Chapter 7

Dimensional Analysis

CHAPTER HIGHLIGHTS

- *Introduction*
- *Dimensional homogeneity*
- *Methods of dimensional analysis*
- *Hydraulic similitude*
- *Dimensionless numbers*
- *Model laws or similarity laws*
- *Reynolds model law*

Introduction

Dimensional analysis is a mathematical technique in which study of dimensions is made use of in solving engineering problems. A physical phenomenon consists of different quantities which can be expressed in terms of fundamental or primary quantities. The fundamental quantities mass, length, time and temperature are designated by M, L, T and θ respectively.

Quantities or variables such as area, velocity, acceleration, viscosity, force, torque, etc., are formed using fundamental quantities. These are called derived quantities. The derived quantities can be expressed in terms of fundamental quantities. These expressions are called dimensions of the derived quantities. In general, the quantities can be classified as fundamental quantities. geometric quantities, kinematic quantities, dynamic quantities, thermodynamic quantities, etc.

Sometimes force (dimension F) is taken as a fundamental quantity instead of mass (dimension M). Accordingly the system becomes FLT instead of MLT.

Dimension of mass in MLT system is M and in FLT system is $FL^{-1}T^2$.

Dimension of force in MLT system is MLT^{-2} and in FLT system is F.

Some important quantities with their units, symbol and dimensions in MLT system are given below.

	Quantity	Unit	Symbol	Dimensions (*MLT* System)
1.	Fundamental quantities:			
	Mass	kg	m	M
	Length	m (metre)	l	L
	Time	s (second)	t	T
	Temperature	K	T	θ
2.	Geometric quantities:			
	Area	m^2	A, a	L^2
	Volume	m^3	V	L^3
3.	Kinematic quantities:			
	Velocity	m/s	V, v, u	LT^{-1}
	Angular velocity	rad/s	ω	T^{-1}
	Acceleration	m/s^2	f, a	LT^{-2}
	Angular acceleration	rad/s^2	α	T^{-2}
	Discharge	m^3/s	Q	L^3T^{-1}
	Kinematic viscosity	m^2/s	n	L^2T^{-1}
4.	Dynamic quantities:			
	Force	N	F	MLT^{-2}
	Density	kg/m^3	ρ	ML^{-3}
	Specific weight	N/m^3	w	$ML^{-2}T^{-2}$

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	Quantity	Unit	Symbol	Dimensions (*MLT* System)
	Dynamic viscosity	Ns/m^2	μ	$ML^{-1}T^{-1}$
	Pressure	N/m^2	p	$ML^{-1}T^{-2}$
	Torque	N-m	T	ML^2T^{-2}
5.	Thermodynamic properties:			
	Thermal conductivity	W/mK	k	$MLT^{-3}\theta^{-1}$
	Enthalpy/unit mass	J/kg	h	L^2T^{-2}
	Entropy	J/K	ϕ, s	$ML^2T^2\theta^{-1}$
	Internal energy	J	E, u	L^2T^{-2}

Dimensional Homogeneity

Concept of dimensional homogeneity is applicable to physical equations. An equation is formed using two or more quantities. An equation is said to be dimensionally homogenous when left hand side of the equation is dimensionally equal to the right hand side of the equation. When each term in the equation is reduced to their dimensions, fundamental dimensions in each side of the equation will have identical powers. According to Fourier's principle of homogeneity, a correct equation expressing a physical relationship between quantities must be dimensionally homogenous and numerically equivalent.

For example, consider the equation $v = \sqrt{2gH}$

Dimensions of left hand side = LT^{-1}

Dimensions of right hand side $\sqrt{LT^{-2}L} = LT^{-1}$

Therefore the equation is dimensionally homogenous.

If the value of $g = 9.81$ m/s^2 is substituted, the equation becomes,

$$V = \sqrt{2 \times 9.81\ \text{H}} = 4.429\sqrt{\text{H}}$$

$$\text{Here, RHS} = \frac{1}{L^2}$$

Therefore the equation is not dimensionally homogenous and the equation cannot be applied for other unit systems such as FPS or CGS.

Methods of Dimensional Analysis

Two important methods used for dimensional analysis are:

1. Rayleighs method
2. Buckinghams π-theorem method

Rayleighs Method

Rayleighs method is suitable when the number of independent variables in the phenomenon is only 3 or 4.

The interrelated variables or quantities having different dimensions are expressed in the form of an exponential equation which must be dimensionally homogenous.

Let $$x = f(x_1, x_2, x_3, \ldots, x_n)$$

Then an exponential equation can be formed as follows:

$$x = C(x_1^a, x_2^b, x_2^c \ldots)$$

Where, C is a non-dimensional factor. Values of a, b, c, … are obtained by comparing left hand side to right hand side of the equation after writing dimensions of the variables.

SOLVED EXAMPLE

Example 1

A fluid of density ρ and viscosity μ flows through a pipe of diameter d. Derive an expression in terms of Reynolds number for resistance per unit area of surface using Rayleigh's method of dimensional analysis.

