CHAPTER

2

Complex Numbers

INTRODUCTION

If a, b are natural numbers such that $a > b$, then the equation $x + a = b$ is not solvable in N, the set of natural numbers, i.e. there is no natural number satisfying the equation $x + a = b$. So, the set of natural numbers is extended to form the set I of integers in which every equation of the form $x + a = b$ such that $a, b \in N$ is solvable.

But equations of the form $xa = b$, where $a, b \in I, a \neq 0$ are not solvable in I also. Therefore the set I of integers is extended to obtain the set Q of all rational numbers in which every equation of the form $xa = b, a \neq 0, a, b \in I$ is uniquely solvable.

The equations of the form $x^2 = 2$, $x^2 = 3$ etc. are not solvable in Q because there is no rational number whose square is 2. Such numbers are known as irrational numbers. The set Q of all rational numbers is extended to obtain the set R which included both rational and irrational numbers. This set is known as the set of real numbers.

The equations of the form $x^2 + 1 = 0$, $x^2 + 4 = 0$, etc. are not solvable in R, i.e. there is no real number whose square is a negative real number. Euler was the first mathematician to introduce the symbol i (iota) for the square root of -1 with the property $i^2 = -1$. He also called this symbol as the imaginary number.

DEFINITION OF COMPLEX NUMBERS

A number of the form $x + iy$, where $x, y \in R$ and $i = \sqrt{-1}$ is called a complex number and 'i' is called iota. A complex number is usually denoted by z and the set of complex numbers is denoted by C.

$$C = \{x + iy; x \in R, y \in R, i = \sqrt{-1}\}$$

For example, $5 + 3i$, $-1 + i$, $0 + 4i$, $4 + 0i$, etc. are complex numbers.

Here 'x' is called the real part of z and 'y' is known as the imaginary part of z. The real part of z is denoted by $\text{Re}(z)$ and the imaginary part by $\text{Im}(z)$. If $z = 3 - 4i$, then $\text{Re}(z) = 3$ and $\text{Im}(z) = -4$.

Note that the sign '+' does not indicate addition as normally understood, nor does the symbol i denote a number. These things are parts of the scheme used to express numbers of a new class and they signify the pair of real numbers (x, y) to form a single complex number.

A complex number z is purely real if its imaginary part is zero, i.e. $\text{Im}(z) = 0$ and purely imaginary if its real part is zero, i.e. $\text{Re}(z) = 0$.

Note:

1. *For any positive real number a, we have* $\sqrt{-a} = \sqrt{-1 \times a} = \sqrt{-1}\,\sqrt{a} = i\sqrt{a}$.
2. *The property* $\sqrt{a}\sqrt{b} = \sqrt{ab}$ *is valid only if at least one of a and b is non-negative. If a and b are both negative, then* $\sqrt{a}\sqrt{b} = -\sqrt{|a||b|}$.
3. *Inequality in complex numbers are never talked. If* $a + ib > c + id$ *has to be meaningful then* $b = d = 0$. *Equalities however in complex numbers are meaningful. Two complex numbers* $z_1 = a_1 + ib_1$ *and* $z_2 = a_2 + ib_2$ *are equal if* $a_1 = a_2$ *and* $b_1 = b_2$, *i.e.* $Re(z_1) = Re(z_2)$ *and* $Im(z_1) = Im(z_2)$.
4. *In real number system* $a^2 + b^2 = 0 \Rightarrow a = 0 = b$. *But if* z_1 *and* z_2 *are complex numbers then* $z_1^2 + z_2^2 = 0$ *does not imply* $z_1 = z_2 = 0$, *e.g.* $z_1 = 1 + i$ *and* $z_2 = 1 - i$. *However, if the product of two complex numbers is zero then at least one of them must be zero, same as in case of real numbers.*

Integral Power of Iota (i)

Since $i = \sqrt{-1}$, we have $i^2 = -1$, $i^3 = -i$ and $i^4 = 1$.
To find the value of i^n $(n > 4)$, first divide n by 4.
Let q be the quotient and r be the remainder,
i.e. $n = 4q + r$ where $0 \le r \le 3$
$i^n = i^{4q+r} = (i^4)^q\,(i)^r = (1)^q(i)^r = i^r$
In general, we have the following results: $i^{4n} = 1$, $i^{4n+1} = i$, $i^{4n+2} = -1$, $i^{4n+3} = -i$, where n is any integer.

Algebraic Operations with Complex Numbers

Let two complex number be $z_1 = a + ib$ and $z_2 = c + id$

Addition $(z_1 + z_2)$**:**

$$(a + ib) + (c + id) = (a + c) + i(b + d)$$

Subtraction $(z_1 - z_2)$**:**

$$(a + ib) - (c + id) = (a - c) + i(b - d)$$

Multiplication $(z_1 z_2)$**:**

$$(a + ib)(c + id) = (ac - bd) + i(ad + bc)$$

Division (z_1/z_2)**:**

$$\frac{a+ib}{c+id} = \frac{(a+ib)}{(c+id)}\frac{(c-id)}{(c-id)} \quad \text{(Rationalization)}$$

(where at least one of c and d is non-zero)

$$\Rightarrow \quad \frac{a+ib}{c+id} = \frac{(ac+bd)}{c^2+d^2} + \frac{i(bc-ad)}{c^2+d^2}$$

Properties of Algebraic Operations on Complex Numbers

Let z_1, z_2 and z_3 are any three complex numbers. Then their algebraic operations satisfy the following properties:

1. Addition of complex numbers satisfies the commutative and associative properties, i.e.
$$z_1 + z_2 = z_2 + z_1 \text{ and } (z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$$
2. Multiplication of complex numbers satisfies the commutative and associative properties, i.e.
$$z_1z_2 = z_2z_1 \text{ and } (z_1z_2)z_3 = z_1(z_2z_3)$$
3. Multiplication of complex numbers is distributive over addition, i.e.
$$z_1(z_2 + z_3) = z_1z_2 + z_1z_3 \text{ and } (z_2 + z_3)z_1 = z_2z_1 + z_3z_1$$

Example 2.1 **Evaluate:**

a. i^{135}

b. $(-\sqrt{-1})^{4n+3}, n \in N$

c. $\sqrt{-25}+3\sqrt{-4}+2\sqrt{-9}$

Sol. a. 135 leaves remainder as 3 when it is divided by 4

$\therefore \quad i^{135} = i^3 = -i$

b. $(-\sqrt{-1})^{4n+3} = (-i)^{4n+3}$
$= (-i)^{4n}(-i)^3$
$= \{(-i)^4\}^n (-i)^3$
$= 1 \times (-i)^3 = i$

c. $\sqrt{-25}+3\sqrt{-4}+2\sqrt{-9} = 5i + 6i + 6i = 17i$

Example 2.2 **Find the value of $i^n + i^{n+1} + i^{n+2} + i^{n+3}$ for all $n \in N$.**

Sol. $i^n + i^{n+1} + i^{n+2} + i^{n+3} = i^n[1 + i + i^2 + i^3]$
$= i^n[1 + i - 1 - i]$
$= i^n(0) = 0$

Example 2.3 **Find the value of $1 + i^2 + i^4 + i^6 + \cdots + i^{2n}$**

Sol. $S = 1 + i^2 + i^4 + \cdots + i^{2n} = 1 - 1 + 1 - 1 + \cdots + (-1)^n$
Obviously it depends on n. Hence it cannot be determined unless n is known.

Example 2.4 **Show that the polynomial $x^{4p} + x^{4q+1} + x^{4r+2} + x^{4s+3}$ is divisible by $x^3 + x^2 + x + 1$ where $p, q, r, s \in N$.**

Sol. Let $f(x) = x^{4p} + x^{4q+1} + x^{4r+2} + x^{4s+3}$. Now,
$x^3 + x^2 + x + 1 = (x^2 + 1)(x + 1)$
$= (x + i)(x - i)(x + 1)$

$f(i) = i^{4p} + i^{4q+1} + i^{4r+2} + i^{4s+3}$
$= 1 + i^1 + i^2 + i^3$
$= 1 + i - 1 - i = 0$

$f(-i) = (-i)^{4p} + (-i)^{4q+1} + (-i)^{4r+2} + (-i)^{4s+3}$
$= 1 + (-i)^1 + (-i)^2 + (-i)^3$
$= 1 - i - 1 + i = 0$

$f(-1) = (-1)^{4p} + (-1)^{4q+1} + (-1)^{4r+2} + (-1)^{4s+3} = 0$

Thus by division theorem $f(x)$ is divisible by $x^3 + x^2 + x + 1$.

Example 2.5 **If $z \neq 0$ is a complex number, then prove that $\text{Re}(z) = 0 \Rightarrow \text{Im}(z^2) = 0$.**

Sol. If $z \neq 0$, let $z = x + iy$. Then,
$z^2 = x^2 - y^2 + i(2xy)$
$\therefore \quad \text{Re}(z) = 0 \Rightarrow x = 0$
$\Rightarrow \quad \text{Im}(z^2) = 2xy = 0$
Thus,
$\text{Re}(z) = 0 \Rightarrow \text{Im}(z^2) = 0$

Example 2.6 **Express each one of the following in the standard form $a + ib$.**

a. $\dfrac{5+4i}{4+5i}$ b. $\dfrac{(1+i)^2}{3-i}$ c. $\dfrac{1}{1-\cos\theta + 2i\sin\theta}$

Sol. a.
$$\frac{5+4i}{4+5i} = \frac{5+4i}{4+5i} \times \frac{4-5i}{4-5i} = \frac{(20+20)+i(16-25)}{16-25i^2} = \frac{40-9i}{41} = \frac{40}{41} - \frac{9}{41}i$$

b.
$$\frac{(1+i)^2}{3-i} = \frac{1+2i+i^2}{3-i} = \frac{2i}{3-i} = \frac{2i}{3-i}\,\frac{3+i}{3+i} = \frac{6i+2i^2}{9-i^2} = \frac{-2+6i}{10} = -\frac{1}{5} + \frac{3}{5}i$$

c.
$$\frac{1}{1-\cos\theta + 2i\sin\theta}$$
$$= \frac{1}{1-\cos\theta+2i\sin\theta}\,\frac{1-\cos\theta-2i\sin\theta}{1-\cos\theta-2i\sin\theta}$$
$$= \frac{1-\cos\theta-2i\sin\theta}{(1-\cos\theta)^2+4\sin^2\theta}$$
$$= \frac{1-\cos\theta-2i\sin\theta}{1-2\cos\theta+\cos^2\theta+4\sin^2\theta}$$
$$= \frac{1-\cos\theta-2i\sin\theta}{2-2\cos\theta+3\sin^2\theta}$$
$$= \left(\frac{1-\cos\theta}{2-2\cos\theta+3\sin^2\theta}\right) + i\left(\frac{-2\sin\theta}{2-2\cos\theta+3\sin^2\theta}\right)$$

Example 2.7 **Solve**
(i) $ix^2 - 3x - 2i = 0$, (ii) $2(1+i)x^2 - 4(2-i)x - 5 - 3i = 0$

Sol. i. $ix^2 - 3x - 2i = 0$,
$\Rightarrow \quad x^2 + 3ix - 2 = 0$ (dividing by i)
$$\Rightarrow \quad x = \frac{-3i \pm \sqrt{-9+4.1.2}}{2.1} = \frac{-3i \pm \sqrt{-1}}{2} = \frac{-3i \pm i}{2}$$
$\Rightarrow \quad x = -i$, or $x = -2i$

(ii) $2(1+i)x^2 - 4(2-i)x - 5 - 3i = 0$
$$\Rightarrow \quad x = \frac{4(2-i) \pm \sqrt{16(2-i)^2 + 4.2.(1+i).(5+3i)}}{2.(1+i)}$$
$$= \frac{4(2-i) \pm \sqrt{16(4-4i-1) + 8(5+3i+5i-3)}}{4(1+i)}$$

$$= \frac{4(2-i) \pm \sqrt{48-64i+16+64i}}{4(1+i)}$$

$$= \frac{4(2-i) \pm 8}{4(1+i)}$$

$$= \frac{(2-i) \pm 2}{1+i}$$

$$= \frac{2-i+2}{1+i}; \frac{2-i-2}{1+i}$$

$$= \frac{(4-i)(1-i)}{2}; \frac{-i(1-i)}{2}$$

$$= \frac{3-5i}{2}; -\left(\frac{1+i}{2}\right)$$

Example 2.8 **If $z = 4 + i\sqrt{7}$, then find the value of $z^3 - 4z^2 - 9z + 91$.**

Sol. $z = 4 + i\sqrt{7}$

$\Rightarrow$ $z - 4 = i\sqrt{7}$

$\Rightarrow$ $z^2 - 8z + 16 = -7$

$\Rightarrow$ $z^2 - 8z + 23 = 0$

Now dividing $z^3 - 4z^2 - 9z + 91$ by $z^2 - 8z + 23$

We get $z^3 - 4z^2 - 9z + 91 = (z^2 - 8z + 23)(z + 4) - 1 = -1$

Equality of Complex Numbers

Two complex numbers $z_1 = a_1 + ib_1$ and $z_2 = a_2 + ib_2$ are equal if $a_1 = a_2$ and $b_1 = b_2$, i.e., $\text{Re}(z_1) = \text{Re}(z_2)$ and $\text{Im}(z_1) = \text{Im}(z_2)$. Thus, $z_1 = z_2 \Leftrightarrow \text{Re}(z_1) = \text{Re}(z_2)$ and $\text{Im}(z_1) = \text{Im}(z_2)$.

Example 2.9 **If $(a + b) - i(3a + 2b) = 5 + 2i$, then find a and b.**

Sol. We have,

$(a + b) - i(3a + 2b) = 5 + 2i$

$\Rightarrow$ $a + b = 5$ and $-(3a + 2b) = 2$

$\Rightarrow$ $a = -12, b = 17$

Example 2.10 **Given that $x, y \in R$, solve:**

$$\frac{x}{1+2i} + \frac{y}{3+2i} = \frac{5+6i}{8i-1}$$

Sol. $\frac{x}{1+2i} + \frac{y}{3+2i} = \frac{5+6i}{8i-1}$

$\Rightarrow$ $\frac{x(1-2i)}{1-4i^2} + \frac{y(3-2i)}{9-4i^2} = \frac{(5+6i)(8i+1)}{(8i)^2-1^2}$

$\Rightarrow$ $\frac{x-2xi}{5} + \frac{3y-2yi}{13} = \frac{40i+5-48+6i}{-64-1}$

$\Rightarrow$ $\frac{13x-26xi+15y-10yi}{65} = \frac{-43+46i}{-65}$

$\Rightarrow$ $(13x + 15y) - i(26x + 10y) = 43 - 46i$

equating real and imaginary parts,

$13x + 15y = 43$ (1)

$13x + 5y = 23$ (2)

Solving for x and y we get $x = 1$ and $y = 2$

Example 2.11 **Find the ordered pair (x, y) for which $x^2 - y^2 - i(2x + y) = 2i$**

Sol. We have

$x^2 - y^2 = 0$ or $x = \pm y$ (1)

and $-(2x + y) = 2$

or $2x + y = -2$ (2)

Solving $x = y$ with (2) we have $x = y = -\frac{2}{3}$

Solving $x = -y$ with (2) we have $x = -2$ and $y = 2$

Example 2.12 **If $\sqrt{x+iy} = \pm(a + ib)$, then find $\sqrt{-x-iy}$.**

Sol. $\sqrt{x+iy} = \pm(a+bi)$

$\Rightarrow$ $x + iy = a^2 - b^2 + 2iab$

$\Rightarrow$ $x = a^2 - b^2, y = 2ab$

$\therefore$ $\sqrt{-x-iy} = \sqrt{-(a^2-b^2)-2iab}$

$= \sqrt{b^2+(ia)^2-2iab}$

$= \sqrt{(b-ia)^2}$

$= \pm(b - ia)$

Example 2.13 **If sum of square of roots of the equation $x^2 + (p + iq)x + 3i = 0$ is 8 then find the value of p and q, where p and q are real.**

Sol. Let the roots are α, β

We have $\alpha + \beta = -(p + iq)$; $\alpha\beta = 3i$

Given: $\alpha^2 + \beta^2 = 8$

$\Rightarrow$ $(\alpha + \beta)^2 - 2\alpha\beta = 8$

$\Rightarrow$ $(p + iq)^2 - 6i = 8$

$\Rightarrow$ $p^2 - q^2 + i(2pq - 6) = 8$

$\Rightarrow$ $p^2 - q^2 = 8$ and $pq = 3$

$\Rightarrow$ $p = 3$ and $q = 1$ or $p = -3$ and $q = -1$

Example 2.14 **If $z = x + iy, z^{1/3} = a - ib$ and $x/a - y/b = k(a^2 - b^2)$, then find the value of k.**

Sol. $(x + iy)^{1/3} = a - ib$

$\Rightarrow$ $x + iy = (a - ib)^3$

$= (a^3 - 3ab^2) + i(b^3 - 3a^2b)$

$\Rightarrow$ $x = a^3 - 3ab^2, y = b^3 - 3a^2b$

$\Rightarrow$ $\frac{x}{a} = a^2 - 3b^2$ and $\frac{y}{b} = b^2 - 3a^2$

$\Rightarrow$ $\frac{x}{a} - \frac{y}{b} = a^2 - 3b^2 - b^2 + 3a^2 = 4(a^2 - b^2)$

$\therefore$ $k = 4$

Example 2.15 **Let z be a complex number satisfying the equation $z^2 - (3 + i)z + m + 2i = 0$, where $m \in R$. Suppose the equation has a real root. Then find the non-real root.**

Sol. Let α be the real root. Then,

$\alpha^2 - (3+i)\alpha + m + 2i = 0$

$\Rightarrow (\alpha^2 - 3\alpha + m) + i(2-\alpha) = 0$

$\Rightarrow \alpha = 2 \Rightarrow 4 - 6 + m = 0 \Rightarrow m = 2$

Product of the roots is $2(1+i)$ with one root as 2. Hence the non-real root is $1+i$.

Square Root of a Complex Number

Let $a + ib$ be a complex number such that $\sqrt{a+ib} = x + iy$, where x and y are real numbers. Then,

$\sqrt{a+ib} = x + iy$

$\Rightarrow (a+ib) = (x+iy)^2$

$\Rightarrow a + ib = (x^2 - y^2) + 2ixy$

On equating real and imaginary parts, we get

$x^2 - y^2 = a$ (i)

$2xy = b$

Now,

$(x^2+y^2)^2 = (x^2-y^2)^2 + 4x^2y^2$

$\Rightarrow (x^2+y^2)^2 = a^2 + b^2$

$\Rightarrow (x^2+y^2) = \sqrt{a^2+b^2}$ $[\because x^2+y^2 \geq 0]$ (ii)

Solving equations (i) and (ii), we get

$$x^2 = \left(\frac{1}{2}\right)\left[\sqrt{a^2+b^2}+a\right] \text{ and } y^2 = \left(\frac{1}{2}\right)\left[\sqrt{a^2+b^2}-a\right]$$

$$\Rightarrow x = \pm\sqrt{\left(\frac{1}{2}\right)\left[\sqrt{a^2+b^2}+a\right]} \text{ and } y = \pm\sqrt{\left(\frac{1}{2}\right)\left[\sqrt{a^2+b^2}-a\right]}$$

If b is positive, then by the relation $2xy = b$, x and y are of the same sign. Hence,

$$\sqrt{a+ib} = \pm\left\{\sqrt{\frac{1}{2}\left[\sqrt{a^2+b^2}+a\right]} + i\sqrt{\frac{1}{2}\left[\sqrt{a^2+b^2}-a\right]}\right\}$$

If b is negative, then by the relation $2xy = b$, x and y are of different signs. Hence,

$$\sqrt{a+ib} = \pm\left\{\sqrt{\frac{1}{2}\left[\sqrt{a^2+b^2}+a\right]} - i\sqrt{\frac{1}{2}\left[\sqrt{a^2+b^2}-a\right]}\right\}$$

Example 2.16 **Find the square roots of the following:**

a. $7 - 24i$ **b. $5 + 12i$** **c. $-15 - 8i$**

Sol. a. Let $\sqrt{7-24i} = x + iy$. Then,

$\sqrt{7-24i} = x + iy$

$\Rightarrow 7 - 24i = (x+iy)^2$

$\Rightarrow 7 - 24i = (x^2-y^2) + 2ixy$

$\Rightarrow x^2 - y^2 = 7$ (i)

and

$2xy = -24$ (ii)

Now,

$(x^2+y^2)^2 = (x^2-y^2)^2 + 4x^2y^2$

$\Rightarrow (x^2+y^2)^2 = 49 + 576 = 625$

$\Rightarrow x^2 + y^2 = 25$ $[\because x^2+y^2 > 0]$ (iii)

On solving (i) and (iii), we get

$x^2 = 16$ and $y^2 = 9 \Rightarrow x = \pm 4$ and $y = \pm 3$

From (ii), $2xy$ is negative. So, x and y are of opposite signs. Hence, $x = 4$ and $y = -3$ or $x = -4$ and $y = 3$.

Hence, $\sqrt{7-24i} = \pm(4-3i)$.

b. Let $\sqrt{5+12i} = x + iy$. Then,

$\sqrt{5+12i} = x + iy$

$\Rightarrow 5 + 12i = (x+iy)^2$

$\Rightarrow 5 + 12i = (x^2-y^2) + 2ixy$

$\Rightarrow x^2 - y^2 = 5$ (i)

and

$2xy = 12$ (ii)

Now,

$(x^2+y^2)^2 = (x^2-y^2)^2 + 4x^2y^2$

$\Rightarrow (x^2+y^2)^2 = 5^2 + 12^2 = 169$

$\Rightarrow x^2 + y^2 = 13$ $(\because x^2+y^2 > 0)$ (iii)

On solving (i) and (iii), we get

$x^2 = 9$ and $y^2 = 4 \Rightarrow x = \pm 3$ and $y = \pm 2$

From (ii), $2xy$ is positive. So, x and y are of the same sign.

$\Rightarrow x = 3$ and $y = 2$ or $x = -3$ and $y = -2$

Hence, $\sqrt{5+12i} = \pm(3+2i)$.

c. Let $\sqrt{-15-8i} = x + iy$. Then,

$\sqrt{-15-8i} = x + iy$

$\Rightarrow -15 - 8i = (x+iy)^2$

$\Rightarrow -15 - 8i = (x^2-y^2) + 2ixy$

$\Rightarrow -15 = x^2 - y^2$ (i)

and

$2xy = -8$ (ii)

Now,

$(x^2+y^2)^2 = (x^2-y^2)^2 + 4x^2y^2$

$\Rightarrow (x^2+y^2)^2 = (-15)^2 + 64 = 289$

$\Rightarrow x^2 + y^2 = 17$

On solving (i) and (iii), we get

$x^2 = 1$ and $y^2 = 16 \Rightarrow x = \pm 1$ and $y = \pm 4$

From (ii), $2xy$ is negative. So, x and y are of opposite signs. Hence, $x = 1$ and $y = -4$ or $x = -1$ and $y = 4$

Hence, $\sqrt{-15-8i} = \pm(1-4i)$.

Example 2.17 **Find all possible values of $\sqrt{i} + \sqrt{-i}$.**

Sol. $\sqrt{i} + \sqrt{-i} = \sqrt{0+1.i} + \sqrt{0-1.i}$

$$\text{Now } \sqrt{a+i.b} = \pm\left\{\sqrt{\frac{1}{2}\left\{\sqrt{a^2+b^2}+a\right\}} + i\sqrt{\frac{1}{2}\left\{\sqrt{a^2+b^2}-a\right\}}\right\}$$

$$\text{and } \sqrt{a-i.b} = \pm\left\{\sqrt{\frac{1}{2}\left\{\sqrt{a^2+b^2}+a\right\}} - i\sqrt{\frac{1}{2}\left\{\sqrt{a^2+b^2}-a\right\}}\right\}$$

$$\Rightarrow \sqrt{0+1.i} = \pm\left\{\sqrt{\frac{1}{2}\left\{\sqrt{0+1^2}+0\right\}} + i\sqrt{\frac{1}{2}\left\{\sqrt{0+1^2}-0\right\}}\right\}$$

$$= \pm\frac{1}{\sqrt{2}}(1+i)$$

and $\sqrt{0-1.i} = \pm\left\{\sqrt{\frac{1}{2}\left\{\sqrt{0+1^2}+0\right\}} - i\sqrt{\frac{1}{2}\left\{\sqrt{0+1^2}-0\right\}}\right\}$

$$= \pm\frac{1}{\sqrt{2}}(1-i)$$

Now, $\sqrt{i}+\sqrt{-i} = \pm\frac{1}{\sqrt{2}}(1+i)\pm\frac{1}{\sqrt{2}}(1-i)$

or $\quad \sqrt{i}+\sqrt{-i} = \pm\sqrt{2}+0.i$ or $0\pm\sqrt{2}i$

Example 2.18 **Solve for z: $z^2-(3-2i)z=(5i-5)$.**

Sol. $z^2-(3-2i)z=(5i-5)$

$\Rightarrow \quad z^2-(3-2i)z-(5i-5)=0$

$$\Rightarrow \quad z = \frac{(3-2i)\pm\sqrt{(3-2i)^2+4(5i-5)}}{2}$$

$$= \frac{(3-2i)\pm\sqrt{9-4-12i+20i-20}}{2}$$

$$= \frac{(3-2i)\pm\sqrt{8i-15}}{2}$$

Now $\sqrt{-15+8i}$

$$= \pm\left\{\sqrt{\frac{1}{2}\left\{\sqrt{(-15)^2+8^2}+(-15)\right\}} + i\sqrt{\frac{1}{2}\left\{\sqrt{(-15)^2+8^2}-(-15)\right\}}\right\}$$

$$= \pm\left\{\sqrt{\frac{1}{2}\left(\sqrt{289}-15\right)}+\sqrt{\frac{1}{2}\left(\sqrt{289}+15\right)}\right\}$$

$$= \pm(1+i4)$$

$$\Rightarrow \quad z = \frac{3-2i\pm(1+4i)}{2}$$

$\Rightarrow \quad z = (2+i)$ and $(1-3i)$

Concept Application Exercise 2.1

1. Is the following computation correct? If not give the correct computation:
   $\sqrt{(-2)}\sqrt{(-3)} = \sqrt{(-2)(-3)} = \sqrt{6}$
2. Find the value of
   a. $\dfrac{i^{592}+i^{590}+i^{588}+i^{586}+i^{584}}{i^{582}+i^{580}+i^{578}+i^{576}+i^{574}}-1$
   b. $(1+i)^6+(1-i)^6$
3. Find the value of $x^4+9x^3+35x^2-x+4$ for $x=-5+2\sqrt{-4}$.
4. The value of $i^{1+3+5+\cdots+(2n+1)}$ is ——.
5. If one root of the equation $z^2-az+a-1=0$ is $(1+i)$, where a is a complex number, then find the other root.
6. If the number $(1-i)^n/(1+i)^{n-2}$ is real and positive, then n is
   a. any integer b. any even integer
   c. any odd integer d. none of these
7. If $(x+iy).(p+iq)=(x^2+y^2)i$, prove that $x=q$, $y=p$.
8. Simplify:
   $\dfrac{\sqrt{5+12i}+\sqrt{5-12i}}{\sqrt{5+12i}-\sqrt{5-12i}}$
9. Find square root of $9+40i$.

GEOMETRICAL REPRESENTATION OF A COMPLEX NUMBER

A complex number $z=x+iy$ can be represented by a point (x,y) on the plane which is known as the Argand plane. To represent $z=x+iy$ geometrically we take two mutually perpendicular straight lines $X'OX$ and $Y'OY$. Now plot a point whose x and y coordinates are respectively the real and the imaginary parts of z. This $P(x, y)$ represents the complex number $z=x+iy$.

If a complex number is purely real, then its imaginary part is zero. Therefore, a purely real number is represented by a point on x-axis. A purely imaginary complex number is represented by a point on y-axis. That is why x-axis is known as the real axis and y-axis, as the imaginary axis.

Conversely, if $P(x, y)$ is a point in the plane, then the point $P(x, y)$ represents a complex number $z=x+iy$. The complex number $z=x+iy$ is known as the affix of the point P.

The plane in which we represent a complex number geometrically is known as the complex plane, the **Argand plane** or the **Gaussian plane**.

