 2xyz^2(2z-3)\mathbf{k}\end{aligned}$$

Thus, we get $\text{curl } \mathbf{F} = \left\langle \frac{y}{z^2}, \frac{z}{x^2}, \frac{x}{y^2} \right\rangle$.

Since $\text{curl } \mathbf{F} \neq 0$, the given vector field is not conservative.

Chapter 16 Vector Calculus Exercise 16.5 16E

Consider the vector field

$$\mathbf{F}(x, y, z) = \mathbf{i} + \sin z\mathbf{j} + y\cos z\mathbf{k}$$

Recall that

If $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ is a vector field on $\mathbb{R}^3$ and curl F = 0, then F is a conservative vector field.

Curl of the vector field

$$\text{curl } \mathbf{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\mathbf{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\mathbf{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathbf{k}.$$

Compare the vector field $\mathbf{F}(x,y,z)=\mathbf{i}+\sin z\mathbf{j}+y\cos z\mathbf{k}$ with $\mathbf{F}=P\mathbf{i}+Q\mathbf{j}+R\mathbf{k}$

Then $P=1, Q=\sin z,$ and $R=y\cos z$

Now find $\frac{\partial P}{\partial y}, \frac{\partial P}{\partial z}, \frac{\partial Q}{\partial x}, \frac{\partial Q}{\partial z}, \frac{\partial R}{\partial x},$ and $\frac{\partial R}{\partial y}$.

Differentiate the function P with respect to $y,$ and z respectively.

$$\begin{aligned}\frac{\partial P}{\partial y} &= \frac{\partial}{\partial y}(1)\\ &= 0\end{aligned}$$

$$\begin{aligned}\frac{\partial P}{\partial z} &= \frac{\partial}{\partial z}(1)\\ &= 0\end{aligned}$$

Differentiate the function Q with respect to $x,$ and z respectively.

$$\begin{aligned}\frac{\partial Q}{\partial x} &= \frac{\partial}{\partial x}(\sin z)\\ &= 0\end{aligned}$$

$$\begin{aligned}\frac{\partial Q}{\partial z} &= \frac{\partial}{\partial z}(\sin z)\\ &= \cos z\end{aligned}$$

Differentiate the function R with respect to $x,$ and y respectively.

$$\begin{aligned}\frac{\partial R}{\partial x} &= \frac{\partial}{\partial x}(y\cos z)\\ &= 0\end{aligned}$$

$$\begin{aligned}\frac{\partial R}{\partial y} &= \frac{\partial}{\partial y}(y\cos z)\\ &= \cos z\end{aligned}$$

Substitute the values of $\frac{\partial P}{\partial y}, \frac{\partial P}{\partial z}, \frac{\partial Q}{\partial x}, \frac{\partial Q}{\partial z}, \frac{\partial R}{\partial x},$ and $\frac{\partial R}{\partial y}$ in

$$\text{curl }\mathbf{F} = \left(\frac{\partial R}{\partial y}-\frac{\partial Q}{\partial z}\right)\mathbf{i}+\left(\frac{\partial P}{\partial z}-\frac{\partial R}{\partial x}\right)\mathbf{j}+\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)\mathbf{k}.$$

$$\begin{aligned}\text{curl }\mathbf{F} &= (\cos z-\cos z)\mathbf{i}+(0-0)\mathbf{j}+(0-0)\mathbf{k}\\ &= 0\mathbf{i}+0\mathbf{j}+0\mathbf{k}\end{aligned}$$

Therefore,

$$\text{curl}\mathbf{F}=\mathbf{0}$$

Hence the vector field $\mathbf{F}(x,y,z)=\mathbf{i}+\sin z\mathbf{j}+y\cos z\mathbf{k}$ is $\boxed{\text{conservative}}$.

To find the function f such that $\mathbf{F}=\nabla f$

Consider the Vector Field

$$\mathbf{F}(x,y,z)=\mathbf{i}+\sin z\mathbf{j}+y\cos z\mathbf{k} \quad \text{......} \quad (1)$$

Let f be function such that $\mathbf{F}=\nabla f$,

$$\nabla f = \mathbf{i}+\sin z\mathbf{j}+y\cos z\mathbf{k}.$$

This implies that

$$f_x(x,y,z) = 1, \text{......} \quad (2)$$

$$f_y(x,y,z) = \sin z, \text{ and } \quad \text{......} \quad (3)$$

$$f_z(x,y,z) = y\cos z. \quad \text{......} \quad (4)$$

Integrate from (2) with respect to *x*,

, ... (5)

Where *g*(*y*, *z*) is a constant with respect to *x*

Differentiate $f(x, y, z) = x + g(y, z)$ with respect to *y*.

$f_y(x, y, z) = g_y(y, z)$

$g_y(y, z) = \sin z$ From (3)

Integrate $g_y(y, z) = \sin z$ with respect to y

$g(y, z) = y\sin z + h(z)$

Substitute $g(y, z) = y\sin z + h(z)$ in equation (5),

$f(x, y, z) = x + y\sin z + h(z)$ ...... (6)

Differentiate $f(x, y, z) = x + y\sin z + h(z)$ with respect to *z*.

$f_z(x, y, z) = 0 + y\cos z + h'(z)$

$y\cos z = y\cos z + h'(z)$ From (4)

$h'(z) = 0$

Integrate $h'(z) = 0$ with respect to z

$h(z) = K$, a constant.

Substitute $h(z) = K$ in equation (6),

$f(x, y, z) = x + y\sin z + K$

Therefore the required function f such that $\mathbf{F} = \nabla f$ is

$\boxed{f(x, y, z) = x + y\sin z + K}$.

Chapter 16 Vector Calculus Exercise 16.5 17E

If $\mathbf{F}$ is a vector field on $\mathbb{R}^3$ and curl $\mathbf{F} = 0$, then $\mathbf{F}$ is a conservative vector field.

We know that curl $\mathbf{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\mathbf{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\mathbf{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathbf{k}$.

Let $P = e^{yz}$, $Q = xze^{yz}$, and $R = xye^{yz}$.