Solution

Let F be the resistance per unit area:

F is a function of the independent variables. ρ, μ, v and d or

$$F = f(\rho, v, d, \mu)$$

This can be written as

$$F = C\,\rho^a\,v^b\cdot d^c\,\mu^d$$

Writing in dimensional from

$$ML^{-1}T^{-2} = [ML^{-3}]^a\,[LT^{-1}]^b\,L^c\,[ML^{-1}T^{-1}]^d$$

Comparing the powers

For M,

$$1 = a + d, \tag{1}$$

For L,

$$-1 = -3a + b + c - d \tag{2}$$

For T,

$$-2 = -b - d \tag{3}$$

There are four unknowns and only three equations. Values cannot be found out. But 3 unknowns can be expressed in terms of the other.

Writing in terms of 'a',

$$d = 1 - a$$

$$\begin{aligned} b &= -d + 2 \\ &= -1 + a + 2 \\ &= 1 + a \end{aligned}$$

$$\begin{aligned} c &= -3a - b + d \\ &= -1 + 3a - 1 + 1 - a \\ &= a - 1 \end{aligned}$$

$$\therefore F = C\rho^a v^{1+a} d^{a-1} \mu^{1-a}$$

$$= C\left(\frac{\rho v d}{\mu}\right)^a \frac{v\mu}{d}\frac{\rho v}{\rho v}$$

$$= C\left(\frac{\rho v d}{\mu}\right)^a \left(\frac{\mu}{\rho v d}\right)\rho v^2$$

$$= C\left(\frac{\rho v d}{\mu}\right)^{a-1} \rho v^2$$

$$= C(R_e)^{a-1}\rho v^2$$

$$= C\ \rho v^2\ \phi\ (R_e)$$

Buckingham's π-theorem Method

From a set of variables, dimensionless groups are formed using this method. According to this method, if there are n variables (dependent and independent) influencing a phenomenon, which can be fully expressed in terms of m fundamental units, then these n variables can be grouped as $(n - m)$ dimensionless terms called π terms.

Method of Forming Dimensionless Constants

1. Number of fundamental units involved (m) is found out and thus number of dimensionless group $(n - m)$ is found out.
2. Repeating variables are selected. Number of repeating variables is equal to number of fundamental units involved, i.e., m number. The repeating variables should together contain all the fundamental dimensions. One geometric characteristic (like l, d) one fluid characteristic (like ρ, μ) and one flow characteristic (like v) are selected as repeating variables. l or d, v and ρ would be the best choice in most cases. As far as possible dependent variable is not selected as a repeating variable.
3. π terms are formed using the repeating variables and one of the non-repeating variables. Repeating variables are raised to indexes.

For example,

Let x_1 be the dependent variable and $x_2, x_3, x_4, \ldots, x_n$ be the independent variables.

Then,

$$x_1 = f(x_2, x_3, x_4, \ldots, x_n)$$

This can be written as

$$f(x_1, x_2, x_3, \ldots, x_n) = 0$$

If there are m fundamental units involved,

Number of π terms $= (n - m)$

If $m = 3$,

Let x_2, x_3 and x_4 be the repeating variables selected.

Then,

$$\pi_1 = x_2^{a_1}\ x_3^{b_1}\ x_4^{c_1} . x_1$$

$$\pi_2 = x_2^{a_2}\ x_3^{b_2}\ x_4^{c_3} . x_5$$

$$\pi_3 = x_2^{a_3}\ x_3^{b_3}\ x_4^{c_3} . x_6$$

$$\pi_{n-m} = x_2^{a_{n-m}}\ x_3^{b_{n-m}}\ x_4^{c_{n-m}} . x_n$$

The π terms are written in the dimensional form and values of powers are found out as in the case of Rayleigh's method. Thus the dimensionless constants are identified. Using the π terms relation between variables are obtained as follows. For example,

$$\pi_1 = f(\pi_2, \pi_3, \ldots)$$

or

$$f_1(\pi_1, \pi_2, \pi_3, \ldots) = 0$$

Example 2

Using Buckinghams π-theorem show that the velocity through a circular orifice is given by:

$$V = \sqrt{2gH}\ \phi\left[\frac{D}{H}, \frac{\mu}{\rho v H}\right]$$

Where

H = Head causing flow

D = Diameters of orifice

μ = Coefficient of viscosity

ρ = Mass density

g = Acceleration due to gravity

Solution

V is a function of H, D, μ, ρ and g

$$\text{or } V = f(H, D, \mu, \rho, g)$$

$$\text{or } f_1(V, H, D, \mu, \rho, g) = 0$$

Total number of variables, $n = 6$

Writing the variables in dimensions

$$V = LT^{-1}$$

$$H = L$$

$$D = L$$

$$\mu = ML^{-1}T^{-1}$$

$$\rho = ML^{-3}$$

$$g = LT^{-2}$$

From the above, number of fundamental dimensions involved,

$$m = 3$$

∴ Number of π terms

$$= n - m$$

$$= 6 - 3 = 3$$

∴ The equations in terms of π terms is $f_1(\pi_1, \pi_2, \pi_3) = 0$.