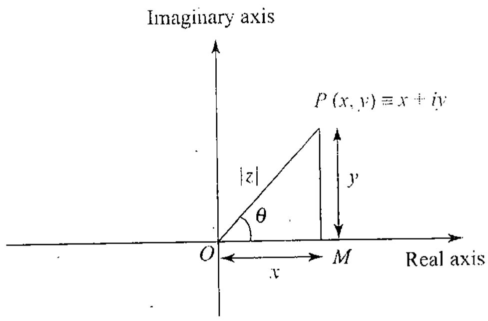


Fig. 2.1

Modulus of Complex Number

The length of the line segment OP is called the modulus of z and is denoted by $|z|$. From Fig. 2.1, we have

$$OP^2 = OM^2 + MP^2$$

$$\Rightarrow \quad OP^2 = x^2+y^2 \Rightarrow OP = \sqrt{x^2+y^2}.$$

Thus,

$$|z| = \sqrt{x^2+y^2} = \sqrt{\{\text{Re}(z)\}^2+\{\text{Im}(z)\}^2}$$

Clearly, $|z|\geq 0$ for all $z\in C$. If $z_1=3-4i$, $z_2=-5+2i$ and $z_3=1+\sqrt{-3}$, then $|z_1|=\sqrt{3^2+(-4)^2}=5$, $|z_2|=\sqrt{(-5)^2+2^2}=\sqrt{29}$ and $|z_3|=|1+i\sqrt{3}|=\sqrt{1^2+(\sqrt{3})^2}=2$.

Remark

In the set C of all complex numbers, the order relation is not defined. As such $z_1 > z_2$ or $z_1 < z_2$ has no meaning but $|z_1| > |z_2|$ or $|z_1| < |z_2|$ has got its meaning since $|z_1|$ and $|z_1|$ are real numbers.
Infinite complex numbers having same $|z|$ lying on circle which has center origin and radius $|z|$

Argument of Complex Number

The angle θ which OP makes with x-axis is called the **argument** or **amplitude** of z and is denoted by arg(z) or amp(z). From Fig. 2.1, we have

$$\tan\theta = \frac{PM}{OM} = \frac{y}{x} = \frac{\text{Im}(z)}{\text{Re}(z)} \Rightarrow \theta = \tan^{-1}\left(\frac{\text{Im}(z)}{\text{Re}(z)}\right)$$

This angle θ has infinitely many values differing by multiples of 2π.

The unique value of θ such that $-\pi < \theta \le \pi$ is called the principal value of the amplitude or **principal argument**. This formula for determining the argument of $z = x + iy$ has severe drawback, because $z_1 = 1 + i\sqrt{3}$ and $z_2 = -1 - i\sqrt{3}$ are two distinct complex numbers represented by two distinct points in the Argand plane but their arguments seem to be $\tan^{-1}\sqrt{3} = \pi/3$ or $4\pi/3$ which is not correct. In fact the argument is the common solution of the simultaneous trigonometric equations

$$\cos\theta = \frac{x}{\sqrt{x^2+y^2}} \text{ and } \sin\theta = \frac{y}{\sqrt{x^2+y^2}}$$

The argument of z depends upon the quadrant in which the point P lies as discussed below:

(i) $z = x + iy$, z lies in first quadrant ($x > 0$ and $y > 0$)
From the figure $\tan\alpha = |y/x|$ and $\theta = \alpha$.
Then $\arg(z) = \tan^{-1}|y/x|$.

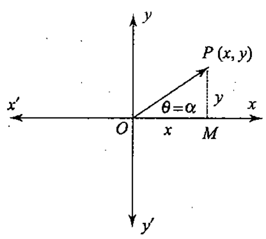


Fig. 2.2

(ii) $z = x + iy$, z lies in second quadrant ($x < 0$ and $y > 0$)
From the figure $\tan\alpha = |y/x|$ and $\theta = \pi - \alpha$.
Then $\arg(z) = \pi - \alpha$, where α is the acute angle given by $\tan^{-1}|y/x|$.

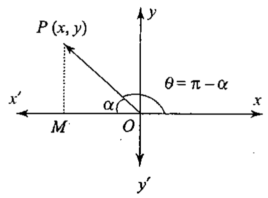


Fig. 2.3

(iii) $z = x + iy$, z lies in third quadrant ($x < 0$ and $y < 0$)
From the figure $\tan\alpha = |y/x|$ and $\theta = -(\pi - \alpha) = -\pi + \alpha$.
Then, $\arg(z) = \alpha - \pi$ where α is the acute angle given by $\tan\alpha = |y/x|$.

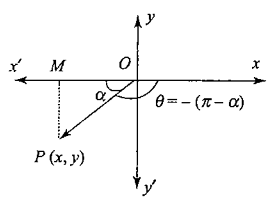


Fig. 2.4

(iv) $z = x + iy$, z lies in fourth quadrant ($x > 0$ and $y < 0$)
From the figure $\tan\alpha = |y/x|$ and $\theta = -\alpha$.
Then $\arg(z) = -\alpha$, where α is the acute angle given by $\tan\alpha = |y/x|$.

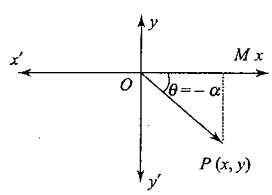


Fig. 2.5

Polar Form of Complex Number

We have,

$$z = x + iy$$

$$= \sqrt{x^2+y^2}\left[\frac{x}{\sqrt{x^2+y^2}} + i\frac{x}{\sqrt{x^2+y^2}}\right] = |z|[\cos\theta + i\sin\theta]$$

where $|z|$ is the modulus of complex number, i.e. the distance of z from origin and θ is the argument or amplitude of the complex number.

Here we should take the principal value of θ. For general values of argument $z = r[\cos(2n\pi + \theta) + i\sin(2n\pi + \theta)]$ (where n is an integer). This is polar form of the complex number.

Euler's Form of Complex Number

$$e^{i\theta} = \cos\theta + i\sin\theta$$

This form makes the study of complex numbers and its properties simple. Any complex number can be expressed as

$z = x + iy$ (Cartesian form)

$= |z|[\cos\theta + i\sin\theta]$ (polar form)

$= |z|e^{i\theta}$

Product of Two Complex Numbers

Let two complex numbers be $z_1 = |z_1|e^{i\theta_1}$ and $z_2 = |z_2|e^{i\theta_2}$. Now,

$$z_1z_2 = |z_1|e^{i\theta_1} \times |z_2|e^{i\theta_2}$$

$$= |z_1||z_2|e^{i(\theta_1+\theta_2)}$$

$$= |z_1||z_2|[\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)]$$

Thus,

$$|z_1z_2| = |z_1||z_2|$$

$$\arg(z_1z_2) = \theta_1 + \theta_2$$

$$= \arg(z_1) + \arg(z_2)$$

Division of Two Complex Numbers

$$\frac{z_1}{z_2} = \frac{|z_1|e^{i\theta_1}}{|z_2|e^{i\theta_2}} = \frac{|z_1|}{|z_2|}e^{i(\theta_1-\theta_2)}$$

$$\Rightarrow \left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$$

and $\arg\left(\frac{z_1}{z_2}\right) = \theta_1 - \theta_2$

$$= \arg(z_1) - \arg(z_2)$$

Logarithm of a complex number is given by

$$\log_e(x+iy) = \log_e(|z|e^{i\theta})$$

$$= \log_e|z| + \log_e e^{i\theta}$$

$$= \log_e|z| + i\theta$$

$$= \log_e\sqrt{(x^2+y^2)} + i\arg(z)$$

$$\therefore \log_e(z) = \log_e|z| + i\arg(z)$$

Example 2.19 **Prove that the triangle formed by the points 1, $\frac{1+i}{\sqrt{2}}$ and i as vertices in the Argand diagram is isosceles.**

Sol. The vertices of the triangle are $A(1, 0)$, $B(1/\sqrt{2}, 1/\sqrt{2})$ and $C(0, 1)$,

$$\therefore AB^2 = 2 - \sqrt{2},\ BC^2 = 2 - \sqrt{2},\ AC^2 = 1 + 1 = 2$$

$$\therefore AB = BC \Rightarrow \text{Triangle is isosceles}$$

Example 2.20 **Write the following complex numbers in polar form:**

a. $-3\sqrt{2} + 3\sqrt{2}i$ **b.** $1 + i$ **c.** $-1 - i$ **d.** $1 - i$ **e.** $\frac{(1+7i)}{(2-i)^2}$

Sol. a. Let $z = -3\sqrt{2} + 3\sqrt{2}\,i$. Then,

$$r = |z| = \sqrt{(-3\sqrt{2})^2 + (3\sqrt{2})^2} = 6$$

Let

$$\tan\alpha = \left|\frac{\text{Im}(z)}{\text{Re}(z)}\right| = 1 \Rightarrow \alpha = \pi/4$$

Since the point representing z lies in the second quadrant, therefore, the argument of z is given by

$$\theta = \pi - \alpha = \pi - \left(\frac{\pi}{4}\right) = \left(\frac{3\pi}{4}\right)$$

So, the polar form of $z = -3\sqrt{2} + 3\sqrt{2}\,i$ is

$$z = r(\cos\theta + i\sin\theta) = 6\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$$

b. Let $z = 1 + i$. Then, $r = |z| = \sqrt{1^2+1^2} = \sqrt{2}$. Let,

$$\tan\alpha = \left|\frac{\text{Im}(z)}{\text{Re}(z)}\right|$$

Then,

$$\tan\alpha = \left|\frac{1}{1}\right| = 1 \Rightarrow \alpha = \frac{\pi}{4}$$

Since the point (1, 1) representing z lies in first quadrant, therefore, the argument of z is given by $\theta = \alpha = \pi/4$. So, the polar form of $z = 1 + i$ is

$$z = r(\cos\theta + i\sin\theta) = \left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$$

c. Let $z = -1 - i$. Then, $r = |z| = \sqrt{(-1)^2 + (-1)^2} = \sqrt{2}$. Let,

$$\tan\alpha = \left|\frac{\text{Im}(z)}{\text{Re}(z)}\right|.$$

Then,

$$\tan\alpha = \left|\frac{-1}{-1}\right| = 1 \Rightarrow \alpha = \frac{\pi}{4}$$

Since the point (−1, −1) representing z lies in the third quadrant, therefore, the argument of z is given by

$$\theta = -(\pi - \alpha) = -\left(\pi - \frac{\pi}{4}\right) = \frac{-3\pi}{4}$$

So, the polar form of $z = -1 - i$ is

$$z = r(\cos\theta + i\sin\theta) = \sqrt{2}\left\{\cos\left(\frac{-3\pi}{4}\right) + i\sin\left(\frac{-3\pi}{4}\right)\right\}$$

d. Let $z = 1 - i$. Then, $|z| = \sqrt{1^2 + (-1)^2} = \sqrt{2}$. Let

$$\tan\alpha = \left|\frac{\text{Im}(z)}{\text{Re}(z)}\right|$$

Then,

$$\tan\alpha = \left|\frac{-1}{1}\right| = 1 \Rightarrow \alpha = \pi/4$$

Since the point (1, −1) lies in the fourth quadrant, therefore the argument of z is given by $\theta = -\alpha = -\pi/4$. So, the polar form of $z = 1 - i$ is

$$r(\cos\theta + i\sin\theta) = \sqrt{2}\left\{\cos\left(\frac{-\pi}{4}\right) + i\sin\left(\frac{-\pi}{4}\right)\right\}$$

$$= \sqrt{2}\left(\cos\frac{\pi}{4} - i\sin\frac{\pi}{4}\right)$$

e. Let $z = (1+7i)/[(2-i)^2]$ Then,

$$z = \frac{1+7i}{4-4i+i^2} = \frac{1+7i}{3-4i} = \left(\frac{1+7i}{3-4i}\right)\left(\frac{3+4i}{3+4i}\right) = \frac{-25+25i}{25} = -1+i$$

$$\therefore \quad r = |z| = \sqrt{(-1)^2 + (1)^2} = \sqrt{2}$$

Let α be the acute angle given by

$$\tan\alpha = \left|\frac{\text{Im}(z)}{\text{Re}(z)}\right| = \left|-\frac{1}{1}\right| = 1$$

Then $\alpha = \pi/4$. Since the point (−1, 1) representing z lies in the second quadrant, therefore $\theta = \arg(z) = \pi - \alpha = \pi - \pi/4 = 3\pi/4$. Hence, z in the polar form is given by

$$z = \sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$$

Example 2.21 **If $z = re^{i\theta}$, then prove that $|e^{iz}| = e^{-r\sin\theta}$.**

Sol. $z = re^{i\theta} = r(\cos\theta + i\sin\theta)$

$$\Rightarrow iz = ir(\cos\theta + i\sin\theta)$$
$$= -r\sin\theta + ir\cos\theta$$
$$\Rightarrow e^{iz} = e^{(-r\sin\theta + ir\cos\theta)}$$
$$= e^{-r\sin\theta} e^{ri\cos\theta}$$
$$\Rightarrow |e^{iz}| = |e^{-r\sin\theta}|\,|e^{r\,i\cos\theta}|$$
$$= e^{-r\sin\theta}|e^{i\alpha}|, \text{ where } \alpha = r\cos\theta$$
$$= e^{-r\sin\theta}[\cos^2\alpha + \sin^2\alpha]^{1/2}$$
$$= e^{-r\sin\theta}$$

Example 2.22 **Prove that**

$$\tan\left(i\log_e\left(\frac{a-ib}{a+ib}\right)\right) = \frac{2ab}{a^2-b^2} \text{ (where } a, b \in R^+)$$

Sol. Let

$$a + ib = re^{i\theta}$$
$$\Rightarrow a - ib = re^{-i\theta}$$
$$\Rightarrow \frac{a-ib}{a+ib} = e^{-i2\theta}$$
$$\Rightarrow \log_e\left(\frac{a-ib}{a+ib}\right) = -i2\theta$$
$$\Rightarrow \tan\left(i\log_e\left(\frac{a-ib}{a+ib}\right)\right) = \tan 2\theta$$
$$= \frac{2\tan\theta}{1-\tan^2\theta}$$
$$= \frac{2b/a}{1-b^2/a^2}$$
$$= \frac{2ab}{a^2-b^2}$$

Example 2.23 **Find the real part of $(1 - i)^{-i}$.**

Sol. Let $z = (1 - i)^{-i}$. Taking log on both sides,

$$\log z = -i\log_e(1-i)$$
$$= -i\log_e\sqrt{2}\left(\cos\frac{\pi}{4} - i\sin\frac{\pi}{4}\right)$$
$$= -i\log_e(\sqrt{2}e^{-i(\pi/4)})$$
$$= -i\left[\frac{1}{2}\log_e 2 + \log_e{}^{-i\pi/4}\right]$$
$$= -i\left[\frac{1}{2}\log_e 2 - \frac{i\pi}{4}\right]$$
$$= -\frac{i}{2}\log_e 2 - \frac{\pi}{4}$$
$$\Rightarrow z = e^{-\pi/4}e^{-i(\log 2)/2}$$
$$\Rightarrow \text{Re}(z) = e^{-\pi/4}\cos\left(\frac{1}{2}\log 2\right)$$

Example 2.24 **Show that $e^{2mi\theta}\left(\dfrac{i\cot\theta+1}{i\cot\theta-1}\right)^m = 1$.**

Sol. Let $\cot^{-1} p = \theta$. Then $\cot\theta = p$. Now,

$$\text{L.H.S.} = e^{2mi\theta}\left(\frac{i\cot\theta+1}{i\cot\theta-1}\right)^m$$
$$= e^{2mi\theta}\left[\frac{i(\cot\theta-i)}{i(\cot\theta+i)}\right]^m$$
$$= e^{2mi\theta}\left(\frac{\cot\theta-i}{\cot\theta+i}\right)^m$$

$$= e^{2m\pi i\theta}\left(\frac{\cos\theta - i\sin\theta}{\cos\theta + i\sin\theta}\right)^m$$

$$= e^{2m\pi i\theta}\left(\frac{e^{-i\theta}}{e^{i\theta}}\right)^m$$

$$= e^{2m\pi i\theta}(e^{-2i\theta})^m$$

$$= e^{2m\pi i\theta}e^{-2m\pi i\theta} = e^0 = 1 = \text{R.H.S.}$$

Conjugate of a Complex Number

For complex number $z = x + iy$, $(x, y) \in R$, its conjugate is defined as $\bar{z} = x - iy$. Clearly, $z = x + iy$ is represented by a point $P(x, y)$ in the Argand plane. Now,

$$z = x + iy \Rightarrow \bar{z} = x - iy = x + i(-y)$$

So, $\bar{z}$ is represented by a point $Q(x, -y)$ in the Argand plane. Clearly, Q is the image of point P on the real axis.

Thus, if a point P represents a complex number z, then its conjugate $\bar{z}$ is represented by the image of P on the real axis.

It is evident from the following figure that $|z| = |\bar{z}|$ and $\arg(\bar{z}) = -\arg(z)$.

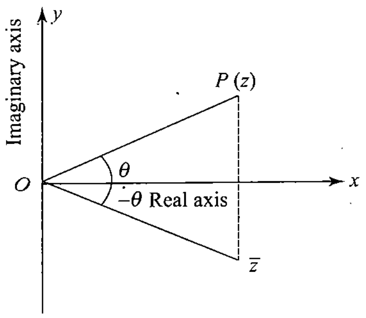


Fig. 2.6

Also, we have $\text{Re}(z) = (z - \bar{z})/2$ and $\text{Im}(z) = (z - \bar{z})/2i$. Thus if $z = |z|e^{i\theta}$, then $\bar{z} = |z|e^{-i\theta}$.

Properties of Conjugate

(i) $\overline{(\bar{z})} = z$

(ii) $z + \bar{z} = 2\text{Re}(z)$

(iii) $z - \bar{z} = 2i\,\text{Im}(z)$

(iv) $z = \bar{z} \Leftrightarrow z$ is purely real

(v) $z + \bar{z} = 0 \Leftrightarrow z$ is purely imaginary

(vi) $z\bar{z} = \{\text{Re}(z)\}^2 + \{\text{Im}(z)\}^2 = |z|^2$

(vii) $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$

(viii) $\overline{z_1 - z_2} = \bar{z}_1 - \bar{z}_2$

(ix) $\overline{z_1 z_2} = \bar{z}_1\bar{z}_2$

From this we can say that $\overline{z^n} = \bar{z}^n$, where $n \in N$.

(x) $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}$, $z_2 \neq 0$

Example 2.25 **Find real values of x and y for which the complex numbers $-3 + ix^2y$ and $x^2 + y + 4i$ are conjugate of each other.**

Sol. Given,

$$-3 + ix^2y = \overline{x^2 + y + 4i}$$

$$\Rightarrow -3 + ix^2y = x^2 + y - 4i$$

$$\Rightarrow -3 = x^2 + y \quad \text{(i)}$$

and

$$x^2y = -4 \quad \text{(ii)}$$

$$\therefore -3 = x^2 - \frac{4}{x^2} \quad [\text{Putting } y = -4/x^2 \text{ from (ii) in (i)}]$$

$$\Rightarrow x^4 + 3x^2 - 4 = 0$$

$$\Rightarrow (x^2 + 4)(x^2 - 1) = 0$$

$$\Rightarrow x^2 - 1 = 0$$

$$\Rightarrow x = \pm 1$$

From (ii), $y = -4$, when $x = \pm 1$. Hence, $x = 1$, $y = -4$ or $x = -1$, $y = -4$.

Example 2.26 **If $(x + iy)^5 = p + iq$, then prove that $(y + ix)^5 = q + ip$.**

Sol. $(x + iy)^5 = p + iq$

$$\Rightarrow \overline{(x + iy)^5} = \overline{p + iq}$$

$$\Rightarrow \overline{(x + iy)}^5 = p - iq$$

$$\Rightarrow (x - iy)^5 = p - iq$$

$$\Rightarrow i^5(x - iy)^5 = pi^5 - i^6q$$

$$\Rightarrow (xi - i^2y)^5 = pi + q$$

$$\Rightarrow (y + ix)^5 = pi + q$$

Example 2.27 **Find the values of θ if $(3 + 2i\sin\theta)/(1 - 2i\sin\theta)$ is purely real or purely imaginary.**

Sol. $z = \dfrac{3 + 2i\sin\theta}{1 - 2i\sin\theta}$

Multiplying numerator and denominator by conjugate,

$$z = \frac{(3 + 2i\sin\theta)(1 + 2i\sin\theta)}{1 + 4\sin^2\theta}$$

$$= \frac{3 - 4\sin^2\theta + 8i\sin\theta}{1 + 4\sin^2\theta}$$

Now z is purely real if $\sin\theta = 0$ or $\theta = n\pi$, $n \in Z$. z is purely imaginary if

$$3 - 4\sin^2\theta = 0$$

$$\Rightarrow \sin\theta = \pm\frac{\sqrt{3}}{2} = \pm\sin\frac{\pi}{3}$$

$$\Rightarrow \theta = n\pi \pm \frac{\pi}{3}, n \in Z$$

Example 2.28 **If z is a complex number such that $z^2 = (\bar{z})^2$, then find the location of z on the Argand plane.**

Sol. Let, $z = x + iy \Rightarrow \bar{z} = x - iy$

Given that

$$z^2 = (\bar{z})^2$$

$$\Rightarrow x^2 - y^2 + 2ixy = x^2 - y^2 - 2ixy$$

$$\Rightarrow 4ixy = 0$$

If $x \neq 0$, then $y = 0$ and if $y \neq 0$, then $x = 0$.

Example 2.29 **Find the least positive integer n which will reduce $\left(\frac{(i-1)}{(i+1)}\right)^n$ to a real number.**

Sol.
$$\left(\frac{i-1}{i+1}\right)^n = \left(\frac{i-1}{i+1} \times \frac{i+1}{i+1}\right)^n$$

$$= \left(\frac{i^2-1}{(i+1)^2}\right)^n$$

$$= \left(\frac{-2}{i^2+2i+1}\right)^n$$

$$= \left(\frac{-2}{2i}\right)^n$$

$$= \left(\frac{-1}{i}\right)^n$$

Hence, the required positive integer is 2.

Example 2.30 **Consider two complex numbers α and β as $\alpha = [(a+bi)/(a-bi)]^2 + [(a-bi)/(a+bi)]^2$, where $a, b \in R$ and $\beta = (z-1)/(z+1)$, where $|z| = 1$, then find the correct statement:**

a. both α and β are purely real
b. both α and β are purely imaginary
c. α is purely real and β is purely imaginary
d. β is purely real and α is purely imaginary

Sol. Note that $\alpha = \bar{\alpha} \Rightarrow \alpha$ is real.

$$\beta + \bar{\beta} = \frac{z-1}{z+1} + \frac{\bar{z}-1}{\bar{z}+1}$$

$$= \frac{(z-1)(\bar{z}+1) + (z+1)(\bar{z}-1)}{(z+1)(\bar{z}+1)}$$

$$= \frac{2z\bar{z} - 2}{(z+1)(\bar{z}+1)}$$

$$= 0 \qquad [\text{as } z\bar{z} = |z|^2 = 1 \text{ (given)}]$$

Hence the correct statement is (c).

Expressing Complex Numbers in *a* + *ib* Form

The following example illustrates how a complex number can be expressed in the standard $a + ib$ form.

Solving Complex Equations

Simple equations in z may be solved by putting $z = x + iy$ in the equation and equating the real part on the L.H.S. with the real part on the R.H.S. and the imaginary part on the L.H.S. with the imaginary part on the R.H.S.

Example 2.31 **Find the complex number 'z' satisfying $\text{Re}(z^2) = 0$, $|z| = \sqrt{3}$.**

Sol. $z = x + iy$

$$\Rightarrow z^2 = x^2 - y^2 + 2ixy$$

$$\Rightarrow \text{Re}(z^2) = x^2 - y^2$$

Also,

$$|z| = \sqrt{x^2 + y^2}$$

$$\Rightarrow x^2 - y^2 = 0, x^2 + y^2 = 3$$

$$\Rightarrow x^2 = y^2 = \frac{3}{2}$$

$$\Rightarrow x = \pm\sqrt{\frac{3}{2}}, y = \pm\sqrt{\frac{3}{2}} \Rightarrow z = \pm\sqrt{\frac{3}{2}} \pm \sqrt{\frac{3}{2}}i$$

Thus there are four complex numbers.

Example 2.32 **Solve the equation $|z| = z + 1 + 2i$.**

Sol. $|z| = z + 1 + 2i$

$$\Rightarrow \sqrt{x^2 + y^2} = x + iy + 1 + 2i$$

$$= x + 1 + (2 + y)i$$

$$\Rightarrow \sqrt{x^2 + y^2} = x + 1 \text{ and } 0 = 2 + y \text{ or } y = -2$$

$$\Rightarrow \sqrt{x^2 + 4} = x + 1$$

$$\Rightarrow x^2 + 4 = x^2 + 2x + 1$$

$$\Rightarrow 2x = 3$$

$$\Rightarrow x = 3/2$$

$$\Rightarrow x + iy = \frac{3}{2} - 2i$$

Concept Application Exercise 2.2

1. Express the following complex numbers in $a + ib$ form:
   a. $\frac{(3-2i)(2+3i)}{(1+2i)(2-i)}$ b. $\frac{2-\sqrt{-25}}{1-\sqrt{-16}}$
2. If $z_1 = 9y^2 - 4 - 10ix$, $z_2 = 8y^2 - 20i$, where $z_1 = \bar{z}_2$, then find $z = x + iy$.
3. Find the least positive integer n such that $\left(\frac{2i}{1+i}\right)^n$ is a positive integer.
4. Find the real part of $e^{e^{i\theta}}$.
5. Prove that $z = i^i$, where $i = \sqrt{-1}$, is purely real.
6. Solve: $z^2 + |z| = 0$.

Geometric Presentation of Various Algebraic Operations

Addition:

Let z_1 and z_2 be two complex numbers represented by points P and Q.

Now $OP = |z_1|$ and $OQ = |z_2|$. Let R represent complex number $z_1 + z_2$. Now,

$$PR = \text{distance between complex numbers } z_1 + z_2 \text{ and } z_1$$
$$= |(z_1 + z_2) - z_1|$$
$$= |z_2| = OQ$$

(For distance formula, see properties of modulus on page 2.10.)

Similarly $\quad QR = |(z_1 + z_2) - z_2| = |z_1| = OP$

Hence points O, P, R, Q complete the parallelogram.

Also in triangle OPR, we have

$$OP + PR \geq OR \Rightarrow |z_1| + |z_2| \geq |z_1 + z_2|$$

(however, equality sign holds when origin, z_1 and z_2 are collinear)

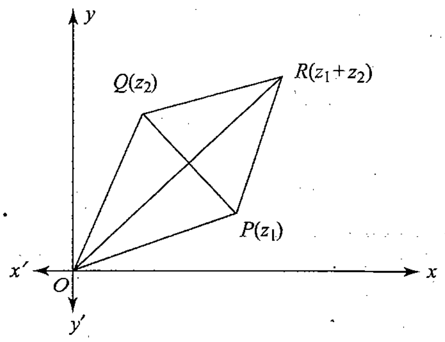


Fig. 2.7

Subtraction

In the adjacent figure $OPRQ'$ is parallelogram.

$$OP = Q'R = |z_1| \text{ and } OQ = OQ' = PR = |z_2|$$

Also in triangle OPR, we have

$$|OP - PR| \leq OR \geq ||z_1| - |z_2|| \leq |z_1 - z_2|$$

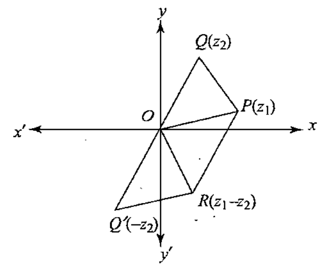


Fig. 2.8

Multiplication

Let $|z_1| = r_1$, $|z_2| = r_2$, $\arg(z_1) = \theta_1$ and $\arg(z_2) = \theta_2$. Then we have

$|z_1 z_2| = r_1 r_2$ and $\arg(z_1 z_2) = \theta_1 + \theta_2$.