Find $\frac{\partial P}{\partial y}$, $\frac{\partial P}{\partial z}$, $\frac{\partial Q}{\partial x}$, $\frac{\partial Q}{\partial z}$, $\frac{\partial R}{\partial x}$, and $\frac{\partial R}{\partial y}$.

$$\frac{\partial P}{\partial y} = \frac{\partial}{\partial y}\left(e^{yz}\right)$$
$$= ze^{yz}$$

$$\frac{\partial P}{\partial z} = \frac{\partial}{\partial z}\left(e^{yz}\right)$$
$$= ye^{yz}$$

$$\frac{\partial Q}{\partial x} = \frac{\partial}{\partial x}\left(xze^{yz}\right)$$
$$= ze^{yz}$$

$$\frac{\partial Q}{\partial z} = \frac{\partial}{\partial z}\left(xze^{yz}\right)$$
$$= xe^{yz} + xyze^{yz}$$
$$\frac{\partial R}{\partial x} = \frac{\partial}{\partial x}\left(xye^{yz}\right)$$
$$= ye^{yz}$$
$$\frac{\partial R}{\partial y} = \frac{\partial}{\partial y}\left(xye^{yz}\right)$$
$$= xe^{yz} + xyze^{yz}$$

Substitute the known values in $\text{curl } \mathbf{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\mathbf{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\mathbf{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathbf{k}$.

$$\text{curl } \mathbf{F} = \left(xe^{yz} + xyze^{yz} - xe^{yz} - xyze^{yz}\right)\mathbf{i} + \left(ye^{yz} - ye^{yz}\right)\mathbf{j} + \left(ze^{yz} - ze^{yz}\right)\mathbf{k}$$
$$= 0\mathbf{i} + 0\mathbf{j} + 0\mathbf{k}$$

Since curl $\mathbf{F} = 0$, the given vector field is conservative.

We need to find a function f such that $\nabla f = \mathbf{F}$. Then, we get
$\nabla f = e^{yz}\mathbf{i} + xze^{yz}\mathbf{j} + xye^{yz}\mathbf{k}$.
This implies that
$f_x(x, y) = e^{xy}$,
$f_y(x, y) = xze^{yz}$,
and $f_z(x, y) = xye^{yz}$
On integrating $f_x(x, y, z)$ with respect to x, we get $f(x, y, z) = xe^{yz} + g(y, z)$, where $g(y, z)$ is a constant with respect to x.

Now, differentiate $f(x, y, z) = xe^{yz} + g(y, z)$ with respect to y.
$f_y(x, y, z) = xze^{yz} + g_y(y, z)$

But, we have $f_y(x, y) = xze^{yz}$. We then get $g(y, z)$ as a constant K.
Thus, we get the function as $f(x, y, z) = xe^{yz} + K$.

Chapter 16 Vector Calculus Exercise 16.5 18E

If $\mathbf{F}$ is a vector field on $\mathbb{R}^3$ and curl $\mathbf{F} = 0$, then $\mathbf{F}$ is a conservative vector field.

We know that $\text{curl } \mathbf{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\mathbf{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\mathbf{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathbf{k}$.

Let $P = e^x \sin yz$, $Q = ze^x \cos yz$, and $R = ye^x \cos yz$.

Find $\frac{\partial P}{\partial y}, \frac{\partial P}{\partial z}, \frac{\partial Q}{\partial x}, \frac{\partial Q}{\partial z}, \frac{\partial R}{\partial x}$, and $\frac{\partial R}{\partial y}$.
$$\frac{\partial P}{\partial y} = \frac{\partial}{\partial y}\left(e^x \sin yz\right)$$
$$= ze^x \cos yz$$
$$\frac{\partial P}{\partial z} = \frac{\partial}{\partial z}\left(e^x \sin yz\right)$$
$$= ye^x \cos yz$$
$$\frac{\partial Q}{\partial x} = \frac{\partial}{\partial x}\left(ze^x \cos yz\right)$$
$$= ze^x \cos yz$$

$$\frac{\partial Q}{\partial z} = \frac{\partial}{\partial z}\left(ze^x \cos yz\right)$$
$$= -yze^x \sin yz + e^x \cos yz$$
$$\frac{\partial R}{\partial x} = \frac{\partial}{\partial x}\left(ye^x \cos yz\right)$$
$$= ye^x \cos yz$$
$$\frac{\partial R}{\partial y} = \frac{\partial}{\partial y}\left(ye^x \cos yz\right)$$
$$= -yze^x \sin yz + e^x \cos yz$$

Substitute the known values in $\text{curl } \mathbf{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\mathbf{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\mathbf{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathbf{k}$.

$$\text{curl } \mathbf{F} = \left(-yze^x \sin yz + e^x \cos yz + yze^x \sin yz - e^x \cos yz\right)\mathbf{i}$$
$$+ \left(ye^x \cos yz - ye^x \cos yz\right)\mathbf{j} + \left(ze^x \cos yz - ze^x \cos yz\right)\mathbf{k}$$
$$= 0\mathbf{i} + 0\mathbf{j} + 0\mathbf{k}$$

Since $\text{curl } \mathbf{F} = 0$, the given vector field is conservative.

We need to find a function f such that $\nabla f = \mathbf{F}$. Then, we get $\nabla f = e^x \sin yz\mathbf{i} + ze^x \cos yz\mathbf{j} + ye^x \cos yz\mathbf{k}$. This implies that

$f_x(x, y) = e^x \sin yz$,

$f_y(x, y) = ze^x \cos yz$

and $f_z(x, y) = ye^x \cos yz$.

On integrating $f_x(x, y, z)$ with respect to x, we get $f(x, y, z) = e^x \sin yz + g(y, z)$, where $g(y, z)$ is a constant with respect to x.

Now, differentiate $f(x, y, z) = e^x \sin yz + g(y, z)$ with respect to y.

$f_y(x, y, z) = ze^x \cos yz + g_y(y, z)$

But, we have $f_y(x, y) = ze^x \cos yz$. We then get $g(y, z)$ as a constant K.

Thus, we get the function as $f(x, y, z) = e^x \sin yz + K$.

Chapter 16 Vector Calculus Exercise 16.5 19E

No. If there is such a $\vec{G}$, then

$$\text{div}\left(\text{curl}\vec{G}\right) = \frac{\partial}{\partial x}\left(x\sin\left(y\right)\right) + \frac{\partial}{\partial y}\left(\cos\left(y\right)\right) + \frac{\partial}{\partial z}\left(z - xy\right) = \sin\left(y\right) - \sin\left(y\right) +$$

This does not agree with Theorem 11.

Chapter 16 Vector Calculus Exercise 16.5 20E

No. Assume there is such a **G**. Then div(curl **G**) = yz - 2yz + 2yz = yz ≠ 0 which contradicts Theorem 11.

Chapter 16 Vector Calculus Exercise 16.5 21E

$\vec{F}(x,y,z) = f(x)\hat{i} + g(y)\hat{j} + h(z)\hat{k}$

Then $curl\,\vec{F} = \vec{\nabla}\times\vec{F}$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f(x) & g(y) & h(z) \end{vmatrix}$$

$$= \hat{i}\left(\frac{\partial}{\partial y}h(z) - \frac{\partial}{\partial z}g(y)\right) - \hat{j}\left(\frac{\partial}{\partial x}h(z) - \frac{\partial}{\partial z}f(x)\right)$$
$$+\hat{k}\left(\frac{\partial}{\partial x}g(y) - \frac{\partial}{\partial y}f(x)\right)$$
$$= \hat{i}(0) - \hat{j}(0) + \hat{k}(0)$$
$$= \vec{0}$$

Since $curl\,\vec{F} = \vec{0}$, then $\vec{F}$ is irrotational.

