Vector Algebra

Vector and Its Related Concepts

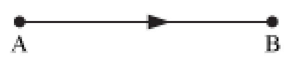
Vector

- The quantity that involves only magnitude (a value) is called a scalar quantity.
  Example: Length, mass, time, distance, etc.

- The quantity that involves both magnitude and direction is called a vector.
  Example: Acceleration, momentum, force, etc.

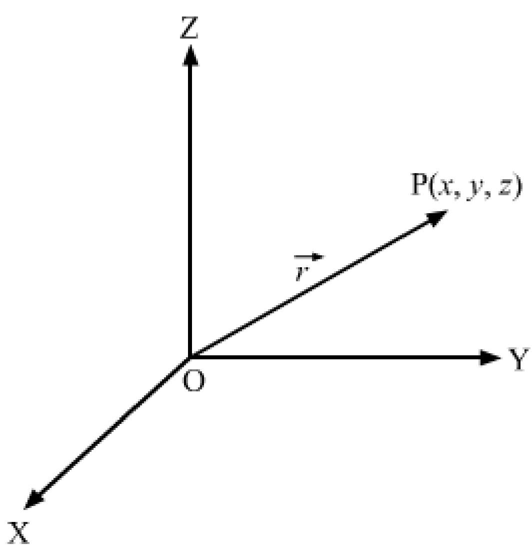
- Vector is represented as a directed line segment (line segment whose direction is given by means of an arrowhead).

- In the following figure, line segment AB is directed towards B.



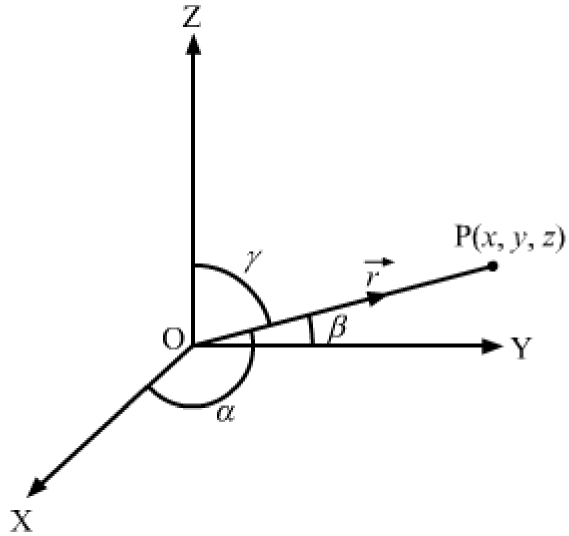
  Hence, the vector representing directed line segment AB is $\overrightarrow{AB}$ or simply $\vec{a}$. Here, the arrow indicates the direction of AB. In $\overrightarrow{AB}$, A is called the initial point and B is called the terminal point.

- The position vector of a point P in space having coordinates (*x, y, z*) with respect to origin O (0, 0, 0) is given by $\overrightarrow{OP}$ or $\vec{r}$.



- Here, the magnitude of $\vec{r}$ i.e., $|\vec{r}|$ is given by $\sqrt{x^2+y^2+z^2}$.

- If a position vector $\vec{r}$ of point P (x, y, z) makes angles α, β, and γ with the positive directions of x–axis, y-axis and z-axis respectively, then these angles are called direction angles.



- The cosine values of direction angles are called direction cosines of $\vec{r}$. This means that direction cosines (d.c.s.) of $\vec{r}$ are $\cos\alpha$, $\cos\beta$, and $\cos\gamma$. We may write the d.c.s of $\vec{r}$ as l, m, n where $l = \cos\alpha$, $m = \cos\beta$ and $n = \cos\gamma$.

- The direction ratios of $\vec{r}$ will be lr, mr, and nr. We may write the direction ratios (d.r.s.) of $\vec{r}$ as a, b, c, where $a = lr$, $b = mr$ and $c = nr$.

- If l, m, n are the d.c.s. of a position vector $\vec{r}$, then
$l^2 + m^2 + n^2 = 1$

Types of Vectors

- A vector whose initial and terminal points coincide is called a zero vector or a null vector.

- It is represented as $\vec{0}$.

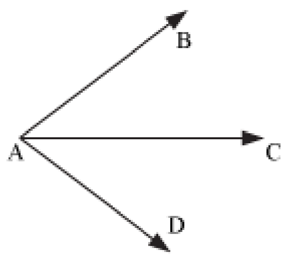
- A zero vector cannot be assigned in a definite direction since its magnitude is zero or it may be regarded as having any direction.

- The vector $\overrightarrow{PP}$, $\overrightarrow{AA}$, etc. represents a zero vector.

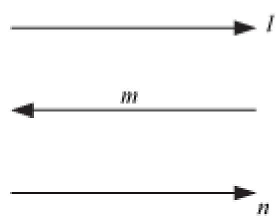
- A vector whose magnitude is unity or 1 unit is called a unit vector.

- A unit vector in the direction of a position vector $\vec{r}$ is given as $\hat{r}$.

- Two or more vectors having the same initial point are called co-initial vectors.
- In the following figure, vectors $\overrightarrow{AB}, \overrightarrow{AC}$ and $\overrightarrow{AD}$ are called initial vectors as each vector has the same initial point i.e., A.



- Two or more vectors are said to be collinear if they are parallel to the same line irrespective of their magnitudes and directions.
- In the following figure, $\vec{l}, \vec{m}$ and $\vec{n}$ are collinear vectors.



- Two vectors are said to be equal if they have the same magnitude and direction regardless of the positions of their initial points.
- For two equal vectors $\vec{a}$ and $\vec{b}$, we write $\vec{a} = \vec{b}$
- A vector whose magnitude is the same as that of a given vector but whose direction is opposite to that of the given vector is called the negative of the given vector.
- The negative vector of $\overrightarrow{PQ}$ is $\overrightarrow{QP}$ and it is written as $\overrightarrow{PQ} = -\overrightarrow{QP}$.

Solved Examples

Example 1

Find the direction cosines and direction ratios of the position vector of point P(8, −4, 1).

Solution:

Let O be the origin. The position vector of point P(8, −4, 1) with respect to origin will be $\overrightarrow{OP}$.