Choosing H, g, ρ as repeating variables, the π terms are as follows.

$$\pi_1 = H^{a_1} g^{b_1} \rho^{c_1} \; v$$

$$\pi_2 = H^{a_2} g^{b_2} \rho^{c_2} \; D$$

$$\pi_3 = H^{a_3} g^{b_3} \rho^{c_3} \; \mu$$

Writing dimensions:

π_1 term:

$$M^\circ L^\circ T^\circ = L^{a_1} (LT^{-2})^{b_1} (ML^{-3})^{c_1} LT^{-1}$$

Equating exponents of M, L and T

$$M: 0 = C_1$$

$$L: 0 = a_1 + b_1 + -3c_1 + 1$$

$$T: 0 = -2b_1 - 1$$

From the above,

$$c_1 = 0$$

$$b_1 = \frac{-1}{2}$$

$$a_1 = -b_1 + 3c_1 - 1$$

$$= \frac{1}{2} + 0 - 1$$

$$= \frac{-1}{2}$$

$$\therefore \; \pi_1 = H^{\frac{-1}{2}} g^{\frac{-1}{2}} \rho^\circ \; v$$

$$= \frac{v}{\sqrt{gH}}$$

π_2 term:

$$M^\circ L^\circ T^\circ = L^{a_2} (LT^{-2})^{b_2} (ML^{-3})^{c_2} \; L$$

$$M: 0 = c_2$$

$$L: 0 = a_2 + b_2 - 3c_2 + 1$$

$$T: 0 = -2b_2$$

$$\therefore \; c_2 = 0$$

$$b_2 = 0$$

$$a_2 = -b_2 + 3c_2 - 1 = -1$$

$$\therefore \; \pi_2 = H^{-1} g^\circ \rho^\circ D$$

$$= \frac{D}{H}$$

π_3 term:

$$M^\circ L^\circ T^\circ = L^{a_3} (LT^{-2})^{b_3} (ML^{-3})^{c_3} ML^{-1}T^{-1}$$

$$M: 0 = c_3 + 1$$

$$L: 0 = a_3 + b_3 + -3c_3 - 1$$

$$T: 0 = -2b_3 - 1$$

$$\therefore \; c_3 = -1$$

$$b_3 = \frac{-1}{2}$$

$$a_3 = -b_3 + 3c_3 + 1$$

$$= \frac{1}{2} - 3 + 1 = \frac{-3}{2}$$

$$\therefore \pi_3 = H^{\frac{-3}{2}} g^{\frac{-1}{2}} \rho^{-1} \; \mu$$

$$= \frac{\mu}{\rho\sqrt{g} \; H^{\frac{3}{2}}}$$

$$= \frac{\mu v}{HV\rho\sqrt{gH}} = \frac{\mu}{H\rho V} \; \pi_1$$

∴ Equation can be written as,

$$f_1\left(\frac{V}{\sqrt{gH}}, \frac{D}{H}, \frac{\mu}{\rho v H} \pi_1\right) = 0$$

$$\text{or} \quad \frac{V}{\sqrt{gH}} = \phi\left[\frac{D}{H}, \frac{\mu}{\rho v H} \pi_1\right]$$

$$\text{or} \quad V = \sqrt{2gH} \; \phi\left[\frac{D}{H}, \frac{\mu}{\rho v H}\right]$$

Because, multiplying or dividing by a constant does not change the character of π terms.

Hydraulic Similitude

Solutions of complicated problems in hydraulic engineering is simplified by model analysis. Model analysis is also required for predicting performance of hydraulic structures like dams and spillways, hydraulic machines such as turbines and pumps, structures, ships, aircrafts etc. The results of model studies represent the behaviour of prototype, if there is similitude or similarities between model and prototype. Three similarities required are:

1. Geometric similarity
2. Kinematic similarity
3. Dynamic similarity

Geometric Similarity

Geometrically similar objects are similar in their shape. They differs only in their size. The ratio of any length of the prototype to the corresponding length in the model is same everywhere. The ratio is known as scale factor.

For perfect geometric similarity, roughness of surface also should be geometrically similar. But this is not easily attained.

If l_m, b_m, d_m, h_m, etc., are certain linear dimensions of the model and c_p, l_p, d_p, h_p, etc., are the corresponding linear dimensions of the prototype, condition for geometric similarity is,

$$\frac{l_m}{l_p} = \frac{b_m}{b_p} = \frac{d_m}{d_p} = \frac{h_m}{h_p} = l_r$$

l_r is called scale ratio or scale factor or model ratio

$$\text{Area ratio, } A_r = \frac{A_m}{A_p} = l_r^2$$

$$\text{Volume ratio, } V_r = \frac{V_m}{V_p} = l_r^3$$

Kinematic Similarity

Similarity of motion is called kinematic similarity. Kinematic similarity between prototype and model exists when the ratios of corresponding kinematic quantities such as velocity, acceleration, etc., at corresponding points are same.

Therefore, velocity ratios,

$$= \frac{V_{m_1}}{V_{p_1}} = \frac{V_{m_2}}{V_{p_2}} = V_r$$

Similarly acceleration ratio,

$$= \frac{a_{m1}}{a_{p1}} = \frac{a_{m2}}{a_{p2}} = a_r$$

In terms of scale ratio,

$$V_r = \frac{l_r}{t_r} \text{ and } a_r = \frac{l_r}{t_r^2}$$

Where, t_r = ratio of corresponding time intervals.

Geometric similarity is a pre-requisite for kinematic similarity. Also the directions of velocities in the model and prototype should be same.