Hence any vector field of the form

$\vec{F}(x,y,z) = f(x)\hat{i} + g(y)\hat{j} + h(z)\hat{k}$, is irrotational.

Chapter 16 Vector Calculus Exercise 16.5 22E

$\vec{F}(x,y,z) = f(y,z)\hat{i} + g(x,z)\hat{j} + h(x,y)\hat{k}$

Then $div(\vec{F}) = \vec{\nabla}\times\vec{F}$

$$= \vec{\nabla}.\left(f(y,z)\hat{i} + g(x,z)\hat{j} + h(x,y)\hat{k}\right)$$
$$= \frac{\partial}{\partial x}f(y,z) + \frac{\partial}{\partial y}g(x,z) + \frac{\partial}{\partial z}h(x,y)$$
$$= 0+0+0$$
$$= 0$$

Since $div(\vec{F}) = 0$, therefore $\vec{F}$ is incompressible.

Chapter 16 Vector Calculus Exercise 16.523E

Consider the identity,

$$\text{div}(\mathbf{F}+\mathbf{G}) = \text{div } \mathbf{F} + \text{div } \mathbf{G}.$$

The objective is to prove this identity.

Suppose $\mathbf{F} = \langle f_1, f_2, f_3\rangle$, $\mathbf{G} = \langle g_1, g_2, g_3\rangle$, then,

$$\mathbf{F}+\mathbf{G} = \langle f_1+g_1, f_2+g_2, f_3+g_3\rangle.$$

Recall that the divergence of the vector field $\mathbf{F} = \langle f_1, f_2, f_3\rangle$ is,

$$\text{div } \mathbf{F} = \frac{\partial(f_1)}{\partial x} + \frac{\partial(f_2)}{\partial y} + \frac{\partial(f_3)}{\partial z}.$$

So, the divergence of the sum of these two vector fields is,

$$\text{div}(\mathbf{F}+\mathbf{G}) = \frac{\partial(f_1+g_1)}{\partial x} + \frac{\partial(f_2+g_2)}{\partial y} + \frac{\partial(f_3+g_3)}{\partial z}$$

$$= \frac{\partial(f_1)}{\partial x} + \frac{\partial(g_1)}{\partial x} + \frac{\partial(f_2)}{\partial y} + \frac{\partial(g_2)}{\partial y} + \frac{\partial(f_3)}{\partial z} + \frac{\partial(g_3)}{\partial z}$$

$$= \left(\frac{\partial(f_1)}{\partial x} + \frac{\partial(f_2)}{\partial y} + \frac{\partial(f_3)}{\partial z}\right) + \left(\frac{\partial(g_1)}{\partial x} + \frac{\partial(g_2)}{\partial y} + \frac{\partial(g_3)}{\partial z}\right)$$

$$= \text{div}(\mathbf{F}) + \text{div}(\mathbf{G})$$

Therefore, the identity is proved.

Chapter 16 Vector Calculus Exercise 16.524E

$$curl\left(\vec{F}+\vec{G}\right)=\vec{\nabla}\times\left(\vec{F}+\vec{G}\right)$$
$$=\left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)\times\left(\vec{F}+\vec{G}\right)$$
$$=\sum\hat{i}\times\frac{\partial}{\partial x}\left(\vec{F}+\vec{G}\right)$$
$$=\sum\hat{i}\times\left(\frac{\partial\vec{F}}{\partial x}+\frac{\partial\vec{G}}{\partial x}\right)$$
$$=\sum\hat{i}\times\frac{\partial\vec{F}}{\partial x}+\sum\hat{i}\times\frac{\partial\vec{G}}{\partial x}$$
$$=\left(\sum\hat{i}\frac{\partial}{\partial x}\right)\times\vec{F}+\left(\sum\hat{i}\frac{\partial}{\partial x}\right)\times\vec{G}$$
$$=\left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)\times\vec{F}$$
$$+\left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)\times\vec{G}$$
$$=curl\left(\vec{F}\right)+curl\left(\vec{G}\right)$$

Hence $curl\left(\vec{F}+\vec{G}\right)=curl\,\vec{F}+curl\,\vec{G}$

Chapter 16 Vector Calculus Exercise 16.5 25E

If f = f(x, y, z) and

$$\vec{F}(x,y,z)=P(x,y,z)\hat{i}+Q(x,y,z)\hat{j}+R(x,y,z)\hat{k}$$

Where P, Q, R are continuous and their partial derivatives exist.

Then,

$$div\left(f\vec{F}\right)=\vec{\nabla}.\left(f\vec{F}\right)$$
$$=\vec{\nabla}.\left(f\,P(x,y,z)\hat{i}+f\,Q(x,y,z)\hat{j}+f\,R(x,y,z)\hat{k}\right)$$
$$=\frac{\partial}{\partial x}\left[f(x,y,z)P(x,y,z)\right]+\frac{\partial}{\partial y}\left[f(x,y,z)Q(x,y,z)\right]$$
$$+\frac{\partial}{\partial z}\left[f(x,y,z)R(x,y,z)\right]$$

$$=f\frac{\partial}{\partial x}P(x,y,z)+P\frac{\partial}{\partial x}f(x,y,z)+f\frac{\partial}{\partial y}Q(x,y,z)+Q\frac{\partial}{\partial y}f(x,y,z)$$
$$+f\frac{\partial}{\partial z}R(x,y,z)+R\frac{\partial}{\partial z}f(x,y,z)$$
$$=f\left[\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}+\frac{\partial R}{\partial z}\right]+P\frac{\partial}{\partial x}f(x,y,z)+Q\frac{\partial}{\partial y}f(x,y,z)+R\frac{\partial}{\partial z}f(x,y,z)$$

$$=f(x,y,z)\left[\vec{\nabla}.\left(P\hat{i}+Q\hat{j}+R\hat{k}\right)\right]+\left(P\hat{i}+Q\hat{j}+R\hat{k}\right).\left(\frac{\partial f}{\partial x}\hat{i}+\frac{\partial f}{\partial y}\hat{j}+\frac{\partial f}{\partial z}\hat{k}\right)$$

$$=f(x,y,z)\left[\vec{\nabla}.\vec{F}\right]+\vec{F}.\vec{\nabla}f$$
$$=f\,div\left(\vec{F}\right)+\vec{F}.\vec{\nabla}f$$

Hence $\boxed{div\left(f\vec{F}\right)=f\,div\left(\vec{F}\right)+\vec{F}.\vec{\nabla}f}$

Chapter 16 Vector Calculus Exercise 16.5 26E

Now $f=f(x,y,z)$

And let $\vec{F}(x,y,z)=P(x,y,z)\hat{i}+Q(x,y,z)\hat{j}+R(x,y,z)\hat{k}$

Where P, Q, R are continuous and their partial derivatives exist.