$\therefore r = \sqrt{x^2 + y^2 + z^2} = \sqrt{(8)^2 + (-4)^2 + (1)^2} = 9$

The direction cosines of $\overrightarrow{OP}$ are

$l, m, n = \frac{x}{r}, \frac{y}{r}, \frac{z}{r} = \frac{8}{9}, \frac{-4}{9}, \frac{1}{9}$

The direction ratios of $\vec{r}$ are

$lr, mr, nr = \frac{8}{9} \times 9, -\frac{4}{9} \times 9, \frac{1}{9} \times 9 = 8, -4, 1$

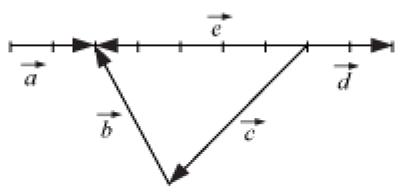
Example 2

In the following figure, which of the vectors are

(i) Collinear

(ii) Equal

(iii) Co-initial



Solution:

(i) Collinear vectors: $\vec{a}, \vec{e}$ and $\vec{d}$

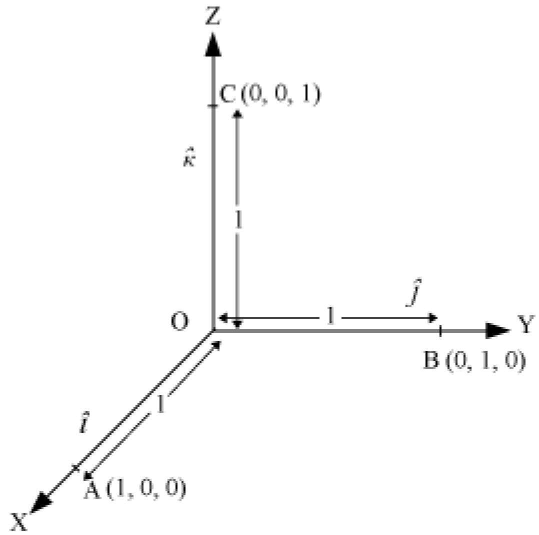
(ii) Equal vectors: $\vec{a}$ and $\vec{d}$

(iii) Co-initial vectors, $\vec{c}$, $\vec{d}$ and $\vec{e}$.

Component Form of a Vector

Key Concept0073

- In a space, the unit vectors along the *x*-axis, *y*-axis and *z*-axis are denoted by $\hat{i}$, $\hat{j}$, and $\hat{k}$ respectively.



It can be noted that $\left|\hat{i}\right| = \left|\hat{j}\right| = \left|\hat{k}\right| = 1$

- The position vector $(\vec{r})$ of any point (*x, y, z*) in a space is given by $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

- The scalar components of $\vec{r}$ are *x, y* and *z*.

- The vector components of $\vec{r}$ are $x\hat{i}, y\hat{j}$ and $z\hat{k}$, which are along the *x, y* and *z*-axes respectively.

- If the position vector $(\vec{r})$ of a point is $x\hat{i} + y\hat{j} + z\hat{k}$, then $|\vec{r}| = \sqrt{x^2 + y^2 + z^2}$

- Two vectors $\vec{a}$ and $\vec{b}$ given by $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ are equal if $a_1 = b_1$, $a_2 = b_2$, and $a_3 = b_3$.

Solved Examples

Example 1

Find the value of *x* if $5|\vec{a}| = 3|\vec{b}|$, where $\vec{a} = 7\hat{i} + (x-2)\hat{j} + 4\hat{k}$ and $\vec{b} = 9\hat{i} - 12\hat{j} + (x+2)\hat{k}$.

Solution:

We have

$\vec{a} = 7\hat{i} + (x-2)\hat{j} + 4\hat{k}$

$\vec{b} = 9\hat{i} - 12\hat{j} + (x+2)\hat{k}$

$\therefore |\vec{a}| = \sqrt{(7)^2 + (x-2)^2 + (4)^2} = \sqrt{x^2 - 4x + 69}, \quad |\vec{b}| = \sqrt{(9)^2 + (-12)^2 + (x+2)^2} = \sqrt{x^2 + 4x + 229}$

It is given that

$5|\vec{a}| = 3|\vec{b}|$

$\Rightarrow 25|\vec{a}|^2 = 9|\vec{b}|^2$

$\Rightarrow 25(x^2 - 4x + 69) = 9(x^2 + 4x + 229)$

$\Rightarrow 25x^2 - 100x + 1725 = 9x^2 + 36x + 2061$

$\Rightarrow 16x^2 - 136x - 336 = 0$

$\Rightarrow 2x^2 - 17x - 42 = 0$

$\Rightarrow 2x^2 + 4x - 21x - 42 = 0$

$\Rightarrow 2x(x + 2) - 21(x + 2) = 0$

$\Rightarrow (x + 2)(2x - 21) = 0$

$\Rightarrow (x + 2) = 0$ or $(2x - 21) = 0$

$\Rightarrow x = -2$ or $x = \frac{21}{2}$

Thus, the required value of x is -2 or $\frac{21}{2}$.

Example 2

Find the values of x, y and z if $\vec{a} = \vec{b}$, where $\vec{a} = (x+y)\hat{i} + (7-2x)\hat{j} + (z-2)\hat{k}$ and $\vec{b} = (2y-1)\hat{i} + (2-y)\hat{j} + 2\hat{k}$.

Solution:

It is given that

$\vec{a} = (x+y)\hat{i} + (7-2x)\hat{j} + (z-2)\hat{k}$

$\vec{b} = (2y-1)\hat{i} + (2-y)\hat{j} + 2\hat{k}$

$\vec{a} = \vec{b}$

$\Rightarrow x + y = 2y - 1$, $7 - 2x = 2 - y$ and $z - 2 = 2$

$\Rightarrow x - y + 1 = 0$ … (1)

$2x - y - 5 = 0$ … (2)

$z = 4$ … (3)

On subtracting (2) from (1), we obtain

$-x + 6 = 0$

$\Rightarrow x = 6$

On substituting $x = 6$ in equation (1), we obtain

$6 - y + 1 = 0$

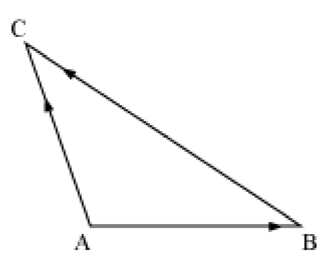
$\Rightarrow y = 7$

Hence, $x = 6$, $y = 7$, and $z = 4$.

Addition and Difference of vectors

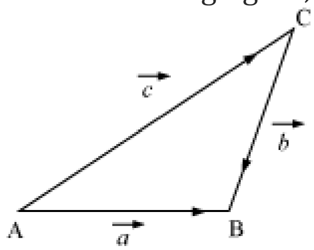
Triangle Law and Parallelogram Law of Vector Addition

- Let points A, B, C form a triangle. If one person goes from A to B (represented by vector $\overrightarrow{AB}$) and B to C (represented by $\overrightarrow{BC}$), then the net displacement of the person from A to C ($\overrightarrow{AC}$) is given by $\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC}$.
This is known as the triangle law of vector addition.



- To add two vectors $\vec{a}$ and $\vec{b}$, they are positioned in such a manner that the initial point of one coincides with the terminal point of the other.