So, to find the point R representing the complex number $z_1 z_2$, we have to construct the point with polar coordinates $(r_1 r_2, \theta_1 + \theta_2)$. For this take a point L on real axis such that $OL = 1$ and draw a triangle OQR similar to the triangle OLP. Since triangles OQR and OLP are similar, therefore

$$\frac{OR}{OQ} = \frac{OP}{OL} \Rightarrow \frac{OR}{r_2} = \frac{r_1}{1} \Rightarrow OR = r_1 r_2$$

and $\qquad \angle QOR = \angle LOP = \theta_1$

so that $\qquad \angle XOR = \angle XOQ + \angle QOR = \theta_2 + \theta_1$

Hence, R has the polar coordinates $(r_1 r_2, \theta_1 + \theta_2)$. Consequently, it represents the complex number $z_1 z_2$.

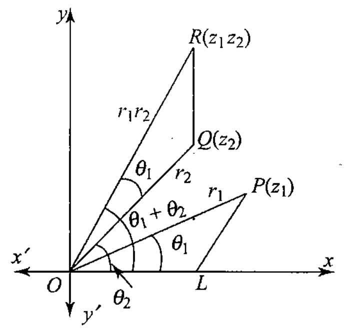


Fig. 2.9

Division

We have $|z_1/z_2| = r_1/r_2$ and $\arg(z_1/z_2) = \theta_1 - \theta_2$. So, to find a point R representing the complex number z_1/z_2, we have to construct the point with polar coordinates $(r_1/r_2, \theta_1 - \theta_2)$. For this take a point L on real axis OX such that $OL = 1$ and draw a triangle OPR similar to OQL. Since triangle OPR and OQL are similar, therefore

$$\frac{OP}{OQ} = \frac{OR}{OL} \Rightarrow \frac{OR}{1} = \frac{r_1}{r_2} \Rightarrow OR = \frac{r_1}{r_2}$$

and

$$\angle XOR = \angle XOP - \angle ROP = \theta_1 - \theta_2$$

Hence, R has the polar coordinates $(r_1/r_2, \theta_1 - \theta_2)$. Consequently, it represents the complex number z_1/z_2.

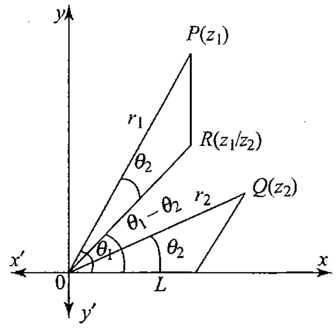


Fig. 2.10

Properties of Modulus

If $z, z_1, z_2 \in C$, then

1. $|z| = 0 \Leftrightarrow z = 0$, i.e., $\text{Re}(z) = \text{Im}(z) = 0$
2. $|z| = |\bar{z}| = |-z| = |-\bar{z}|$
3. $-|z| \le \text{Re}(z) \le |z|$; $-|z| \le \text{Im}(z) \le |z|$
4. $z\bar{z} = |z|^2$
5. $|z_1 z_2| = |z_1|\,|z_2|$, in general $|z_1 z_2 z_3 \cdots z_n| = |z_1|\,|z_2||z_3| \cdots |z_n|$
6. $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}; z_2 \ne 0$
7. $|z_1 + z_2|^2 = (z_1 + z_2)(\bar{z}_1 + \bar{z}_2)$
   $= |z_1|^2 + |z_2|^2 + z_1\bar{z}_2 + z_2\bar{z}_1$
   $= |z_1|^2 + |z_2|^2 + 2\,\text{Re}(z_1\bar{z}_2)$
8. $|z_1 - z_2|^2 = (z_1 - z_2)(\bar{z}_1 - \bar{z}_2)$
   $= |z_1|^2 + |z_2|^2 - z_1\bar{z}_2 - z_2\bar{z}_1$
   $= |z_1|^2 + |z_2|^2 - 2\,\text{Re}(z_1\bar{z}_2)$
9. $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2(|z_1|^2 + |z_2|^2)$
10. $|z_1 - z_2| = |(a_1 + ib_1) - (a_2 + ib_2)|$
    $= |(a_1 - a_2) + i(b_1 - b_2)|$
    $= \sqrt{(a_1 - a_2)^2 + (b_1 - b_2)^2}$

    This is the distance between the points (a_1, b_1) and (a_2, b_2) which is the distance between z_1 and z_2.
11. $|z^n| = |z|^n$, where $n \in Q$
12. $|z_1 \pm z_2| \le |z_1| + |z_2|$, in general $|z_1 + z_2 + z_3 + \cdots + z_n| \le |z_1| + |z_2| + |z_3| + \cdots + |z_n|$
13. $|z_1 \pm z_2| \ge ||z_1| - |z_2||$

Example 2.33 **If $(1 + i)(1 + 2i)(1 + 3i) \cdots (1 + ni) = (x + iy)$, then show that $2 \times 5 \times 10 \times \cdots \times (1 + n^2) = x^2 + y^2$.**

Sol. We have,

$$|(1 + i)(1 + 2i)(1 + 3i) \cdots (1 + ni) = x + iy$$
$$\Rightarrow (1 + i)(1 + 2i) \cdots (1 + ni)| = |x + iy|$$
$$\Rightarrow |1 + i|\;|1 + 2i| \cdots |1 + ni| = |x + iy|$$
$$[\because |z_1 z_2 \cdots z_n| = |z_1|\,|z_2| \cdots |z_n|]$$
$$\Rightarrow \sqrt{1+1}\,\sqrt{1+4} \cdots \sqrt{1+n^2} = \sqrt{x^2 + y^2}$$
$$\Rightarrow 2 \times 5 \times 10 \cdots (1 + n^2) = (x^2 + y^2) \quad \text{[On squaring both sides]}$$

Example 2.34 **If $z = x + iy$ and $w = (1 - iz)/(z - i)$, then show that $|w| = 1 \Rightarrow z$ is purely real.**

Sol. We have,

$$|w| = 1$$
$$\Rightarrow \left|\frac{1 - iz}{z - i}\right| = 1$$
$$\Rightarrow \frac{|1 - iz|}{|z - i|} = 1$$
$$\Rightarrow |1 - iz| = |z - i| \quad \text{(i)}$$
$$\Rightarrow |1 - i(x + iy)| = |x + iy - i|, \text{ where } z = x + iy$$
$$\Rightarrow |1 + y - ix| = |x + i(y - 1)|$$
$$\Rightarrow \sqrt{(1+y)^2 + (-x)^2} = \sqrt{x^2 + (y-1)^2}$$
$$\Rightarrow (1 + y)^2 + x^2 = x^2 + (y - 1)^2$$
$$\Rightarrow y = 0$$
$$\Rightarrow z = x + i0 = x, \text{ which is purely real}$$

Example 2.35 **If α and β are different complex numbers with $|\beta| = 1$, then find the value of $|(\beta - \alpha)/(1 - \bar{\alpha}\beta)|$.**

Sol. Given,

$$|\beta| = 1 \Rightarrow \beta\bar{\beta} = 1 \Rightarrow \beta = \frac{1}{\bar{\beta}} \quad (1)$$

Now,

$$\left|\frac{\beta - \alpha}{1 - \bar{\alpha}\beta}\right| = \left|\frac{\frac{1}{\bar{\beta}} - \alpha}{1 - \bar{\alpha}\beta}\right|$$
$$= \left|\frac{1}{\bar{\beta}}\right|\left|\frac{1 - \alpha\bar{\beta}}{1 - \bar{\alpha}\beta}\right|$$
$$= \frac{1}{|\beta|}\left|\frac{\overline{1 - \alpha\bar{\beta}}}{1 - \bar{\alpha}\beta}\right|$$
$$= \left|\frac{1 - \bar{\alpha}\beta}{1 - \bar{\alpha}\beta}\right| = 1$$

Example 2.36 **Solve the equation $z^3 = \bar{z}$ $(z \ne 0)$.**

Sol. As $z \ne 0$,

$$|z^3| = |\bar{z}|$$
$$\Rightarrow |z| = 1$$
$$\therefore z^3 = \bar{z} \Rightarrow z^4 = z\bar{z} = |z|^2 = 1$$
$$\Rightarrow z = \pm 1, \pm i$$

Example 2.37 **If z_1, z_2, z_3, z_4 are the affixes of four points in the Argand plane, z is the affix of a point such that $|z - z_1| = |z - z_2| = |z - z_3| = |z - z_4|$, then prove that z_1, z_2, z_3, z_4 are concyclic.**

Sol. We have,

$$|z - z_1| = |z - z_2| = |z - z_3| = |z - z_4|$$

Therefore, the point having affix z is equidistant from the four points having affixes z_1, z_2, z_3, z_4. Thus, z is the affix of either the centre of a circle or the point of intersection of diagonals of a square. Therefore z_1, z_2, z_3, z_4 are either concyclic.

Example 2.38 **If $|z| = 1$ and let $\omega = \frac{(1 - z)^2}{1 - z^2}$, then prove that the locus of ω is equivalent to $|z - 2| = |z + 2|$.**

Sol. Given $\omega = \frac{(1 - z)^2}{1 - z^2} = \frac{1 - z}{1 + z}$

$\Rightarrow \quad = \dfrac{z\bar{z}-z}{z\bar{z}+z}$

$= \dfrac{\bar{z}-1}{\bar{z}+1} = -\left(\overline{\dfrac{1-z}{1+z}}\right) = -\bar{\omega}$

$\therefore \quad \omega + \bar{\omega} = 0 \Rightarrow \omega$ is purely imaginary. Hence ω lies on y axis.

Also $|z-2| = |z+2| \Rightarrow z$ lies on perpendicular bisector of 2 and -2, which is imaginary axis.

Example 2.39 **Let $z = x + iy$ then find the locus of $P(z)$ such that $\dfrac{1+\bar{z}}{z} \in R$.**

Sol. Given $\dfrac{1+\bar{z}}{z}$ is real

$\Rightarrow \quad \dfrac{1+\bar{z}}{z} = \dfrac{1+z}{\bar{z}}$

$\Rightarrow \quad \bar{z} + \bar{z}^2 = z + z^2$

$\Rightarrow \quad (\bar{z} - z) + (\bar{z} - z)(\bar{z} + z) = 0$

$\Rightarrow \quad (\bar{z} - z)(1 + \bar{z} + z) = 0$

$\Rightarrow$ either $\bar{z} = z$ ($z \neq 0$) or $z + \bar{z} + 1 = 0$

$\Rightarrow \quad y = 0$ or $x = \dfrac{-1}{2}$ but excluding origin.

Example 2.40 **Identify the locus z if $\text{Re}(z + 1) = |z - 1|$.**

Sol. $\text{Re}(z + 1) = |z - 1|$

$\Rightarrow \quad x + 1 = \sqrt{(x-1)^2 + y^2}$

$\Rightarrow \quad (x + 1)^2 = (x - 1)^2 + y^2$

$\Rightarrow \quad y^2 = 4x$

Example 2.41 **If $|z_1| = 1$, $|z_2| = 2$, $|z_3| = 3$ and $|9z_1z_2 + 4z_1z_3 + z_2z_3| = 12$, then find the value of $|z_1 + z_2 + z_3|$.**

Sol. $|z_1| = 1 \Rightarrow z_1\bar{z}_1 = 1$, $|z_2| = 2 \Rightarrow z_2\bar{z}_2 = 4$, $|z_3| = 3 \Rightarrow z_3\bar{z}_3 = 9$

Also,

$|9z_1z_2 + 4z_1z_3 + z_2z_3| = 12$

$\Rightarrow \quad |z_1z_2z_3\bar{z}_3 + z_1z_2z_3\bar{z}_2 + z_1\bar{z}_1z_2z_3| = 12$

$\Rightarrow \quad |z_1z_2z_3|\,|\bar{z}_1 + \bar{z}_2 + \bar{z}_3| = 12$

$\Rightarrow \quad |z_1|\,|z_2|\,|z_3|\,|\bar{z}_1 + \bar{z}_2 + \bar{z}_3| = 12$

$\Rightarrow \quad 6\,|\bar{z}_1 + \bar{z}_2 + \bar{z}_3| = 12$

$\Rightarrow \quad |\bar{z}_1 + \bar{z}_2 + \bar{z}_3| = 2$

$\Rightarrow \quad |z_1 + z_2 + z_3| = 2$

Example 2.42 **If $|z - i\,\text{Re}(z)| = |z - \text{Im}(z)|$, then prove that z lies on the bisectors of the quadrants.**

Sol. $z = x + iy \Rightarrow \text{Re}(z) = x$, $\text{Im}(z) = y$

$|z - i\text{Re}(z)| = |z - \text{Im}(z)|$

$\Rightarrow \quad |x + iy - ix| = |x + iy - y|$

$\Rightarrow \quad x^2 + (x - y)^2 = (x - y)^2 + y^2$

$\Rightarrow \quad x^2 = y^2$

$\Rightarrow \quad |x| = |y|$

$\Rightarrow \quad z$ lies on the bisectors the quadrants.

Example 2.43 **Show that $(x^2 + y^2)^4 = (x^4 - 6x^2y^2 + y^4)^2 + (4x^3y - 4xy^3)^2$.**

Sol. $(x^2 + y^2)^4 = |x + iy|^8$

$= |(x + iy)^2|^4$

$= |(x^2 - y^2) + 2ixy)|^4$

$= |[(x^2 - y^2) + 2ixy)]^2|^2$

$= |(x^2 - y^2)^2 + (2ixy)^2 + 2(x^2 - y^2)(2ixy)|^2$

$= |x^4 + y^4 - 2x^2y^2 - 4x^2y^2 + i(4x^3y - 4xy^3)|^2$

$= |x^4 + y^4 - 6x^2y^2 + i(4x^3y - 4xy^3)|^2$

$= (x^4 - 6x^2y^2 + y^4)^2 + (4x^3y - 4xy^3)^2$

Example 2.44 **Let $|(\bar{z}_1 - 2\bar{z}_2)/(2 - z_1\bar{z}_2)| = 1$ and $|z_2| \neq 1$, where z_1 and z_2 are complex numbers. Show that $|z_1| = 2$.**

Sol. $\left|\dfrac{\bar{z}_1 - 2\bar{z}_2}{2 - z_1\bar{z}_2}\right| = 1$

$\Rightarrow \quad |\bar{z}_1 - 2\bar{z}_2|^2 = |2 - z_1\bar{z}_2|^2$

$\Rightarrow \quad (\bar{z}_1 - 2\bar{z}_2)(\overline{\bar{z}_1 - 2\bar{z}_2}) = (2 - z_1\bar{z}_2)(\overline{2 - z_1\bar{z}_2})$

$\Rightarrow \quad (\bar{z}_1 - 2\bar{z}_2)(z_1 - 2z_2) = (2 - z_1\bar{z}_2)(2 - \bar{z}_1z_2)$

$\Rightarrow \quad z_1\bar{z}_1 - 2\bar{z}_1z_2 - 2z_1\bar{z}_2 + 4z_2\bar{z}_2 = 4 - 2\bar{z}_1z_2 - 2z_1\bar{z}_2 + z_1\bar{z}_1z_2\bar{z}_2$

$\Rightarrow \quad |z_1|^2 + 4|z_2|^2 = 4 + |z_1|^2|z_2|^2$

$\Rightarrow \quad |z_1|^2 - |z_1|^2|z_2|^2 + 4|z_2|^2 - 4 = 0$

$\Rightarrow \quad |z_1|^2(1 - |z_2|^2) + 4(|z_2|^2 - 1) = 0$

$\Rightarrow \quad (|z_2|^2 - 1)(|z_1|^2 - 4) = 0$

$\Rightarrow \quad |z_1| = 2$ (as $|z_2| \neq 1$)

Example 2.45 **If z_1 and z_2 are complex numbers and $u = \sqrt{z_1z_2}$, then prove that**

$$|z_1| + |z_2| = \left|\frac{z_1 + z_2}{2} + u\right| + \left|\frac{z_1 + z_2}{2} - u\right|$$

Sol. $\left|\dfrac{z_1 + z_2}{2} + u\right| + \left|\dfrac{z_1 + z_2}{2} - u\right|$

$= \left|\dfrac{z_1 + z_2}{2} + \sqrt{z_1z_2}\right| + \left|\dfrac{z_1 + z_2}{2} - \sqrt{z_1z_2}\right|$

$= \left|\dfrac{(\sqrt{z_1} + \sqrt{z_2})^2}{2}\right| + \left|\dfrac{(\sqrt{z_1} - \sqrt{z_2})^2}{2}\right|$

$= \left|\dfrac{(p+q)^2}{2}\right| + \left|\dfrac{(p-q)^2}{2}\right|$ (where $p = \sqrt{z_1}$ and $q = \sqrt{z_2}$)

$= \dfrac{1}{2}\left[|p+q|^2 + |p-q|^2\right]$

$= \dfrac{1}{2}\left[2|p|^2 + 2|q|^2\right]$

$= |p|^2 + |q|^2$

$= |p^2| + |q^2|$

$= |z_1| + |z_2|$

Example 2.46 **If $(\sqrt{8}+i)^{50} = 3^{49}(a+ib)$, then find the value of $a^2 + b^2$.**

Sol. Given that

$$(\sqrt{8}+i)^{50} = 3^{49}(a+ib)$$

Taking modulus and squaring both sides, we get

$(8 + 1)^{50} = 3^{98}(a^2 + b^2)$

$\Rightarrow 9^{50} = 3^{98}(a^2 + b^2)$

$\Rightarrow 3^{100} = 3^{98}(a^2 + b^2)$

$\Rightarrow (a^2 + b^2) = 9$

Example 2.47 **Find complex number satisfying the system of equations $z^3 + \bar{\omega}^7 = 0$ and $z^5\omega^{11} = 1$.**

Sol. $z^3 + \bar{\omega}^7 = 0$

$\Rightarrow z^3 = -\bar{\omega}^7$

$\Rightarrow |z|^3 = |\bar{\omega}|^7 = |\omega|^7$

$\Rightarrow |z|^{15} = |\omega|^{35}$ (1)

Again,

$z^5\omega^{11} = 1$

$\Rightarrow |z|^5 |\omega|^{11} = 1$

$\Rightarrow |z|^{15} |\omega|^{33} = 1$ (2)

From (1) and (2), we have

$|z| = |\omega| = 1$

Again,

$\bar{\omega}^7 = -z^3$ and $\omega^{11} = z^{-5}$

$\Rightarrow \bar{\omega}^{77} \cdot \omega^{77} = -z^{33} \cdot z^{-35}$

$\Rightarrow z^2 = -1 = i^2$

$\Rightarrow z = +i$

Example 2.48 **If $|z_1 - 1| \le 1$, $|z_2 - 2| \le 2$, $|z_3 - 3| \le 3$, then find the greatest value of $|z_1 + z_2 + z_3|$.**

Sol. $|z_1 + z_2 + z_3| = |(z_1 - 1) + (z_2 - 2) + (z_3 - 3) + 6|$

$\le |z_1 - 1| + |z_2 - 2| + |z_3 - 3| + 6$

$\le 1 + 2 + 3 + 6 = 12$

Hence the greatest value is 12.

Example 2.49 **For any complex number z, find the minimum value of $|z| + |z - 2i|$.**

Sol. We have, for $z \in C$

$|2i| = |z + (2i - z)|$

$\le |z| + |2i - z|$

$\Rightarrow 2 \le |z| + |z - 2i|$

Thus, minimum value of $|z| + |z - 2i|$ is 2.

Example 2.50 **If z is any complex number such that $|z + 4| \le 3$, then find the greatest value of $|z + 1|$.**

Sol. $|z + 1| = |z + 4 - 3|$

$= |(z + 4) + (-3)|$

$\le |z + 4| + |-3|$

$= |z + 4| + 3$

$\le 3 + 3 = 6$ $[\because |z + 4| \le 3]$

Hence, the greatest value of $|z + 1|$ is 6.

Example 2.51 **Prove that the distance of the roots of the equation $|\sin\theta_1| z^3 + |\sin\theta_2| z^2 + |\sin\theta_3| z + |\sin\theta_4| = 3$ from $z = 0$ is greater than 2/3.**

Sol. We know that $|\sin\theta_k| < 1$. Given,

$|\sin\theta_1| z^3 + |\sin\theta_2| z^2 + |\sin\theta_3| z + |\sin\theta_4| = 3$

$\Rightarrow |3| = ||\sin\theta_1| z^3 + |\sin\theta_2| z^2 + |\sin\theta_3| z + |\sin\theta_4||$

$\le 1 |z^3 + z^2 + z + 1|$

$< |z|^3 + |z|^2 + |z| + 1$

$< 1 + |z| + |z|^2 + |z|^3 + |z|^4 + \cdots \infty$ $(\because |z| < 1)$

$\Rightarrow 3 < \dfrac{1}{1 - |z|}$

$\Rightarrow 3 - 3|z| < 1$

$\Rightarrow 2 < 3|z|$

$\Rightarrow |z| > \dfrac{2}{3}$

Example 2.52 **If z_1 and z_2 are two complex numbers and $c > 0$, then prove that $|z_1 + z_2|^2 \le (1 + c)|z_1|^2 + (1 + c^{-1})|z_2|^2$.**

Sol. We have to prove

$|z_1 + z_2|^2 \le (1 + c)|z_1|^2 + (1 + c^{-1})|z_2|^2$

$\Rightarrow |z_1|^2 + |z_2|^2 + z_1\bar{z}_2 + \bar{z}_2 z_2 \le (1 + c)|z_1|^2 + (1 + c^{-1})|z_2|^2$

$\Rightarrow z_1\bar{z}_2 + \bar{z}_2 z_2 \le c|z_1|^2 + c^{-1}|z_2|^2$

$\Rightarrow c|z_1|^2 + \dfrac{1}{c}|z_2|^2 - z_1\bar{z}_2 - \bar{z}_2 z_2 \ge 0$ [using Re $(z_1\bar{z}_2) \le |z_1\bar{z}_2|$]

$\Rightarrow \left(\sqrt{c}|z_1| - \dfrac{1}{\sqrt{c}}|z_2|\right)^2 \ge 0$

which is always true.

Example 2.53 **Find the greatest and the least value of $|z_1 + z_2|$ if $z_1 = 24 + 7i$ and $|z_2| = 6$.**

Sol. $|z_1 + z_2| \le |z_1| + |z_2|$

$= |24 + 7i| + 6$

$= 25 + 6 = 31$

Also,

$|z_1 + z_2| = |z_1 - (-z_2)| \ge ||z_1| - |z_2||$

$\Rightarrow |z_1 + z_2| \ge |25 - 6| = 19$

Hence the least value of $|z_1 + z_2|$ is 19 and the greatest value is 25.

Properties of Arguments

1. $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$.
   In general,
   $$\arg(z_1 z_2 z_3 \ldots z_n) = \arg(z_1) + \arg(z_2) + \arg(z_3) + \cdots + \arg(z_n)$$
2. $\arg\left(\frac{z_1}{z_2}\right) = \arg z_1 - \arg z_2$
3. $\arg \bar{z} = -\arg z$
4. $\arg\left(\frac{1}{z}\right) = \arg\left(\frac{z\bar{z}}{z}\right)$
   $= \arg\left(\frac{|z|^2}{z}\right)$
   $= \arg(|z|^2) - \arg z = 0 - \arg z$
5. $\arg\left(\frac{z}{\bar{z}}\right) = \arg(z) - \arg(\bar{z}) = \theta - (-\theta) = 2\theta = 2\arg(z)$,
   where $\theta = \arg(z)$.
6. $\arg(z^n) = n \arg z$, when $n \in \theta$.
7. $z_1\bar{z}_2 + \bar{z}_1 z_2 = 2|z_1||z_2|\cos(\theta_1 - \theta_2)$, where $\theta_1 = \arg(z_1)$ and $\theta_2 = \arg(z_2)$.
8. If z is purely imaginary, then $\arg(z) = \pm\pi/2$.
9. If z is purely real, then $\arg(z) = 0$ or π.
10. Locus of z, if $\arg(z) = \theta$ (= constant) is ray excluding origin

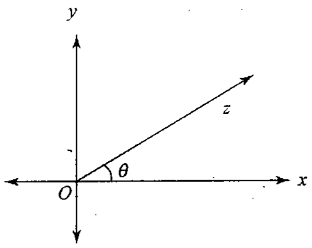


Fig. 2.11

11. Locus of z, if $\arg(z - a) = \theta$ (= constant) and $a > 0$

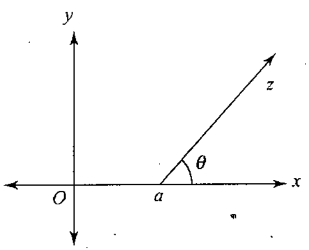


Fig. 2.12.

12. Angle between two lines,
   $= \alpha - \beta$
   $= \arg(z_3 - z_1) - \arg(z_2 - z_1)$
   $= \arg\left(\frac{z_3 - z_1}{z_2 - z_1}\right)$

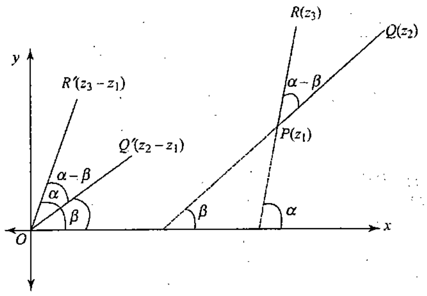


Fig. 2.13

13. Angle between lines joining z_1, z_2 and z_3, z_4:
   $$\theta = \arg\left(\frac{z_4 - z_3}{z_1 - z_2}\right)$$

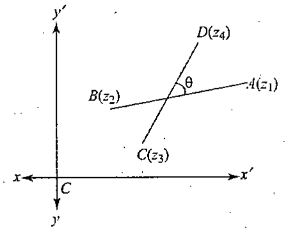


Fig. 2.14

Note: *By specifying the modulus and argument, a complex number is completely defined. However, for the complex number 0 + 0i the argument is not defined and this is the only complex number which is completely defined by taking in terms of its modulus.*

Important Results

1. If z_1, z_2, z_3 be the vertices of an equilateral triangle, then
   $$\frac{1}{z_1 - z_2} + \frac{1}{z_2 - z_3} + \frac{1}{z_3 - z_1} = 0$$
   or
   $$z_1^2 + z_2^2 + z_3^2 = z_1 z_2 + z_2 z_3 + z_3 z_1$$

Proof:

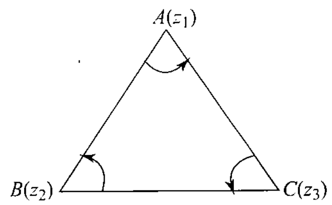


Fig. 2.15

Since ΔABC is equilateral, therefore
$AB = BC = CA$
$\therefore \quad |z_1 - z_2| = |z_2 - z_3| = |z_3 - z_1|$
$$\therefore \quad \left|\frac{z_1 - z_2}{z_3 - z_2}\right| = \left|\frac{z_2 - z_3}{z_1 - z_3}\right| \qquad \text{(i)}$$
Also,
$\angle CBA = \angle ACB$

$\therefore \quad \arg\left(\frac{z_1 - z_2}{z_3 - z_2}\right) = \arg\left(\frac{z_2 - z_3}{z_1 - z_3}\right)$ (ii)

From (i) and (ii), it follows that

$$\frac{z_1 - z_2}{z_3 - z_2} = \frac{z_2 - z_3}{z_1 - z_3}$$

$\Rightarrow \quad (z_1 - z_2)(z_1 - z_3) = -(z_2 - z_3)^2$

$\Rightarrow \quad z_1^2 - z_1z_2 - z_1z_3 + z_2z_3 = -(z_2^2 + z_3^2 - 2z_2z_3)$

$\Rightarrow \quad z_1^2 + z_2^2 + z_3^2 = z_1z_2 + z_2z_3 + z_3z_1$

2. If a, b, c and u, v, w are complex numbers representing the vertices of two triangles such that they are similar, then $\begin{vmatrix} a & u & 1 \\ b & v & 1 \\ c & w & 1 \end{vmatrix} = 0$ or $\frac{a-c}{a-b} = \frac{u-w}{u-v}$

Proof:

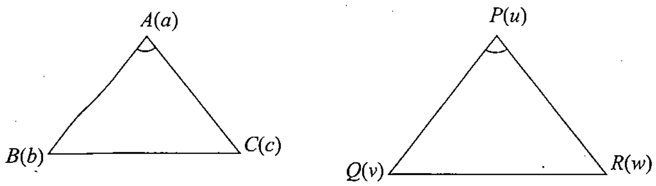


Fig. 2.16

In the figure triangles are similar. Hence

$$\frac{AC}{PR} = \frac{AB}{PQ}$$

$\Rightarrow \quad \frac{AC}{AB} = \frac{PR}{PQ}$

$\Rightarrow \quad \left|\frac{a-c}{a-b}\right| = \left|\frac{u-w}{u-v}\right|$ (1)

Also,

$\angle BAC = \angle QPR$

$\Rightarrow \quad \arg\left(\frac{a-c}{a-b}\right) = \arg\left(\frac{u-w}{u-v}\right)$ (2)

From (1) and (2), we have

$$\frac{a-c}{a-b} = \frac{u-w}{u-v}$$

Simplifying, we get

$$\begin{vmatrix} a & u & 1 \\ b & v & 1 \\ c & w & 1 \end{vmatrix} = 0$$

Example 2.54 **Find the amplitude of**

a. $\frac{1+\sqrt{3}i}{\sqrt{3}+i}$ **b.** $-1 - i\sqrt{3}$ **c.** $\sin\alpha + i(1 - \cos\alpha), 0 < \alpha < \pi$

Sol.

a. $\text{amp}\left(\frac{1+\sqrt{3}\,i}{\sqrt{3}+i}\right) = \text{amp}(1 + \sqrt{3}\,i) - \text{amp}(\sqrt{3} + i)$

$$= \frac{\pi}{3} - \frac{\pi}{6} = \frac{\pi}{6}$$

b. Let, $z = -1 - i\sqrt{3}$.