Dynamic Similarity

Dynamic similarity is the similarity of masses and forces. The ratios of masses of corresponding fluid particles should be same. Similarly magnitudes of forces at corresponding points in each system should be in a fixed ratio. Therefore the ratio of magnitudes of any two forces in the prototype should be same as the magnitude ratio of the corresponding forces in the model. The different forces that may act on a fluid element are:

1. Viscous force (F_v)
2. Pressure force (F_p)
3. Gravity force (F_g)
4. Surface tension or capillary force (F_s)
5. Elastic force (F_e) (due to compressibility)

Resultant of these forces causes acceleration of the fluid element which is opposed by the inertia force (F_i),

$$\therefore F_R = F_v + F_p + F_g + F_s + F_e$$
$$= -F_i \text{ and } F_v + F_p + F_g + F_s + F_e + F_i = 0$$

For dynamic similarity, the ratio of these forces should be same for prototype and model. Generally F_i is taken as the common one to describe ratios.

For example, $\dfrac{F_v}{F_i}, \dfrac{F_p}{F_i}$, etc.

For absolute dynamic similarity the ratios corresponding to all the forces should be same for model and prototype, but it is not possible to satisfy all these equations simultaneously. Therefore for practical cases ratio of the predominant force with inertia force is considered for dynamic similarity.

Various forces acting on a fluid element as mentioned above are functions of certain variables which can be classified into three:

1. Linear dimension (l, d)
2. Fluid properties (ρ, μ, σ, E)
3. Kinematic and dynamic characteristics (v, p, g)

Various forces can be expressed in terms of the above as follows:

1. Viscous force,

$$F_v = \text{Shear stress} \times \text{Area}$$
$$= \mu \times \text{Velocity gradient} \times \text{Area}$$
$$= \mu \frac{v}{L} \times L^2 = \mu v L$$

2. Pressure force,

$$F_p = \text{Pressure intensity} \times \text{Area}$$
$$= pL^2$$

3. Gravity force,

$$F_g = mg = \rho L^3 g$$

4. Surface tension force,

$$F_s = \text{Surface tension} \times \text{Length}$$
$$= \sigma L$$

5. Elastic force,

$$F_e = \text{Stress} \times \text{Area}$$
$$= \text{Strain} \times \text{Modulus of elasticity} \times \text{Area}$$
$$= EL^2$$

6. Inertia force,

$$F_i = \text{Mass} \times \text{Acceleration}$$
$$= \rho L^3 \times v\frac{dv}{ds}$$
$$= \rho L^3 \times \frac{v^2}{L} = \rho L^2 v^2$$

Dimensionless Numbers

Dimensionless parameters or numbers are obtained by dividing the inertia force by a force like viscous or gravity force.

Important dimensionless numbers are:

1. Reynold's number
2. Froude's number
3. Euler's number
4. Weber's number
5. Mach's number

Reynolds number (*Re*): It is the ratio of inertia force to viscous force.

$$\therefore\ Re = \frac{\rho L^2 v^2}{\mu v L} = \frac{\rho v L}{\mu}$$

Froude's number (*Fr*): It is the square root of the ratio of the inertia force and gravity force.

$$\therefore\ Fr = \sqrt{\frac{F_i}{F_g}}$$
$$= \sqrt{\frac{\rho L^2 v^2}{\rho L^3 g}} = \frac{v}{\sqrt{Lg}}$$

Euler's number (*Eu*): It is the square root of the ratio of inertia force to pressure force.

$$\therefore\ Eu = \sqrt{\frac{F_i}{F_p}} = \sqrt{\frac{\rho L^2 v^2}{pL^2}} = \frac{v}{\sqrt{\frac{p}{\rho}}}$$

Weber's number (*We*): It is the square root of ratio of inertia force to surface tension force.

$$\therefore\ We = \sqrt{\frac{F_i}{F_s}}$$
$$= \sqrt{\frac{\rho L^2 v^2}{\sigma\ L}} = \frac{v}{\sqrt{\frac{\sigma}{\rho L}}}$$

Mach's number (*M*): It is the square root of the ratio of inertia force to the elastic force

$$\therefore\ M = \sqrt{\frac{F_i}{F_e}}$$
$$= \sqrt{\frac{\rho L^2 v^2}{KL^2}}$$
$$= \frac{V}{\sqrt{\frac{K}{\rho}}} = \frac{V}{C}$$

Where, C = Velocity of sound in the fluid.

Model Laws or Similarity Laws

For dynamic similarity ratio of corresponding forces at the corresponding points should be same. This means that the various dimensionless numbers mentioned above should be same. Based on these numbers the model or similarity laws are:

1. Reynolds model law
2. Froude model law
3. Euler model law
4. Weber model law
5. Mach model law

Reynolds Model Law

When the predominant force is viscous force, Reynolds number should be same for model and prototype.

$$\therefore\ (Re)_m = (Re)_p$$

or
$$\left(\frac{\rho VL}{\mu}\right)_m = \left(\frac{\rho VL}{\mu}\right)_p$$

or
$$\frac{\rho_p}{\rho_m} \times \frac{V_p}{V_m} \times \frac{L_p}{L_m} \times \frac{1}{\left(\frac{\mu_p}{\mu_m}\right)} = 1$$

or
$$\frac{\rho_r V_r\ L_r}{\mu_r} = 1$$

Time scale ratio, $T_r = \frac{L_r}{V_r}\left[\because V = \frac{L}{T}\right]$

Acceleration scale ratio, $a_r = \frac{V_r}{T_r}$

Discharge scale ratio,

$$Q_r = \rho_r A_r V_r = \rho_r L_r^{\,2} V_r$$

Force scale ratio $F_r = m_r \times a_r$

where quantities with subscript r represent the corresponding scale ratios.