- In the following figure, the initial point of $\vec{b}$ and the final point of $\vec{c}$ coincide at *C*.

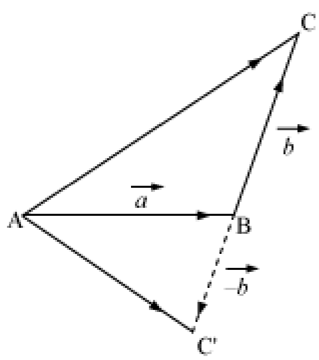


Hence, we write: $\vec{c}+\vec{b}=\vec{a}$

- In the following figure,

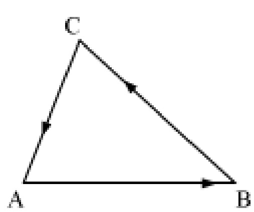
$$\overrightarrow{AC}=\overrightarrow{AB}+\overrightarrow{BC}=\vec{a}+\vec{b}$$
$$\overrightarrow{AC}'=\overrightarrow{AB}+\overrightarrow{BC}'=\vec{a}+(-\vec{b})=\vec{a}-\vec{b}$$

Here, the vector $\overrightarrow{AC}'$ is said to represent the difference between vectors $\vec{a}$ and $\vec{b}$.

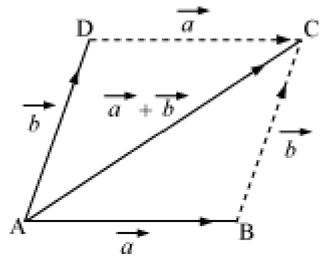


- When the sides of a triangle are taken in order, then their resultant vector is $\vec{0}$ because the initial and terminal points coincide.

- In Δ*ABC*, if $\overrightarrow{AB}, \overrightarrow{BC},$ and $\overrightarrow{CA}$ are in the same order, then $\overrightarrow{AB}+\overrightarrow{BC}+\overrightarrow{CA}=\vec{0}$



- If two vectors $\vec{a}$ and $\vec{b}$ represent the adjacent sides of a parallelogram in magnitude and direction, then their sum i.e., $\vec{a}+\vec{b}$ represents the magnitude and direction of the vector through their common point. This is known as the parallelogram law of vector addition.



Addition and Difference of Vectors

If $\overrightarrow{a} = a_1\hat{\text{i}} + a_2\hat{\text{j}} + a_3\widehat{\text{k}}$ and $\overrightarrow{b} = b_1\hat{\text{i}} + b_2\hat{\text{j}} + b_3\widehat{\text{k}}$, then

$$\overrightarrow{a} + \overrightarrow{b} = (a_1 + b_1)\hat{\text{i}} + (a_2 + b_2)\hat{\text{j}} + (a_3 + b_3)\widehat{\text{k}}$$

$$\overrightarrow{a} - \overrightarrow{b} = (a_1 - b_1)\hat{\text{i}} + (a_2 - b_2)\hat{\text{j}} + (a_3 - b_3)\widehat{\text{k}}$$

Here, $\vec{a}+\vec{b}$ and $\vec{a}-\vec{b}$ are called the sum and difference of vectors $\vec{a}$ and $\vec{b}$ respectively.

- Properties of vector addition

- Commutative Law: For any two vectors $\vec{a}$ and $\vec{b}$,
  $\vec{a}+\vec{b}=\vec{b}+\vec{a}$

- Associative Law: For any three vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$
  $(\vec{a}+\vec{b})+\vec{c}=\vec{a}+(\vec{b}+\vec{c})$

- The existence of additive identity: For any vector $\vec{a}$, we find $-\vec{a}$ (negative of vector $\vec{a}$) such that
  $\vec{a}+(-\vec{a})=0$
  Here, $(-\vec{a})$ is called the additive inverse of $\vec{a}$.

Solved Examples

Example 1

If $\vec{a}=9\hat{i}-3\hat{j}-3\hat{k}$ and $\vec{b}=-\hat{i}+7\hat{j}+2\hat{k},$ then find the value of $\left\|\vec{a}-\vec{b}\right|-2\left|\vec{a}+\vec{b}\right\|$.

Solution:

We have, $\vec{a} = 9\hat{i} - 3\hat{j} - 3\hat{k}$ and $\vec{b} = -\hat{i} + 7\hat{j} + 2\hat{k}$.

$\therefore \vec{a} + \vec{b} = [9 + (-1)]\hat{i} + [(-3) + 7]\hat{j} + [(-3) + (2)]\hat{k}$

$= 8\hat{i} + 4\hat{j} - \hat{k}$

And, $\vec{a} - \vec{b} = [9 - (-1)]\hat{i} + [(-3) - 7]\hat{i} + [(-3) - 2]\hat{k}$

$= 10\hat{i} - 10\hat{j} - 5\hat{k}$

Hence, $\left|\vec{a} + \vec{b}\right| = \sqrt{(8)^2 + (4)^2 + (-1)^2} = 9$

And, $\left|\vec{a} - \vec{b}\right| = \sqrt{(10)^2 + (-10)^2 + (-5)^2} = 15$

Hence, $\left|\left|\vec{a} - \vec{b}\right| - 2\left|\vec{a} + \vec{b}\right|\right| = |15 - 2 \times 9|$

$= |15 - 18|$

$= |-3|$

$= 3$

Example 2

Find $x + 5y$ if $\left|\vec{a} + \vec{b}\right| = 6\sqrt{10}$ and $\vec{a} = x\hat{i} + 3x\hat{j} + \sqrt{xy}\hat{k}$ and $\vec{b} = 9y\hat{i} + 13y\hat{j} + \sqrt{xy}\hat{k}$ such that both x and y are positive integers.

Solution:

We have:

$\vec{a} = x\hat{i} + 3x\hat{j} + \sqrt{xy}\hat{k}$

and $\vec{b} = 9y\hat{i} + 13y\hat{j} + \sqrt{xy}\hat{k}$

Hence, $\vec{a} + \vec{b} = (x + 9y)\hat{i} + (3x + 13y)\hat{k} + 2\sqrt{xy}\hat{k}$

$\therefore \left|\vec{a} + \vec{b}\right| = \sqrt{(x + 9y)^2 + (3x + 13y)^2 + \left(2\sqrt{xy}\right)^2}$

$= \sqrt{(x^2 + 81y^2 + 18xy) + (9x^2 + 169y^2 + 78xy) + 4xy}$

$= \sqrt{10(x^2 + 25y^2 + 10xy)}$

$= \sqrt{10}\sqrt{(x + 5y)^2}$

$= \sqrt{10}(x + 5y)$

It is given that

$\left|\vec{a} + \vec{b}\right| = 6\sqrt{10}$

$\Rightarrow \sqrt{10}(x + 5y) = 6\sqrt{10}$

$\Rightarrow x + 5y = 6$

`Multiplication of a Vector with a Scalar

Key Concepts:

- If λ is any scalar and $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ is any vector, then the multiplication of scalar λ with this vector $\vec{a}$, denoted by λ $\vec{a}$, is given by:
  $\lambda\vec{a} = \lambda\left(a_1\hat{i} + a_2\hat{j} + a_3\hat{k}\right) = \left(\lambda a_1\right)\hat{i} + \left(\lambda a_2\right)\hat{j} + \left(\lambda a_3\right)\hat{k}$
- For example: $5\left(2\hat{i} - \hat{j} + 4\hat{k}\right)$ = (5 × 2) $\hat{i} - 5\hat{j} + (5\times4)\hat{k}$ = $10\hat{i} - 5\hat{j} + 20\hat{k}$
- If $\vec{a}$ is any vector and λ is any scalar, then $|\lambda\vec{a}| = |\lambda||\vec{a}|$.
- The unit vector in the direction of vector $\vec{a}$ is denoted by $\hat{a}$ and it is given by: $\hat{a} = \frac{1}{|\hat{a}|}\vec{a}$
- If $\vec{a} = 9\hat{i} - 8\hat{j} - 12\hat{k}$, then $\hat{a} = \frac{1}{|\vec{a}|}\vec{a}$
  $= \frac{1}{\sqrt{(9)^2 + (-8)^2 + (-12)^2}}\left(9\hat{i} - 8\hat{j} - 12\hat{k}\right)$
  $= \frac{1}{17}\left(9\hat{i} - 8\hat{j} - 12\hat{k}\right)$
  $= \frac{9}{17}\hat{i} - \frac{8}{17}\hat{j} - \frac{12}{17}\hat{k}$
- Two vectors $\vec{a}$ and $\vec{b}$ are said to be collinear vectors, if there exists a scalar λ such that $\vec{b} = \lambda\vec{a}$
- In this case, if $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$, then $\vec{a}$ and $\vec{b}$ are collinear, provided $\frac{b_1}{a_1} = \frac{b_2}{a_2} = \frac{b_3}{a_3} = \lambda$
- If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$, then a_1, a_2, a_3 are called direction ratios of $\vec{a}$.
- $\frac{a_1}{|\vec{a}|}, \frac{a_2}{|\vec{a}|}, \frac{a_3}{|\vec{a}|}$ are called direction cosines of $\vec{a}$.

- Some properties of multiplication of a scalar with a vector:
  If $\vec{a}$ and $\vec{b}$ are any two vectors and *k* and *m* are any scalars, then
- $(k+m)\vec{a} = k\vec{a} + m\vec{a}$
- $k\left(\vec{a}+\vec{b}\right) = k\vec{a} + k\vec{b}$
- $k\left(m\vec{a}\right) = \left(km\right)\vec{a}$

Solved Examples:

Example 1:

If $\vec{a} = 5\hat{i} - 3\hat{j} + \hat{k}$ and $\vec{b} = -2\hat{i} + \hat{j} - \hat{k}$, then find the value of λ, such that

$$\lambda\left|\frac{\left(2\vec{a}+3\vec{b}\right)}{\left|\vec{a}+\vec{b}\right|}\right| + \mu\left|2\sqrt{35}\hat{a} + \sqrt{6}\hat{b}\right| = 4\sqrt{2} + 3\sqrt{10}$$

Solution:

$$\vec{a} = 5\hat{i} - 3\hat{j} + \hat{k}$$

$$\vec{b} = -2\hat{i} + \hat{j} - \hat{k}$$

$$\therefore \left|\vec{a}\right| = \sqrt{(5)^2 + (-3)^2 + (1)^2} = \sqrt{35}$$
$$\left|\vec{b}\right| = \sqrt{(-2)^2 + (1)^2 + (-1)^2} = \sqrt{6}$$

Now,

$$2\vec{a} + 3\vec{b} = 2\left(5\hat{i} - 3\hat{j} + \hat{k}\right) + 3\left(-2\hat{i} + \hat{j} - \hat{k}\right) = (10-6)\hat{i} + (-6+3)\hat{j} + (2-3)\hat{k} = 4\hat{i} - 3\hat{j} - \hat{k}$$
$$\vec{a} + \vec{b} = \left(5\hat{i} - 3\hat{j} + \hat{k}\right) + \left(-2\hat{i} + \hat{j} - \hat{k}\right) = (5-2)\hat{i} + (-3+1)\hat{j} + (1-1)\hat{k} = 3\hat{i} - 2\hat{j}$$
$$\therefore \left|\vec{a} + \vec{b}\right| = \sqrt{(3)^2 + (-2)^2} = \sqrt{13}$$

Therefore,

$$\left|\frac{2\vec{a}+3\vec{b}}{|\vec{a}+\vec{b}|}\right| = \left|\frac{4\hat{i}-3\hat{j}-\hat{k}}{\sqrt{13}}\right| = \left|\frac{1}{\sqrt{13}}\right|\left|4\hat{i}-3\hat{j}-\hat{k}\right| = \frac{1}{\sqrt{13}}\sqrt{(4)^2+(-3)^2+(-1)^2}$$

$$= \frac{1}{\sqrt{13}}\sqrt{26} = \sqrt{2}$$

Then,

$$2\sqrt{35}\hat{a}+\sqrt{6}\hat{b} = 2\sqrt{35}\times\frac{1}{|\vec{a}|}\vec{a}+\sqrt{6}\times\frac{1}{|\vec{b}|}\vec{b} = 2\sqrt{35}\times\frac{1}{\sqrt{35}}\left(5\hat{i}-3\hat{j}+\hat{k}\right)+\sqrt{6}\times\frac{1}{\sqrt{6}}.\left(-2\hat{i}+\hat{j}-\hat{k}\right)$$

$$= 2\left(5\hat{i}-3\hat{j}+\hat{k}\right)+\left(-2\hat{i}+\hat{j}-\hat{k}\right)$$

$$= (10-2)\hat{i}+(-6+1)\hat{j}+(2-1)\hat{k}$$

$$= 8\hat{i}-5\hat{j}+\hat{k}$$

$$\therefore \left|2\sqrt{35}\hat{a}+\sqrt{6}\hat{b}\right| = \sqrt{(8)^2+(-5)^2+(1)^2} = \sqrt{90} = 3\sqrt{10}$$

It is given that,

$$\lambda\left|\frac{2\vec{a}+3\vec{b}}{|\vec{a}+\vec{b}|}\right|+\mu\left|2\sqrt{35}\hat{a}+\sqrt{6}\hat{b}\right| = 4\sqrt{2}+3\sqrt{10}$$

$$\Rightarrow \lambda\sqrt{2}+\mu\times3\sqrt{10} = 4\sqrt{2}+3\sqrt{10}$$

Comparing the co-efficients of $\sqrt{2}$ and $\sqrt{10}$, we obtain $\lambda = 4$ and $3\mu = 3$

Thus, $\lambda = 4$ and $\mu = 1$

Example 2:

If for a vector $\vec{a}$, $\hat{a} = \frac{1}{\sqrt{2}}\hat{i}-\frac{1}{\sqrt{3}}\hat{j}+x\hat{k}$ and $|\vec{a}| = 12$, then find $\vec{a}$.