Then $\alpha = \tan^{-1}|b/a| = \tan^{-1}|\sqrt{3}/1| = \pi/3$

Here, z is in third quadrant. Therefore argument is $\theta = -(\pi - \alpha) = -(\pi - \pi/3) = -2\pi/3$

c. $z = \sin\alpha + i(1 - \cos\alpha), 0 < \alpha < \pi = \sin\alpha + i(1 - \cos\alpha)$

$\Rightarrow \quad \text{amp}(z) = \tan^{-1}\left(\frac{1-\cos\alpha}{\sin\alpha}\right)$ ($\because$ z lies in first quadrant)

$$= \tan^{-1}\left(\frac{2\sin^2\frac{\alpha}{2}}{2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}}\right) = \tan^{-1}\tan\left(\frac{\alpha}{2}\right) = \frac{\alpha}{2}$$

Example 2.55 **Find the modulus, argument and the principal argument of the complex numbers.**

(i) $(\tan 1 - i)^2$

(ii) $\dfrac{i-1}{i\left(1-\cos\frac{2\pi}{5}\right)+\sin\frac{2\pi}{5}}$

Sol. (i) $z = (\tan 1 - i)^2 = (\tan^2 1 - 1) - (2\tan 1)i$

$|z| = \sqrt{(\tan^2 1 - 1)^2 + 4\tan^2 1}$

$= \sqrt{(\tan^2 1 + 1)^2} = \tan^2 1 + 1$

Since $\tan^2 1 - 1 < 0$ and $-2\tan 1 < 0$

$\Rightarrow$ z lies on third quadrant

$\Rightarrow \arg(z) = -\pi + \tan^{-1}\left|\frac{2\tan 1}{1-\tan^2 1}\right|$

$= -\pi + \tan^{-1}|\tan 2|$

$= 2 - \pi$

(ii) $z = \dfrac{i-1}{i\left(1-\cos\frac{2\pi}{5}\right)+\sin\frac{2\pi}{5}}$

$= \dfrac{i-1}{i2\sin^2\frac{\pi}{5}+2\sin\frac{\pi}{5}\cos\frac{\pi}{5}}$

$= \dfrac{i-1}{\left(2\sin\frac{\pi}{5}\right)\left(\cos\frac{\pi}{5}+i\sin\frac{\pi}{5}\right)}$

$|z| = \dfrac{|i-1|}{\left(2\sin\frac{\pi}{5}\right)\left|\left(\cos\frac{\pi}{5}+i\sin\frac{\pi}{5}\right)\right|}$

$$= \frac{\sqrt{2}}{\left(2\sin\frac{\pi}{5}\right)\left|\left(\cos\frac{\pi}{5}+i\sin\frac{\pi}{5}\right)\right|}$$

$$= \frac{1}{\sqrt{2}}\operatorname{cosec}\frac{\pi}{5}$$

$$\arg z = \arg\left[\frac{i-1}{\left(2\sin\frac{\pi}{5}\right)\left(\cos\frac{\pi}{5}+i\sin\frac{\pi}{5}\right)}\right]$$

$$= \arg(-1+i)-\arg\left(2\sin\frac{\pi}{5}\right)-\arg\left(\cos\frac{\pi}{5}+i\sin\frac{\pi}{5}\right)$$

$$= \frac{3\pi}{4}-0-\frac{\pi}{5}=\frac{11\pi}{20}$$

Example 2.56 **If z_1, z_2 and z_3, z_4 are two pairs of conjugate complex numbers, then find the value of $\arg(z_1/z_4)+\arg(z_2/z_3)$.**

Sol. We have $z_2 = \bar{z}_1$ and $z_4 = \bar{z}_3$. Therefore,

$z_1z_2 = |z_1|^2$ and $z_3z_4 = |z_3|^2$

Now,

$$\arg\left(\frac{z_1}{z_4}\right)+\arg\left(\frac{z_2}{z_3}\right)=\arg\left(\frac{z_1z_2}{z_4z_3}\right)$$

$$= \arg\left(\frac{|z_1|^2}{|z_3|^2}\right)$$

$$= \arg\left(\left|\frac{z_1}{z_3}\right|^2\right)=0$$

Example 2.57 **Prove that $|z_1 + z_2| = |z_1| + |z_2| \Rightarrow \arg(z_1) = \arg(z_2)$.**

Sol. $|z_1 + z_2| = |z_1| + |z_2|$

$\Rightarrow |z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 + 2|z_1||z_2|$

$\Rightarrow |z_1|^2 + |z_2|^2 + 2\operatorname{Re}(z_1\bar{z}_2) = |z_1|^2 + |z_2|^2 + 2|z_1||z_2|$

$\Rightarrow 2\operatorname{Re}(z_1\bar{z}_2) = 2|z_1||z_2|$

$\Rightarrow \cos(\theta_1 - \theta_2) = 1$

$\Rightarrow \theta_1 - \theta_2 = 0$

$\Rightarrow \arg(z_1) = \arg(z_2)$

Example 2.58 **If $\arg(z_1) = 170°$ and $\arg(z_2) = 70°$, then find the principal argument of z_1z_2.**

Sol. $\arg(z_1z_2) = \arg(z_1) + \arg(z_2) = 170° + 70° = 240°$

Thus z_1z_2 lies in third quadrant. Hence its principal argument is $-120°$.

Example 2.59 **If z_1 and z_2 are conjugate to each other, then find $\arg(-z_1z_2)$.**

Sol. z_1 and z_2 are conjugate to each other, i.e., $z_2 = \bar{z}_1$. Therefore,

$\arg(-z_1z_2) = \arg(-z_1\bar{z}_1)$

$= \arg(-|z_1|^2)$

$= \arg(\text{negative real number})$

$= \pi$

Example 2.60 **If $\pi/2 < \alpha < 3\pi/2$, then find the modulus and argument of $(1 + \cos 2\alpha) + i\sin 2\alpha$.**

Sol. $z = (1 + \cos 2\alpha) + i\sin 2\alpha$

$= 2\cos^2\alpha + 2i\sin\alpha\cos\alpha$

$= 2\cos\alpha[\cos\alpha + i\sin\alpha]$

$= -2\cos\alpha[-\cos\alpha - i\sin\alpha]$

$= -2\cos\alpha[\cos(\alpha-\pi) + i\sin(\alpha-\pi)]$

$[\because \pi/2 < \alpha < 3\pi/2]$

Thus, $|z| = -2\cos\alpha$ and $\arg(z) = \alpha - \pi$.

Example 2.61 **Find the point of intersection of the curves $\arg(z - 3i) = 3\pi/4$ and $\arg(2z + 1 - 2i) = \pi/4$.**

Sol. Given loci are as follows:

$$\arg(z-3i) = \frac{3\pi}{4}$$

which is a ray that starts from $3i$ and makes an angle $3\pi/4$ with positive real axis as shown in the figure.

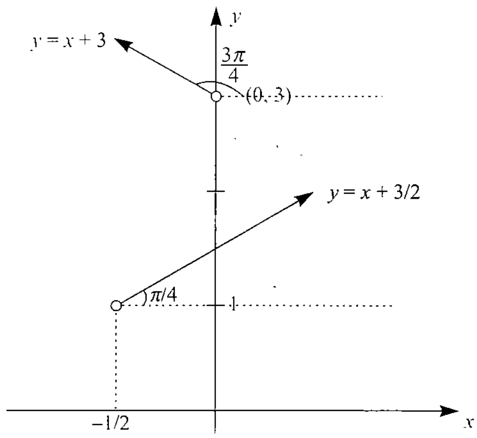


Fig. 2.17

$$\arg(2z+1-2i) = \frac{\pi}{4}$$

$$\Rightarrow \arg\left[2\left(z+\frac{1}{2}-i\right)\right]=\frac{\pi}{4}$$

$$\Rightarrow \arg 2+\arg\left[z-\left(-\frac{1}{2}+i\right)\right]=\frac{\pi}{4}$$

$$\Rightarrow 0+\arg\left[z-\left(-\frac{1}{2}+i\right)\right]=\frac{\pi}{4}$$

$$\Rightarrow \arg\left[z-\left(-\frac{1}{2}+i\right)\right]=\frac{\pi}{4}$$

This is a ray that starts from point $-1/2 + i$ and makes an angle $\pi/4$ with positive real axis as shown in the figure. From the figure, it is obvious that the system of equations has no solution.

Example 2.62 **Let z and w be two non-zero complex numbers such that $|z| = |w|$ and $\arg(z) + \arg(w) = \pi$. Then prove that $z = -\bar{w}$.**

Sol. Let $\arg(w) = \theta$. Then $\arg(z) = \pi - \theta$. Therefore,

$$w = |w|(\cos\theta + i\sin\theta)$$

and

$$z = |z|(\cos(\pi-\theta) + i\sin(\pi-\theta))$$

$$= |w|(-\cos\theta + i\sin\theta) \qquad [\because |z| = |w|]$$

$$= -|w|(\cos\theta - i\sin\theta) = -\bar{w}.$$

Example 2.63 **Let $z = x + iy$ be a complex number, where x and y are real numbers. Let A and B be the sets defined by $A = \{z : |z| \le 2\}$ and $B = \{z : (1-i)z + (1+i)\bar{z} \ge 4\}$. Find the area of region $A \cap B$.**

Sol. $z = x + iy$

$A = \{z : |z| \le 2\}$

$\Rightarrow \sqrt{x^2+y^2} \le 2$

$\Rightarrow x^2 + y^2 \le 4$

$\Rightarrow$ z lies on or inside the circle $x^2 + y^2 = 4$

$B = \{z : (1-i)z + (1+i)\bar{z} \ge 4\}$

$\Rightarrow (1-i)(x+iy) + (1+i)(x-iy) \ge 4$

$\Rightarrow x + iy - ix + y + x - iy + ix + y \ge 4$

$\Rightarrow x + y \ge 2$

Area of region $A \cap B$ is shaded region the diagram.

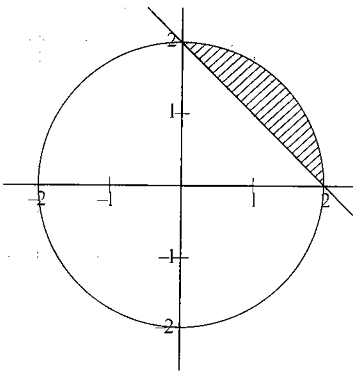


Fig. 2.18

$$\text{Area} = \frac{\pi(2)^2}{4} - \frac{1}{2}\times 2\times 2 = \pi - 2$$

Example 2.64 **Find the area bounded by $\arg|z| \le \pi/4$ and $|z - 1| < |z - 3|$.**

Sol. $|\arg z| < \pi/4$

$-\pi/4 < \arg z < \pi/4$ (i)

Which represents the region given in the following diagram.

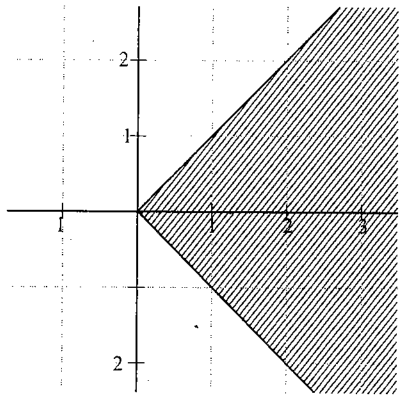


Fig. 2.19

$|z - 1| < |z - 3|$

$\Rightarrow (x-1)^2 + y^2 < (x-3)^2 + y^2$

$\Rightarrow x < 2$ (ii)

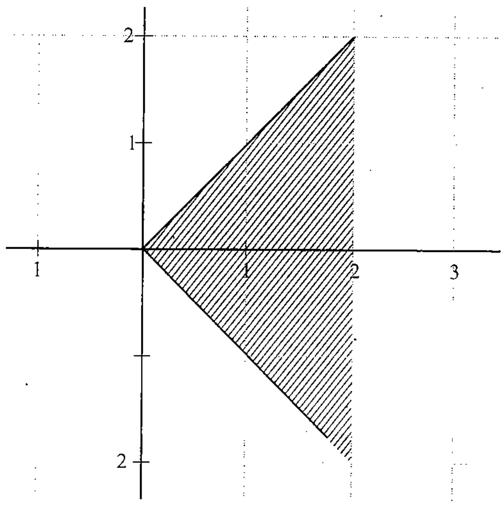


Fig. 2.20

Common region of (i) and (ii) is shown in above figure.

Area of the shaded region is $\frac{1}{2}(4)(2) = 4$ square units.

Example 2.65 **If $z + 1/z = 2\cos\theta$, prove that**

$$\left|(z^{2n}-1)/(z^{2n}+1)\right| = |\tan n\theta|$$

Sol. $z + \dfrac{1}{z} = 2\cos\theta$

$\Rightarrow z^2 - 2\cos\theta\, z + 1 = 0$

$$\Rightarrow z = \frac{2\cos\theta \pm \sqrt{4\cos^2\theta - 4}}{2}$$

$= \cos\theta + i\sin\theta$

Taking positive sign,

$$z = \cos\theta + i\sin\theta,\ \frac{1}{z} = (\cos\theta - i\sin\theta)$$

$$\therefore \quad \frac{z^{2n}-1}{z^{2n}+1} = \frac{z^n - \frac{1}{z^n}}{z^n + \frac{1}{z^n}}$$

$$= \frac{(\cos\theta + i\sin\theta)^n - (\cos\theta - i\sin\theta)^n}{(\cos\theta + i\sin\theta)^n + (\cos\theta - i\sin\theta)^n}$$

$$= \frac{2i\sin n\theta}{2\cos n\theta}$$

$$= i\tan n\theta$$

Taking negative sign, we get

$$\frac{z^{2n}-1}{z^{2n}+1} = \frac{-2i\sin n\theta}{2\cos n\theta} = -\tan n\theta$$

$$\Rightarrow \quad \left|\frac{z^{2n}-1}{z^{2n}+1}\right| = |\pm i\tan\theta| = \tan n\theta$$

Concept Application Exercise 2.3

1. For $z_1 = \sqrt[6]{(1-i)/(1+i\sqrt{3})}$, $z_2 = \sqrt[6]{(1-i)/(\sqrt{3}+i)}$, $z_3 = \sqrt[6]{(1+i)/(\sqrt{3}-i)}$, prove that $|z_1| = |z_2| = |z_3|$.
2. Let z be a complex number satisfying the equation $(z^3+3)^2 = -16$, then find the value of $|z|$.
3. Show that a real value of x will satisfy the equation $(1-ix)/(1+ix) = a-ib$ if $a^2+b^2=1$, where a, b are real.
4. Find non-zero integral solutions of $|1-i|^x = 2^x$.
5. Let z is not a real number such that $(1+z+z^2)/(1-z+z^2) \in R$, then prove that $|z| = 1$.
6. If a, b, c are non-zero complex numbers of equal moduli and satisfy $az^2+bz+c=0$, then prove that $(\sqrt{5}-1)/2 \le |z| \le (\sqrt{5}+1)/2$.
7. If z_1, z_2, z_3 are distinct non-zero complex numbers and $a, b, c \in R^+$ such that
$$\frac{a}{|z_1-z_2|} = \frac{b}{|z_2-z_3|} = \frac{c}{|z_3-z_1|}$$
then find the value of
$$\frac{a^2}{z_1-z_2} + \frac{b^2}{z_2-z_3} + \frac{c^2}{z_3-z_1}$$
8. Prove that $|z_1+z_2|^2 = |z_1|^2 + |z_2|^2$, if z_1/z_2 is purely imaginary.
9. If θ is real and z_1, z_2 are connected by $z_1^2 + z_2^2 + 2z_1z_2\cos\theta = 0$, then prove that the triangle formed by vertices O, z_1 and z_2 is isosceles.
10. If $2z_1/3z_2$ is a purely imaginary number, then find the value of $|(z_1-z_2)/(z_1+z_2)|$.
11. If $|z| \le 4$ then find the maximum value of $|iz+3-4i|$.
12. If $\sqrt{3}+i = (a+ib)(c+id)$, then find the value of $\tan^{-1}(b/a) + \tan^{-1}(d/c)$.
13. Let z be any non-zero complex number, then find $\arg(z) + \arg(\bar{z})$.
14. Find the area bounded by the curves $\arg z = \pi/3$, $\arg z = 2\pi/3$ and $\arg(z - 2 - 2\sqrt{3}\,i) = \pi$ on the complex plane.
15. If $\sqrt{ab} = \sqrt{a}\sqrt{b}$ and $a < 0$, then find $\arg(b + ai)$.
16. If $|z_1 - z_0| = |z_2 - z_0| = a$ and $\text{amp}\left((z_2-z_0)/(z_0-z_1)\right) = \pi/2$, then find z_0.
17. Let $z_1, z_2, z_3, \ldots, z_n$ are the complex numbers such that $|z_1| = |z_2| = \cdots = |z_n| = 1$. If
$$z = \left(\sum_{k=1}^{n} z_k\right)\left(\sum_{k=1}^{n}\frac{1}{z_k}\right)$$
then prove that
   a. z is a real number
   b. $0 < z \le n^2$

DE MOIVRE'S THEOREM

Statement:

(i) If $n \in Z$ (the set of integers), then
$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$

(ii) If $n \in Q$ (the set of rational numbers), then $\cos n\theta + i\sin n\theta$ is one of the values of $(\cos\theta + i\sin\theta)^n$.

Proof:

(i) When $n \in Z$, we know that

$e^{i\theta} = \cos\theta + i\sin\theta$

$\Rightarrow (e^{i\theta})^n = (\cos\theta + i\sin\theta)^n$

$\Rightarrow e^{i(n\theta)} = (\cos\theta + i\sin\theta)^n$

$\Rightarrow \cos(n\theta) + i\sin(n\theta) = (\cos\theta + i\sin\theta)^n$

(ii) Let n be rational number. Let $n = p/q$, where p, q are integers and $q \ne 0$. From part (i), we have

$$\left(\cos\frac{p\theta}{q} + i\sin\frac{p\theta}{q}\right)^q = \cos\left(\left(\frac{p\theta}{q}\right)q\right) + i\sin\left(\left(\frac{p\theta}{q}\right)q\right)$$
$$= \cos p\theta + i\sin p\theta$$

$\Rightarrow \cos\frac{p\theta}{q} + i\sin\frac{p\theta}{q}$ is one of the values of $(\cos p\theta + i\sin p\theta)^{1/q}$

$\Rightarrow \cos\frac{p\theta}{q} + i\sin\frac{p\theta}{q}$ is one of the values of $[(\cos\theta + i\sin\theta)^p]^{1/q}$

$\Rightarrow \cos\frac{p\theta}{q} + i\sin\frac{p\theta}{q}$ is one of the values of $(\cos\theta + i\sin\theta)^{p/q}$

Note:

1. *De Moivre's theorem is also true for* $(\cos\theta - i\sin\theta)$, *i.e.*,
$(\cos\theta - i\sin\theta)^n = \cos n\theta - i\sin n\theta$, *because*
$$(\cos\theta - i\sin\theta)^n = [\cos(-\theta) + i\sin(-\theta)]^n = \cos(-n\theta) + i\sin(-n\theta) = \cos n\theta - i\sin n\theta.$$
2. $\dfrac{1}{\cos\theta + i\sin\theta} = (\cos\theta + i\sin\theta)^{-1} = \cos\theta - i\sin\theta$
3. $(\sin\theta \pm i\cos\theta)^n \neq \sin n\theta \pm i\cos n\theta$.
4. $(\sin\theta + i\cos\theta)^n = [\cos(\pi/2 - \theta) + i\sin(\pi/2 - \theta)]^n = \cos(n\pi/2 - n\theta) + i\sin(n\pi/2 - n\theta)$
5. $(\cos\theta + i\sin\phi)^n \neq \cos n\theta + i\sin n\phi$

Important Result

Writing the binomial expression of $(\cos\theta + i\sin\theta)^n$ and equating the real part to $\cos n\theta$ and the imaginary part to $\sin n\theta$, we get

$$\cos(n\theta) = \cos^n\theta - {}^nC_2\cos^{n-2}\theta\sin^2\theta + {}^nC_4\cos^{n-4}\theta\sin^4\theta + \cdots$$

$$\sin(n\theta) = {}^nC_1\cos^{n-1}\theta\sin\theta - {}^nC_3\cos^{n-3}\theta\sin^3\theta + {}^nC_5\cos^{n-5}\theta\sin^5\theta - \cdots$$

$$\tan(n\theta) = \frac{{}^nC_1\tan\theta - {}^nC_3\tan^3\theta + {}^nC_5\tan^5\theta - {}^nC_7\tan^7\theta + \cdots}{1 - {}^nC_2\tan^2\theta + {}^nC_4\tan^4\theta - {}^nC_6\tan^6\theta + \cdots}$$

Example 2.66 **Express the following in $a + ib$ form:**

a. $\left(\dfrac{\cos\theta + i\sin\theta}{\sin\theta + i\cos\theta}\right)^4$

b. $\dfrac{(\cos 2\theta - i\sin 2\theta)^4(\cos 4\theta + i\sin 4\theta)^{-5}}{(\cos 3\theta + i\sin 3\theta)^{-2}(\cos 3\theta - i\sin 3\theta)^{-9}}$

c. $\dfrac{(\sin\pi/8 + i\cos\pi/8)^8}{(\sin\pi/8 - i\cos\pi/8)^8}$

Sol. a. $\left(\dfrac{\cos\theta + i\sin\theta}{\sin\theta + i\cos\theta}\right)^4$

$$= \frac{(\cos\theta + i\sin\theta)^4}{i^4(\cos\theta - i\sin\theta)^4}$$

$$= \frac{(\cos\theta + i\sin\theta)^4}{(\cos\theta + i\sin\theta)^{-4}}$$

$$= (\cos 8\theta + i\sin\theta)^8$$

$$= \cos 8\theta + i\sin 8\theta$$

b. $\dfrac{(\cos 2\theta - i\sin 2\theta)^4(\cos 4\theta + i\sin 4\theta)^{-5}}{(\cos 3\theta + i\sin 3\theta)^{-2}(\cos 3\theta - i\sin 3\theta)^{-9}}$

$$= \frac{\left[(\cos\theta + i\sin\theta)^{-2}\right]^4\left[(\cos\theta + i\sin\theta)^4\right]^{-5}}{\left[(\cos\theta + i\sin\theta)^3\right]^{-2}\left[(\cos\theta + i\sin\theta)^{-3}\right]^{-9}}$$

$$= \frac{(\cos\theta + i\sin\theta)^{-8}(\cos\theta + i\sin\theta)^{-20}}{(\cos\theta + i\sin\theta)^{-6}(\cos\theta + i\sin\theta)^{27}}$$

$$= (\cos\theta + i\sin\theta)^{-8-20+6-27}$$

$$= (\cos\theta + i\sin\theta)^{-49}$$

$$= \cos 49\theta - i\sin 49\theta$$

c. $\dfrac{(\sin\pi/8 + i\cos\pi/8)^8}{(\sin\pi/8 - i\cos\pi/8)^8}$

$$\frac{i^8(\cos\pi/8 - i\sin\pi/8)^8}{(-i)^8(\cos\pi/8 + i\sin\pi/8)^8} = \frac{\cos\pi - i\sin\pi}{\cos\pi + i\sin\pi} = 1$$

Example 2.67 **If $z = \left(\dfrac{\sqrt{3}}{2} + \dfrac{i}{2}\right)^5 + \left(\dfrac{\sqrt{3}}{2} - \dfrac{i}{2}\right)^5$, then prove that Im(z) = 0.**

Sol. Given that

$$z = \left(\frac{\sqrt{3}}{2} + i\frac{1}{2}\right)^5 + \left(\frac{\sqrt{3}}{2} - i\frac{1}{2}\right)^5$$

$$= \left[\cos\left(\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{6}\right)\right]^5 + \left[\cos\left(\frac{\pi}{6}\right) - i\sin\left(\frac{\pi}{6}\right)\right]^5$$

$$= \cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6} + \cos\frac{5\pi}{6} - i\sin\frac{5\pi}{6}$$

Hence Im(z) = 0.

Example 2.68 **Find the value of the expression**

$$\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)\left(\cos\frac{\pi}{2^2} + i\sin\frac{\pi}{2^2}\right)\ldots \text{ to } \infty$$

Sol. $\left(\cos\dfrac{\pi}{2} + i\sin\dfrac{\pi}{2}\right)\left(\cos\dfrac{\pi}{2^2} + i\sin\dfrac{\pi}{2^2}\right)\ldots$ to ∞

$$= \cos\left(\frac{\pi}{2} + \frac{\pi}{2^2} + \cdots\right) + i\sin\left(\frac{\pi}{2} + \frac{\pi}{2^2} + \cdots\right)$$

$$= \cos\left[\frac{\pi}{2}\left(1 + \frac{1}{2} + \frac{1}{2^2} + \cdots\right)\right] + i\sin\left[\frac{\pi}{2}\left(1 + \frac{1}{2} + \frac{1}{2^2} + \cdots\right)\right]$$

$$= \cos\left[\frac{\pi}{2}\left(\frac{1}{1 - \frac{1}{2}}\right)\right] + i\sin\left[\frac{\pi}{2}\left(\frac{1}{1 - \frac{1}{2}}\right)\right] = \cos\pi + i\sin\pi = -1$$

Example 2.69 **If $z = (\sqrt{3} + i)^{17}/(1 - i)^{50}$, then find amp(z).**

Sol. $z = \dfrac{(\sqrt{3} + i)^{17}}{(1 - i)^{50}} = \dfrac{\left[2\left(\dfrac{\sqrt{3}}{2} + \dfrac{i}{2}\right)\right]^{17}}{\left[\sqrt{2}\left(\dfrac{1}{\sqrt{2}} - \dfrac{i}{\sqrt{2}}\right)\right]^{50}}$

$$= \frac{2^{17}\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)^{17}}{2^{25}\left(\cos\frac{\pi}{4} - i\sin\frac{\pi}{4}\right)^{50}}$$

$$= \frac{\left(\cos\frac{17\pi}{6} + i\sin\frac{17\pi}{6}\right)}{2^8\left(\cos\frac{50\pi}{4} - i\sin\frac{50\pi}{4}\right)}$$

$$= \frac{\left[\cos\left(2\pi + \frac{5\pi}{6}\right) + i\sin\left(2\pi + \frac{5\pi}{6}\right)\right]}{2^8\left[\cos\left(12\pi + \frac{\pi}{2}\right) - i\sin\left(12\pi + \frac{\pi}{2}\right)\right]}$$

$$= \frac{\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right)}{2^8\left(\cos\frac{\pi}{2} - i\sin\frac{\pi}{2}\right)}$$

$$= \frac{\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right)}{2^8\left[\cos\left(-\frac{\pi}{2}\right) + i\sin\left(-\frac{\pi}{2}\right)\right]}$$

Hence,

$$\arg(z) = \frac{5\pi}{6} - \left(-\frac{\pi}{2}\right) = \frac{4\pi}{3}$$

Thus z lies in the third quadrant and principal argument is $-2\pi/3$.

