2.5 LIQUIDS. CAPILLARY EFFECTS

2.160 (a) $\Delta p = \alpha\left(\frac{1}{d/2}+\frac{1}{d/2}\right) = \frac{4\alpha}{d}$

$$= \frac{4\times 490\times 10^{-3}}{1{\cdot}5\times 10^{-6}}\frac{\text{N}}{\text{m}^2} = 1{\cdot}307\times 10^6\,\frac{\text{N}}{\text{m}^2} = 13 \text{ atmosphere}$$

(b) The soap bubble has two surfaces

so $$\Delta p = 2\,\alpha\left(\frac{1}{d/2}+\frac{1}{d/2}\right) = \frac{8\alpha}{d}$$

$$= \frac{8\times 45}{3\times 10^{-3}}\times 10^{-3} = 1{\cdot}2\times 10^{-3} \text{ atomsphere.}$$

2.161 The pressure just inside the hole will be less than the outside pressure by $4\,\alpha/d$. This can support a height h of Hg where

$$\rho\, g\, h = \frac{4\alpha}{d} \quad \text{or} \quad h = \frac{4\alpha}{\rho\, g\, d}$$

$$= \frac{4\times 490\times 10^{-3}}{13{\cdot}6\times 10^3\times 9{\cdot}8\times 70\times 10^{-6}} = \frac{200}{13{\cdot}6\times 70} \approx {\cdot}21 \text{ m of Hg}$$

2.162 By Boyle's law

$$\left(p_0+\frac{8\alpha}{d}\right)\frac{4\,\pi}{3}\left(\frac{d}{2}\right)^3 = \left(\frac{p_0}{n}+\frac{8\alpha}{\eta d}\right)\frac{4\pi}{3}\left(\frac{\eta\, d}{2}\right)^3$$

or $$p_0\left(1-\frac{\eta^3}{n}\right) = \frac{8\alpha}{d}(\eta^2-1)$$

Thus $$\alpha = \frac{1}{8}p_0\, d\left(1-\frac{\eta^3}{n}\right)(\eta^2-1)$$

2.163 The pressure has terms due to hydrostatic pressure and capillarity and they add

$$p = p_0+\rho g h+\frac{4\alpha}{d}$$

$$= \left(1+\frac{5\times 9{\cdot}8\times 10^3}{10^5}+\frac{4\times{\cdot}73\times 10^{-3}}{4\times 10^{-6}}\times 10^{-5}\right)\text{atoms} = 2{\cdot}22 \text{ atom.}$$

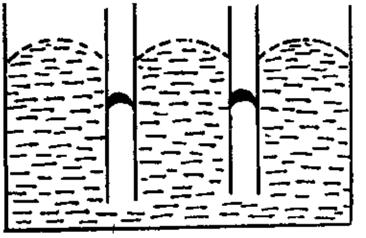
2.164 By Boyle's law

$$\left(p_0+h\,g\,\rho+\frac{4\alpha}{d}\right)\frac{\pi}{6}d^3 = \left(p_0+\frac{4\alpha}{nd}\right)\frac{\pi}{6}n^3\,d^3$$

or $$\left[hg\rho-p_0\,(n^3-1)\right] = \frac{4\alpha}{d}(n^2-1)$$

or $$h = \left[p_0\,(n^3-1)+\frac{4\alpha}{d}(n^2-1)\right]\Big/ g\,\rho = 4{\cdot}98 \text{ meter of water}$$

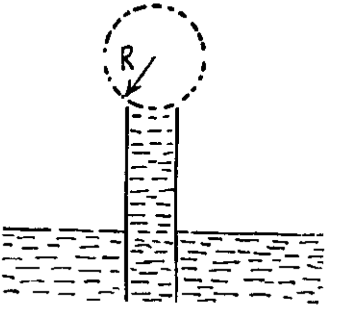
2.165 Clearly



$$\Delta h\,\rho\, g = 4\,\alpha\,|\cos\theta|\left(\frac{1}{d_1} - \frac{1}{d_2}\right)$$

$$\Delta h = \frac{4\,\alpha\,|\cos\theta|\,(d_2 - d_1)}{d_1\, d_2\, \rho g} = 11 \text{ mm}$$