Solution:

Since $\hat{a} = \frac{1}{\sqrt{2}}\hat{i}-\frac{1}{\sqrt{3}}\hat{j}+x\hat{k}$ is a unit vector, we must have $|\vec{a}| = 1$

$$|\vec{a}| = 1$$

$$\Rightarrow \sqrt{\left(\frac{1}{\sqrt{2}}\right)^2 + \left(-\frac{1}{\sqrt{3}}\right)^2 + x^2} = 1$$

$$\Rightarrow \sqrt{\frac{1}{2} + \frac{1}{3} + x^2} = 1$$

$$\Rightarrow \frac{5}{6} + x^2 = 1$$

$$\Rightarrow x^2 = \frac{1}{6}$$

$$\Rightarrow x = \pm\frac{1}{\sqrt{6}}$$

$$\therefore \hat{a} = \frac{1}{\sqrt{2}}\hat{i} - \frac{1}{\sqrt{3}}\hat{j} \pm \frac{1}{\sqrt{6}}\hat{k}$$

We know that,

$$\hat{a} = \frac{1}{|\vec{a}|}\vec{a}$$

$$\therefore \vec{a} = |\vec{a}|\hat{a}$$

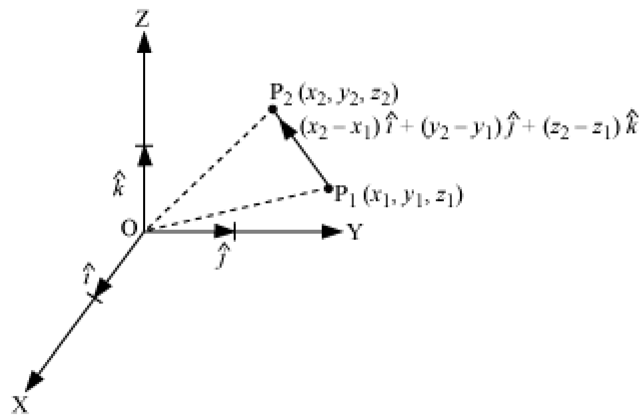
$$\Rightarrow \vec{a} = 12\left(\frac{1}{\sqrt{2}}\hat{i} - \frac{1}{\sqrt{3}}\hat{j} \pm \frac{1}{\sqrt{6}}\hat{k}\right) = 6\sqrt{2}\hat{i} - 4\sqrt{3}\hat{j} \pm 2\sqrt{6}\hat{k}$$

Vector Joining Two Points and Section Formula

Vector Joining Two Points

- The vector joining two points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$, represented as $\overrightarrow{P_1P_2}$, is calculated as

$$\overrightarrow{P_1P_2} = (x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j} + (z_2 - z_1)\hat{k}$$



- The magnitude of $\overrightarrow{P_1P_2}$ is given by $\left|\overrightarrow{P_1P_2}\right| = \sqrt{\left(x_2 - x_1\right)^2 + \left(y_2 - y_1\right)^2 + \left(z_2 - z_1\right)^2}$

Section Formula

- If point *R* (position vector $\vec{r}$) lies on the vector $\overrightarrow{PQ}$ joining two points *P* (position vector $\vec{a}$) and *Q* (position vector $\vec{b}$) such that *R* divides $\overrightarrow{PQ}$ in the ratio *m: n* $\left[\text{i.e.} \frac{\overrightarrow{PR}}{\overrightarrow{RQ}} = \frac{m}{n}\right]$

- Internally, then $\vec{r} = \frac{m\vec{b} + n\vec{a}}{m+n}$

- Externally, then $\vec{r} = \frac{m\vec{b} - n\vec{a}}{m-n}$

- If the position vectors of points *P*, *Q* and *R* are $\vec{a}, \vec{b}$ and $\vec{r}$ respectively such that *R* is the mid-point of $\overrightarrow{PQ}$, then $\vec{r} = \frac{\vec{a} + \vec{b}}{2}$.

Solved Examples

Example 1

Using the concept of vectors, find the area of ΔABC formed by the vertices

$A\left(-2\hat{i} - 4\hat{j} + 2\hat{k}\right), B\left(-3\hat{i} - 3\hat{j} + 4\hat{k}\right)$ and $C\left(2\hat{j} + 3\hat{k}\right)$.

Solution:

We have

$$\overrightarrow{AB} = \left[-3-(-2)\right]\hat{i} + \left[(-3)-(-4)\right]\hat{j} + (4-2)\hat{k}$$

$$= -\hat{i} + \hat{j} + 2\hat{k}$$

$$\overrightarrow{BC} = \left[0-(-3)\right]\hat{i} + \left[2-(-3)\right]\hat{j} + (3-4)\hat{k}$$
$$= 3\hat{i} + 5\hat{j} - \hat{k}$$
$$\overrightarrow{CA} = (-2-0)\hat{i} + (-4-2)\hat{j} + (2-3)\hat{k}$$
$$= -2\hat{i} - 6\hat{j} - \hat{k}$$

$$\therefore \left|\overrightarrow{AB}\right| = \sqrt{(-1)^2 + (1)^2 + (2)^2} = \sqrt{6}$$
$$\left|\overrightarrow{BC}\right| = \sqrt{(3)^2 + (5)^2 + (-1)^2} = \sqrt{35}$$
$$\left|\overrightarrow{CA}\right| = \sqrt{(-2)^2 + (-6)^2 + (-1)^2} = \sqrt{41}$$

We know that, $\left(\sqrt{41}\right)^2 = \left(\sqrt{35}\right)^2 + \left(\sqrt{6}\right)^2$

$$\Rightarrow \left|\overrightarrow{CA}\right|^2 = \left|\overrightarrow{BC}\right|^2 + \left|\overrightarrow{AB}\right|^2$$

Hence, ΔABC is right-angled at B.

Thus, area (ΔABC) $= \frac{1}{2}\text{BC} \cdot \text{AB}$

$$= \frac{1}{2}\sqrt{35} \cdot \sqrt{6}$$
$$= \frac{1}{2}\sqrt{210} \text{ sq.units}$$

Example 2

If *P* is the mid-point of the line segment joining the points $A\left(3\hat{i} - 6\hat{j} + 5\hat{k}\right)$ and $B\left(-7\hat{i} + \hat{k}\right)$, *Q* is the mid-point of the line segment joining the points $C\left(-i - 2\hat{j} + 3\hat{k}\right)$ and $D\left(\hat{i} + 5\hat{k}\right)$, and *R* is a point on $\overrightarrow{PQ}$ such that it divides *PQ* externally in the ratio 2:3, then find the position vector of point *R*.