Then $curl\left(f\vec{F}\right)=\vec{\nabla}\times\left(f\vec{F}\right)$

$$=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ fP(x,y,z) & fQ(x,y,z) & fR(x,y,z)\end{vmatrix}$$

$$=\hat{i}\left(\frac{\partial}{\partial y}fR-\frac{\partial}{\partial z}fQ\right)+\hat{j}\left(\frac{\partial}{\partial z}fP-\frac{\partial}{\partial x}fR\right)$$
$$+\hat{k}\left(\frac{\partial}{\partial x}fQ-\frac{\partial}{\partial y}fP\right)$$

$$=\hat{i}\left(f\frac{\partial R}{\partial y}+R\frac{\partial f}{\partial y}-f\frac{\partial Q}{\partial z}-Q\frac{\partial f}{\partial z}\right)+\hat{j}\left(f\frac{\partial P}{\partial z}+P\frac{\partial f}{\partial z}-f\frac{\partial R}{\partial x}-R\frac{\partial f}{\partial x}\right)$$
$$+\hat{k}\left(f\frac{\partial Q}{\partial x}+Q\frac{\partial f}{\partial x}-f\frac{\partial P}{\partial y}-P\frac{\partial f}{\partial y}\right)$$

$$=f\left[\hat{i}\left(\frac{\partial R}{\partial y}-\frac{\partial Q}{\partial z}\right)+\hat{j}\left(\frac{\partial P}{\partial z}-\frac{\partial R}{\partial x}\right)+\hat{k}\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)\right]$$
$$+\hat{i}\left(R\frac{\partial f}{\partial y}-Q\frac{\partial f}{\partial z}\right)+\hat{j}\left(P\frac{\partial f}{\partial z}-R\frac{\partial f}{\partial x}\right)+\hat{k}\left(Q\frac{\partial f}{\partial x}-P\frac{\partial f}{\partial y}\right)$$

$$=f\left(\vec{\nabla}\times\vec{F}\right)+\left(\vec{\nabla}f\right)\times\vec{F}$$

$$=f\,curl\,\vec{F}+\vec{\nabla}f\times\vec{F}$$

Hence $\boxed{curl\left(f\vec{F}\right)=f\,curl\,\vec{F}+\vec{\nabla}f\times\vec{F}}$

Since, $\vec{\nabla}f\times\vec{F}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z}\\ P & Q & R\end{vmatrix}$

$$=\hat{i}\left(R\frac{\partial f}{\partial y}-Q\frac{\partial f}{\partial z}\right)+\hat{j}\left(P\frac{\partial f}{\partial z}-R\frac{\partial f}{\partial x}\right)+\hat{k}\left(Q\frac{\partial f}{\partial x}-P\frac{\partial f}{\partial y}\right)$$

Chapter 16 Vector Calculus Exercise 16.5 27E

Consider the following identity:

$$\operatorname{div}(\mathbf{F}\times\mathbf{G})=\mathbf{G}\cdot\operatorname{curl}\mathbf{F}-\mathbf{F}\cdot\operatorname{curl}\mathbf{G}$$

The objective is to prove this identity.

Examine a single component $\mathbf{i}$. Here $\mathbf{i},\mathbf{j}$ and $\mathbf{k}$ can be any three components such that the cross-product of $\mathbf{i}$ and $\mathbf{j}$ is equal to $\mathbf{k}$.

$$\operatorname{div}(\boldsymbol{F}\times\boldsymbol{G})=\left(\frac{\partial\left(\mathbf{F}_j\mathbf{G}_k-\mathbf{F}_k\mathbf{G}_j\right)}{\partial i}\right)$$
$$=\left(\frac{\partial\left(\mathbf{F}_j\mathbf{G}_k\right)}{\partial i}-\frac{\partial\left(\mathbf{F}_k\mathbf{G}_j\right)}{\partial i}\right)$$
$$=\left(\mathbf{G}_k\frac{\partial(\mathbf{F}_j)}{\partial i}-\mathbf{G}_j\frac{\partial(\mathbf{F}_k)}{\partial i}+\mathbf{F}_j\frac{\partial(\mathbf{G}_k)}{\partial i}-\mathbf{F}_k\frac{\partial(\mathbf{G}_j)}{\partial i}\right)$$

Sum over all indices:

$$\operatorname{div}(\mathbf{F}\times\mathbf{G})=\sum_{i=x,y,z}\operatorname{div}(\mathbf{F}\times\mathbf{G})_i$$
$$=\sum_{i=x,y,z}\left(\mathbf{G}_k\frac{\partial(\mathbf{F}_j)}{\partial i}-\mathbf{G}_j\frac{\partial(\mathbf{F}_k)}{\partial i}+\mathbf{F}_j\frac{\partial(\mathbf{G}_k)}{\partial i}-\mathbf{F}_k\frac{\partial(\mathbf{G}_j)}{\partial i}\right)_i$$
$$=\mathbf{G}\cdot\operatorname{curl}\mathbf{F}-\mathbf{F}\cdot\operatorname{curl}\mathbf{G}$$

Therefore, $\boxed{\operatorname{div}(\mathbf{F}\times\mathbf{G})=\mathbf{G}\cdot\operatorname{curl}\mathbf{F}-\mathbf{F}\cdot\operatorname{curl}\mathbf{G}}$.