Areas where Reynolds model law can be applied are,

1. Motions of air planes
2. Flow of incompressible fluids in closed pipes
3. Motion of Submarines completely under water, etc.

Froude Model Law

Froude model law is applied for comparison of model and prototype where the predominant force is the gravity force. According to this,

$$(F_r)_m = (F_r)_p \text{ or } \frac{V_m}{\sqrt{g_m L_m}} = \frac{V_p}{\sqrt{g_p L_p}}$$

When, $g_m = g_p$,

$$\frac{V_m}{\sqrt{L_m}} = \frac{V_p}{\sqrt{L_p}}$$

or

$$\frac{V_p}{V_m} = \sqrt{\frac{L_p}{L_m}} \text{ or } V_r = \sqrt{Lr}$$

Now,

$$T_r = \frac{L_r}{V_r} = \frac{L_r}{\sqrt{L_r}} = \sqrt{L_r}$$

$$a_r = \frac{V_r}{T_r} = \frac{\sqrt{L_r}}{\sqrt{L_r}} = 1$$

$$Q_r = A_r \cdot V_r = L_r^2 \times \sqrt{L_r} = L_r^{2.5}$$

$$F_r = m_r \times a_r = \rho_r L_r^3 \times 1 = \rho_r L_r^3$$

If same fluid is used, $F_r = L_r^3$

Areas of application are:

1. Flow over spill ways, sluices, etc.
2. Flow of jet from nozzle or orifice
3. Flow through channels
4. Water bodies where surface waves are formed

Euler Model Law

When pressure forces are dominant forces, Euler's number is the criterion for comparison. According to this,

$$(E_u)_m = (E_u)_p \text{ or } \frac{V_m}{\sqrt{\frac{P_m}{\rho_m}}} = \frac{V_p}{\sqrt{\frac{P_p}{\rho_p}}}$$

If same fluid is used,

$$\frac{V_m}{\sqrt{P_m}} = \frac{V_p}{\sqrt{P_p}}$$

Areas of applications are:

1. Enclosed fluid systems, where turbulence is fully developed and viscous forces are negligible.
2. Where cavitation is present
3. Discharge through orifices, sluices, etc.

Weber Model Law

Weber model law based on Weber's number is used when surface tension effects predominate. According to this,

$$(We)_m = (We)_p$$

or

$$\frac{V_m}{\sqrt{\frac{\sigma_m}{\rho_m L_m}}} = \frac{V_p}{\sqrt{\frac{\sigma_p}{\rho_p L_p}}}$$

Areas of application are:

1. Capillary movement of water in soils
2. Flow over weirs at very small heads
3. Liquid spraying systems
4. Capillary rise in narrow passages, etc.

Mach Model Law

When elastic forces are predominant Mach model law is applied. According to this,

$$(M)_m = (M)_p \text{ or } \frac{V_m}{\sqrt{\frac{K_m}{\rho_m}}} = \frac{V_p}{\sqrt{\frac{K_p}{\rho_p}}}$$

Areas of application are:

1. Aerodynamic testing
2. Flow of gases with supersonic velocities
3. Unsteady flows with water hammer problems.
4. Testing of torpedoes under water, etc.

Example 3

A geometrically similar model of a spillway is to be made to a scale of 1 : 49. Determine velocity ratio, timescale ratio, acceleration ratio and discharge ratio.

Solution

For spillways Froude law can be applied.

Velocity ratio,

$$V_r = \frac{V_m}{V_p} = \sqrt{\frac{L_m}{L_p}} = \sqrt{L_r} = \sqrt{\frac{1}{49}} = \frac{1}{7}$$

Time scale ratio,

$$T_r = \frac{L_r}{V_r} = \frac{L_r}{\sqrt{L_r}}$$

$$= \sqrt{Lr} = \frac{1}{7}$$

Acceleration ratio, $a = \dfrac{V}{T}$

$$a_r = \frac{V_r}{T_r} = \frac{\sqrt{L_r}}{\sqrt{L_r}} = 1$$

Discharge ratio,

$$Q_r = A_r\, V_r = Lr^2\ Vr$$

$$= Lr^2 .\sqrt{Lr} = Lr^{2.5}$$

$$= \left(\frac{1}{49}\right)^{2.5} = \frac{1}{16807}.$$

Example 4

Model of a spillway is to be built to a scale ratio 1 : 50 across a flume of 70 cm width. The prototype has a height of 16 m and expected maximum head is 1.6 m. Determine

(i) height and head required for the model.
(ii) flow per metre length of the prototype if the model has a flow of 15 litres per second at a particular head.
(iii) value of negative pressure in the prototype, if the negative pressure in the model is 15 cm.