Example 2.70 **If $z = x + iy$ is a complex number with $x, y \in Q$ and $|z| = 1$, then show that $|z^{2n} - 1|$ is a rational number for every $n \in N$.**

Sol. $|z| = 1 \Rightarrow z = e^{i\theta} = x + iy$

$\Rightarrow \quad x = \cos\theta, y = \sin\theta$

Now $\cos\theta$ and $\sin\theta \in Q$. Also,

$$|z^{2n} - 1|^2 = (z^{2n} - 1)(\bar{z}^{2n} - 1)$$
$$= (z\bar{z})^{2n} - z^{2n} - \bar{z}^{2n} + 1$$
$$= 2 - (z^{2n} + \bar{z}^{2n})$$
$$= 2 - 2\cos 2n\theta = 4\sin^2 n\theta$$

$\Rightarrow \quad |z^{2n} - 1| = 2|\sin n\theta|$

Now,

$$\sin n\theta = {}^nC_1 \cos^{n-1}\theta \sin\theta - {}^nC_3 \cos^{n-3}\theta \sin^3\theta + \cdots$$
$= $ Rational number $\quad$ ($\because \sin\theta, \cos\theta$ are rationals)

$\Rightarrow \quad |z^{2n} - 1| = $ Rational number

Example 2.71 **If $z = \cos\theta + i\sin\theta$ be a root of the equation $a_0 z^n + a_1 z^{n-1} + a_2 z^{n-2} + \cdots + a_{n-1} z + a_n = 0$, then prove that**

(i) $a_0 + a_1\cos\theta + a_2\cos 2\theta + \cdots + a_n\cos n\theta = 0$

(ii) $a_1\sin\theta + a_2\sin 2\theta + \cdots + a_n\sin n\theta = 0$

Sol. Dividing the given equation by z^n, we get

$$a_0 + a_1 z^{-1} + a_2 z^{-2} + \cdots + a_{n-1} z^{1-n} + a_n z^{-n} = 0$$

Now, $z = \cos\theta + i\sin\theta = e^{i\theta}$ satisfies the above equation. Hence,

$$a_0 + a_1 e^{-i\theta} + a_2 e^{-2i\theta} + \cdots + a_{n-1} e^{-i(n-1)\theta} + a_n e^{-in\theta} = 0$$

$\Rightarrow \quad (a_0 + a_1\cos\theta + a_2\cos 2\theta + \cdots + a_n\cos n\theta)$
$+ i(a_1\sin\theta + a_2\sin 2\theta + \cdots + a_n\sin n\theta) = 0$

$\Rightarrow a_0 + a_1\cos\theta + a_2\cos 2\theta + \cdots + a_n\cos n\theta = 0$

and

$$a_1\sin\theta + \theta_2\sin 2\theta + \cdots + a_n\sin n\theta = 0$$

Concept Application Exercise 2.4

1. Express the following in $a + ib$ form
   a. $\dfrac{(\cos\alpha + i\sin\alpha)^4}{(\sin\beta + i\cos\beta)^5}$
   b. $\left(\dfrac{1+\cos\phi + i\sin\phi}{1+\cos\phi - i\sin\phi}\right)^n$
   c. $\dfrac{(\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta)}{(\cos\gamma + i\sin\gamma)(\cos\delta + i\sin\delta)}$
2. If $(1/x) + x = 2\cos\theta$, then prove that $x^n + 1/x^n = 2\cos n\theta$.
3. Find the value of the following expression:
   $$\left[\frac{1-\cos\frac{\pi}{10} + i\sin\frac{\pi}{10}}{1-\cos\frac{\pi}{10} - i\sin\frac{\pi}{10}}\right]^{10}$$
4. If $iz^4 + 1 = 0$, then prove that z can take the value $\cos\pi/8 + i\sin\pi/8$.
5. If n is a positive integer, then prove that $(1 + i)^n + (1 - i)^n = (\sqrt{2})^{n+2}\cos\left(\frac{n\pi}{4}\right)$
6. If $\cos\alpha + \cos\beta + \cos\gamma = 0$ and also $\sin\alpha + \sin\beta + \sin\gamma = 0$, then prove that
   a. $\cos 2\alpha + \cos 2\beta + \cos 2\gamma = \sin 2\alpha + \sin 2\beta + \sin 2\gamma = 0$
   b. $\sin 3\alpha + \sin 3\beta + \sin 3\gamma = 3\sin(\alpha + \beta + \gamma)$
   c. $\cos 3\alpha + \cos 3\beta + \cos 3\gamma = 3\cos(\alpha + \beta + \gamma)$

CUBE ROOTS OF UNITY

Let $z = 1^{1/3}$. Then,

$$z^3 = 1$$
$\Rightarrow \quad z^3 - 1 = 0$
$\Rightarrow \quad (z - 1)(z^2 + z + 1) = 0$
$\Rightarrow \quad z - 1 = 0$, or $z^2 + z + 1 = 0$
$\Rightarrow \quad z = 1$ or $z = \dfrac{-1 \pm \sqrt{1-4}}{2}$
$\Rightarrow \quad z = 1$ or $z = \dfrac{-1 \pm i\sqrt{3}}{2}$

So, the cube roots of unity are 1, $(-1+i\sqrt{3})/2$ and $(-1-i\sqrt{3})/2$. Clearly, one of the roots of unity is real and the other two are complex.

Properties of Cube Roots of Unity

1. Each complex cube root of unity is the square of the other.

Proof:

Complex cube roots of unity are $(-1+i\sqrt{3})/2$ and $(-1-i\sqrt{3})/2$.

Now,

$$\left(\frac{-1+i\sqrt{3}}{2}\right)^2 = \frac{1-2i\sqrt{3}+3i^2}{4} = \frac{1-2i\sqrt{3}-3}{4} = \left(\frac{-1-i\sqrt{3}}{2}\right)$$

$$\left(\frac{-1-i\sqrt{3}}{2}\right)^2 = \frac{1+2i\sqrt{3}+3i^2}{4} = \frac{1+2i\sqrt{3}-3}{4} = \left(\frac{-1+i\sqrt{3}}{2}\right)$$

Hence, each complex cube root of unity is the square of the other.

It follows from the above discussion that if we denote one of the complex cube roots of unity by ω (omega), then the other complex cube root of unity is ω^2. Let,

$$\omega = (-1+i\sqrt{3})/2$$

Then

$$\omega^2 = (-1-i\sqrt{3})/2$$

Clearly, $\bar{\omega} = \omega^2$ and $\overline{\omega^2} = \omega$.

2. Integral powers of ω

Since ω is a root of the equation $z^3 - 1 = 0$, so, it satisfies the equation $z^3 - 1 = 0$. Therefore,

$$\omega^3 - 1 = 0 \Rightarrow \omega^3 = 1$$

Since $\omega^3 = 1$, therefore $\omega^n = \omega^r$, where r is the least non-negative remainder obtained by dividing n by 3. For example, $\omega^{18} = (\omega^3)^6 = 1^6 = 1$, $\omega^{20} = (\omega^3)^6\omega^2 = 1^6\omega^2 = \omega^2$, $\omega^{-30} = (\omega^3)^{-10} = 1^{-10} = 1$, $\omega^{28} = (\omega^{27})\,\omega = \omega$.

3. The sum of three cube roots of unity is zero, i.e., $1 + \omega + \omega^2 = 0$.

Proof:

We have,

$$1 + \omega + \omega^2 = 1 + \left(\frac{-1+i\sqrt{3}}{2}\right) + \left(\frac{-1-i\sqrt{3}}{2}\right)$$

$$= \frac{2-1+i\sqrt{3}-1-i\sqrt{3}}{2} = 0$$

4. The product of three cube roots of unity is 1.

Proof:

Three cube roots of unity are 1, ω and ω^2. So, product of cube roots of unity is $1 \times \omega \times \omega^2 = \omega^3 = 1$.

5. Each complex cube root of unity is the reciprocal of the other.

Proof:

We have,

$$\omega \times \omega^2 = \omega^3 = 1 \Rightarrow \omega = \frac{1}{\omega^2} \text{ and } \omega^2 = \frac{1}{\omega}$$

6. Cube roots of -1 are -1, $-\omega$ and $-\omega^2$.

Proof:

$$z = (-1)^{1/3}$$

$$\Rightarrow z^3 = -1$$

$$\Rightarrow (-z)^3 = 1$$

$$\Rightarrow -z = 1, \omega, \omega^2$$

$$\Rightarrow z = -1, -\omega, -\omega^2$$

The idea of finding cube roots of 1 and -1 can be extended to find cube roots of any real number. If a is any positive real number then $a^{1/3}$ has values $a^{1/3}$, $a^{1/3}\,\omega$ and $a^{1/3}\,\omega^2$. If a is a negative real number, then $a^{1/3}$ has values $-|a|^{1/3}$, $-|a|^{1/3}\,\omega$ and $-|a|^{1/3}\,\omega^2$. For example, $8^{1/3}$ has values 2, 2ω and $2\omega^2$ whereas $(-8)^{1/3}$ attains values -2, -2ω and $-2\omega^2$.

7. If $1, \omega, \omega^2$ be cube roots of unity and n is a positive integer, then

$$1 + \omega^n + \omega^{2n} = \begin{cases} 3, & \text{when } n \text{ is a multiple of 3} \\ 0, & \text{when } n \text{ is not a multiple of 3} \end{cases}$$

Proof:

Case I: n is a multiple of 3.

In this case, $n = 3m$ for some positive integer m.

$$\therefore \quad 1 + \omega^n + \omega^{2n} = 1 + \omega^{3m} + \omega^{6m} = 1 + (\omega^3)^m + (\omega^3)^{2m}$$

$$= 1 + 1 + 1 = 3 \quad [\because (\omega^3)^m = 1^m = 1 \text{ and } (\omega^3)^{2m} = 1^{2m} = 1]$$

Case II: n is not a multiple of 3.

In this case, $n = 3m + 1$ or $n = 3m + 2$ for some positive integer m. When $n = 3m + 1$,

$$\begin{aligned} 1 + \omega^n + \omega^{2n} &= 1 + \omega^{3m+1} + \omega^{6m+2} \\ &= 1 + \omega^{3m}\omega + \omega^{6m}\,\omega^2 \\ &= 1 + (\omega^3)^m\omega + (\omega^3)^{2m}\,\omega^2 \\ &= 1 + \omega + \omega^2 = 0 \end{aligned}$$

When $n = 3m + 2$,

$$\begin{aligned} 1 + \omega^n + \omega^{2n} &= 1 + \omega^{3m+2} + \omega^{6m+4} \\ &= 1 + \omega^{3m}\omega^2 + \omega^{6m}\,\omega^4 \\ &= 1 + (\omega^3)^m\,\omega^2 + (\omega^3)^{2m}\omega^3\omega \\ &= 1 + \omega^2 + \omega = 0 \end{aligned}$$

8. Factorization of $a^3 + b^3$ and $a^3 - b^3$

$$\begin{aligned} a^3 + b^3 &= (a + b)(a^2 - ab + b^2) \\ &= (a + b)(a + b\omega)(a + b\omega^2) \end{aligned}$$

and

$$\begin{aligned} a^3 - b^3 &= (a - b)(a^2 + ab + b^2) \\ &= (a - b)(a - b\omega)(a - b\omega^2) \end{aligned}$$

9. Factorization of $a^3 + b^3 + c^3 - 3abc$

$$a^3 + b^3 + c^3 - 3abc = (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca) = (a + b + c)\,(a + b\omega + c\omega^2)\,(a + b\omega^2 + c\omega)$$

10. Cube roots of unity represent vertices of equilateral triangle on the Argand plane.

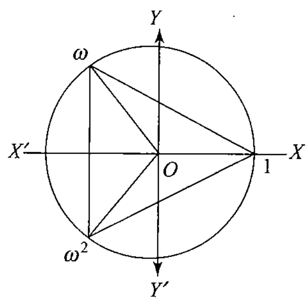


Fig. 2.21

Example 2.72 **If ω is a cube root of unity, then find the value of the following:**

(i) $(1+\omega-\omega^2)(1-\omega+\omega^2)$

(ii) $(1-\omega)(1-\omega^2)(1-\omega^4)(1-\omega^8)$

(iii) $\dfrac{a+b\omega+c\omega^2}{b+c\omega+a\omega^2}+\dfrac{a+b\omega+c\omega^2}{c+a\omega+b\omega^2}$

Sol.

(i) If ω is a complex cube root of unity, then $\omega^3=1$ and $1+\omega+\omega^2=0$.

$\therefore\quad (1+\omega-\omega^2)(1-\omega+\omega^2)=(-2\omega^2)(-2\omega)=4$

(ii) $(1-\omega)(1-\omega^2)(1-\omega^4)(1-\omega^8)$
$=(1-\omega)(1-\omega^2)(1-\omega)(1-\omega^2)$
$=(1-\omega)^2(1-\omega^2)^2$
$=(1-2\omega+\omega^2)(1-2\omega^2+\omega^4)$
$=(1-2\omega+\omega^2)(1-2\omega^2+\omega)$
$=(-3\omega)(-3\omega^2)=9\omega^3=9$

(iii) Multiplying the numerator and denominator of expressions I and II by ω and ω^2, respectively, we have

$$\frac{a+b\omega+c\omega^2}{b+c\omega+a\omega^2}+\frac{a+b\omega+c\omega^2}{c+a\omega+b\omega^2}$$
$$=\frac{\omega(a+b\omega+c\omega^2)}{(b\omega+c\omega^2+a)}+\frac{\omega^2(a+b\omega+c\omega^2)}{(c\omega^2+a+b\omega)}$$
$$=\omega+\omega^2=-1$$

Example 2.73 **Solve the equation $(x-1)^3+8=0$ in the set C of all complex numbers.**

Sol. We have,

$(x-1)^3+8=0\quad\Rightarrow(x-1)^3=-8\quad\Rightarrow x-1=(-8)^{1/3}$
$\Rightarrow\quad x-1=2(-1)^{1/3}$
$\Rightarrow\quad x-1=2(-1)$ or $x-1=2(-\omega)$ or $x-1=2(-\omega^2)$
$[\because (-1)^{1/3}=-1 \text{ or } -\omega \text{ or } -\omega^2]$
$\Rightarrow\quad x-1=-2$ or $x-1=-2\omega$ or $x-1=-2\omega^2$
$\Rightarrow\quad x=-1$ or $x=1-2\omega$ or $x=1-2\omega^2$

Hence, the solutions of the equation $(x-1)^3+8=0$ are -1, $1-2\omega$ and $1-2\omega^2$.

Example 2.74 **If n is an odd integer that is greater than or equal to 3 but not a multiple of 3, then prove that $(x+1)^n-x^n-1$ is divisible by x^3+x^2+x.**

Sol. Let

$f(x)=(x+1)^n-x^n-1$
$x^3+x^2+x=x(x^2+x+1)=x(x-\omega)(x-\omega^2)$
$f(0)=(0+1)^n-0^n-1=0$

$f(\omega)=(\omega+1)^n-\omega^n-1=(-\omega^2)^n-\omega^n-1=-(\omega^{2n}+\omega^n+1)=0$
when n is not a multiple of 3.

$f(\omega^2)=(\omega^2+1)^n-\omega^{2n}-1=(-\omega)^n-\omega^{2n}-1=-(\omega^n+\omega^{2n}+1)=0$
when n is not a multiple of 3.

Example 2.75 **ω is an imaginary root of unity. Prove that**

(i) $(a+b\omega+c\omega^2)^3+(a+b\omega^2+c\omega)^3=(2a-b-c)(2b-a-c)(2c-a-b)$,

(ii) If $a+b+c=0$, then prove that $(a+b\omega+c\omega^2)^3+(a+b\omega^2+c\omega)^3=27abc$.

Sol.

(i) Let $a+b\omega+c\omega^2=x$ and $a+b\omega^2+c\omega=y$.

$\therefore\quad (a+b\omega+c\omega^2)^3+(a+b\omega^2+c\omega)^3=x^3+y^3$
$=(x+y)(x+\omega y)(x+\omega^2 y)$

Now,

$x+y=(a+b\omega+c\omega^2)+(a+b\omega^2+c\omega)$
$=2a+b(\omega+\omega^2)+c(\omega+\omega^2)$
$=2a-b-c$
$x+\omega y=(a+b\omega+c\omega^2)+\omega(a+b\omega^2+c\omega)$
$=(1+\omega)a+(1+\omega)b+2\omega^2c$
$=\omega^2(2c-a-b)$

Similarly,
$x+\omega^2y=\omega(2b-a-c)$
$\Rightarrow\quad (x+y)(x+\omega y)(x+\omega^2y)$
$=\omega^3(2a-b-c)(2c-a-b)(2b-a-c)$
$=(2a-b-c)(2c-a-b)(2b-a-c)$

(ii) $a+b+c=0\Rightarrow b+c=-a,\ c+a=-b$ and $a+b=-c$

Putting these values on the R.H.S. of result (i), we get
$(a+b\omega+c\omega^2)^3+(a+b\omega^2+c\omega)^3=27abc$

Example 2.76 **Find the complex number ω satisfying the equation $z^3=8i$ and lying in the second quadrant on the complex plane.**

Sol. $z^3=8i$

$\Rightarrow\quad z^3=-8i^3$

$\Rightarrow\quad \left(\dfrac{z}{-2i}\right)^3=1$

$\Rightarrow\quad \dfrac{z}{-2i}=1$ or ω or ω^2

$\Rightarrow\quad z=-2i$ or $-2i\omega$ or $-2i\omega^2$

$\Rightarrow\quad z=-2i$ or $-2i\left(\dfrac{-1+\sqrt{3}i}{2}\right)$ or $-2i\left(\dfrac{-1-\sqrt{3}i}{2}\right)$

Hence, $z=-2i$ or $i+\sqrt{3}$ or $i-\sqrt{3}$ out of which $i-\sqrt{3}$ lies in second quadrant.

Example 2.77 **When the polynomial $5x^3+Mx+N$ is divided by x^2+x+1, the remainder is 0. Then find the value of $M+N$.**

Sol. Let $f(x)=5x^3+Mx+N$.

Also, $x^2+x+1=(x-\omega)(x-\omega^2)$

$f(x)$ is divisible by x^2+x+1. Hence, $f(\omega)=5+M\omega+N=0$

and

$$f(\omega^2) = 5 + M\omega^2 + N = 0$$

$\Rightarrow$ $M = 0; N = -5$

$\Rightarrow$ $M + N = -5$

Example 2.78 **If ω and ω^2 are the non-real cube roots of unity and $[1/(a+\omega)] + [1/(b+\omega)] + [1/(c+\omega)] = 2\omega^2$ and $[1/(a+\omega)^2] + [1/(b+\omega)^2] + [1/(c+\omega)^2] = 2\omega$, then find the value of $[1/(a+1)] + [1/(b+1)] + [1/(c+1)]$.**

Sol. The given relations can be rewritten as

$$\frac{1}{a+\omega} + \frac{1}{b+\omega} + \frac{1}{c+\omega} = \frac{2}{\omega}$$

and

$$\frac{1}{a+\omega^2} + \frac{1}{b+\omega^2} + \frac{1}{c+\omega^2} = \frac{2}{\omega^2}$$

$\Rightarrow$ ω and ω^2 are roots of $\frac{1}{a+x} + \frac{1}{b+x} + \frac{1}{c+x} = \frac{2}{x}$

$$\Rightarrow \frac{3x^2 + 2(a+b+c)x + bc + ca + ab}{(a+x)(b+x)(c+x)} = \frac{2}{x}$$

$$\Rightarrow x^3 + (bc + ca + ab)x - 2abc = 0 \quad (1)$$

Two roots of the Eq. (1) are ω and ω^2. Let the third root be α. Then,

$$\alpha + \omega + \omega^2 = 0 \Rightarrow \alpha = -\omega - \omega^2 = 1$$

Therefore, $\alpha = 1$ will satisfy Eq. (1). Hence,

$$\frac{1}{a+1} + \frac{1}{b+1} + \frac{1}{c+1} = 2$$

Concept Application Exercise 2.5

1. If α and β are imaginary cube roots of unity, then find the value of $\alpha^4 + \beta^4 + 1/(\alpha\beta)$.
2. If ω is a complex cube root of unity, then find the value of $(1+\omega)(1+\omega^2)(1+\omega^4)(1+\omega^8)\cdots$ to $2n$ factors.
3. Write the complex number in $a + ib$ form using cube roots of unity:

   a. $\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)^{1000}$

   b. If $z = \frac{(\sqrt{3}+i)^{17}}{(1-i)^{50}}$

   c. $(i+\sqrt{3})^{100} + (i-\sqrt{3})^{100} + 2^{100}$
4. If $z + z^{-1} = 1$, then find the value of $z^{100} + z^{-100}$.
5. Find the common roots of $x^{12} - 1 = 0, x^4 + x^2 + 1 = 0$.
6. If ω $(\neq 1)$ is a cube root of unity, then find the value of

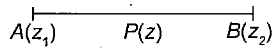
$$\begin{vmatrix} 1 & 1+\omega^2 & \omega^2 \\ 1-i & -1 & \omega^2 - 1 \\ -i & -1+\omega & -1 \end{vmatrix}$$

GEOMETRY WITH COMPLEX NUMBERS

Section Formula

If $P(z)$ divides the line segment joining $A(z_1)$ and $B(z_2)$ internally in the ratio $m:n$, then

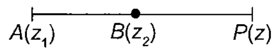
$$z = \frac{mz_2 + nz_1}{m+n}$$



$$\frac{AP}{PB} = \frac{m}{n}$$

If division is external, then

$$z = \frac{mz_2 - nz_1}{m-n}$$



Explanation:

Let $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$. Then, $A \equiv (x_1, y_1)$ and $B \equiv (x_2, y_2)$. Let $z = x + iy$. Then $P \equiv (x, y)$. We have from coordinate geometry.

$$x = \frac{mx_2 + nx_1}{m+n} \text{ and } y = \frac{my_2 + ny_1}{m+n}$$

Hence complex number of P is

$$z = \frac{mx_2 + nx_1}{m+n} + i\frac{my_2 + ny_1}{m+n}$$

$$= \frac{m(x_2 + iy_2) + n(x_1 + iy_1)}{m+n}$$

$$= \frac{mz_2 + nz_1}{m+n}$$

Note:

1. *In acute triangle, orthocentre* (H), *centroid* (G) *and circumcentre* (O) *are collinear and* $HG:GO \equiv 2:1$.
2. *Centroid of the triangle formed by points* $A(z_1)$, $B(z_2)$ *and* $C(z_3)$ *is* $(z_1 + z_2 + z_3)/3$.
3. *If circumcentre of the triangle is origin, then its orthocentre is* $z_1 + z_2 + z_3$ (using 1).

Example 2.79 **Find the relation if z_1, z_2, z_3, z_4 are the affixes of the vertices of a parallelogram taken in order.**

Sol. As the diagonals of a parallelogram bisect each other, affix of the mid-point of AC is same as the affix of the mid-point of BD, i.e.,

$$\frac{z_1 + z_3}{2} = \frac{z_2 + z_4}{2}$$

$$\Rightarrow z_1 + z_3 = z_2 + z_4$$

Example 2.80 **If z_1, z_2, z_3 are three non-zero complex numbers such that $z_3 = (1-\lambda)z_1 + \lambda z_2$ where $\lambda \in R - \{0\}$, then points corresponding to z_1, z_2 and z_3.**

a. **lie on a circle** **b.** **are vertices of a triangle**
c. **are collinear** **d.** **none of these**

Sol. $z_3 = (1 - \lambda)z_1 + z_2$

$$= \frac{(1-\lambda)z_1 + \lambda z_2}{1 - \lambda + \lambda}$$

Hence, z_3 divides the line joining $A(z_1)$ and $B(z_2)$ in the ratio $\lambda: (1 - \lambda)$. Thus the given points are collinear.

Example 2.81 **In ΔABC, $A(z_1)$, $B(z_2)$ and $C(z_3)$ are inscribed in the circle $|z| = 5$. If $H(z_H)$ be the orthocentre of triangle ABC, then find z_H.**

Sol. Circumcentre of ΔABC is clearly origin. Let $G(z_G)$ be its centroid. Then,

$$z_G = \frac{1}{3}(z_1 + z_2 + z_3)$$

Now we know that

$OG:GH = 1:2$

$$\Rightarrow \quad z_G = \frac{2\times 0 + 1\times z_H}{3}$$

$$\Rightarrow \quad z_H = 3z_G = z_1 + z_2 + z_3$$

Example 2.82 **Let z_1, z_2, z_3 be three complex numbers and a, b, c be real numbers not all zero, such that $a + b + c = 0$ and $az_1 + bz_2 + cz_3 = 0$. Show that z_1, z_2, z_3 are collinear.**

Sol. Given,

$$a + b + c = 0 \qquad \text{(i)}$$

and

$$az_1 + bz_2 + cz_3 = 0 \qquad \text{(ii)}$$

Since a, b, c are not all zero, from (ii), we have

$$az_1 + bz_2 - (a + b)z_3 = 0 \qquad \text{[From (i), } c = -(a + b)\text{]}$$

$$\Rightarrow \quad az_1 + bz_2 = (a + b)z_3$$

$$\Rightarrow \quad z_3 = \frac{az_1 + bz_2}{a + b} \qquad \text{(iii)}$$

From (iii), it follows that z_3 divides the line segment joining z_1 and z_2 internally in the ratio $b:a$.

If a and b are of the same sign, then division is in fact internal and if a and b are of opposite sign, then division is external in the ratio $|b|:|a|$.

Equation of the line passing through the points z_1 and z_2

Such equations are given by

$$\begin{vmatrix} z & \bar{z} & 1 \\ z_1 & \bar{z}_1 & 1 \\ z_2 & \bar{z}_2 & 1 \end{vmatrix} = 0$$

or

$$\frac{z - z_1}{\bar{z} - \bar{z}_1} = \frac{z_1 - z_2}{\bar{z}_1 - \bar{z}_2}$$

Explanation:

Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$. Let A and B be the points representing z_1 and z_2, respectively.

Let $P(z)$ be any point on the line joining A and B. Let $z = x + iy$. Then $P \equiv (x, y)$, $A \equiv (x_1, y_1)$ and $B \equiv (x_2, y_2)$. Points P, A, B are collinear.

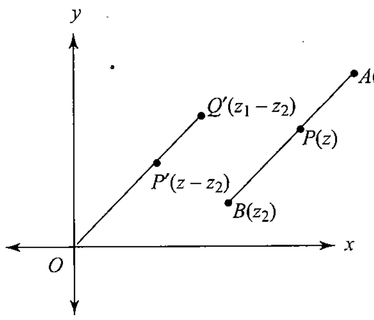


Fig. 2.22

From the diagram, points A, P and B are collinear.

Shifting the line AB at origin as shown in the figure, points O, P' and Q' are collinear. Hence,

$$\arg(z - z_2) = \arg(z_1 - z_2)$$

$$\Rightarrow \quad \arg\left(\frac{z - z_2}{z_1 - z_2}\right) = 0$$

$$\Rightarrow \quad \frac{z - z_2}{z_1 - z_2} \text{ is purely real}$$

$$\Rightarrow \quad \frac{z - z_2}{z_1 - z_2} = \overline{\left(\frac{z - z_2}{z_1 - z_2}\right)}$$

$$\Rightarrow \quad \frac{z - z_2}{z_1 - z_2} = \frac{\bar{z} - \bar{z}_2}{\bar{z}_1 - \bar{z}_2} \qquad (1)$$

$$\Rightarrow \quad \begin{vmatrix} z & \bar{z} & 1 \\ z_1 & \bar{z}_1 & 1 \\ z_2 & \bar{z}_2 & 1 \end{vmatrix} = 0 \qquad (2)$$

From (2), if points z_1, z_2, z_3 are collinear, then

$$\begin{vmatrix} z_1 & \bar{z}_1 & 1 \\ z_2 & \bar{z}_2 & 1 \\ z_3 & \bar{z}_3 & 1 \end{vmatrix} = 0$$

Equation (2) can also be written as

$$(\bar{z}_1 - \bar{z}_2)z - (z_1 - z_2)\bar{z} + z_1\bar{z}_2 - z_2\bar{z}_1 = 0$$

$$\Rightarrow \quad i(\bar{z}_1 - \bar{z}_2)z - i(z_1 - z_2)\bar{z} + i(z_1\bar{z}_2 - z_2\bar{z}_1) = 0$$

$$\Rightarrow \quad \bar{a}z + a\bar{z} + b = 0 \qquad (3)$$

where

$$a = -i(z_1 - z_2)$$

and

$$b = i(z_1\bar{z}_2 - z_2\bar{z}_1)$$
$$= i\,2i \times \text{Im}(z_1\bar{z}_2)$$
$$= -2 \times \text{Im}(z_1\bar{z}_2)$$
$$= \text{a real number}$$

Slope of the Given Line

In Eq. (3), replacing z by $x + iy$, we get

$$(x + iy)\bar{a} + (x - iy)a + b = 0$$

$$\Rightarrow (a + \bar{a})x + iy(\bar{a} - a) + b = 0$$

$$\therefore \text{ Slope} = \frac{a+\bar{a}}{i(a-\bar{a})} = \frac{2\text{Re}(a)}{2i^2 \times \text{Im}(a)} = -\frac{\text{Re}(a)}{\text{Im}(a)}$$

Equation of a line parallel to the line $z\bar{a} + \bar{z}a + b = 0$ is $z\bar{a} + \bar{z}a + \lambda = 0$ (where λ is a real number).