Chapter 16 Vector Calculus Exercise 16.5 28E

$$div\left(\vec{\nabla} f \times \vec{\nabla} g\right) = \vec{\nabla}.\left(\vec{\nabla} f \times \vec{\nabla} g\right)$$

Since $div\left(\vec{F} \times \vec{G}\right) = \vec{G}.curl\,\vec{F} - \vec{F}.curl\,\vec{G}$

Then $div\left(\vec{\nabla} f \times \vec{\nabla} g\right) = \left(\vec{\nabla} g\right).curl\left(\vec{\nabla} f\right) - \left(\vec{\nabla} g\right).curl\left(\vec{\nabla} g\right)$

>

Consider $curl\left(\vec{\nabla} f\right) = \vec{\nabla} \times \left(\vec{\nabla} f\right)$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \times \left(\hat{i}\frac{\partial f}{\partial x} + \hat{j}\frac{\partial f}{\partial y} + \hat{k}\frac{\partial f}{\partial z}\right)$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{vmatrix}$$

$$= \left(\frac{\partial^2 f}{\partial y\,\partial z} - \frac{\partial^2 f}{\partial z\,\partial y}\right)\hat{i} + \left(\frac{\partial^2 f}{\partial z\,\partial x} - \frac{\partial^2 f}{\partial x\,\partial z}\right)\hat{j} + \left(\frac{\partial^2 f}{\partial x\,\partial y} - \frac{\partial^2 f}{\partial y\,\partial x}\right)\hat{k}$$

$$= 0\hat{i} + 0\hat{j} + 0\hat{k} \text{ , \{Using Clairauts's theorem\}}$$
$$= \vec{0}$$

Similarly $curl\left(\vec{\nabla} g\right) = \vec{0}$

Therefore $div\left(\vec{\nabla} f \times \vec{\nabla} g\right) = \vec{\nabla} g.\vec{0} - \vec{\nabla} f.\vec{0}$

$$= 0 - 0$$
$$= 0$$

i.e. $\boxed{div\left(\vec{\nabla} f \times \vec{\nabla} g\right) = 0}$

Chapter 16 Vector Calculus Exercise 16.5 29E

Let $\vec{F} = P(x,y,z)\hat{i} + Q(x,y,z)\hat{j} + R(x,y,z)\hat{k}$

Where P, Q, R are continuous and their second order partial derivatives exist

Then $\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$

$$= \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\hat{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\hat{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\hat{k}$$

$$\text{Therefore } \vec{\nabla}\left(\vec{\nabla}\times\vec{F}\right)=\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial R}{\partial y}-\frac{\partial Q}{\partial z} & \frac{\partial P}{\partial z}-\frac{\partial R}{\partial x} & \frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y} \end{vmatrix}$$

$$=\sum\left[\left\{\frac{\partial}{\partial y}\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)-\frac{\partial}{\partial z}\left(\frac{\partial P}{\partial z}-\frac{\partial R}{\partial x}\right)\right\}\hat{i}\right]$$

$$=\sum\left[\left\{\frac{\partial^2 Q}{\partial y\,\partial x}-\frac{\partial^2 P}{\partial y^2}-\frac{\partial^2 P}{\partial z^2}+\frac{\partial^2 R}{\partial z\,\partial x}\right\}\hat{i}\right]$$

$$=\sum\left[\left\{\left(\frac{\partial^2 Q}{\partial y\,\partial x}+\frac{\partial^2 R}{\partial z\,\partial x}\right)-\left(\frac{\partial^2 P}{\partial y^2}+\frac{\partial^2 P}{\partial z^2}\right)\right\}\hat{i}\right]$$

$$=\sum\left[\left\{\frac{\partial}{\partial x}\left(\frac{\partial Q}{\partial y}+\frac{\partial R}{\partial z}\right)-\left(\frac{\partial^2 P}{\partial y^2}+\frac{\partial^2 P}{\partial z^2}\right)\right\}\hat{i}\right]$$

$$=\sum\left[\left\{\frac{\partial}{\partial x}\left(\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}+\frac{\partial R}{\partial z}\right)-\left(\frac{\partial^2 P}{\partial x^2}+\frac{\partial^2 P}{\partial y^2}+\frac{\partial^2 P}{\partial z^2}\right)\right\}\hat{i}\right]$$

$$=\sum\left[\left\{\frac{\partial}{\partial x}\left(\vec{\nabla}\,.\,\vec{F}\right)-\left(\vec{\nabla}^2 P\right)\right\}\hat{i}\right]$$

$$\text{i.e. } \vec{\nabla}\left(\vec{\nabla}\times\vec{F}\right)=\sum\left[\left\{\frac{\partial}{\partial x}\left(\vec{\nabla}.\vec{F}\right)\right\}\hat{i}\right]-\vec{\nabla}^2\sum P\hat{i}$$

$$=\vec{\nabla}\left(\vec{\nabla}.\vec{F}\right)-\vec{\nabla}^2\vec{F}$$

$$\text{Hence } \vec{\nabla}\times\left(\vec{\nabla}\times\vec{F}\right)=\vec{\nabla}\left(\vec{\nabla}\times\vec{F}\right)-\vec{\nabla}^2\vec{F}$$

$$\text{i.e. } curl\left(curl\,\vec{F}\right)=grad\left(div\,\vec{F}\right)-\vec{\nabla}^2\vec{F}$$

Chapter 16 Vector Calculus Exercise 16.5 30E

$$\vec{r}=x\hat{i}+y\hat{j}+z\hat{k}$$

$$\text{And } \left|\vec{r}\right|=\left(x^2+y^2+z^2\right)^{\frac{1}{2}}=r$$

(A)

$$\vec{\nabla}.\vec{r}=\left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right).\left(x\hat{i}+y\hat{j}+z\hat{k}\right)$$

$$=\frac{\partial}{\partial x}(x)+\frac{\partial}{\partial y}(y)+\frac{\partial}{\partial z}(z)$$

$$=1+1+1=3$$

$$\text{Hence } \boxed{\vec{\nabla}.\vec{r}=3}$$

(B)

As we know $\vec{\nabla}.\left(f\vec{F}\right)=f\left(\vec{\nabla}.\vec{F}\right)+\vec{F}.\vec{\nabla}f$

Then $\vec{\nabla}.\left(r\vec{r}\right)=r\left(\vec{\nabla}.\vec{r}\right)+\vec{r}.\left(\vec{\nabla}r\right)$

Using part (A),

$$\vec{\nabla}.\left(r\vec{r}\right)=r\left(3\right)+\left(x\hat{i}+y\hat{j}+z\hat{k}\right).\left(\hat{i}\frac{\partial r}{\partial x}+\hat{j}\frac{\partial r}{\partial y}+\hat{k}\frac{\partial r}{\partial z}\right)$$

$$=3r+\left(x\hat{i}+y\vec{j}+z\vec{k}\right).\left(\frac{x}{\left(x^2+y^2+z^2\right)^{1/2}}\hat{i}+\frac{y}{\left(x^2+y^2+z^2\right)^{1/2}}\hat{j}+\frac{z}{\left(x^2+y^2+z^2\right)^{1/2}}\hat{k}\right)$$

$$=3r+\frac{x^2}{\left(x^2+y^2+z^2\right)^{1/2}}+\frac{y^2}{\left(x^2+y^2+z^2\right)^{1/2}}+\frac{z^2}{\left(x^2+y^2+z^2\right)^{1/2}}$$