2.166 In a capillary with diameter $d = 0{\cdot}5$ mm water will rise to a height



$$\frac{2\alpha}{\rho\, g\, r} = \frac{4\alpha}{\rho\, g\, d}$$

$$= \frac{4 \times 73 \times 10^{-3}}{10^3 \times 9{\cdot}8 \times 0{\cdot}5 \times 10^{-3}} = 59{\cdot}6 \text{ mm}$$

Since this is greater than the height (= 25 mm) of the tube, a meniscus of radius R will be formed at the top of the tube, where

$$R = \frac{2\alpha}{\rho\, gh} = \frac{2 \times 73 \times 10^{-3}}{10^3 \times 9{\cdot}8 \times 25 \times 10^{-3}} \approx 0{\cdot}6 \text{ mm}$$

2.167 Initially the pressure of air in the cppillary is p_0 and it's length is l. When submerged under water, the pressure of air in the portion above water must be $p_0 + 4\frac{\alpha}{d}$, since the level of water inside the capillary is the same as the level outside. Thus by Boyle's law

$$\left(p_0 + \frac{4\alpha}{d}\right)(l - x) = p_0\, l$$

or $$\frac{4\alpha}{d}(l - x) = p_0\, x \quad \text{or} \quad x = \frac{l}{1 + \frac{p_0\, d}{4\alpha}}$$

2.168 We have by Boyle's law

$$\left(p_0 - \rho g h + \frac{4\,\alpha\cos\theta}{d}\right)(l - h) = p_0\, l$$

or, $$\frac{4\,\alpha\cos\theta}{d} = \rho\, gh + \frac{p_0\, h}{l - h}$$

Hence, $$\alpha = \left(\rho\, gh + \frac{p_0\, h}{l - h}\right)\frac{d}{4\cos\theta}$$

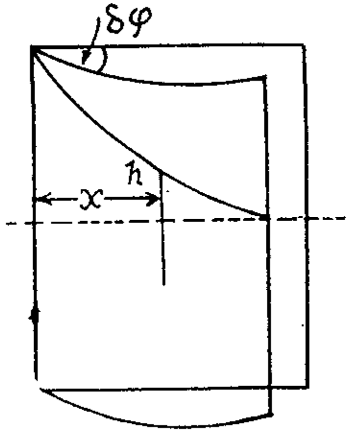
2.169 Suppose the liquid rises to a height h. Then the total energy of the liquid in the capillary is

$$E\,(h) = \frac{\pi}{4}\,(d_2^2 - d_1^2)\, h \times \rho g \times \frac{h}{2} - \pi\,(d_2 - d_1)\,\alpha h$$

Minimising E we get

$$h = \frac{4\alpha}{\rho g\,(d_2 - d_1)} = 6 \text{ cm.}$$

2.170 Let h be the height of the water level at a distance x from the edge. Then the total energy of water in the wedge above the level outside is.



$$E = \int x\,\delta\varphi \cdot dx \cdot h \cdot \rho g \frac{h}{2} - 2\int dx \cdot h \cdot \alpha \cos 0$$

$$= \int dx \frac{1}{2} x \rho g \delta\varphi \left(h^2 - 2\frac{2\alpha\cos\theta}{x\rho g \delta\varphi} h\right)$$

$$= \int dx \frac{1}{2} x \rho g \delta\varphi \left[\left(h - \frac{2\alpha\cos\theta}{x\rho g\delta\varphi}\right)^2 - \frac{4\alpha^2\cos^2\theta}{x^2\rho^2 g^2 \delta\varphi^2}\right]$$