Solution:

It is given that *P* is the mid-point of the line segment joining the points $A\left(3\hat{i}-6\hat{j}+5\hat{k}\right)$ and $B\left(-7\hat{i}+\hat{k}\right)$.

∴ Position vector of $P(\vec{p})$ is given by

$$\vec{p}=\frac{\left(3\hat{i}-6\hat{j}+5\hat{k}\right)+\left(-7\hat{i}+\hat{k}\right)}{2}$$
$$=\frac{-4\hat{i}-6\hat{j}+6\hat{k}}{2}$$
$$=-2\hat{i}-3\hat{j}+3\hat{k}$$

It is also given that *Q* is the mid-point of the line segment joining the points $C\left(-i-2\hat{j}+3\hat{k}\right)$ and $D\left(\hat{i}+5\hat{k}\right)$.

∴ Position vector of $Q(\vec{q})=\frac{\left(-i-2\hat{j}+3\hat{k}\right)+\left(\hat{i}+5\hat{k}\right)}{2}$

$$=\frac{-2\hat{j}+8\hat{k}}{2}$$
$$=-\hat{j}+4\hat{k}$$

Now, *R* divides line segment *PQ* externally in the ratio 2:3.

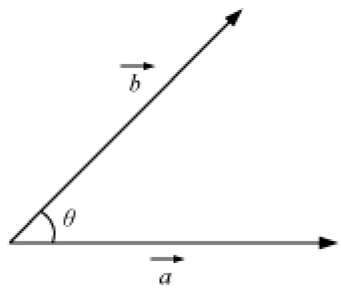
∴ Position vector $\vec{r}$ of point *R* is given by

$$\vec{r}=\frac{2\vec{q}-3\vec{p}}{2-3}$$
$$=\frac{2\left(-\hat{j}+4\hat{k}\right)-3\left(-2\hat{i}-3\hat{j}+3\hat{k}\right)}{2-3}$$
$$=\frac{-\widehat{2j}+8\hat{k}+6\hat{i}+9\hat{j}-9\hat{k}}{-1}$$
$$=-6\hat{i}-7\hat{j}+\hat{k}$$

Scalar (or Dot) Product of Vectors and Projection of a Vector on a Line

Scalar (or Dot) Product of Vectors

- The scalar (or dot) product of two non-zero vectors $\vec{a}$ and $\vec{b}$ (denoted by $\vec{a}\cdot\vec{b}$) is given by the formula $\vec{a}\cdot\vec{b}=|\vec{a}||\vec{b}|\cos\theta$, where θ is the angle between $\vec{a}$ and $\vec{b}$, $0 \le \theta \le \pi$.



- If $\vec{a}=a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$ and $\vec{b}=b_1\hat{i}+b_2\hat{j}+b_3\hat{k}$, then $\vec{a}\cdot\vec{b}=a_1b_1+a_2b_2+a_3b_3$.
- Some observations on scalar product of $\vec{a}$ and $\vec{b}$:
- $\vec{a}\cdot\vec{b}$ is a real number
- The angle between $\vec{a}$ and $\vec{b}$ is given by

$$\theta=\cos^{-1}\left(\frac{\vec{a}\,.\,\vec{b}}{\left|\vec{a}\right|\left|\vec{b}\right|}\right) \text{ or } \theta=\cos^{-1}\left(\frac{a_1b_1+a_2b_2+a_3b_3}{\sqrt{{a_1}^2+{a_2}^2+{a_3}^2}\sqrt{{b_1}^2+{b_2}^2+{b_3}^2}}\right)$$

$$where,\ \vec{a}=a\hat{i}+a\hat{j}+a\hat{k},\ \vec{b}=b\hat{i}+b\hat{j}+b\hat{k}$$

- If $\vec{a}\cdot\vec{b}=0$, then $\vec{a}\perp\vec{b}$
- If the angle between $\vec{a}$ and $\vec{b}$ is 0, then $\vec{a}\cdot\vec{b}=|\vec{a}||\vec{b}|$
- If the angle between $\vec{a}$ and $\vec{b}$ is π, then $\vec{a}\cdot\vec{b}=-|\vec{a}||\vec{b}|$
- $\hat{i}\cdot\hat{i}=\hat{j}\cdot\hat{j}=\hat{k}\cdot\hat{k}=1,\ \hat{i}\cdot\hat{j}=\hat{j}\cdot\hat{k}=\hat{k}\cdot\hat{i}=0$
- Some important properties of scalar product:

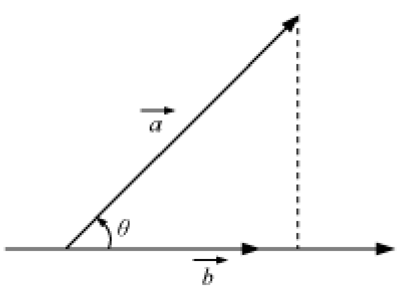
- Commutativity: If $\vec{a}$ and $\vec{b}$ are two vectors, then $\vec{a}\cdot\vec{b}=\vec{b}\cdot\vec{a}$
- Distributivity: If $\vec{a}$, $\vec{b}$ and $\vec{c}$ are any three vectors, then $\vec{a}\cdot\left(\vec{b}+\vec{c}\right)=\vec{a}\cdot\vec{b}+\vec{a}\cdot\vec{c}$
- If $\vec{a}$ and $\vec{b}$ are any two vectors and λ be any scalar, then $(\lambda\vec{a})\cdot\vec{b}=\vec{a}\cdot\left(\lambda\vec{b}\right)=\lambda\left(\vec{a}\cdot\vec{b}\right)$
- (Triangle inequality): For any two vectors $\vec{a}$ and $\vec{b}$, $\left|\vec{a}+\vec{b}\right|\le|\vec{a}|+\left|\vec{b}\right|$
- (Cauchy-Schwarz inequality): For any two vectors $\vec{a}$ and $\vec{b}$, $\left|\vec{a}\cdot\vec{b}\right|\le|\vec{a}|\left|\vec{b}\right|$

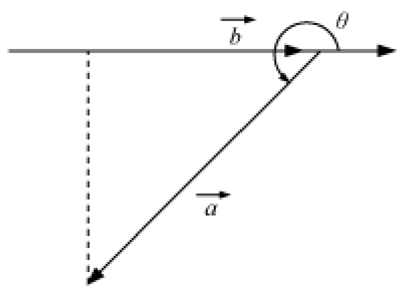
Projection of a Vector on a Line

- If $\hat{b}$ is the unit vector along $\vec{b}$, then the projection of a vector $\vec{a}$ on $\vec{b}$ is given by

$$\vec{a}\cdot\hat{b} \text{ or } \vec{a}\cdot\left(\frac{\vec{b}}{\left|\vec{b}\right|}\right) \text{ or } \frac{1}{\left|\vec{b}\right|}\left(\vec{a}\cdot\vec{b}\right) \text{ or } \frac{|\vec{a}|\left|\vec{b}\right|\cos\theta}{\left|\vec{b}\right|} \text{ or } |\vec{a}|\cos\theta$$
,

where θ is the angle between $\vec{a}$ and $\vec{b}$ measured anti-clockwise, $0 \le \theta < 2\pi$.