$$=3r+\frac{\left(x^2+y^2+z^2\right)}{\left(x^2+y^2+z^2\right)^{1/2}}$$

$$=3r+\left(x^2+y^2+z^2\right)^{1/2}$$

$$=3r+r$$

$$=4r$$

Hence $\boxed{\vec{\nabla}.\left(r\vec{r}\right)=4r}$

(C)

$$\vec{\nabla}^2 r^3=\vec{\nabla}.\left(\vec{\nabla}.r^3\right)$$

Now

$$\vec{\nabla}r^3=\hat{i}\frac{\partial}{\partial x}\left(x^2+y^2+z^2\right)^{3/2}+\hat{j}\frac{\partial}{\partial y}\left(x^2+y^2+z^2\right)^{3/2}+\hat{k}\frac{\partial}{\partial z}\left(x^2+y^2+z^2\right)^{3/2}$$

$$=3x\left(x^2+y^2+z^2\right)^{1/2}\hat{i}+3y\left(x^2+y^2+z^2\right)^{1/2}\hat{j}+3z\left(x^2+y^2+z^2\right)^{1/2}\hat{k}$$

Then

$$\vec{\nabla}.\left(\vec{\nabla}r^3\right)=\left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right).\left(3x\left(x^2+y^2+z^2\right)^{1/2}\hat{i}\right.$$

$$\left.+3y\left(x^2+y^2+z^2\right)^{1/2}\hat{j}+3z\left(x^2+y^2+z^2\right)^{1/2}\hat{k}\right)$$

$$=3\left(x^2+y^2+z^2\right)^{1/2}+\frac{3x^2}{\left(x^2+y^2+z^2\right)^{1/2}}+3\left(x^2+y^2+z^2\right)^{1/2}$$

$$+\frac{3y^2}{\left(x^2+y^2+z^2\right)^{1/2}}+3\left(x^2+y^2+z^2\right)^{1/2}+\frac{3z^2}{\left(x^2+y^2+z^2\right)^{1/2}}$$

$$=9\left(x^2+y^2+z^2\right)^{1/2}+3\left(x^2+y^2+z^2\right)/\left(x^2+y^2+z^2\right)^{1/2}$$

$$=9r+3r$$

$$=12r$$

Hence $\boxed{\vec{\nabla}^2 r^3=12r}$

Chapter 16 Vector Calculus Exercise 16.5 31E

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

And $\quad |\vec{r}| = \left(x^2+y^2+z^2\right)^{1/2} = r$

(A)

$$\vec{\nabla} r = \hat{i}\frac{\partial}{\partial x}\left(x^2+y^2+z^2\right)^{1/2} + \hat{j}\frac{\partial}{\partial y}\left(x^2+y^2+z^2\right)^{1/2} + \hat{k}\frac{\partial}{\partial z}\left(x^2+y^2+z^2\right)^{1/2}$$

$$= \hat{i}\frac{x}{\left(x^2+y^2+z^2\right)^{1/2}} + \hat{j}\frac{y}{\left(x^2+y^2+z^2\right)^{1/2}} + \hat{k}\frac{z}{\left(x^2+y^2+z^2\right)^{1/2}}$$

$$= \frac{x\hat{i}+y\hat{j}+z\hat{k}}{\left(x^2+y^2+z^2\right)^{1/2}}$$

$$= \frac{\vec{r}}{r}$$

Hence $\boxed{\vec{\nabla} r = \vec{r}/r}$

(B)

$$\vec{\nabla}\times\vec{r} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix}$$

$$= \hat{i}\left(\frac{\partial(z)}{\partial y} - \frac{\partial(y)}{\partial z}\right) + \hat{j}\left(\frac{\partial(x)}{\partial z} - \frac{\partial(z)}{\partial x}\right) + \hat{k}\left(\frac{\partial(y)}{\partial x} - \frac{\partial(x)}{\partial y}\right)$$

$$= \hat{i}(0) + \hat{j}(0) + \hat{k}(0)$$

$$= \vec{0}$$

Hence $\boxed{\vec{\nabla}\times\vec{r} = \vec{0}}$

(C)

$$\vec{\nabla}\left(\frac{1}{r}\right) = \hat{i}\frac{\partial}{\partial x}\left(x^2+y^2+z^2\right)^{1/2} + \hat{j}\frac{\partial}{\partial y}\left(x^2+y^2+z^2\right)^{1/2} + \hat{k}\frac{\partial}{\partial z}\left(x^2+y^2+z^2\right)^{1/2}$$

$$= -\left[\hat{i}\frac{(x)}{\left(x^2+y^2+z^2\right)^{1/2}} + \hat{j}\frac{(y)}{\left(x^2+y^2+z^2\right)^{1/2}} + \hat{k}\frac{(z)}{\left(x^2+y^2+z^2\right)^{1/2}}\right]$$

$$= -\frac{\left[x\hat{i}+y\hat{j}+z\hat{k}\right]}{\left(x^2+y^2+z^2\right)^{3/2}}$$

$$= \frac{-\vec{r}}{r^3}$$

Hence $\vec{\nabla}\left(\frac{1}{r}\right) = -\vec{r}/r^3$

(D)

Now $\quad r = \left(x^2+y^2+z^2\right)^{1/2}$

Then $\quad \ln r = \frac{1}{2}\ln\left(x^2+y^2+z^2\right)$

And

$$\vec{\nabla}\ln r = \frac{1}{2}\left[\hat{i}\frac{\partial}{\partial x}\ln\left(x^2+y^2+z^2\right) + \hat{j}\frac{\partial}{\partial y}\ln\left(x^2+y^2+z^2\right) + \hat{k}\frac{\partial}{\partial z}\ln\left(x^2+y^2+z^2\right)\right]$$

$$= \frac{1}{2}\left[\frac{2x\hat{i}}{x^2+y^2+z^2} + \frac{2y\,\hat{j}}{x^2+y^2+z^2} + \frac{2z\hat{k}}{x^2+y^2+z^2}\right]$$

$$= \frac{x\hat{i}+y\hat{j}+z\hat{k}}{x^2+y^2+z^2}$$

$$= \frac{\vec{r}}{r^2}$$

Hence $\vec{\nabla}\ln r = \vec{r}/r^2$

Chapter 16 Vector Calculus Exercise 16.5 32E

We know $div\left(g\vec{G}\right) = g\, div\vec{G} + \vec{G}.\vec{\nabla}g$

Putting $\vec{G} = \vec{r}$ and $g = r^{-p}$ in this identity, we get

$$div\left(\vec{r}\,r^{-p}\right) = r^{-p} div\,\vec{r} + \vec{r}\,.\vec{\nabla}\,r^{-p}$$

$$= r^{-p}\,(3) + \vec{r}.\left(-p\,r^{-p-1}\vec{\nabla}\,r\right)$$

(Because $div\,\vec{r} = 3$ and $grad\ f\left(u\right) = f\,'\left(u\right) grad\,u$)