This is minimum when $h = \dfrac{2\alpha\cos\theta}{x\rho g\delta\varphi}$

2.171 From the equation of continuity

$$\frac{\pi}{4} d^2 \cdot v = \frac{\pi}{4}\left(\frac{d}{n}\right)^2 \cdot V \quad \text{or} \quad V = n^2 v.$$

We then apply Bernoulli's theorem

$$\frac{p}{\rho} + \frac{1}{2} v^2 + \Phi = \text{constant}$$

The pressure p differs from the atmospheric pressure by capillary effects. At the upper section

$$p = p_0 + \frac{2\alpha}{d}$$

neglecting the curvature in the vertical plane. Thus,

$$\frac{p_0 + \frac{2\alpha}{d}}{\rho} + \frac{1}{2} v^2 + gl = \frac{p_0 + \frac{2n\alpha}{d}}{\rho} + \frac{1}{2} n^4 v^2$$

or
$$v = \sqrt{\frac{2gl - \frac{4\alpha}{\rho d}(n-1)}{n^4 - 1}}$$

Finally, the liquid coming out per second is,

$$V = \frac{1}{4}\pi d^2 \sqrt{\frac{2gl - \frac{4\alpha}{\rho d}(n-1)}{n^4 - 1}}$$

2.172 The radius of curvature of the drop is R_1 at the upper end of the drop and R_2 at the lower end. Then the pressure inside the drop is $p_0 + \dfrac{2\alpha}{R_1}$ at the top end and $p_0 + \dfrac{2\alpha}{R_2}$ at the bottom end. Hence

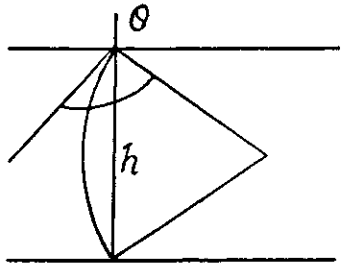
$$p_0 + \frac{2\alpha}{R_1} = p_0 + \frac{2\alpha}{R_2} + \rho g h \quad \text{or} \quad \frac{2\alpha(R_2 - R_1)}{R_1 R_2} = \rho g h$$

To a first approximation $R_1 \approx R_2 \approx \dfrac{h}{2}$ so $R_2 - R_1 \approx \dfrac{1}{8}\rho g h^3/\alpha. \approx 0{\cdot}20$ mm

if $\qquad h = 2{\cdot}3$ mm, $\alpha = 73$ mN/m

2.173 We must first calculate the pressure difference inside the film from that outside. This is

$$p = \alpha\left(\frac{1}{r_1} + \frac{1}{r_2}\right).$$



Here $2\,r_1\,|\cos\theta| = h$ and $r_2 \sim -R$ the radius of the tablet and can be neglected. Thus the total force exerted by mercury drop on the upper glass plate is

$$\frac{2\pi R^2 \alpha\,|\cos\theta|}{h} \text{ typically}$$

We should put h/n for h because the tablet is compressed n times. Then since Hg is nearly, incompressible, $\pi R^2 h$ = constants so $R \to R\sqrt{n}$. Thus,

$$\text{total force} = \frac{2\pi R^2 \alpha\,|\cos\theta|}{h}\,n^2$$

Part of the force is needed to keep the Hg in the shape of a table rather than in the shape of infinitely thin sheet. This part can be calculated being putting $n = 1$ above. Thus

$$mg + \frac{2\pi R^2 \alpha\,|\cos\theta|}{h} = \frac{2\pi R^2 \alpha\,|\cos\theta|}{h}\,n^2$$

or
$$m = \frac{2\pi R^2 \alpha\,|\cos\theta|}{hg}(n^2 - 1) = 0{\cdot}7\text{ kg}$$

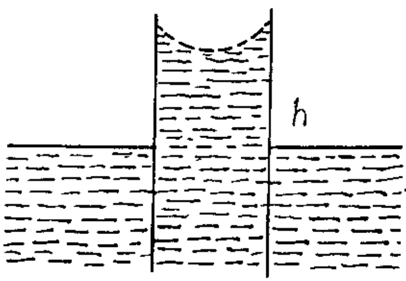
2.174 The pressure inside the film is less than that outside by an amount $\alpha\left(\frac{1}{r_1} + \frac{1}{r_2}\right)$ where r_1 and r_2 are the principal radii of curvature of the meniscus. One of these is small being given by $h = 2\,r_1\cos\theta$ while the other is large and will be ignored. Then $F \approx \frac{2A\cos 0}{h}\alpha$ where A = area of the water film between the plates.