Equation of a line perpendicular to the line $z\bar{a} + \bar{z}a + b = 0$ is $z\bar{a} + \bar{z}a + i\lambda = 0$ (where λ is a real number).

Equation of Perpendicular Bisector

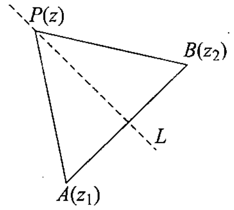


Fig. 2.23

Consider a line segment joining $A(z_1)$ and $B(z_2)$. Let the line 'L' be its perpendicular bisector. If $P(z)$ be any point on 'L', then we have

$$PA = PB \Rightarrow |z - z_1| = |z - z_2|$$

$$\Rightarrow |z - z_1|^2 = |z - z_2|^2$$

$$\Rightarrow (z - z_1)(\bar{z} - \bar{z}_1) = (z - z_2)(\bar{z} - \bar{z}_2)$$

$$\Rightarrow z\bar{z} - z\bar{z}_1 - z_1\bar{z} + z_1\bar{z}_1 = z\bar{z} - z\bar{z}_2 - z_2\bar{z} + z_2\bar{z}_2$$

$$\Rightarrow z(\bar{z}_2 - \bar{z}_1) + \bar{z}(z_2 - z_1) + z_1\bar{z}_1 - z_2\bar{z}_2 = 0$$

Here, $a = z_2 - z_1$ and $b = z_1\bar{z}_1 - z_2\bar{z}_2$.

Distance of a Given Point from a Given Line

Let the given line be $z\bar{a} + \bar{z}a + b = 0$ and the given point be z_c

$z_c = x_c + iy_c$

Replacing z by $x + iy$ in the given equation, we get

$$x(a + \bar{a}) + iy(\bar{a} - a) + b = 0$$

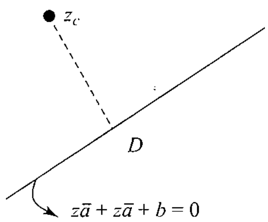


Fig. 2.24

Distance of (x_c, y_c) from this line is

$$\frac{|x_c(a+\bar{a}) + iy_c(\bar{a} - a) + b|}{\sqrt{(a+\bar{a})^2 - (a-\bar{a})^2}} = \frac{|z_c\bar{a} + \bar{z}_c a + b|}{\sqrt{4(\text{Re}(a))^2 + 4(\text{Im}(a))^2}}$$

$$= \frac{|z_c\bar{a} + \bar{z}_c a + b|}{2|a|}$$

Example 2.83 **Let $A(z_1)$ and (z_2) represent two complex numbers on the complex plane. Suppose the complex slope of the line joining A and B is defined as $(z_1 - z_2)/(\bar{z}_1 - \bar{z}_2)$. If the line l_1 with complex slope ω_1 and l_2 with complex slope ω_2 on the complex plane are perpendicular then prove that $\omega_1 + \omega_2 = 0$.**

Sol. l_1 is perpendicular to l_2. Hence, $(z_1 - z_2)/(\bar{z}_1 - \bar{z}_2)$ is purely imaginary.

$$\frac{z_1 - z_2}{z_3 - z_4} + \frac{\bar{z}_1 - \bar{z}_2}{\bar{z}_3 - \bar{z}_4} = 0$$

$$\frac{z_1 - z_2}{\bar{z}_1 - \bar{z}_2} + \frac{z_3 - z_4}{\bar{z}_3 - \bar{z}_4} = 0$$

$$\Rightarrow \omega_1 + \omega_2 = 0$$

Note:

If l_1 is parallel to l_2, then

$$\frac{z_1 - z_2}{z_3 - z_4} = \frac{\bar{z}_1 - \bar{z}_2}{\bar{z}_3 - \bar{z}_4}$$

$$\omega_1 = \omega_2$$

Example 2.84 **If z_1, z_2, z_3 are three complex numbers such that $5z_1 - 13z_2 + 8z_3 = 0$, then prove that**

$$\begin{vmatrix} z_1 & \bar{z}_1 & 1 \\ z_2 & \bar{z}_2 & 1 \\ z_3 & \bar{z}_3 & 1 \end{vmatrix} = 0$$

Sol. $5z_1 - 13z_2 + 8z_3 = 0$

$$\Rightarrow \frac{5z_1 + 8z_2}{5+8} = z_3$$

$\Rightarrow$ z_1, z_2, z_3 are collinear

$$\Rightarrow \begin{vmatrix} z_1 & \bar{z}_1 & 1 \\ z_2 & \bar{z}_2 & 1 \\ z_3 & \bar{z}_3 & 1 \end{vmatrix} = 0 \text{ (condition of collinear points)}$$

Example 2.85 **If $\bar{z} = \bar{z}_0 + A(z - z_0)$, where A is a constant, then prove that locus of z is a straight line.**

Sol. $\bar{z} = \bar{z}_0 + A(z - z_0)$

$$Az - \bar{z} - Az_0 + \bar{z}_0 = 0 \quad (1)$$

$$\bar{A}\bar{z} - z - \bar{A}\bar{z}_0 + z_0 = 0 \quad (2)$$

Adding (1) and (2),

$$(A - 1)z + (\bar{A} - 1)\bar{z} - (Az_0 + \bar{A}\bar{z}_0) + z_0 + \bar{z}_0 = 0$$

This is of the form $\bar{a}z + a\bar{z} + b = 0$, where $a = \bar{A} - 1$ and $b = -(Az_0 + \bar{A}\bar{z}_0) + z_0 + \bar{z}_0 \in R$. Hence locus of z is a straight line.

Equation of a Circle

Consider a fixed complex number z_0 and let z be any complex number which moves in such a way that its distance from z_0 is always to 'r'. This implies z would lie on a circle whose centre is z_0 and radius r. And its equation would be

$$|z - z_0| = r$$

$$\Rightarrow |z - z_0|^2 = r^2$$

$\Rightarrow \quad (z - z_0)(\bar{z} - \bar{z}_0) = r^2$

$\Rightarrow \quad z\bar{z} - z\bar{z}_0 - \bar{z}z_0 + z_0\bar{z}_0 - r^2 = 0$

Let $-a = z_0$ and $z_0\bar{z}_0 - r^2 = b$. Then,

$$z\bar{z} + a\bar{z} + \bar{a}z + b = 0$$

It represents the general equation of a circle in the complex plane.

Remark

$z\bar{z} + a\bar{z} + \bar{a}z + b = 0$ represents a circle whose centre is $-a$ and radius is $\sqrt{a\bar{a} - b}$. Thus $z\bar{z} + a\bar{z} + \bar{a}z + b = 0$ $(b \in R)$ represents a real circle if and only if $a\bar{a} - b \geq 0$.

Now, let us consider a circle described on a line segment AB $(A(z_1), B(z_2))$ as its diameter. Let $P(z)$ be any point on the circle. As the angle in the semicircle is $\pi/2$, so

$$\angle APB = \pi/2$$

$$\Rightarrow \quad \arg\left(\frac{z_1 - z}{z_2 - z}\right) = \pm\pi/2$$

$$\Rightarrow \quad \frac{z - z_1}{z - z_2} \text{ is purely imaginary}$$

$$\Rightarrow \quad \frac{z - z_1}{z - z_2} + \frac{\bar{z} - \bar{z}_1}{\bar{z} - \bar{z}_2} = 0$$

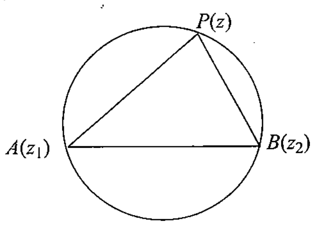


Fig. 2.25

$$\Rightarrow (z - z_1)(\bar{z} - \bar{z}_2) + (z - z_2)(\bar{z} - \bar{z}_1) = 0$$

Example 2.86 **Let vertices of acute angled triangle are $A(z_1)$, $B(z_2)$ and $C(z_3)$. If the origin 'O' is the orthocentre of the triangle, then prove that $z_1\bar{z}_2 + \bar{z}_1 z_2 = {}_2\bar{z}_3 + \bar{z}_2 z_3 = z_3\bar{z}_1 + \bar{z}_3 z_1$.**

Sol.

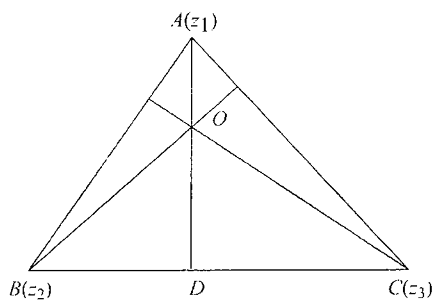


Fig. 2.26

Here O is orthocenter then

(AD=)OA ⊥ BC

$$\therefore \quad \arg\left(\frac{z_1 - 0}{z_2 - z_3}\right) = \frac{\pi}{2}$$

$$\therefore \quad \frac{z_1 - 0}{z_2 - z_3} \text{ is purely imaginary.}$$

$$\Rightarrow \quad \frac{z_1 - 0}{z_2 - z_3} + \overline{\left(\frac{z_1 - 0}{z_2 - z_3}\right)} = 0$$

$$\Rightarrow \quad \frac{z_1}{z_2 - z_3} + \frac{\bar{z}_1}{\bar{z}_2 - \bar{z}_3} = 0$$

$$\Rightarrow \quad z_1(\bar{z}_2 - \bar{z}_3) + \bar{z}_1(z_2 - z_3) = 0$$

$$\Rightarrow \quad z_1\bar{z}_2 + \bar{z}_1 z_2 = z_1\bar{z}_3 + \bar{z}_1 z_3 \qquad (1)$$

similarly OB ⊥ AC

$$\Rightarrow \quad z_1\bar{z}_2 + \bar{z}_1\bar{z}_2 = z_2\bar{z}_3 + \bar{z}_2 z_3 \qquad (2)$$

From (1) and (2) $z_1\bar{z}_2 + \bar{z}_1 z_2 = z_2\bar{z}_3 + \bar{z}_2 z_3 = z_1\bar{z}_3 + \bar{z}_1 z_3$

Example 2.87 **Show that the equation of a circle passing through the origin and having intercepts and on real and imaginary axis respectively on the argand plane is given by = (Re) + (Im).**

Sol. From figure,

$$\arg\left(\frac{z - a}{z - ib}\right) = \pm\frac{\pi}{2}$$

$$\Rightarrow \quad \frac{z - a}{z - ib} + \frac{\bar{z} - a}{\bar{z} + ib} = 0$$

$$\Rightarrow \quad z\bar{z} - a\left(\frac{z + \bar{z}}{2}\right) - b\left(\frac{z - \bar{z}}{2i}\right) = 0$$

$$\Rightarrow \quad z\bar{z} = a(\text{Re } z) + b(\text{Im } z)$$

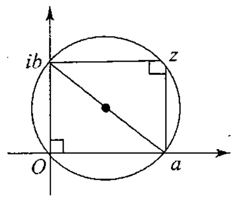


Fig. 2.27

Example 2.88 **Intercept made by the circle $z\bar{z} + \bar{a} + a\bar{z} + r = 0$ on the real axis on complex plane is**

a. $\sqrt{(a + \bar{a}) - r}$ **b.** $\sqrt{(a + \bar{a})^2 - 2r}$

c. $\sqrt{(a + \bar{a})^2 - 4r}$ **d.** $\sqrt{(a + \bar{a})^2 - 4r}$

Sol. Points where the circle cuts the x-axis $z = \bar{z}$.

Hence substituting $z = \bar{z}$ in the equation of circle, we get

$$z^2 + \bar{\alpha}z + \alpha z + r = 0$$

$$\Rightarrow \quad z^2 + (\alpha + \bar{\alpha})z + r = 0$$

$\Rightarrow \quad AB = |z_1 - z_2|$ (where A and B are points of intersection of circle with x-axis)

$$= \sqrt{(z_1 + z_2)^2 - 4z_1z_2}$$

$$= \sqrt{(\alpha + \bar{\alpha})^2 - 4r}$$

Example 2.89 **Prove that $|Z - Z_1|^2 + |Z - Z_2|^2 = a$ will represent a real circle [with centre $(Z_1 + Z_2)/2$] on the Argand plane if $2a \geq |Z_1 - Z_2|^2$.**

Sol. $|Z - Z_1|^2 + |Z - Z_2|^2 = a$

$$\Rightarrow \quad (Z - Z_1)(\bar{Z} - \bar{Z}_1) + (Z - Z_2)(\bar{Z} - \bar{Z}_2) = a$$

$$\Rightarrow \quad 2Z\bar{Z} - Z(\bar{Z}_1 + \bar{Z}_2) - \bar{Z}(Z_1 + Z_2) + Z_1\bar{Z}_1 + Z_2\bar{Z}_2 = a$$

$$\Rightarrow \quad z\bar{z} - \left(\frac{\bar{Z}_1 + \bar{Z}_2}{2}\right) Z - \left(\frac{Z_1 + Z_2}{2}\right)\bar{Z} + \frac{Z_1\bar{Z}_1 + Z_2\bar{Z}_2 - a}{2} = 0 \quad (1)$$

Equation (1) is of the form of $z\bar{z} + \bar{\alpha}Z + \alpha\bar{Z} + r = 0$. Hence centre $= -$ coefficient of $\bar{Z}$; which is given by $(Z_1 + Z_2)/2$. Also, Eq. (1) will represent a real circle if $\alpha\bar{\alpha} - r > 0$

$$\Rightarrow \quad \frac{(Z_1 + Z_2)(\bar{Z}_1 + \bar{Z}_2)}{4} \geq \frac{Z_1\bar{Z}_1 + Z_2\bar{Z}_2 - a}{2}$$

$$\Rightarrow \quad Z_1\bar{Z}_1 + Z_1\bar{Z}_2 + \bar{Z}_1Z_2 + Z_2\bar{Z}_2 > 2(Z_1\bar{Z}_1 + Z_2\bar{Z}_2) - 2a$$

$$\Rightarrow \quad 2a \geq Z_1\bar{Z}_1 + Z_2\bar{Z}_2 - Z_1\bar{Z}_2 - Z_2\bar{Z}_1$$

$$= Z_1(\bar{Z}_1 - \bar{Z}_2) - Z_2(\bar{Z}_1 - \bar{Z}_2)$$

$$= (Z_1 - Z_2)(\bar{Z}_1 - \bar{Z}_2)$$

$$= (Z_1 - Z_2)\overline{(Z_1 - Z_2)}$$

$$\Rightarrow \quad 2a \geq |Z_1 - Z_2|^2$$

Example 2.90 **Two different non-parallel lines cut the circle $|z| = r$ at points a, b, c and d, respectively. Prove that these lines meet at the point given by**

$$\frac{a^{-1} + b^{-1} - c^{-1} - d^{-1}}{a^{-1}b^{-1} - c^{-1}d^{-1}}$$

Sol.

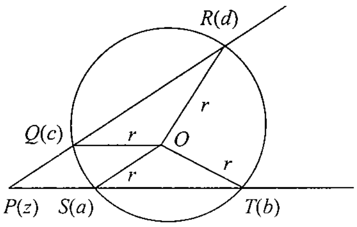


Fig. 2.28

Since P, Q, R are collinear, so

$$\begin{vmatrix} z & \bar{z} & 1 \\ c & \bar{c} & 1 \\ d & \bar{d} & 1 \end{vmatrix} = 0$$

$$\Rightarrow \quad z(\bar{c} - \bar{d}) - \bar{z}(c - d) + (c\bar{d} - \bar{c}d) = 0 \quad (1)$$

Similarly,

$$z(\bar{a} - \bar{b}) - \bar{z}(a - b) + (a\bar{b} - \bar{a}b) = 0 \quad (2)$$

From $\{(1) \times (a - b)\} - \{(2) \times (c - d)\}$,

$$z[(\bar{c} - \bar{d})(a - b) - (\bar{a} - \bar{b})(c - d)]$$

$$= (a\bar{b} - b\bar{a})(c - d) - (c\bar{d} - \bar{c}d)(a - b) \quad (3)$$

Now,

$$|a|^2 = a\bar{a} = r^2 \Rightarrow \bar{a} = \frac{r^2}{a}$$

Similarly,

$$\bar{b} = \frac{r^2}{b}, \bar{c} = \frac{r^2}{c}, \bar{d} = \frac{r^2}{d}$$

From (3),

$$z\left[\left(\frac{r^2}{c} - \frac{r^2}{d}\right)(a - b) - \left(\frac{r^2}{a} - \frac{r^2}{b}\right)(c - d)\right]$$

$$= \left(\frac{ar^2}{b} - \frac{br^2}{a}\right)(c - d) - \left(\frac{cr^2}{d} - \frac{dr^2}{c}\right)(a - b)$$

$$\Rightarrow \quad z\left[-\frac{1}{cd} + \frac{1}{ab}\right] = \frac{(a + b)}{ab} - \frac{c + d}{cd}$$

$$\Rightarrow \quad z = \frac{a^{-1} + b^{-1} - c^{-1} - d^{-1}}{a^{-1}b^{-1} - c^{-1}d^{-1}}$$

Example 2.91 **Prove that the circles $z\bar{z} + z\bar{a}_1 + \bar{z}a_1 + b_1 = 0$, $b_1 \in R$ and $z\bar{z} + z\bar{a}_2 + \bar{z}a_2 + b_2 = 0$, $b_2 \in R$ will intersect orthogonally if $2\,\mathrm{Re}(a_1\bar{a}_2) = b_1 + b_2$.**

Sol. Centre and radius of $z\bar{z} + z\bar{a}_1 + \bar{z}a_1 + b_1 = 0$ are $-a_1$ and $\sqrt{a_1\bar{a}_1 - b_1}$, respectively, and that for other circle are $-a_2$ and $\sqrt{a_2\bar{a}_2 - b_2}$, respectively. These circles will intersect orthogonally, if sum of squares of radii is equal to square of distance between their centres. Therefore,

$$|a_1 - a_2|^2 = a_1\bar{a}_1 - b_1 + a_2\bar{a}_2 - b_2$$

$$\Rightarrow \quad a_1\bar{a}_1 + a_2\bar{a}_2 - a_1\bar{a}_2 - \bar{a}_1a_2 = a_1\bar{a}_1 + a_2\bar{a}_2 - b_1 - b_2$$

$$\Rightarrow \quad a_1\bar{a}_2 + \bar{a}_1a_2 = b_1 + b_2$$

$$\Rightarrow \quad 2\mathrm{Re}(a_1\bar{a}_2) = b_1 + b_2$$

Condition for Four Points to be Concyclic

Let $ABCD$ be a cyclic quadrilateral such that $A(z_1)$, $B(z_2)$, $C(z_3)$ and $D(z_4)$ lie on a circle. Clearly,

$$\arg\left(\frac{z_4 - z_1}{z_2 - z_1}\right) + \arg\left(\frac{z_2 - z_3}{z_4 - z_3}\right) = \pi$$

$$\Rightarrow \quad \arg\left(\frac{z_4 - z_1}{z_2 - z_1}\right)\left(\frac{z_2 - z_3}{z_4 - z_3}\right) = \pi$$

$$\Rightarrow \quad \left(\frac{z_4 - z_1}{z_2 - z_1}\right)\left(\frac{z_2 - z_3}{z_4 - z_3}\right) \text{ is purely real}$$

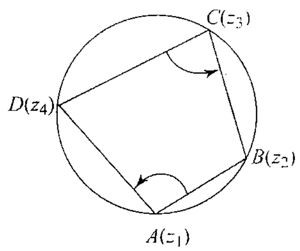


Fig. 2.29

Thus points $A(z_1)$, $B(z_2)$, $C(z_3)$ and $D(z_4)$ (taken in order) would be concyclic if $[(z_4 - z_1)(z_2 - z_3)]/[(z_2 - z_1)(z_4 - z_3)]$ is purely real.

Example 2.92 **If z_1, z_2, z_3 are complex numbers such that $(2/z_1) = (1/z_2) + (1/z_3)$, then show that the points represented by z_1, z_2, z_3 lie on a circle passing through the origin.**

Sol.

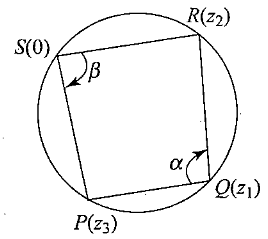


Fig. 2.30

$$\frac{2}{z_1} = \frac{1}{z_2} + \frac{1}{z_3}$$

$\Rightarrow \quad \frac{1}{z_1} - \frac{1}{z_2} = \frac{1}{z_3} - \frac{1}{z_1}$

$\Rightarrow \quad \frac{z_2 - z_1}{z_1 z_2} = \frac{z_1 - z_3}{z_3 z_1}$

$\Rightarrow \quad \frac{z_2 - z_1}{z_3 - z_1} = -\frac{z_2}{z_3}$

$\Rightarrow \quad \arg\left(\frac{z_2 - z_1}{z_3 - z_1}\right) = \arg\left(-\frac{z_2}{z_3}\right)$

$\Rightarrow \quad \arg\left(\frac{z_2 - z_1}{z_3 - z_1}\right) = \pi + \arg\left(\frac{z_2}{z_3}\right)$

$\Rightarrow \quad \arg\left(\frac{z_2 - z_1}{z_3 - z_1}\right) = \pi - \arg\left(\frac{z_3}{z_2}\right)$

$\Rightarrow \quad \alpha = \pi - \beta$

$\Rightarrow \quad \alpha + \beta = \pi$

Hence, the said points are concyclic.

Concept of Rotation

If z and z' are two complex numbers, then argument of z/z' is the angle through which Oz' must be turned in order that it may lie along Oz.

$$\frac{z}{z'} = \frac{|z|e^{i\theta}}{|z'|e^{i\theta'}} = \frac{|z|}{|z'|}e^{i(\theta - \theta')} = \frac{|z|}{|z'|}e^{i\alpha}$$

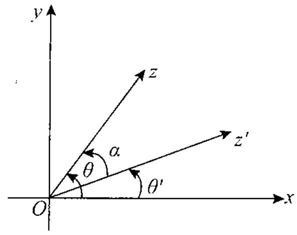


Fig. 2.31

In general, let z_1, z_2, z_3 be the three vertices of a triangle ABC described in the counterclockwise sense. Draw OP and OQ parallel and equal to AB and AC, respectively. Then the point P is $z_2 - z_1$ and Q is $z_3 - z_1$ and

$$\frac{z_3 - z_1}{z_2 - z_1} = \frac{OQ}{OP}(\cos\alpha + i\sin\alpha) = \frac{CA}{BA}e^{i\alpha} = \frac{|z_3 - z_1|}{|z_2 - z_1|}e^{i\alpha}$$

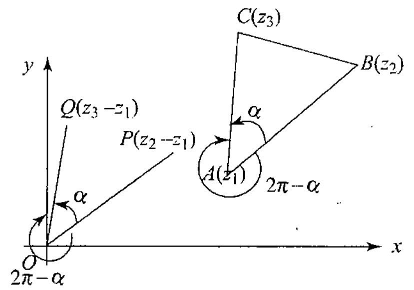


Fig. 2.32

Note that $\arg(z_3 - z_1) - \arg(z_2 - z_1) = \alpha$ the angle through which OP must be rotated in the anticlockwise direction so that it becomes parallel to OQ.

Also in this case we are rotating OP in clockwise direction by an angle $2\pi - \alpha$. Since the rotation is in clockwise direction, we are taking negative sign with angle $2\pi - \alpha$. Here, we can write

$$\frac{z_3 - z_1}{z_2 - z_1} = \frac{|z_3 - z_1|}{|z_2 - z_1|}e^{-i(2\pi - \alpha)}$$

Example 2.93 **$A(z_1)$, $B(z_2)$, $C(z_3)$ are the vertices of the triangle ABC (in anticlockwise order). If $\angle ABC = \pi/4$ and $AB = \sqrt{2}(BC)$, then prove that $z_2 = z_3 + i(z_1 - z_3)$.**

Sol.

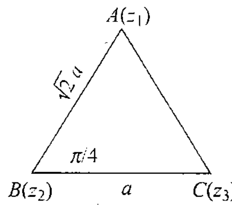


Fig. 2.33

Rotating about the point 'B', we get

$$\frac{z_1 - z_2}{z_3 - z_2} = \frac{a\sqrt{2}}{a}e^{i\pi/4} = \sqrt{2}\left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right) = (1 + i)$$

$\Rightarrow \quad z_1 - z_2 = (z_3 - z_2)(1 + i)$

$\Rightarrow \quad z_2 = z_1 - (z_3 - z_2)(1 + i)$

$\Rightarrow \quad z_2(1 - (1 + i)) = z_1 - z_3(1 + i)$

$\Rightarrow \quad z_2 = \frac{z_1}{-i} - \frac{z_3}{-i}(1 + i) = (iz_1 - iz_3(1 + i))$

$= z_3 + i(z_1 - z_3)$

Example 2.94 **If one vertex of a square whose diagonals intersect at the origin is $3(\cos\theta + i\sin\theta)$, then find the two adjacent vertices.**

Sol.

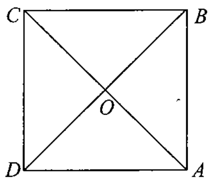


Fig. 2.34

Let the vertex A be $3(\cos\theta + i\sin\theta)$, then OB and OD can be obtained by rotating OA through $\pi/2$ and $-\pi/2$. Thus,

$$\overrightarrow{OB} = (\overrightarrow{OA})\, e^{i\pi/2} \text{ and } \overrightarrow{OD} = \overrightarrow{OA}\, e^{-i\pi/2}$$

$$\Rightarrow \quad \overrightarrow{OB} = 3(\cos\theta + i\sin\theta)i \text{ and } \overrightarrow{OD} = 3(\cos\theta + i\sin)(-i)$$

$$\Rightarrow \quad \overrightarrow{OB} = 3(-\sin\theta + i\cos\theta) \text{ and } \overrightarrow{OD} = 3(\sin\theta - i\cos\theta)$$

Thus, vertices B and D are represented by $\pm 3(\sin\theta - i\cos\theta)$.

Example 2.95 **Find the centre of the arc represented by $\arg[(z-3i)/(z-2i+4)] = \pi/4$.**

Sol.

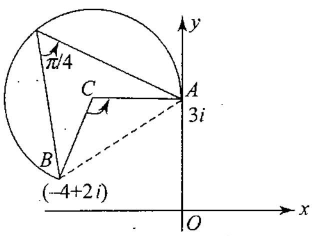


Fig. 2.35

If 'C' be the centre of the arc, then $\angle BCA = \pi/2$. Let C be z_c. Then,

$$\frac{z_c - 3i}{z_c - 2i + 4} = e^{i\pi/2} = i$$

$$\Rightarrow \quad z_c = 3i + i(z_c - 2i + 4)$$

$$\Rightarrow \quad z_c = \frac{7i+2}{(1-i)} = \frac{1}{2}(9i-5)$$

Example 2.96 **z_1 and z_2 are the roots of $3z^2 + 3z + b = 0$. If $O(0)$, (z_1), (z_2) form an equilateral triangle, then find the value of b.**

Sol. $z_1 + z_2 = -1,\ z_1 z_2 = \frac{b}{3}$

Triangle OAB is equilateral. So,

$$0^2 + z_1^2 + z_2^2 = 0 \times z_1 + 0 \times z_2 + z_1 z_2$$

$$\Rightarrow \quad (z_1 + z_2)^2 - 2z_1 z_2 = z_1 z_2$$

$$\Rightarrow \quad 1 = 3z_1 z_2 = 3\frac{b}{3}$$

$$\Rightarrow \quad b = 1$$

Example 2.97 **Let z_1, z_2 and z_3 represent the vertices A, B and C of the triangle ABC in the Argand plane, such that $|z_1| = |z_2| = |z_3| = 5$. Prove that $z_1 \sin 2A + z_2 \sin 2B + z_3 \sin 2C = 0$.**

Sol.