Then $\quad div\left(\vec{r}\ r^{-p}\right) = 3r^{-p} - p\,r^{-p-1}\left(\vec{r}\,.\vec{\nabla}r\right)$

But $\quad \vec{\nabla}r = \dfrac{\vec{r}}{r}$

Therefore $\quad div\left(\vec{r}\,r^{-p}\right) = 3r^{-p} - p\,r^{-p-1}.\dfrac{\left(\vec{r}.\vec{r}\right)}{r}$

$$= 3r^{-p} - p\,r^{-p-2}\left(\vec{r}.\vec{r}\right)$$

$$= 3r^{-p} - pr^{-p-2}r^2$$

$$= 3r^{-p} - p\,r^{-p}$$

$$= \left(3-p\right)r^{-p}$$

Hence $\boxed{div\left(\vec{r}/r^{p}\right) = \left(3-p\right)/r^{p}}$

Or $\quad div\left(\vec{F}\right) = \left(3-p\right)/r^{p}$

(As $\vec{F} = \vec{r}/r^{p}$)

For $div\,\vec{F} = 0$, p must be equal to 3.

That is $\boxed{\text{when p} = 3,\ div\,\vec{F} = 0}$

Chapter 16 Vector Calculus Exercise 16.5 33E

Prove the Green's first identity by using the second vector form of Green's theorem.

Consider the following second vector form of Green's theorem. $\oint_C \mathbf{F}\cdot\mathbf{n}\ ds = \iint_D \text{div}\,\mathbf{F}\left(x,y\right)\ dA$

...... (1)

It can be written as follows:

$$\oint_C \mathbf{F}\cdot\mathbf{n}\ ds = \iint_D \nabla\cdot\mathbf{F}\ dA$$

Put $\mathbf{F} = f\,\nabla g$ in $\nabla\cdot\mathbf{F}$ and simplify.

$$\nabla\cdot\mathbf{F} = \nabla\cdot\left(f\ \nabla g\right)$$

$$= f\left(\nabla\cdot\nabla g\right) + \left(\nabla f\right)\cdot\left(\nabla g\right)$$

$$= f\nabla^2 g + \nabla f\cdot\nabla g$$

Also, $\mathbf{F}\cdot\mathbf{n} = f\left(\nabla g\right)\cdot\mathbf{n}.$

Then Green's theorem (1) gives,

$$\oint_C f\left(\nabla g\right).\mathbf{n}\,ds = \iint_D\left[f\nabla^2 g + \nabla f.\nabla g\right]dA$$

$$\oint_C f\left(\nabla g\right).\mathbf{n}\,ds = \iint_D f\,\nabla^2\,g\,dA + \iint_D \nabla f.\nabla g\,dA$$

$$\iint_D f\nabla^2\,g\,dA = \oint_C\ f\left(\nabla g\right).\mathbf{n}\,ds - \iint_D \nabla f.\nabla g\ dA$$

Therefore, Green's first identity is $\boxed{\iint_D f\nabla^2\,g\,dA = \oint_C\ f\left(\nabla g\right).\mathbf{n}\,ds - \iint_D \nabla f.\nabla g\ dA}$.

Chapter 16 Vector Calculus Exercise 16.5 34E

From Green's first identity

$$\iint_D f\vec{\nabla}^2 g\, dA = \oint_C f\left(\vec{\nabla} g\right).\hat{n}\, ds - \iint_D \vec{\nabla} f . \vec{\nabla} f\, dA \qquad \text{-------------- (1)}$$

Interchanging f and g

$$\iint_D g\vec{\nabla}^2 f\, dA = \oint_C g\left(\vec{\nabla} f\right).\hat{n}\, ds - \iint_D \vec{\nabla} g . \vec{\nabla} f\, dA \qquad \text{-------------- (2)}$$

Subtracting (2) from (1)

$$\iint_D \left(f\vec{\nabla}^2 g - g\vec{\nabla}^2 f\right) dA = \oint_C \left(f\vec{\nabla} g - g\vec{\nabla} f\right).\hat{n}\, ds$$

Which is Green's second identity

Hence proved

Chapter 16 Vector Calculus Exercise 16.5 35E

The function g is harmonic on D. This implies that $\nabla^2 g = 0$ on D.

Use Green's first identity to solve the problem. This is mathematically expressed as follows:

$$\oint_C \mathbf{F}\cdot\mathbf{n}\, ds = \iint_D div\, \mathbf{F}(x,y)\, dA.$$

The objective is to show that $\oint_C D_n g ds = 0$, if g is harmonic on D.

Here, $D_n g$ is the normal derivative of g and $D_n g = \nabla g \cdot \mathbf{n}.$

$$\oint_C D_n g ds = \oint_C \nabla g \cdot \mathbf{n} ds$$

$$= \iint_D div(\nabla g)\, dA \qquad (\text{Use Green's first identity})$$

$$= \iint_D \nabla\cdot(\nabla g)\, dA$$

$$= \iint_D (\nabla^2 g)\, dA$$

$$= \iint_D (0)\, dA \qquad (\text{Since } \nabla^2 g = 0 \text{ on } D)$$

$$= 0$$

Hence $\oint_C D_n g ds = 0.$

Chapter 16 Vector Calculus Exercise 16.5 36E

The function f is harmonic on D that is $\nabla^2 f = 0$ on D.

And $f(x,y) = 0$ on the boundary curve $C.$

Need to show that $\iint_D |\nabla f|^2 dA = 0.$

Since we know that

$$\iint_D \left(f\nabla^2 g\right) dA = \oint_C f(\nabla g)\cdot\mathbf{n}\, ds - \iint_D \nabla f\cdot\nabla g\, dA$$

$$\iint_D \nabla f\cdot\nabla g\, dA = \oint_C f(\nabla g)\cdot\mathbf{n}\, ds - \iint_D \left(f\nabla^2 g\right) dA$$

Replace ∇g by ∇f, get

$$\iint_D |\nabla f|^2 dA = \iint_D \nabla f\cdot\nabla f\, dA$$

$$= \oint_C f(\nabla f)\cdot\mathbf{n}\, ds - \iint_D \left(f\nabla^2 f\right) dA$$

$$= \oint_C f(\nabla f)\cdot\mathbf{n}\, ds - \iint_D f\left(\nabla^2 f\right) dA$$