Solution

$$L_m = 70 \text{ cm} = 0.7 \text{ m}$$

$$Y_p = 16 \text{ m}$$

$$H_p = 1.6 \text{ m}$$

$$Q_m = 15 \text{ lit/s}$$

$$h_m = -15 \text{ cm} = -0.15 \text{ m}$$

$$\frac{L_m}{L_p} = L_r = \frac{1}{50}$$

(i) Height of model,

$$Y_m = Y_p \times L_r$$

$$= 16 \times \frac{1}{50} = 0.32 \text{ m}$$

Head on model,

$$H_m = H_p \times L_r$$

$$= 1.6 \times \frac{1}{50} = 0.032 \text{ m}$$

(ii) Discharge over prototype,

$$Q_r = L_r^{2.5}$$

$$\frac{Q_m}{Q_p} = \left(\frac{1}{50}\right)^{2.5}$$

$$Q_p = Q_m \times 50^{2.5} = 15 \times 20^{2.5}$$

$$= 265{,}165 \text{ lit/s}$$

Length of prototype,

$$L_p = \frac{L_m}{L_r} = L_m \times 50$$

$$= 0.7 \times 50 = 35 \text{ m}$$

Discharge per metre length of prototype,

$$= \frac{Q_p}{L_p}$$

$$= \frac{265{,}165}{35} = 7576 \text{ lit/s}$$

(iii) Negative pressure in the prototype,

$$h_p = \frac{h_m}{L_r}$$

$$= h_m \times 50$$

$$= -0.15 \times 50 \text{ m}$$

$$= -7.5 \text{ m.}$$

Example 5

Water flows through a 200 mm diameter pipe at a velocity of 3 m/s. If oil flows through a pipe of 80 mm diameter, under dynamically similar conditions, its velocity (in m/s) will be______. (Kinematic viscosity of oil and water are 0.03 stoke and 0.01 stoke respectively)

Solution

For water pipe,

$$d_1 = 200 \text{ mm}$$

$$v_1 = 3 \text{ m/s}$$

$$\nu = 0.01 \text{ stoke}$$

For oil pipe,

$$d_2 = 80 \text{ mm}$$

$$\nu = 0.03 \text{ stoke}$$

For dynamic similarity, Reynolds number in both cases should be same.

That is, $\dfrac{v_1 d_1}{\nu_1} = \dfrac{v_2 d_2}{\nu_2}$

$$\therefore v_2 = v_1 \frac{d_1}{d_2}\frac{\nu_2}{\nu_1}$$

$$= 3 \times \frac{200}{80} \times \frac{0.03}{0.01}$$

$$= 22.5 \text{ m/s.}$$

Example 6

A geometrically similar model of an air duct is built in $\frac{1}{30}$ scale and tested with water which is 50 times more viscous and 800 times denser than air. When tested under dynamically similar conditions, pressure drop was 2.5 bar in the model. Corresponding pressure drop in the prototype (in mm of water) will be ______.

Solution

$$\frac{L_p}{L_m} = 30, \ \frac{\mu_p}{\mu_m} = \frac{1}{50},$$

$$\frac{\rho_p}{\rho_m} = \frac{1}{800}, \ \Delta p_m = 2.5 \text{ bar}$$

For dynamic similarity Reynolds number should be same.

$$\frac{\rho_m V_m L_m}{\mu_m} = \frac{\rho_p V_p L_p}{\mu_p}$$

$$\Rightarrow \frac{V_p}{V_m} = \frac{\rho_m}{\rho_p} \times \frac{L_m}{L_p} \times \frac{\mu_p}{\mu_m}$$

$$= 800 \times \frac{1}{30} \times \frac{1}{50}$$

$$\frac{\rho_m V_m L_m}{\mu_m} = \frac{\rho_p V_p L_p}{\mu_p} = \frac{8}{15}$$

$$\text{Pressure} = \frac{\text{Force}}{\text{Area}}$$

$$= \frac{\rho L^2 V^2}{L^2} = \rho V^2$$

$$\therefore \frac{(\Delta P)_p}{(\Delta P)_m} = \frac{\rho_p V_p^2}{\rho_m V_m^2}$$

$$= \frac{1}{800} \times \left(\frac{8}{15}\right)^2$$

$$\Rightarrow (4P)_p = 2.5 \times \frac{1}{800} \times \left(\frac{8}{15}\right)^2 \text{ bar}$$

$$= 8.889 \times 10^{-4} \text{ bar}$$

$$= 8.889 \times 10^{-4} \times 10^5 \text{ N/m}^2$$

$$= 88.89 \text{ N/m}^2 = \rho_w g h_w$$

$$\Rightarrow h_w = \frac{88.89}{9810} \text{ m of water}$$

$$= \frac{88.89 \times 10^3}{9810} = 9.06 \text{ mm water.}$$

Example 7

One steel ball (specific gravity 7.85) and one aluminium ball (specific gravity 2.7) were dropped in water to reach terminal velocity under dynamically similar conditions. The ratio of diameters of aluminium ball to steel ball is _______.

Solution

For dynamic similarity,

Reynolds number should be same for both cases.

$$\therefore \frac{\rho V_s D_s}{\mu} = \frac{\rho V_a D_a}{\mu}$$

$$\Rightarrow V_s D_s = V_a D_a$$

$$\Rightarrow V_s D_s = V_a D_a$$

$$\Rightarrow \frac{D_a}{D_s} = \frac{V_s}{V_a} \quad (1)$$

But terminal velocity of a sphere in water is given by,

$$V_s = \frac{(w_s - w) D_s^2}{18\mu} \text{ for steel ball.}$$

$$V_a = \frac{(w_s - w) D_s^2}{(w_a - w) D_a^2} \quad (2)$$

From Eqs. (1) and (2), we have

$$\frac{D_a}{D_s} = \frac{w_s - w}{w_a - w}\left(\frac{D_s}{D_a}\right)^2$$

$$\Rightarrow \left(\frac{D_a}{D_s}\right)^3 = \frac{w_s - w}{w_a - w}$$

$$= \frac{7.85 - 1}{2.7 - 1}$$

$$= \frac{6.85}{1.7} = 4.0294$$

$$\Rightarrow \frac{D_a}{D_s} = 1.591.$$

Example 8

A model of a rectangular pipe 1.6 m wide and 4.6 m long in the river is built at a scale ratio of 1 : 25. Average depth of water in the river is 3 m. Velocity of flow for the model was 0.6 m/s. Force acting on the model was 3.6 N and height of the standing wave was 30 mm. Determine the following for the prototype:

(i) Velocity of flow
(ii) Force acting
(iii) Height of standing wave
(iv) Coefficient of drag resistance

Solution

Scale ratio $\frac{L_p}{L_m} = 25$

$$V_m = 0.6 \text{ m/s}; \qquad F_m = 3.6 \text{ N}$$

Height of standing wave $H_m = 30$ mm

In a river free surface flow is caused by gravity. So Froude's model is applied.

(i) $$V_r = \sqrt{Lr}$$

$$\Rightarrow \frac{V_p}{V_m} = \sqrt{25} = 5$$

$$\Rightarrow V_p = 0.6 \times 5 = 3 \text{ m/s}$$

(ii) Force acting,

$$F_r = Lr^3$$

$$\frac{F_p}{F_m} = \left(\frac{L_p}{L_m}\right)^3 = 25^3$$

$$\Rightarrow F_p = 3.6 \times 253 = 56250 \text{ N}$$

(iii) Height of standing wave,

$$\frac{H_p}{H_m} = \frac{L_p}{L_m} = 25$$

$$\Rightarrow H_p = 30 \times 25 = 750 \text{ mm}$$

(iv) Coefficient of drag resistance

$$F = C_D \frac{\rho A V^2}{2}$$

$$\Rightarrow (C_D)_p = \frac{2F_p}{\rho_p A_p V_p^2}$$

A = Depth of water × Width of pipe

$= 3 \times 1.6$

$$\Rightarrow C_D = \frac{2 \times 56250}{1000 \times (3 \times 1.6) \times 3^2}$$

$= 2.604.$

Example 9

When the model of scale ratio 1 : 40, of an aeroplane was tested in water the pressure drop was 7 kN/m^2. Density and viscosity of air is 1.24 kg/m^3 and 0.00018 poise respectively and those of water are 1000 kg/m^3 and 0.01 poise respectively, corresponding pressure drop in the prototype would be ______.

Solution

In the problem, viscous force and pressure force is involved. Therefore Reynolds model law and Euler model Law should be applied.

Applying Reynolds model law,

$$\frac{\rho_m V_m L_m}{\mu_m} = \frac{\rho_p V_p L_p}{\mu_p}$$

$$\therefore \frac{V_m}{V_p} = \frac{\rho_p}{\rho_m} \times \frac{L_p}{L_m} \times \frac{\mu_m}{\mu_p}$$

$$= \frac{1.24}{1000} \times 40 \times \frac{0.01}{0.00018} = 2.756$$

Applying Euler model law,

$$\frac{V_m}{\sqrt{\dfrac{(\Delta p)_m}{\rho_m}}} = \frac{V_p}{\sqrt{\dfrac{(\Delta p)_p}{\rho_p}}}$$

$$\Rightarrow \frac{V_m}{V_p} = \left[\frac{(\Delta p)_m}{(\Delta p)_p}\right]^{1/2} \times \left(\frac{\rho_p}{\rho_m}\right)^{1/2}$$

$$\Rightarrow 2.756^2 = \frac{7 \times 10^3}{(\Delta p)_p} \times \frac{1.24}{1000}$$

$$\Rightarrow (\Delta P)_P = 1.143 \text{ N/m}^2.$$

Exercises

1. An 1 : 50 model of an ogee spillway crest records an acceleration of 1.5 m/s^2 at a certain location. The homologous value of acceleration in the prototype is ______.
2. The number of π parameters needed to express the function $F(A, V, t, \mu, L) = 0$ are
   (A) 5 (B) 4
   (C) 3 (D) 2
3. An 1 : 30 model of an ogee spillway crest records an acceleration of 1.3 m/s^2 at a certain location. The homologous value of acceleration in the prototype in m/s^2, is ______.
   (A) 0.043 (B) 0.237
   (C) 1.300 (D) 7.120
4. The flow in a river is 1500 cumecs. A distorted model is built with horizontal scale of 1/150 and vertical scale of 1/25. The flow rate in the model should be.
   (A) 0.04 m^3/s
   (B) 0.06 m^3/s
   (C) 0.08 m^3/s
   (C) 0.10 m^3/s
5. The repeating variables in dimensional analysis should
   (A) include the dependent variable.
   (B) have amongst themselves all the basic dimensions.
   (C) be derivable from one another.
   (D) exclude one of the basic dimensions.