Now $A = \frac{m}{\rho h}$ so $F = \frac{2m\alpha}{\rho h^2}$ when θ (the angle of contact) = 0

2.175 This is analogous to the previous problem except that : $A = \pi R^2$

So
$$F = \frac{2\pi R^2 \alpha}{h} = 0{\cdot}6\text{ kN}$$

2.176 The energy of the liquid between the plates is



$$E = l\,d\,h\,\rho\,g\,\frac{h}{2} - 2\,\alpha\,l\,h = \frac{1}{2}\rho\,g\,l\,d\,h^2 - 2\,\alpha\,l\,h$$

$$= \frac{1}{2}\rho\,g\,l\,d\left(h - \frac{2\alpha}{\rho g d}\right)^2 - \frac{2\alpha^2 l}{\rho g d}$$

This energy is minimum when, $h = \frac{2\alpha}{\rho g d}$ and

the minimum potential energy is then $E_{min} = -\frac{2\alpha^2 l}{\rho g d}$

The force of attraction between the plates can be obtained from this as

$$F = \frac{-\partial E_{min}}{\partial d} = -\frac{2\alpha^2 l}{\rho g d^2} \text{ (minus sign means the force is attractive.)}$$

Thus
$$F = -\frac{\alpha l h}{d} = 13\text{ N}$$

2.177 Suppose the radius of the bubble is x at some instant. Then the pressure inside is $p_0 + \frac{4\alpha}{x}$. The flow through the capillary is by Poiseuille's equation,

$$Q = \frac{\pi r^4}{8\eta l}\frac{4\alpha}{x} = -4\pi^2\frac{dx}{dt}$$

Integrating $\frac{\pi r^4 \alpha}{2\eta l}t = \pi(R^4 - x^4)$ where we have used the fact that $t = 0$ where $x = R$.

This gives $t = \frac{2\eta l R^4}{\alpha r^4}$ as the life time of the bubble corresponding to $x = 0$

2.178 If the liquid rises to a height h, the energy of the liquid column becomes

$$E = \rho g \pi r^2 h \cdot \frac{h}{2} - 2\pi r h \alpha = \frac{1}{2}\rho g \pi\left(r h - 2\frac{\alpha}{\rho g}\right)^2 - \frac{2\pi\alpha^2}{\rho g}$$

This is minimum when $rh = \frac{2\alpha}{\rho g}$ and that is relevant height to which water must rise.

At this point, $$E_{min} = -\frac{2\pi\alpha^2}{\rho g}$$

Since $E = 0$ in the absence of surface tension a heat $Q = \frac{2\pi\alpha^2}{\rho g}$ must have been liberated.

2.179 (a) The free energy per unit area being α,

$$F = \pi\alpha d^2 = 3\ \mu J$$

(b) $F = 2\pi\alpha d^2$ because the soap bubble has two surfaces. Substitution gives $F = 10\ \mu J$

2.180 When two mercury drops each of diameter d merge, the resulting drop has diameter d_1

where $$\frac{\pi}{6}d_1^3 = \frac{\pi}{6}d^3 \times 2 \quad \text{or,} \quad d_1 = 2^{1/3}d$$

The increase in free energy is

$$\Delta F = \pi 2^{2/3} d^2\alpha - 2\pi d^2\alpha = 2\pi d^2\alpha(2^{-1/3} - 1) = -1{\cdot}43\ \mu J$$