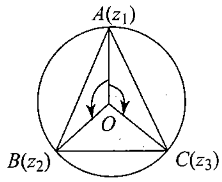


Fig. 2.36

$|z_1| = |z_2| = |z_3| = 5$

$\Rightarrow$ $|z| = 5$ is the circumcircle of triangle ABC

$\Rightarrow$ $\angle AOB = 2C$, $\angle BOC = 2A$ and $\angle COA = 2B$

We have,

$$\frac{z_2}{z_1} = \frac{OB}{OA} e^{i2C} = e^{i2C}$$

Similarly,

$$\frac{z_3}{z_1} = \frac{OC}{OA} e^{-i2B} = e^{-i2B}$$

Now,

$$z_1 \sin 2A + z_2 \sin 2B + z_3 \sin 2C$$

$$= z_1\left(\sin 2A + \frac{z_2}{z_1}\sin 2B + \frac{z_3}{z_1}\sin 2C\right)$$

$$= z_1(\sin 2A + \sin 2B e^{i2C} + \sin 2C\, e^{-2iB})$$

$$= z_1(\sin 2A + \sin 2B \cos 2C + i \sin 2B \sin 2C + \sin 2C \cos 2B - i \sin 2C \sin 2B)$$

$$= z_1(\sin 2A + \sin(2B + 2C))$$

$$= z_1(\sin 2A + \sin(2\pi - 2A))$$

$$= 0$$

Example 2.98 **On the Argand plane z_1, z_2 and z_3 are, respectively, the vertices of an isosceles triangle ABC with $AC = BC$ and equal angles are θ. If z_4 is the incentre of the triangle, then prove that $(z_2 - z_1)(z_3 - z_1) = (1 + \sec\theta)(z_4 - z_1)^2$.**

Sol. $$\frac{z_2 - z_1}{|z_2 - z_1|} = \frac{z_4 - z_1}{|z_4 - z_1|} e^{-i\theta/2} \text{ (clockwise)} \qquad (1)$$

$$\frac{z_3 - z_1}{|z_3 - z_1|} = \frac{z_4 - z_1}{|z_4 - z_1|} e^{i\theta/2} \text{ (anticlowkwise)} \qquad (2)$$

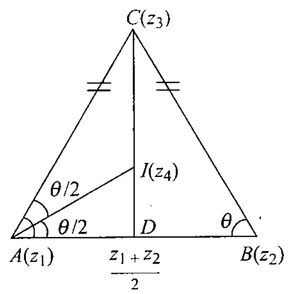


Fig. 2.37

Multiplying (1) and (2), we get

$$\frac{(z_2 - z_1)(z_3 - z_1)}{(z_4 - z_1)^2} = \frac{|(z_2 - z_1)||(z_3 - z_1)|}{|z_4 - z_1|^2}$$

$$= \frac{(AB)(AC)}{(AI)^2}$$

$$= \frac{2(AD)(AC)}{(AI)^2}$$

$$= \frac{2(AD)^2}{(AI)^2} \cdot \frac{AC}{AD}$$

$$= 2\cos^2\frac{\theta}{2}\sec\theta = (1 + \cos\theta)\sec\theta$$

Concept Application Exercise 2.6

1. The centre of a regular polygon of n sides is located at the point $z = 0$, and one of its vertices z_1 is known. If z_2 be the vertex adjacent to z_1, then find z_2.
2. Let z_1 and z_2 be two complex numbers such that $z_1/z_2 + z_2/z_1 = 1$. Prove that z_1, z_2 are the origin form an equilateral triangle.
3. If one vertex of the triangle having maximum area that can be inscribed in the circle $|z - i| = 5$ is $3 - 3i$, then find the other vertices of the triangle.
4. Consider the circle $|z| = r$ in the Argand plane, which is in fact the incircle of triangle ABC. If contact points opposite to the vertices A, B, C are $A_1(z_1)$, $B_1(z_2)$ and $C_1(z_3)$, obtain the complex numbers associated with the vertices A, B, C in terms of z_1, z_2 and z_3.
5. P is a point on the Argnad plane. On the circle with OP as diameter, two points Q and R are taken such that $\angle POQ = \angle QOR = \theta$. If O is the origin and P, Q and R are represented by the complex numbers z_1, z_2 and z_3, respectively, show that $z_2^2 \cos 2\theta = z_1 z_3 \cos^2\theta$.

Standard Loci in the Argand Plane

If $P(z)$ is a variable point and $A(z_1)$, $B(z_2)$ are two fixed points in the Argand plane, then

1. $|z - z_1| = |z - z_2|$

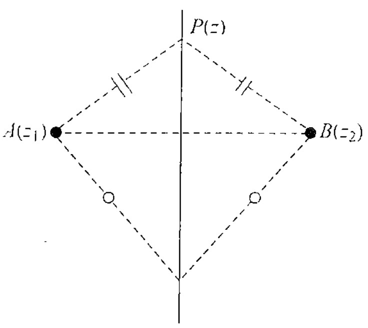
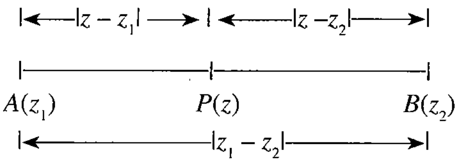


Fig. 2.38

$\Rightarrow AP = BP$

That is, the distance of z from two fixed points z_1 and z_2 is same. Hence, locus of z is the perpendicular bisector of the line segment joining z_1 and z_2.

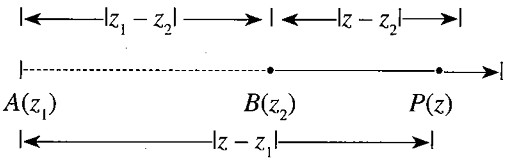
2. $|z - z_1| + |z - z_2| = |z_1 - z_2|$



$AP + BP = AB$

Hence the locus of z is the line segment joining z_1 and z_2.

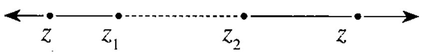
3. $|z - z_1| - |z - z_2| = |z_1 - z_2|$



$\Rightarrow AP - BP = AB$

Hence, the locus of z is a ray as shown in the figure.

4. $||z - z_1| - |z - z_2|| = |z_1 - z_2|$



$\Rightarrow$ Locus of z is a straight line joining z_1 and z_2 but z does not lie between z_1 and z_2 or locus of z is two rays.

5. $|z - z_1| + |z - z_2| = k$ (= **constant** $> |z_1 - z_2|$)

$\Rightarrow PA + PB =$ constant

Hence, the locus of z is an ellipse
(as in ellipse $SP + S'P = 2a$, where S, S' are foci, P is any point on ellipse and a is semi-major axis)

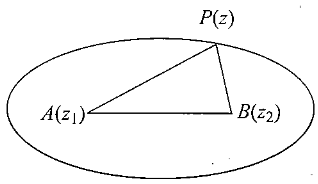


Fig. 2.39

Eccentricity of ellipse is

$$\frac{AB}{AP + BP} = \frac{|z_1 - z_2|}{|z - z_1| + |z - z_2|} = \frac{|z_1 - z_2|}{k}$$

6. $||z - z_1| - |z - z_2|| = k$ (= **constant** $< |z_1 - z_2|$)

$\Rightarrow |AP - BP| =$ constant

Hence, the locus of z is a hyperbola.

(as in hyperbola $S'P - SP = 2a$, where S, S' are foci, P is any point on the hyperbola and a is semi-transverse axis)

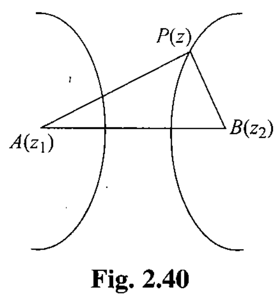


Fig. 2.40

Eccentricity of hyperbola is

$$\frac{AB}{|AP-BP|}=\frac{|z_1-z_2|}{||z-z_1|-|z-z_2||}=\frac{|z_1-z_2|}{k}$$

7. $|z-z_1|^2+|z-z_2|^2=|z_1-z_2|^2$

$\Rightarrow$ $AP^2+BP^2=AB^2$

Hence, the locus of z is a circle with z_1 and z_2 as the extremities of diameter.

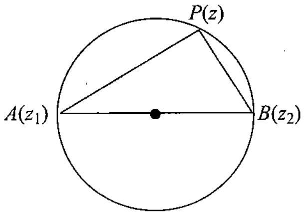


Fig. 2.41

8. $|z-z_1|=k|z-z_2|\ (\neq 1)$

$\Rightarrow$ Locus of z is circle

9. $\arg\left(\frac{z-z_1}{z-z_2}\right)=\alpha$ **(fixed)**

Hence, the locus of z is a segment of circle.

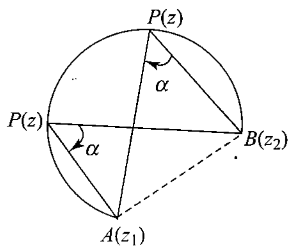


Fig. 2.42

10. $\arg\left(\frac{z-z_1}{z-z_2}\right)=\pm\pi/2$

Hence, the locus of z is a circle with z_1 and z_2 as the vertices of diameter.

11. $\arg\left(\frac{z-z_1}{z-z_2}\right)=0$ **or** π

Hence, the locus of z is a straight line passing through z_1 and z_2

12. $|z-z_0|=\left|\frac{\bar{\alpha}z_0+\alpha\bar{z}_0+r}{2|\alpha|}\right|$

Hence, the locus of z is a parabola whose focus is z_0 and directrix is the line $\bar{\alpha}z+\bar{\alpha}z+r=0$

Example 2.99 **Find the locus of the points representing the complex number z for which $|z+5|^2-|z-5|^2=10$.**

Sol. $|z+5|^2-|z-5|^2=10$

$\Rightarrow$ $(z+5)(\bar{z}+5)-(z-5)(\bar{z}-5)=10$

$\Rightarrow$ $5(z+\bar{z})+25+5(z+\bar{z})-25=10$

$\Rightarrow$ $2\times 2x=2$

$\Rightarrow$ $x=\frac{1}{2}$,

which is the equation of a straight line.

Example 2.100 **Identify the locus of z if $\bar{z}=\bar{a}+\frac{r^2}{z-a}$, >0.**

Sol. $\bar{z}=\bar{a}+\frac{r^2}{z-a}$

$\Rightarrow$ $\bar{z}-\bar{a}=\frac{r^2}{z-a}$

$\Rightarrow$ $(z-a)(\bar{z}-\bar{a})=r^2$

$\Rightarrow$ $|z-a|^2=r^2$

$\Rightarrow$ $|z-a|=r$

Hence, locus of z is circle having center a and radius r.

Example 2.101 **Let z be a complex number having the argument θ, $0<\theta<\pi/2$ and satisfying the equation $|z-3i|=3$. Then find the value of $\cot\theta-6/z$.**

Sol. Let $z=r(\cos\theta+i\sin\theta)$. Now,

$r=OA\sin\theta=6\sin\theta$

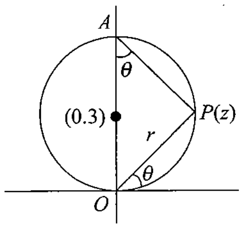


Fig. 2.43

$\Rightarrow$ $z=6\sin\theta(\cos\theta+i\sin\theta)$

$\Rightarrow$ $\frac{6}{z}=\frac{1}{\sin\theta(\cos\theta+i\sin\theta)}$

$=\frac{\cos\theta-i\sin\theta}{\sin\theta}$

$=-i+\cot\theta$

$\Rightarrow$ $\cot\theta-\frac{6}{z}=i$

Example 2.102 **If 'z' be any complex number such that $|3-2|+|3+2|=4$, then identify the locus of 'z'.**

Sol. $|3z-2|+|3z+2|=4$

$\Rightarrow$ $\left|z-\frac{2}{3}\right|+\left|z+\frac{2}{3}\right|=\frac{4}{3}$ (1)

If $P(z)$ be any point $A\equiv(2/3,0)$, $B\equiv(-2/3,0)$, then (i) represents

$PA+PB=4$

Clearly, $AB=4/3\Rightarrow PA+PB=AB\Rightarrow$ 'P' is any point on the line segment AB.

Example 2.103 **$|z-2-3|^2+|z-4-3|^2=\lambda$ represents the equation of a circle with least radius. Find the value of 'λ'.**

Sol.

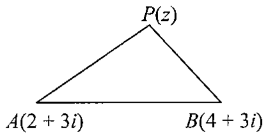


Fig. 2.44

If $P(z)$ lies on a circle and $PB^2 + PA^2 = \lambda$ (constant), then $\lambda = AB^2 \Rightarrow \lambda = 4$.

Example 2.104 **Find the number of complex numbers which satisfy both the equations $|z - 1 - i| = \sqrt{2}$ and $|z + 1 + i| = 2$.**

Sol. The given equations represent circles

$$(x-1)^2 + (y-1)^2 = 2 \text{ and } (x+1)^2 + (y+1)^2 = 4$$

$$\Rightarrow \quad C_1(1, 1),\ r_1 = \sqrt{2},\ C_2(-1, -1),\ r_2 = 2$$

$$C_1C_2 = \sqrt{8} = 2.8,\ r_1 + r_2 = 2 + \sqrt{2} = 3.41$$

$C_1C_2 < r_1 + r_2$ and also $C_1C_2 > r_2 - r_1$ and hence the two circles are intersecting at two points. The common two points will satisfy both.

Example 2.105 **If the imaginary part of $(2z + 1)/(iz + 1)$ is -2, then find the locus of the point representing in the complex plane.**

Sol. Let,

$$z = x + iy$$

$$\Rightarrow \quad \frac{2z+1}{iz+1} = \frac{2(x+iy)+1}{i(x+iy)+1}$$

$$= \frac{(2x+1)+i2y}{1-y+ix}$$

$$= \frac{(2x+1)+i2y}{(1-y)+ix}\,\frac{(1-y)-ix}{(1-y)-ix}$$

$$= \frac{(2x+1)(1-y)+2xy+i\left[-x(2x+1)+2y(1-y)\right]}{(1-y)^2+x^2}$$

Since imaginary part of $(2z + 1)/(iz + 1) = -2$, hence

$$\frac{-x(2x+1)+2y(1-y)}{(1-y)^2+x^2} = -2$$

$$\Rightarrow \quad -2x^2 - x + 2y - 2y^2 = -2[1 + y^2 - 2y + x^2]$$

$$\Rightarrow \quad x + 2y - 2 = 0, \text{ which is a straight line}$$

Example 2.106 **If $|(z - 2)/(z - 3)| = 2$ represents a circle, then find its radius.**

Sol. $$\left|\frac{z-2}{z-3}\right| = 2$$

$$\Rightarrow \quad |z-2|^2 = 4|z-3|^2$$

$$\Rightarrow \quad |x - 2 + iy|^2 = 4|x - 3 + iy|^2$$

$$\Rightarrow \quad (x-2)^2 + y^2 = 4[(x-3)^2 + y^2]$$

$$\Rightarrow \quad 3x^2 + 3y^2 - 24x + 4x + 36 - 4 = 0$$

$$\Rightarrow \quad x^2 + y^2 - \frac{20}{3}x + \frac{32}{3} = 0$$

This represents a circle with centre $[(10/3, 0)]$ and radius $\sqrt{(100/9) - (32/3)} = \sqrt{(4/9)} = 2/3$.

Example 2.107 **If $z_1 + z_2 + z_3 + z_4 = 0$ where $b_i \in R$ such that the sum of no two values being zero and $b_1z_1 + b_2z_2 + b_3z_3 + b_4z_4 = 0$ where z_1, z_2, z_3, z_4 are arbitrary complex numbers such that no three of them are collinear, prove that the four complex numbers would be concyclic if $|b_1b_2||z_1 - z_2|^2 = |b_3b_4||z_3 - z_4|^2$.**

Sol.

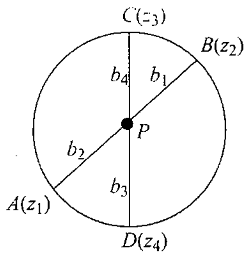


Fig. 2.45

$$b_1 + b_2 = -(b_3 + b_4),\ b_1z_1 + b_2z_2 = -(b_3z_3 + b_4z_4)$$

$$\Rightarrow \quad \frac{b_1z_1 + b_2z_2}{b_1 + b_2} = \frac{b_3z_3 + b_4z_4}{b_3 + b_4}$$

Hence, the line joining the complex numbers $A(z_1)$, $B(z_2)$ and $C(z_3)$, $D(z_4)$ meet.

Let $P(z)$ be the point of intersection. These points will be concyclic, if $PA \times PB = PC \times PD$. Now,

$$PA = \left|\frac{b_2}{b_1+b_2}\right||z_1 - z_2|,\ PB = \left|\frac{b_1}{b_1+b_2}\right||z_1 - z_2|$$

$$PC = \left|\frac{b_4}{b_3+b_4}\right||z_3 - z_4|,\ PD = \left|\frac{b_3}{b_3+b_4}\right||z_3 - z_4|$$

$$\Rightarrow \quad |b_1b_2|\,|z_1 - z_2|^2 = |b_3\,b_4|\,|z_3 - z_4|^2$$

$$(\because |b_1 + b_2| = |b_3 + b_4|)$$

Example 2.108 **Consider an ellipse having its foci at $A(z_1)$ and $B(z_2)$ in the Argand plane. If the eccentricity of the ellipse be 'e' and it is known that origin is an interior point of the ellipse, then prove that**

$$\in \left(0, \frac{|z_1 - z_2|}{|z_1| + |z_2|}\right)$$

Sol. Let $P(z)$ be any point on the ellipse. Then equation of the ellipse is

$$|z - z_1| + |z - z_2| = \frac{|z_1 - z_2|}{e} \qquad (1)$$

If we replace z by z_1 or z_2, L.H.S. of (1) becomes $|z_1 - z_2|$. Thus for any interior point of the ellipse, we have

$$|z - z_1| + |z - z_2| < \frac{|z_1 - z_2|}{a}$$

It is given that origin is an interior point of the ellipse

$$|0 - z_1| + |0 - z_2| < \frac{|z_1 - z_2|}{e}$$

$$\Rightarrow \quad e \in \frac{|z_1 - z_2|}{|z_1|+|z_2|}$$

Example 2.109 **Find the locus of point z if z, i and iz are collinear.**

Sol. If z_1, z_2, z_3 are collinear, then

$$\begin{vmatrix} z_1 & \bar{z}_1 & 1 \\ z_2 & \bar{z}_2 & 1 \\ z_3 & \bar{z}_3 & 1 \end{vmatrix} = 0$$

Given z, i and iz are collinear. Hence,

$$\begin{vmatrix} z & \bar{z} & 1 \\ i & -i & 1 \\ iz & -i\bar{z} & 1 \end{vmatrix} = 0$$

$$\Rightarrow \quad -iz + iz\bar{z} + \bar{z} - z - i\bar{z} + iz\bar{z} = 0$$

$$\Rightarrow \quad z - 2z\bar{z} + i\bar{z} - iz + \bar{z} = 0 \qquad \text{(multiplying with } i\text{)}$$

$$\Rightarrow \quad 2z\bar{z} - (z + \bar{z}) + i(z - \bar{z}) = 0$$

$$\Rightarrow \quad x^2 + y^2 - 2x - 2y = 0 \quad \text{(putting } z = x + iy\text{)}$$

Example 2.110 **If the equation $|z - a| + |z - b| = 3$ represents an ellipse, and $a, b \in C$, where a is fixed, then find the locus of b.**

Sol. $|z - a| + |z - b| = 3$ represents an ellipse. Now,

$$|a - b| < 3$$

$$\Rightarrow \quad |b - a| < 3$$

Hence, b lies inside the circle having centre a and radius 3.

Concept Application Exercise 2.7

1. If $\log_{\sqrt{3}}\left(\frac{|z|^2 - |z| + 1}{2 + |z|}\right) > 2$, then locate the region in the Argand plane which represent z.
2. Identify the region of the Argand diagram defined by $|z - 1| + |z + 1| \le 4$.
3. If $w = z/[z - (1/3)i]$ and $|w| = 1$, then find the locus of z.
4. Let $z\ (\neq 2)$ be a complex number such that $\log_{1/2}|z - 2| > \log_{1/2}|z|$. Then prove that $\text{Re}(z) > 1$.
5. Locate the region in the Argand plane determined by $z^2 + \bar{z}^2 + 2|z|^2 < (8i(\bar{z} - z))$.
6. If $|z - 1| + |z + 3| \le 8$, then find the range of values of $|z - 4|$.

THE n^{th} ROOT OF UNITY

Let x be n^{th} root of unity. Then,

$$\begin{aligned} x^n &= 1 \\ &= 1 + i(0) \\ &= \cos 0 + i \sin 0 \\ &= \cos(2k\pi + 0) + i \sin(2k\pi + 0) \\ &= \cos 2k\pi + i \sin 2k\pi \qquad \text{(where } k \text{ is an integer)} \end{aligned}$$

$$\Rightarrow \quad x = \cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n},\ k = 0, 1, 2, \ldots, n - 1$$

Let $\alpha = \cos(2\pi/n) + i\sin(2\pi/n)$. Then the n^{th} roots of unity are α^t ($t = 0, 1, 2, \ldots, n - 1$), i.e., the n^{th} roots of unity are $1, \alpha, \alpha^2, \ldots, \alpha^{n-1}$.

Sum of the Roots

$$1 + \alpha + \alpha^2 + \cdots + \alpha^{n-1} = \frac{1 - \alpha^n}{1 - \alpha}$$

$$\frac{1 - \left(\cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}\right)^n}{1 - \alpha} = \frac{1 - (\cos 2\pi + i\sin 2\pi)}{1 - \alpha} = 0$$

$$\Rightarrow \sum_{k=0}^{n-1}\cos\frac{2k\pi}{n} = 0 \text{ and } \sum_{k=0}^{n-1}\sin\frac{2k\pi}{n} = 0$$

Thus the sum of the roots of unity is zero.

Product of the Roots

$$1 \times \alpha \times \alpha^2 \times \cdots \times \alpha^{n-1} = \alpha^{\frac{n(n-1)}{2}} = \left(\cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}\right)^{n\left(\frac{n-1}{2}\right)}$$

$$= (\cos\pi + i\sin\pi)^{n-1}$$

If n is even, the product is $(-1)^{n-1}$. If n is odd, the product is 1.

Note: *The points represented by the n^{th} roots of unity are located at the vertices of a regular polygon of n sides inscribed in a unit circle having centre at the origin, one vertex being on the positive real axis (geometrically represented as shown).*

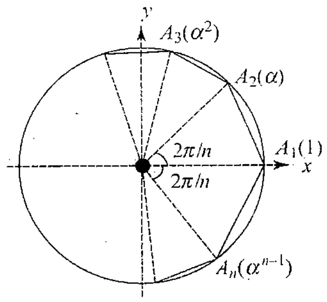


Fig. 2.46

Example 2.111 **If $a = \cos(2\pi/7) + i\sin(2\pi/7)$, then find the quadratic equation whose roots are $\alpha = a + a^2 + a^4$ and $\beta = a^3 + a^5 + a^7$.**

Sol. $a = \cos(2\pi/7) + i\sin(2\pi/7)$

$\Rightarrow \quad a^7 = [\cos(2\pi/7) + i\sin(2\pi/7)]^7$

$= \cos 2\pi + i\sin 2\pi = 1 \qquad \text{(i)}$

$S = \alpha + \beta = (a + a^2 + a^4) + (a^3 + a^5 + a^6)$

$= a + a^2 + a^3 + a^4 + a^5 + a^6 = \dfrac{a(1-a^6)}{1-a}$

$= \dfrac{a - a^7}{1-a} = \dfrac{a-1}{1-a} = -1 \qquad \text{(ii)}$

$P = \alpha\beta = (a + a^2 + a^4)(a^3 + a^5 + a^6)$

$= a^4 + a^6 + a^7 + a^5 + a^7 + a^8 + a^7 + a^9 + a^{10}$

$= a^4 + a^6 + 1 + a^5 + 1 + a + 1 + a^2 + a^3$ [From Eq. (i)]

$= 3 + (a + a^2 + a^3 + a^4 + a^5 + a^6)$

$= 3 + S$

$= 3 - 1 = 2$ [From Eq. (ii)]

Therefore the required equation is

$x^2 - Sx + P = 0$

$\Rightarrow \quad x^2 + x + 2 = 0$

Example 2.112 **If ω is an imaginary fifth root of unity, then find the value of $\log_2 |1 + \omega + \omega^2 + \omega^3 - 1/\omega|$.**

Sol. Here $\omega^5 = 1 \therefore \omega^{-1} = \omega^4$

Also,

$1 + \omega + \omega^2 + \omega^3 + \omega^4 = 0$

$\therefore \quad \log_2 \left|1 + \omega + \omega^2 + \omega^3 - \dfrac{1}{\omega}\right|$

$= \log_2 |1 + \omega + \omega^2 + \omega^3 - \omega^4| \qquad (\because |\omega| = 1)$

$= \log_2 |-2\omega^4| = \log_2 2 = 1$

Example 2.113 **If $1, z_1, z_2, z_3, \ldots, z_{n-1}$ are the n^{th} roots of unity, then prove that $(1 - z_1)(1 - z_2) \cdots (1 - z_{n-1}) = n$.**

Sol. $z^n - 1 = (z - 1)(z - z_1)(z - z_2) \cdots (z - z_{n-1})$

$\Rightarrow \dfrac{z^n - 1}{z - 1} = (z - z_1)(z - z_2) \cdots (z - z_{n-1})$

$\Rightarrow 1 + z + z^2 + \cdots + z^{n-1} = (z - z_1)(z - z_2) \cdots (z - z_{n-1})$

Putting $z = 1$, we get

$(1 - z_1)(1 - z_2) \cdots (1 - z_{n-1}) = 1 + 1 + \cdots + 1 = n$

Example 2.114 **If $\alpha = e^{i2\pi/7}$ and $f(x) = a_0 + \sum_{k=1}^{20} a_k x^k$, then prove that the value of $f(x) + f(\alpha x) + \cdots + f(\alpha^6 x)$ is independent of α.**

Sol. $\alpha = e^{i2\pi/7} = 7^{\text{th}}$ root of unity or root of the equation $z^7 - 1 = 0$. Its other roots are $1, \alpha^2, \alpha^3, \alpha^4, \alpha^5, \alpha^6$. Now,

$f(x) + f(\alpha x) + \cdots + f(\alpha^6 x)$

$= 7a_0 + \sum_{k=1}^{20} a_k x^k + \sum_{k=1}^{20} a_k (\alpha x)^k + \cdots + \sum_{k=1}^{20} a_k (\alpha^6 x)^k$

$= 7a_0 + \sum_{k=1}^{20} a_k (x^k + \alpha^k x^k + \alpha^{2k} x^k + \cdots + \alpha^{6k} x^k)$

$= 7a_0 + \sum_{k=1}^{20} a_k (x^k + \alpha^k x^k + \alpha^{2k} x^k + \cdots + \alpha^{6k} x^k)$

$= 7a_0 + \sum_{k=1}^{20} a_k \left(x^k \dfrac{(\alpha^k)^7 - 1}{\alpha^k - 1} \right)$

$= 7a_0 + \sum_{k=1}^{20} a_k \left(x^k \dfrac{(\alpha^7)^k - 1}{\alpha^k - 1} \right)$

$= 7a_0 + \sum_{k=1}^{20} a_k \left(x^k \dfrac{1 - 1}{\alpha^k - 1} \right)$

$(\because \alpha \text{ is a root of } z^7 - 1 = 0 \Rightarrow \alpha^7 - 1 = 0)$

$= 7a_0$

Example 2.115 **If $n \geq 3$ and $1, \alpha_1, \alpha_2, \ldots, \alpha_{n-1}$ are n^{th} roots of unity, then find the sum $\sum_{1 \leq i < j \leq n-1} \alpha_i \alpha_j$.**

Sol. We have,

$z^n - 1 = (z - 1)(z - \alpha_1)(z - \alpha_2) \ldots (z - \alpha_{n-1})$

Now sum of products taken two at a time,

$S = \sum_{1 \leq i < j \leq n-1} \alpha_i \alpha_j + \alpha_1 + \alpha_2 + \cdots + \alpha_{n-1}$

Now $S = 0$ as coefficient of z^{n-1} is zero and

$\alpha_1 + \alpha_2 + \cdots + \alpha_{n-1} = -1$

$\therefore \quad 0 = \sum_{1 \leq i < j \leq n-1} \alpha_i \alpha_j - 1$

$\Rightarrow \quad \sum_{1 \leq i < j \leq n-1} \alpha_i \alpha_j = 1$

Concept Application Exercise 2.8

1. Given α, β, respectively, the fifth and the fourth non-real roots of unity, then find the value of $(1 + \alpha)(1 + \beta)(1 + \alpha^2)(1 + \beta^2)(1 + \alpha^4)(1 + \beta^4)$
2. If the six solutions of $x^6 = -64$ are written in the form $a + bi$, where a and b are real, then find the product of those solutions with $a > 0$.
3. If z_r: $r = 1, 2, 3, \ldots, 50$ are the roots of the equation $\sum_{r=0}^{50} z^r = 0$, then find the value of $\sum_{r=1}^{50} 1/(z_r - 1)$.
4. If $1, \alpha_1, \alpha_2, \ldots, \alpha_{n-1}$ are the n^{th} roots of unity, prove that $(1 - \alpha_1)(1 - \alpha_2) \cdots (1 - \alpha_{n-1}) = n$ Deduce that

   $\sin\dfrac{\pi}{n} \sin\dfrac{2\pi}{n} \cdots \sin\dfrac{(n-1)\pi}{n} = \dfrac{n}{2^{n-1}}$
5. If $n > 1$, show that the roots of the equation $z^n = (z + 1)^n$ are collinear.