6. Match List I with List II and select the correct answer using the codes given

	List I		List II
a.	Froude number	**1.**	Related to inertia force and elastic force
b.	Euler number	**2.**	Related to inertia force and viscous force
c.	Reynolds number	**3.**	Related to inertia force and pressure force
d.	Mach number	**4.**	Related to inertia force and gravity force

Codes:

	a	b	c	d		a	b	c	d
(A)	4	1	3	2	(B)	2	3	4	1
(C)	4	3	2	1	(D)	2	1	4	3

7. Both Reynold's and Froude numbers assume significance in one of the following examples:
(A) Motion of submarine at large depths
(B) Motion of ship in deep seas
(C) Cruising of a missile in air
(D) Flow over spillways

8. In a 1/50 model of a spillway, the discharge was measured to be 0.3 m^3/s. the corresponding prototype discharge in m^3/s is
(A) 2.0 (B) 15.0
(C) 106.0 (D) 5303.0

9. In order to estimate the energy loss in a pipeline of 1 m diameter through which kerosene of specific gravity 0.80 and dynamic viscosity of 0.02 poise is to be transported at the rate of 2 m^3/s, model tests were conducted on a 0.1 m diameter pipe using water at 20°C. If the absolute viscosity of water at 20°C is 1.00×10^{-2} poise, then the discharge required for the model pipe would be
(A) 60 lit/s (B) 80 lit/s
(C) 120 lit/s (D) 160 lit/s

10. A laboratory model of a river is built to a geometric scale of 1 : 100. The fluid used in the model is oil of mass density 900 kg/m^3. The highest flood in the river is 10,000 m^3/s. The corresponding discharge in the model shall be
(A) 0.095 m^3/s
(B) 0.100 m^3/s
(C) 0.105 m^3/s
(D) 10.5 m^3/s

11. A model of a boat is built to a scale of 1/100. It experiences a resistance of 0.10 N when simulating a speed of 5 m/s of the prototype. Water is the fluid in both the cases. Neglecting frictional forces, corresponding resistance in the prototype is:
(A) 1000 kN (B) 100 kN
(C) 10 kN (D) 1 kN

12. The height of a hydraulic jump in the stilling pool of 1 : 25 scale model was observed to be 10 cm. The corresponding prototype height of the jump is:
(A) Cannot be determined
(B) 2.5 m
(C) 0.5 m
(D) 0.1 m

13. For a homologous model of a pump built to a scale ratio of 1 : 2, fluid and speed being the same in model and prototype, the ratio of model power to prototype power is
(A) 1/2.82 (B) 1/4
(C) 1/8 (D) 1/32

14. The flow of glycerin (kinematic viscosity $v = 5 \times 10^{-4}$ m^2/s) in an open channel is to be modeled in a laboratory flume using water ($v = 10^{-6}$ m^2/s) as the flowing fluid. If both gravity and viscosity are important, what should be the length scale (i.e., ratio of prototype to model dimensions) for maintaining dynamic similarity?
(A) 1 (B) 22
(C) 63 (D) 500

Previous Years' Questions

1. A 1 : 50 scale model of a spillway is to be tested in the laboratory. The discharge in the prototype is 1000 m^3/s. The discharge to be maintained in the model test is: **[GATE, 2007]**
(A) 0.057 m^3/s
(B) 0.08 m^3/s
(C) 0.57 m^3/s
(D) 5.7 m^3/s

2. A river reach of 2.0 km long with maximum flood discharge of 10000 m^3/s is to be physically modeled in the laboratory where maximum available discharge is 0.20 m^3/s. For a geometrically similar model based on equality of Froude number, the length of the river reach (m) in the model is: **[GATE, 2008]**
(A) 26.4 (B) 25.0
(C) 20.5 (D) 18.0

3. List I contains dimensionless parameter and List II contains ratio.

Select the correct matching using the codes given: **[GATE, 2013]**

	List I		List II
P.	Match number	**1.**	Ratio of inertial force and gravity force.
Q.	Reynolds number	**2.**	Ratio of fluid velocity and velocity of sound.
R.	Weber number	**3.**	Ratio of inertial force and viscous force.
S	Froude number	**4.**	Ratio of inertial force and surface tension force.

Codes:

	P	Q	R	S
(A)	3	2	4	1
(B)	3	4	2	1
(C)	2	3	4	1
(D)	1	3	2	4

4. The drag force, F_D, on a sphere kept in a uniform flow field depends on the diameter of the sphere, D; flow velocity, V; fluid density, ρ; and dynamic represents the non-dimensional parameters which could be used to analyze this problem? **[GATE, 2015]**

(A) $\dfrac{F_D}{VD}$ and $\dfrac{\mu}{\rho VD}$

(B) $\dfrac{F_D}{\rho VD^2}$ and $\dfrac{\rho VD}{\mu}$

(C) $\dfrac{F_D}{\rho V^2 D^2}$ and $\dfrac{\rho VD}{\mu}$

(D) $\dfrac{F_D}{\rho V^3 D^3}$ and $\dfrac{\mu}{\rho VD}$

5. The relationship between the length scale ratio (L_r) and the velocity scale ratio (V_r) in hydraulic models, in which Froude dynamic similarity is maintained, is **[GATE, 2015]**

(A) $V_r = L_r$
(B) $L_r = \sqrt{V_r}$
(C) $V_r = L_r^{1.5}$
(D) $V_r = \sqrt{L_r}$

Answer Keys

Exercises

1. 1.5 **2.** D **3.** C **4.** C **5.** B **6.** C **7.** C **8.** D **9.** B **10.** B
11. B **12.** B **13.** D **14.** C

Previous Years' Questions

1. A **2.** A **3.** C **4.** C **5.** D