2.181 Work must be done to stretch the soap film and compress the air inside. The former is simply $2\alpha \times 4\pi R^2 = 8\pi R^2\alpha$, there being two sides of the film. To get the latter we note that the compression is isothermal and work done is

$$-\int_{V_i = V_0}^{V_f = V} p\,dV \quad \text{where} \quad V_0 p_0 = \left(p_0 + \frac{4\alpha}{R}\right)\cdot V,\ V = \frac{4\pi}{3}R^3$$

or $$V_0 = \frac{pV}{p_0},\ p = p_0 + \frac{4\alpha}{R}$$

and minus sign is needed becaue we are calculating work done on the system. Thus since pV remains constants, the work done is

$$pV\ln\frac{V_0}{V} = pV\ln\frac{p}{p_0}$$

So $$A' = 8\pi R^2\alpha + pV\ln\frac{p}{p_0}$$

2.182 When heat is given to a soap bubble the temperature of the air inside rises and the bubble expands but unless the bubble bursts, the amount of air inside does not change. Further we shall neglect the variation of the surface tension with temperature. Then from the gas equations

$$\left(p_0 + \frac{4\alpha}{r}\right)\frac{4\pi}{3}r^3 = \nu R T, \ \nu = \text{Constant}$$

Differentiating

$$\left(p_0 + \frac{8\alpha}{3r}\right)4\pi r^2 \, dr = \nu R \, d T$$

or
$$dV = 4\pi r^2 \, dr = \frac{\nu R dT}{p_0 + \dfrac{8\alpha}{3r}}$$

Now from the first law

$$\bar{d} \, Q = \nu C dT = \nu C_V dT + \frac{\nu R \, dT}{p_0 + \dfrac{8\alpha}{3r}} \cdot \left(p_0 + \frac{4\alpha}{r}\right)$$

or
$$C = C_V + R \frac{p_0 + \dfrac{4\alpha}{r}}{p_0 \dfrac{8\alpha}{3r}}$$

using
$$C_p = C_V + R \, , \ C = C_p + \frac{\dfrac{1}{2}R}{1 + \dfrac{3 p_0 r}{8\alpha}}$$

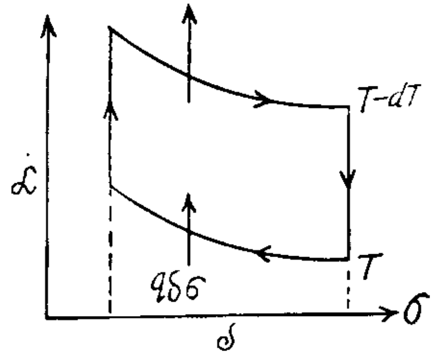
2.183 Consider an infinitesimal Carnot cycle with isotherms at $T - dT$ and T. Let A be the work done during the cycle. Then

$$A = [\alpha(T - dT) - \alpha(T)]\,\delta\sigma = -\frac{d\alpha}{dT} dT \, \delta\sigma$$

Where $\delta\sigma$ is the change in the area of film (we are considering only one surface).

Then $\eta = \dfrac{A}{Q_1} = \dfrac{dT}{T}$ by Carnot therom.

or $$\frac{-\dfrac{d\alpha}{dT} dT \, \delta\sigma}{q\,\delta\sigma} = \frac{dT}{T} \quad \text{or} \quad q = -T\frac{d\alpha}{dT}$$



2.184 As before we can calculate the heat required. It, is taking into account two sides of the soap film

$$\delta q = -T\frac{d\alpha}{dT}\delta\sigma \times 2$$

Thus
$$\Delta S = \frac{\delta q}{T} = -2\frac{d\alpha}{dT}\delta\sigma$$

Now $\Delta F = 2\alpha\,\delta\sigma$ so, $\Delta U = \Delta F + T\Delta S = 2\left(\alpha - T\dfrac{d\alpha}{dT}\right)\delta\sigma$