- Some observations on projection of vector $\vec{a}$ on $\vec{b}$:
- If $\theta = 0$, then projection of $\vec{a}$ on $\vec{b}$ is $|\vec{a}|\cos 0=|\vec{a}|$
- If $\theta=\frac{\pi}{2}\text{or}\frac{3\pi}{2}$, then projection of $\vec{a}$ on $\vec{b}$ is $|\vec{a}|\cos\frac{\pi}{2}\ \left(\text{or}\,|\vec{a}|\cos\frac{3\pi}{2}\right)=\vec{0}$
- If $\theta = \pi$, then projection of $\vec{a}$ on $\vec{b}$ is $\vec{a}\cos\pi=\vec{a}(-1)=-\vec{a}$

Solved Examples

Example 1

If vectors $\vec{a}=2\hat{i}-(x-2)\hat{j}-\hat{k}$ and $\vec{b}=(x-1)\hat{i}-3\hat{j}+3x\hat{k}$ are perpendicular to each other, then find the value of *x*.

Also, find $\dfrac{\left|\vec{a}+\vec{b}\right|}{\left|\vec{a}\right|+\left|\vec{b}\right|}$.

Solution:

The given vectors are

$$\vec{a}=2\hat{i}-(x-2)\hat{j}-\hat{k}$$
$$\vec{b}=(x-1)\hat{i}-3\hat{j}+3x\hat{k}$$
$$\text{Since } \vec{a}\perp\vec{b},\ \vec{a}\cdot\vec{b}=0,$$
$$\Rightarrow\left[2\hat{i}-(x-2)\hat{j}-\hat{k}\right]\cdot\left[(x-1)\hat{i}-3\hat{j}+3x\hat{k}\right]=0$$
$$\Rightarrow 2(x-1)+3(x-2)-3x=0$$
$$\Rightarrow 2x-8=0$$
$$\Rightarrow x=4$$

$$\vec{a}=2\hat{i}-2\hat{j}-\hat{k}$$
$$\vec{b}=3\hat{i}-3\hat{j}+12\hat{k}$$
$$\therefore \vec{a}+\vec{b}=5\hat{i}-5\hat{j}+11\hat{k}$$
$$\left|\vec{a}\right|=\sqrt{(2)^2+(-2)^2+(-1)^2}=3$$
$$\left|\vec{b}\right|=\sqrt{(3)^2+(-3)^2+(12)^2}=9\sqrt{2}$$
$$\left|\vec{a}+\vec{b}\right|=\sqrt{(5)^2+(-5)^2+(11)^2}=3\sqrt{19}$$
$$\text{Thus, } \frac{\left|\vec{a}+\vec{b}\right|}{\left|\vec{a}\right|+\left|\vec{b}\right|}=\frac{3\sqrt{19}}{3+9\sqrt{2}}=\frac{\sqrt{19}}{1+3\sqrt{2}}$$

Example 2

For two vectors $\vec{a}$ and $\vec{b}$, the product of the projection of $\vec{a}$ on $\vec{b}$ with the projection of $\vec{b}$ on $\vec{a}$ is half of the dot product of $\vec{a}$ and $\vec{b}$. Find the angle between $\vec{a}$ and $\vec{b}$.

Solution:

Let θ be the angle between $\vec{a}$ and $\vec{b}$.

Then, the projection $\vec{a}$ on $\vec{b}$ is $\frac{1}{|\vec{b}|}\left(\vec{a}\cdot\vec{b}\right)$.

The projection of $\vec{b}$ on $\vec{a}$ is $\frac{1}{|\vec{a}|}\left(\vec{a}\cdot\vec{b}\right)$.

According to the given condition,

$$\left[\frac{1}{|\vec{b}|}\left(\vec{a}\cdot\vec{b}\right)\right]\left[\frac{1}{|\vec{a}|}\left(\vec{a}\cdot\vec{b}\right)\right]=\frac{1}{2}\left(\vec{a}\cdot\vec{b}\right)$$

$$\Rightarrow \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}=\frac{1}{2}$$

$$\Rightarrow \frac{|\vec{a}||\vec{b}|\cos\theta}{|\vec{a}||\vec{b}|}=\frac{1}{2}$$

$$\Rightarrow \cos\theta=\frac{1}{2}$$

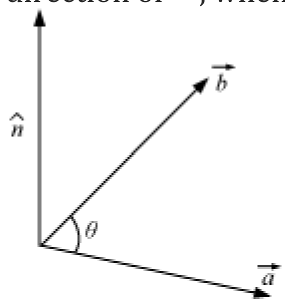
$$\Rightarrow \theta=\cos^{-1}\left(\frac{1}{2}\right)=60^\circ$$

Thus, the angle between the vectors $\vec{a}$ and $\vec{b}$ is 60°.

Vector Product of Vectors

- The vector product (cross product) of two non-zero vectors $\vec{a}$ and $\vec{b}$ is denoted by $\vec{a}\times\vec{b}$ and defined as $\vec{a}\times\vec{b}=|\vec{a}||\vec{b}|\sin\theta\hat{n}$
  where, θ is the angle between $\vec{a}$ and $\vec{b}$, $0 \le \theta \le \pi$ and $\hat{n}$ is a unit vector, which is perpendicular to both $\vec{a}$ and $\vec{b}$ such that $\vec{a}$, $\vec{b}$, and $\hat{n}$ form a right hand system (i.e., the system moves in the

direction of $\hat{n}$, when it is rotated from $\vec{a}$ to $\vec{b}$)



- Some observations of vector product of $\vec{a}$ and $\vec{b}$:

(a) $\vec{a}\times\vec{b}$ is a vector.

(b) If $\vec{a}=\vec{0}$ or $\vec{b}=\vec{0}$, then θ is not defined. In this case, we define $\vec{a}\times\vec{b}$ as 0.