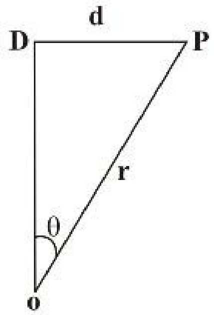
$$= \oint_C (0)(\nabla f)\cdot\mathbf{n}\, ds - \iint_D f(0)\, dA \qquad \left(\text{Since } \nabla^2 f = 0 \text{ on } D, f(x,y) = 0 \text{ on } C\right)$$

$$= \oint_C 0\cdot\mathbf{n}\, ds - \iint_D 0\, dA$$

$$= 0$$

Hence, $\iint_D |\nabla f|^2 dA = 0.$

Chapter 16 Vector Calculus Exercise 16.5 37E



(A)

In right angled triangle ODP

$$\sin\theta = \frac{d}{r}$$

i.e. $d = r\sin\theta$

Now by the given definition of angular speed,

$$w = \frac{v}{d}$$

i.e. $w = \dfrac{v}{r\sin\theta}$

Or $v = wr\sin\theta$

Or $\vec{v} = \vec{w}\times\vec{r}$

($\vec{v}$ at P is directed in negative x – direction and $\vec{r}\times\vec{w}$ is directed in positive x – direction but $\vec{w}\times\vec{r}$ is directed in negative x – direction. So we have $\vec{v} = \vec{w}\times\vec{r}$)

(B)

From part (A) $\vec{v} = \vec{w}\times\vec{r}$

And $\vec{w} = w\hat{k}$, $r = x\hat{i} + y\hat{j} + z\hat{k}$

Then $\vec{v} = \left(w\hat{k}\right)\times\left(x\hat{i} + y\hat{j} + z\hat{k}\right)$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 0 & w \\ x & y & z \end{vmatrix}$$

$$= \hat{i}\,(-wy) - \hat{j}(-xw) + \hat{k}(0)$$

$$= -wy\,\hat{i} + wx\hat{j}$$

(C)

$$curl\,\vec{v} = \vec{\nabla}\times\vec{v}$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ -wy & wx & 0 \end{vmatrix}$$

(Using part (B))

$$= \hat{i}\left(-\frac{\partial}{\partial z}(wx)\right) - \hat{j}\left(\frac{\partial}{\partial z}(wy)\right) + \hat{k}\left(\frac{\partial}{\partial x}(wx) + \frac{\partial}{\partial y}(wy)\right)$$

$$= \hat{i}\,(0) - \hat{j}(0) + \hat{k}(w + w)$$

$$= 2w\hat{k}$$

$$= 2\vec{w}$$

Hence $\boxed{curl\,\vec{v} = 2\vec{w}}$

Chapter 16 Vector Calculus Exercise 16.5 38E

It is given

$$div\,\vec{E} = 0 \qquad div\,\vec{H} = 0$$

$$curl\,\vec{E} = \frac{-1}{C}\frac{\partial\vec{H}}{\partial t} \qquad curl\,\vec{H} = \frac{1}{c}\frac{\partial\vec{E}}{\partial t}$$

(A)

$$\vec{\nabla}\times\left(\vec{\nabla}\times\vec{E}\right)=curl\left(curl\,\vec{E}\right)$$
$$=curl\left(\frac{-1}{C}\frac{\partial\vec{H}}{\partial t}\right)$$
$$=\frac{-1}{C}\vec{\nabla}\times\left(\frac{\partial\vec{H}}{\partial t}\right)$$
$$=\frac{-1}{C}\frac{\partial}{\partial t}\left(\vec{\nabla}\times\vec{H}\right)$$
$$=\frac{-1}{C}\frac{\partial}{\partial t}\left(\frac{1}{C}\frac{\partial\vec{E}}{\partial t}\right)$$
$$=\frac{-1}{C^2}\frac{\partial^2}{\partial t^2}\vec{E}$$

(B)

$$\vec{\nabla}\times\left(\vec{\nabla}\times\vec{H}\right)=curl\left(curl\,\vec{H}\right)$$
$$=curl\left(\frac{1}{C}\frac{\partial\vec{E}}{\partial t}\right)$$
$$=\frac{1}{C}\vec{\nabla}\times\left(\frac{\partial\vec{E}}{\partial t}\right)$$
$$=\frac{1}{C}\frac{\partial}{\partial t}\left(\vec{\nabla}\times\vec{E}\right)$$
$$=\frac{1}{C}\frac{\partial}{\partial t}\left(-\frac{1}{C}\frac{\partial\vec{H}}{\partial t}\right)$$
$$=\frac{-1}{C^2}\frac{\partial^2\vec{H}}{\partial t^2}$$

(C)

We know $curl\left(curl\,\vec{F}\right)=grad\left(div\,\vec{F}\right)-\vec{\nabla}^2\,\vec{F}$

Then $\vec{\nabla}^2\vec{F}=grad\left(div\,\vec{F}\right)-curl\left(curl\,\vec{F}\right)$

Therefore $\vec{\nabla}^2\vec{E}=grad\left(div\vec{E}\right)-curl\left(curl\,\vec{E}\right)$

$$=grad\,(0)-\left[\frac{-1}{C^2}\frac{\partial^2\vec{E}}{\partial t^2}\right]$$

(Using (A))

$$=0+\frac{1}{C^2}\frac{\partial^2\vec{E}}{\partial t^2}$$

i.e. $\vec{\nabla}^2\,\vec{E}=\frac{1}{C^2}\frac{\partial^2\vec{E}}{\partial t^2}$

(D)

$$\vec{\nabla}^2\vec{H}=grad\left(div\,\vec{H}\right)-curl\left(curl\,\vec{H}\right)$$
$$=grad\,(0)-\left[\frac{-1}{C^2}\frac{\partial^2\vec{H}}{\partial t^2}\right]$$
$$=\frac{1}{C^2}\frac{\partial^2\vec{H}}{\partial t^2}$$

Chapter 16 Vector Calculus Exercise 16.539E

Let f be a continuous function defined on R^3 and we define a vector field

$\vec{G}(x,y,z)=\,<g(x,y,z),0,0>$ such that

$g(x,y,z)=\int_0^x f(t,y,z)\,dt.$ --(i)

Now $div\vec{G} = \nabla.\vec{G}$

$$= <\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}> . < g(x,y,z), 0, 0 >$$

$$= \frac{\partial}{\partial x} g(x,y,z) + 0 + 0$$

$$= \frac{\partial}{\partial x} g(x,y,z)$$

$$= \frac{\partial}{\partial x} \int_0^x f(t,y,z)\,dt \qquad \{\text{from } (i)\}$$

$$= f(x,y,z) \qquad \{\text{By fundamental theorem of calculus}\}$$

This implies $div\vec{G}$ is itself the function f. Therefore, Every continuous function f on R^3 is the divergence of some vector field.