(c) If $\left|\vec{a}\times\vec{b}\right|$ are non-zero vectors such that $\vec{a} \parallel \vec{b}$ or $\vec{a}$ and $\vec{b}$ are collinear, then $\vec{a}\times\vec{b}$ = 0. In particular, $\vec{a}\times\vec{a}=\vec{0}$ and $\vec{a}\times(-\vec{a})=\vec{0}$

(d) If $\theta=\frac{\pi}{2}$, then $\vec{a}\times\vec{b}=|\vec{a}||\vec{b}|\hat{n}$

(e) For mutually perpendicular unit vectors $\hat{i},\hat{j},\text{and }\hat{k}$,

$\hat{i}\times\hat{i}=\hat{j}\times\hat{j}=\hat{k}\times\hat{k}=\vec{0}$

$\hat{i}\times\hat{j}=\hat{k},\hat{j}\times\hat{k}=\hat{i},\hat{k}\times\hat{i}=\hat{j}$

$\hat{j}\times\hat{i}=-\hat{k},\hat{k}\times\hat{j}=-\hat{i},\hat{i}\times\hat{k}=-\hat{j}$

(f) $\vec{a}\times\vec{b}=-\vec{b}\times\vec{a}$

(g) The angle between $\vec{a}$ and $\vec{b}$ in terms of vector product is given as

$$\sin\theta=\frac{\left|\vec{a}\times\vec{b}\right|}{|\vec{a}||\vec{b}|}$$

(h) If $\vec{a}=a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$ and $\vec{b}=b_1\hat{i}+b_2\hat{j}+b_3\hat{k}$, then

$$\vec{a}\times\vec{b}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{vmatrix}$$

(i) If $\vec{a}$ and $\vec{b}$ represent the adjacent sides of a triangle, then its area is given as $\frac{1}{2}\left|\vec{a}\times\vec{b}\right|$.

(j) If $\vec{a}$ and $\vec{b}$ represent the adjacent sides of a parallelogram, then its area is given as $\left|\vec{a}\times\vec{b}\right|$.

- Distributive property of vector product over addition − If $\vec{a}$, $\vec{b}$, and $\vec{c}$ are any three vectors and λ be a scalar, then
  (i) $\vec{a}\times\left(\vec{b}+\vec{c}\right)=\vec{a}\times\vec{b}+\vec{a}\times\vec{c}$
  (ii) $\lambda\left(\vec{a}\times\vec{b}\right)=\left(\lambda\vec{a}\right)\times\vec{b}=\vec{a}\times\left(\lambda\vec{b}\right)$

Solved Examples

Example 1:

For what integral value of *x*, 0 ≤ *x* ≤ 5, the area of the parallelogram whose adjacent sides are determined by the vectors, $\vec{a}=\hat{i}-3\hat{j}-\hat{k}$ and $\vec{b}=7\hat{i}+\left(1-x\right)\hat{j}-x\hat{k}$, is $15\sqrt{2}$ square units? Also, find the angle between $\vec{a}$ and $\vec{b}$.

Solution:

The adjacent sides of the parallelogram are determined by the vectors

$\vec{a}=\hat{i}-3\hat{j}-\hat{k}$ and

$\vec{b}=7\hat{i}+\left(1-x\right)\hat{j}-x\hat{k}$

∴ Area of the parallelogram $\left|\vec{a}\times\vec{b}\right|$

Now,

$$\vec{a}\times\vec{b}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & -3 & -1\\ 7 & 1-x & -x\end{vmatrix}=\left(3x+1-x\right)\hat{i}+\left(-7+x\right)\hat{j}+\left(1-x+21\right)\hat{k}$$

$$=\left(2x-1\right)\hat{i}+\left(-7+x\right)\hat{j}+\left(22-x\right)\hat{k}$$

Therefore, area of the parallelogram is

$$\left|\vec{a}\times\vec{b}\right| = \sqrt{(2x+1)^2+(-7+x)^2+(22-x)^2}$$

$$\left|\vec{a}\times\vec{b}\right| = \sqrt{4x^2+4x+1+49+x^2-14x+484+x^2-44x}$$

$$\Rightarrow 15\sqrt{2} = \sqrt{6x^2-54x+534}$$

$$\Rightarrow 450 = 6x^2-54x+534$$

$$\Rightarrow 6x^2-54x+84=0$$

$$\Rightarrow x^2-9x+14=0$$

$$\Rightarrow (x-2)(x-7)=0$$

$$\Rightarrow x=2, x=7$$

Since $0 \le x \le 5$,

$x = 2$

Now,

$$\vec{a} = \hat{i}-3\hat{j}-\hat{k}$$

And, $\vec{b} = 7\hat{i}-\hat{j}-2\hat{k}$

$$\therefore \left|\vec{a}\right| = \sqrt{(1)^2+(-3)^2+(-1)^2} = \sqrt{11}$$

$$\left|\vec{b}\right| = \sqrt{(7)^2+(-1)^2+(-2)^2} = 3\sqrt{6}$$

$$\therefore \sin\theta = \frac{\left|\vec{a}\times\vec{b}\right|}{\left|\vec{a}\right|\left|\vec{b}\right|} = \frac{15\sqrt{2}}{\sqrt{11}\times 3\sqrt{6}} = \frac{5}{\sqrt{33}}$$

$$\Rightarrow \theta = \sin^{-1}\left(\frac{5}{\sqrt{33}}\right)$$

Therefore, angle between $\vec{a}$ and $\vec{b}$ is $\sin^{-1}\left(\frac{5}{\sqrt{33}}\right)$.

Example 2:

For two vectors $\vec{a}$ and $\vec{b}$, if $\left|\vec{a}\right| = \sqrt{29}, \left|\vec{b}\right| = \sqrt{13}$, and $\left|\vec{a}\cdot\vec{b}\right| = 16$, then find $\left|\vec{b}\times 2\vec{a}\right|$.

Solution:

$$\left|\vec{a}\cdot\vec{b}\right| = |\vec{a}|\left|\vec{b}\right||\cos\theta|$$

$$\Rightarrow |\cos\theta| = \frac{\left|\vec{a}\cdot\vec{b}\right|}{|\vec{a}|\left|\vec{b}\right|} = \frac{16}{\sqrt{29}\times\sqrt{13}} = \frac{16}{\sqrt{377}}$$

$$\therefore |\sin\theta| = \sqrt{1-\cos^2\theta} = \sqrt{1-\left(\frac{16}{\sqrt{377}}\right)^2}$$

$$= \sqrt{1-\frac{256}{377}}$$

$$= \sqrt{\frac{121}{377}}$$

$$= \frac{11}{\sqrt{377}}$$

Now,

$$\left|\vec{b}\times 2\vec{a}\right| = 2\left|\vec{b}\times\vec{a}\right| = 2\left|\vec{b}\right||\vec{a}|\cdot|\sin\theta|$$

$$= 2\times\sqrt{13}\times\sqrt{29}\times\frac{11}{\sqrt{377}} = 22$$