EXERCISES

Subjective Type

Solutions on page 2.52

1. If $\left|z/|\bar{z}| - \bar{z}\right| = 1 + |z|$, then prove that z is a purely imaginary number.

2. z_1, z_2 and z_3 are the vertices of an isosceles triangle in anticlockwise direction with origin as in centre then prove that z_2, z_1 and kz_3 are in G.P. where $k \in R^+$.

3. For $|z - 1| = 1$, show that $\tan\{[\arg(z-1)]/2\} - (2i/z) = -i$.

4. The altitude from the vertices A, B and C of the triangle ABC meet its circumcircle at D, E and F, respectively. The complex numbers representing the points D, E and F are z_1, z_2 and z_3, respectively. If $(z_3 - z_1)/(z_2 - z_1)$ is purely real, then show that triangle ABC is right angled at A.

5. Let A, B, C, D be four concyclic points in order in which $AD{:}AB = CD{:}CB$. If A, B, C are represented by complex numbers a, b, c, find the complex number associated with point D.

6. For $x \in (0, 1)$, prove that
$$i^{2i+3} \ln\left(\frac{i^3x^2 + 2x + i}{ix^2 + 2x + i^3}\right) = \frac{1}{e^{\pi}}(\pi - 4\tan^{-1} x)$$

7. If a, b are complex numbers and one of the roots of the equation $x^2 + ax + b = 0$ is purely real whereas the other is purely imaginary, prove that $a^2 - \bar{a}^2 = 4b$.

8. Solve for z, i.e. find all complex numbers z which satisfy $|z| - 2iz + 2c(1 + i) = 0$ where c is real.

9. If 'a' is a complex number such that $|a| = 1$, find $\arg(a)$, so that equation $az^2 + z + 1 = 0$ has one purely imaginary root.

10. Prove the following inequalities:

 a. $\left|\frac{z}{|z|} - 1\right| \le |\arg z|$ b. $|z - 1| \le |z|\,|\arg z| + ||z| - 1|$

11. If n is a positive integer, prove that $|\text{Im}(z^n)| \le n|\text{Im}(z)|\,|z|^{n-1}$.

12. Let z and z_0 be two complex numbers. It is given that $|z| = 1$ and the numbers z, z_0, $z\bar{z}_0$, 1 and 0 are represented in an Argand diagram by the points P, P_0, Q, A and the origin, respectively. Show that the triangles POP_0 and AOQ are congruent. Hence, or otherwise, prove that $|z - z_0| = |z\bar{z}_0 - 1|$.

13. Show that the equation $az^3 + bz^2 + \bar{b}z + \bar{a} = 0$ has a root α, such that $|\alpha| = 1$. a, b, z and α belong to the set of complex numbers.

14. Let $z = t^2 - 1 + \sqrt{t^4 - t^2}$, where $t \in R$ is a parameter. Find the locus of 'z' depending upon t, and draw the locus of 'z' in the Argand plane.

15. If $|z| = 1$, then prove that points represented by $\sqrt{(1+z)/(1-z)}$ lie on one or other of two fixed perpendicular straight lines.

16. If $\alpha = (z-i)/(z+i)$, show that, when z lies above the real axis, α will lie within the unit circle which has centre at the origin. Find the locus of α as z travels on the real axis from $-\infty$ to $+\infty$.

17. Let x_1, x_2 are the roots of the quadratic equation $x^2 + ax + b = 0$ where a, b are complex numbers and y_1, y_2 are the roots of the quadratic equation $y^2 + |a|y + |b| = 0$. If $|x_1| = |x_2| = 1$ then prove that $|y_1| = |y_2| = 1$.

18. Plot the region represented by $\pi/3 \le \arg[(z + 1)/(z - 1)] \le 2\pi/3$ in the Argand plane.

19. Consider an equilateral triangle having vertices at the points
$$A\left(\frac{2}{\sqrt{3}}e^{i\frac{\pi}{2}}\right),\ B\left(\frac{2}{\sqrt{3}}e^{-i\frac{\pi}{6}}\right),\ C\left(\frac{2}{\sqrt{3}}e^{-i\frac{5\pi}{6}}\right).$$
Let P be any point on its incircle. Prove that $AP^2 + BP^2 + CP^2 = 5$.

20. Prove that the locus of mid-point of line segment intercepted between real and imaginary axes by the line $a\bar{z} + \bar{a}z + b = 0$, where b is a real parameter and a is a fixed complex number with non-zero real and imaginary parts, is $az + \overline{az} = 0$.

Objective Type

Solutions on page 2.56

Each question has four choices a, b, c and d, out of which *only one* is correct.

1. If $a < 0$, $b > 0$ then $\sqrt{a}\sqrt{b}$ is equal to

 a. $-\sqrt{|a|b}$ **b.** $\sqrt{|a|b}\,i$ **c.** $\sqrt{|a|b}$ **d.** none of these

2. If $x = 9^{1/3}\, 9^{1/9}\, 9^{1/27} \cdots\infty$, $y = 4^{1/3}\, 4^{-1/9}\, 4^{1/27} \cdots\infty$, and $z = \sum_{r=1}^{\infty}(1 + i)^{-r}$, then $\arg(x + yz)$ is equal to

 a. 0 **b.** $\pi - \tan^{-1}\left(\frac{\sqrt{2}}{3}\right)$ **c.** $-\tan^{-1}\left(\frac{\sqrt{2}}{3}\right)$ **d.** $-\tan^{-1}\left(\frac{2}{\sqrt{3}}\right)$

3. Consider the equation $10z^2 - 3iz - k = 0$, where z is a complex variable and $i^2 = -1$. Which of the following statements is true?

 a. For real positive numbers k, both roots are purely imaginary.

 b. For all complex numbers k, neither root is real.

 c. For all purely imaginary numbers k, both roots are real and irrational.

 d. For real negative numbers k, both roots are purely imaginary.

4. The number of solutions of the equation $z^2 + \bar{z} = 0$ is

 a. 1 **b.** 2 **c.** 3 **d.** 4

5. If $a^2 + b^2 = 1$, then $(1 + b + ia)/(1 + b - ia) =$

 a. 1 **b.** 2 **c.** $b + ia$ **d.** $a + ib$

6. The expression $\left[\dfrac{1 + \sin\frac{\pi}{8} + i\cos\frac{\pi}{8}}{1 + \sin\frac{\pi}{8} - i\cos\frac{\pi}{8}}\right]^8 =$

 a. 1 **b.** -1 **c.** i **d.** $-i$

7. If z_1 and z_2 be complex numbers such that $z_1 \ne z_2$ and $|z_1| = |z_2|$. If z_1 has positive real part and z_2 has negative imaginary part, then $[(z_1 + z_2)/(z_1 - z_2)]$ may be

a. Purely imaginary b. Real and positive
c. Real and negative d. None of these

8. If $|z_1| = |z_2|$ and $\arg(z_1/z_2) = \pi$, then $z_1 + z_2$ is equal to
a. 0 b. purely imaginary
c. purely real d. none of these

9. If $\cos\alpha + 2\cos\beta + 3\cos\gamma = \sin\alpha + 2\sin\beta + 3\sin\gamma = 0$, then the value of $\sin 3\alpha + 8\sin 3\beta + 27\sin 3\gamma$ is
a. $\sin(a + b + \gamma)$ b. $3\sin(\alpha + \beta + \gamma)$
c. $18\sin(\alpha + \beta + \gamma)$ d. $\sin(\alpha + 2\beta + 3)$

10. If centre of a regular hexagon is at origin and one of the vertices on Argand diagram is $1 + 2i$, then its perimeter is
a. $2\sqrt{5}$ b. $6\sqrt{2}$ c. $4\sqrt{5}$ d. $6\sqrt{5}$

11. If $z(1 + a) = b + ic$ and $a^2 + b^2 + c^2 = 1$, then $[(1 + iz)/(1 - iz) =$
a. $\frac{a+ib}{1+c}$ b. $\frac{b-ic}{1+a}$ c. $\frac{a+ic}{1+b}$ d. none of these

12. If z_1, z_2, z_3 are three complex numbers and

$$A = \begin{vmatrix} \arg z_1 & \arg z_2 & \arg z_3 \\ \arg z_2 & \arg z_3 & \arg z_1 \\ \arg z_3 & \arg z_1 & \arg z_2 \end{vmatrix}$$

then A is divisible by
a. $\arg(z_1 + z_2 + z_3)$ b. $\arg(z_1 z_2 z_3)$
c. all numbers d. cannot say

13. Let z, w be complex numbers such that $\bar{z} + i\bar{w} = 0$ and $\arg zw = \pi$. Then $\arg z$ equals
a. $\frac{\pi}{4}$ b. $\frac{\pi}{2}$ c. $\frac{3\pi}{4}$ d. $\frac{5\pi}{4}$

14. If for complex numbers z_1 and z_2, $\arg(z_1) - \arg(z_2) = 0$, then $|z_1 - z_2|$ is equal to
a. $|z_1| + |z_2|$ b. $|z_1| - |z_2|$ c. $||z_1| - |z_2||$ d. 0

15. If $k > 0$, $|z| = |w| = k$ and $\alpha = \frac{z - \bar{w}}{k^2 + z\bar{w}}$, then $\text{Re}(\alpha)$ equals
a. 0 b. $k/2$ c. k d. none of these

16. If $z = x + iy$ and $x^2 + y^2 = 16$, then the range of $||x| - |y||$ is
a. [0, 4] b. [0, 2] c. [2, 4] d. none of these

17. If $k + |k + z^2| = |z|^2$ $(k \in R^-)$, then possible argument of z is
a. 0 b. π c. $\pi/2$ d. none of these

18. If $z = x + iy$ $(x, y \in R, x \neq -1/2)$, the number of values of z satisfying $|z|^n = z^2|z|^{n-2} + z|z|^{n-2} + 1 \cdot (n \in N, n > 1)$ is
a. 0 b. 1 c. 2 d. 3

19. If x and y are complex numbers, then the system of equations $(1 + i)x + (1 - i)y = 1$, $2ix + 2y = 1 + i$ has
a. unique solution b. no solution
c. infinite number of solutions d. none of these

20 Number of solutions of the equation $z^3 + [3(\bar{z})^2]/|z| = 0$ where z is a complex number is
a. 2 b. 3 c. 6 d. 5

21. The principal argument of the complex number $[(1+i)^5(1+\sqrt{3}i)^2]/[-2i(-\sqrt{3}+i)]$ is
a. $\frac{19\pi}{12}$ b. $-\frac{7\pi}{12}$ c. $-\frac{5\pi}{12}$ d. $\frac{5\pi}{12}$

22. The polynomial $x^6 + 4x^5 + 3x^4 + 2x^3 + x + 1$ is divisible by where w is cube root of units
a. $x + \omega$ b. $x + \omega^2$
c. $(x + \omega)(x + \omega^2)$ d. $(x - \omega)(x - \omega^2)$
where ω is one of the imaginary cube roots of unity.

23. If $(\cos\theta + i\sin\theta)(\cos 2\theta + i\sin 2\theta) \cdots (\cos n\theta + i\sin n\theta) = 1$, then the value of θ is, $m \in N$
a. $4m\pi$ b. $\frac{2m\pi}{n(n+1)}$ c. $\frac{4m\pi}{n(n+1)}$ d. $\frac{m\pi}{n(n+1)}$

24. Given $z = (1 + i\sqrt{3})^{100}$, then $[\text{RE}(z)/\text{IM}(z)]$ equals
a. 2^{100} b. 2^{50} c. $\frac{1}{\sqrt{3}}$ d. $\sqrt{3}$

25. If $z = (i)^{(i)^{(i)}}$ where $i = \sqrt{-1}$, then $|z|$ is equal to
a. 1 b. $e^{-\pi/2}$ c. $e^{-\pi}$ d. none of these

26. If $z = i\log(2 - \sqrt{-3})$, then $\cos z =$
a. -1 b. $-1/2$ c. 1 d. 1/2

27. If the equation $z^4 + a_1z^3 + a_2z^2 + a_3z + a_4 = 0$, where a_1, a_2, a_3, a_4 are real coefficients different from zero, has a purely imaginary root, then the expression $a_3/(a_1a_2) + (a_1a_4)/(a_2a_3)$ has the value equal to
a. 0 b. 1 c. -2 d. 2

28. Suppose A is a complex number and $n \in N$, such that $A^n = (A + 1)^n = 1$, then the least value of n is
a. 3 b. 6 c. 9 d. 12

29. Number of complex numbers z such that $|z| = 1$ and $|z/\bar{z} + \bar{z}/z| = 1$ is $(\arg(z) \in [0, 2\pi))$
a. 4 b. 6 c. 8 d. more than 8

30. If α, β be the roots of the equation $u^2 - 2u + 2 = 0$ and if $\cot\theta = x + 1$, then $[(x + \alpha)^n - (x + \beta)^n]/[\alpha - \beta]$ is equal to
a. $\frac{\sin n\theta}{\sin^n\theta}$ b. $\frac{\cos n\theta}{\cos^n\theta}$ c. $\frac{\sin n\theta}{\cos^n\theta}$ d. $\frac{\cos n\theta}{\sin^n\theta}$

31. Dividing $f(z)$ by $z - i$, we obtain the remainder i and dividing it by $z + i$, we get the remainder $1 + i$, then remainder upon the division of $f(z)$ by $z^2 + 1$ is
a. $\frac{1}{2}(z + 1) + i$ b. $\frac{1}{2}(iz + 1) + i$
c. $\frac{1}{2}(iz - 1) + i$ d. $\frac{1}{2}(z + i) + 1$

32. If $z_1, z_2 \in C$, $z_1^2 + z_2^2 \in R$, $z_1(z_1^2 - 3z_2^2) = 2$ and $z_2(3z_1^2 - z_2^2) = 11$, then the value of $z_1^2 + z_2^2$ is
a. 10 b. 12 c. 5 d. 8

33. z_1 and z_2 are two distinct points in an Argand plane. If $a|z_1| = b|z_2|$ (where $a, b \in R$), then the point $(az_1/bz_2) + (bz_2/az_1)$ is a point on the
a. line segment [– 2, 2] of the real axis
b. line segment [– 2, 2] of the imaginary axis
c. unit circle $|z| = 1$
d. the line with $\arg z = \tan^{-1} 2$

34. If $x^2 + x + 1 = 0$, then the value of $(x + 1/x)^2 + (x^2 + 1/x^2)^2 + \cdots + (x^{27} + 1/x^{27})^2$ is

a. 27 **b.** 72 **c.** 45 **d.** 54

35. If ω be a complex n^{th} root of unity, then $\sum_{r=1}^{n} (ar + b)\, \omega^{r-1}$ is equal to

a. $\frac{n(n+1)a}{2}$ **b.** $\frac{nb}{1-n}$ **c.** $\frac{na}{\omega - 1}$ **d.** none of these

36. The roots of the cubic equation $(z + ab)^3 = a^3$, such that $a \neq 0$, represent the vertices of a triangle of sides of length

a. $\frac{1}{\sqrt{3}}|ab|$ **b.** $\sqrt{3}|a|$ **c.** $\sqrt{3}|b|$ **d.** $|a|$

37. Sum of common roots of the equations $z^3 + 2z^2 + 2z + 1 = 0$ and $z^{1985} + z^{100} + 1 = 0$ is

a. -1 **b.** 1 **c.** 0 **d.** 1

38. If $\left|\frac{z_1}{z_2}\right| = 1$ and $\arg(z_1 z_2) = 0$, then

a. $z_1 = z_2$ **b.** $|z_2|^2 = z_1 z_2$ **c.** $z_1 z_2 = 1$ **d.** none of these

39. If z_1 and z_2 are the complex roots of the equation $(x - 3)^3 + 1 = 0$, then $z_1 + z_2$ equals to

a. 1 **b.** 3 **c.** 5 **d.** 7

40. Which of the following is equal to $\sqrt[3]{-1}$?

a. $\frac{\sqrt{3} + \sqrt{-1}}{2}$ **b.** $\frac{-\sqrt{3} + \sqrt{-1}}{\sqrt{-4}}$

c. $\frac{\sqrt{3} - \sqrt{-1}}{\sqrt{-4}}$ **d.** $-\sqrt{-1}$

41. If $|z - 1| \leq 2$ and $|\omega z - 1 - \omega^2| = a$ (where ω is a cube root of unity) then complete set of values of a is

a. $0 \leq a \leq 2$ **b.** $\frac{1}{2} \leq a \leq \frac{\sqrt{3}}{2}$

c. $\frac{\sqrt{3}}{2} - \frac{1}{2} \leq a \leq \frac{1}{2} + \frac{\sqrt{3}}{2}$ **d.** $0 \leq a \leq 4$

42. If $|z^2 - 3| = 3|z|$ then the maximum value of $|z|$ is

a. 1 **b.** $\frac{3 + \sqrt{21}}{2}$ **c.** $\frac{\sqrt{21} - 3}{2}$ **d.** none of these

43. If $|2z - 1| = |z - 2|$ and z_1, z_2, z_3 are complex numbers such that $|z_1 - \alpha| < \alpha, |z_2 - \beta| < \beta$, then $\left|\frac{z_1 + z_2}{\alpha + \beta}\right|$

a. $<|z|$ **b.** $<2|z|$ **c.** $>|z|$ **d.** $>2|z|$

44. If z_1 is a root of the equation $a_0 z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n = 3$, where $|a_i| < 2$ for $i = 0, 1, \ldots, n$. Then

a. $|z_1| > \frac{1}{3}$ **b.** $|z_1| < \frac{1}{4}$ **c.** $|z_1| > \frac{1}{4}$ **d.** $|z| < \frac{1}{3}$

45. If $8iz^3 + 12z^2 - 18z + 27i = 0$, then

a. $|z| = \frac{3}{2}$ **b.** $|z| = \frac{2}{3}$ **c.** $|z| = 1$ **d.** $|z| = \frac{3}{4}$

46. If $|z| < \sqrt{2} - 1$, then $|z^2 + 2z\cos\alpha|$ is

a. less than 1 **b.** $\sqrt{2} + 1$
c. $\sqrt{2} - 1$ **d.** none of these

47. If the complex number z satisfies the condition $|z| \geq 3$, then the least value of $[z + (1/z)]$ is equal to

a. 5/3 **b.** 8/3 **c.** 11/3 **d.** none of these

48. Let $|z_r - r| \leq r, \forall r = 1, 2, 3, \ldots, n$. Then $\left|\sum_{r=1}^{n} z_r\right|$ is less than

a. n **b.** $2n$ **c.** $n(n + 1)$ **d.** $\frac{n(n+1)}{2}$

49. If $|z^2 - 1| = |z|^2 + 1$, then z lies on

a. a circle **b.** a parabola **c.** an ellipse **d.** none of these

50. If $|z| = 1$ then the point representing the complex number $-1 + 3z$ will lie on

a. a circle **b.** a straight line **c.** a parabola **d.** a hyperbola

51. If $z = (\lambda + 3) - i\sqrt{5 - \lambda^2}$, then the locus of z is

a. ellipse **b.** semicircle **c.** parabola **d.** straight line

52. If $A(z_1), B(z_2), C(z_3)$ are the vertices of the triangle ABC such that $(z_1 - z_2)/(z_3 - z_2) = (1/\sqrt{2}) + (i/\sqrt{2})$, the triangle ABC is

a. equilateral **b.** right angled
c. isosceles **d.** obtuse angled

53. If z_1, z_2, z_3 are the vertices of an equilateral triangle ABC such that $|z_1 - i| = |z_2 - i| = |z_3 - i|$, then $|z_1 + z_2 + z_3|$ equals to

a. $3\sqrt{3}$ **b.** $\sqrt{3}$ **c.** 3 **d.** $\frac{1}{3\sqrt{3}}$

54. The greatest positive argument of complex number satisfying $|z - 4| = \text{Re}(z)$ is

a. $\frac{\pi}{3}$ **b.** $\frac{2\pi}{3}$ **c.** $\frac{\pi}{2}$ **d.** $\frac{\pi}{4}$

55. The complex number associated with the vertices A, B, C of ΔABC are $e^{i\theta}, \omega, \bar{\omega}$, respectively [where $\omega, \bar{\omega}$ are the complex cube roots of unity and $\cos\theta > \text{Re}(\omega)$], then the complex number of the point where angle bisector of A meets the circumcircle of the triangle, is

a. $e^{i\theta}$ **b.** $e^{-i\theta}$ **c.** $\omega\bar{\omega}$ **d.** $\omega + \bar{\omega}$

56. The maximum area of the triangle formed by the complex coordinates z, z_1, z_2 which satisfy the relations $|z - z_1| = |z - z_2|$ and $\left|z - (z_1 + z_2)/2\right| \leq r$, where $r > |z_1 - z_2|$ is

a. $\frac{1}{2}|z_1 - z_2|^2$ **b.** $\frac{1}{2}|z_1 - z_2|r$

c. $\frac{1}{2}|z_1 - z_2|^2 r^2$ **d.** $\frac{1}{2}|z_1 - z_2|r^2$

57. Locus of z if $\arg[z - (1 + i)] = \begin{cases} \frac{3\pi}{4} & \text{when } |z| \leq |z - 2| \\ \frac{-\pi}{4} & \text{when } |z| > |z - 4| \end{cases}$ is

a. straight lines passing through (2, 0)
b. straight lines passing through (2, 0), (1, 1)
c. a line segment
d. a set of two rays

58. If z is a complex number such that $-\pi/2 \le \arg z \le \pi/2$, then which of the following inequality is true?

a. $|z-\bar{z}| \le |z|(\arg z - \arg \bar{z})$ b. $|z-\bar{z}| \ge |z|(\arg z - \arg \bar{z})$
c. $|z-\bar{z}| < (\arg z - \arg \bar{z})$ d. none of these

59. If z is a complex number lying in the fourth quadrant of Argand plane and $|[kz/(k+1)] + 2i| > \sqrt{2}$ for all real value of k $(k \ne -1)$, then range of arg(z) is

a. $\left(-\frac{\pi}{8}, 0\right)$ b. $\left(-\frac{\pi}{6}, 0\right)$
c. $\left(-\frac{\pi}{4}, 0\right)$ d. none of these

60 If 'z' is complex number then the locus of 'z' satisfying the condition $|2z-1| = |z-1|$ is

a. perpendicular bisector of line segment joining 1/2 and 1
b. circle
c. parabola
d. none of the above curves

61. If $|z_2 + iz_1| = |z_1| + |z_2|$ and $|z_1| = 3$ and $|z_2| = 4$, then area of ΔABC, if affixes of A, B and C are z_1, z_2 and $[(z_2 - iz_1)/(1-i)]$ respectively, is,

a. $\frac{5}{2}$ b. 0 c. $\frac{25}{2}$ d. $\frac{25}{4}$

62. If a complex number z satisfies $|2z + 10 + 10i| \le 5\sqrt{3} - 5$, then the least principal argument of z is

a. $-\frac{5\pi}{6}$ b. $-\frac{11\pi}{12}$ c. $-\frac{3\pi}{4}$ d. $-\frac{2\pi}{3}$

63. If t and c are two complex numbers such that $|t| \ne |c|$, $|t| = 1$ and $z = (at+b)/(t-c)$, $z = x + iy$. Locus of z is (where a, b are complex numbers)

a. line segment b. straight line
c. circle d. none of these

64. The number of complex numbers z satisfying $|z - 3 - i| = |z - 9 - i|$ and $|z - 3 + 3i| = 3$ are

a. one b. two c. four d. none of these

65. Let a be a complex number such that $|a| < 1$ and $z_1, z_2, z_3, \ldots$ be the vertices of a polygon such that $z_k = 1 + a + a^2 + \cdots + a^{k-1}$ for all $k = 1, 2, 3, \ldots$ then $z_1, z_2, \ldots$ lie within the circle

a. $\left|z - \frac{1}{1-a}\right| = \frac{1}{|a-1|}$ b. $\left|z + \frac{1}{a+1}\right| = \frac{1}{|a+1|}$

c. $\left|z - \frac{1}{1-a}\right| = |a-1|$ d. $\left|z + \frac{1}{a+1}\right| = |a+1|$

66. Let $\lambda \in R$, the origin and the non-real roots of $2z^2 + 2z + \lambda = 0$ form the three vertices of an equilateral triangle in the Argand plane then λ is

a. 1 b. $\frac{2}{3}$ c. 2 d. −1

67. Let $z = 1 - t + i\sqrt{t^2 + t + 2}$, where t is a real parameter. The locus of z in the Argand plane is

a. a hyperbola b. an ellipse
c. a straight line d. none of these

68. If $z^2 + z|z| + |z^2| = 0$, then the locus of z is

a. a circle b. a straight line
c. a pair of straight lines d. none of these

69. The roots of the equation $t^3 + 3at^2 + 3bt + c = 0$ are z_1, z_2, z_3 which represent the vertices of an equilateral triangle, then

a. $a^2 = 3b$ b. $b^2 = a$
c. $a^2 = b$ d. $b^2 = 3a$

70. If 'z' lies on the circle $|z - 2i| = 2\sqrt{2}$ then the value of $\arg[(z-2)/(z+2)]$ is equal to

a. $\frac{\pi}{3}$ b. $\frac{\pi}{4}$ c. $\frac{\pi}{6}$ d. $\frac{\pi}{2}$

71. $P(z)$ be a variable point in the Argand plane such that $|z| = \text{minimum}\ \{|z-1|, |z+1|\}$ then $z + \bar{z}$ will be equal to

a. −1 or 1 b. 1 but not equal to −1
c. −1 but not equal to 1 d. none of these

72. The locus of point z satisfying Re $\left(\frac{1}{z}\right) = k$, where k is a non-zero real number, is

a. a straight line b. a circle
c. an ellipse d. a hyperbola

73. z_1 and z_2 lie on a circle with centre at the origin. The point of intersection z_3 of the tangents at z_1 and z_2 is given by

a. $\frac{1}{2}(\bar{z}_1 + \bar{z}_2)$ b. $\frac{2z_1z_2}{z_1 + z_2}$

c. $\frac{1}{2}\left(\frac{1}{z_1} + \frac{1}{z_2}\right)$ d. $\frac{z_1 + z_2}{\bar{z}_1\bar{z}_2}$

74. If $|z_1| = |z_2| = |z_3| = 1$ and $z_1 + z_2 + z_3 = 0$, then area of the triangle whose vertices are z_1, z_2, z_3 is

a. $3\sqrt{3}/4$ b. $\sqrt{3}/4$ c. 1 d. 2

75. z_1, z_2, z_3, z_4 are distinct complex numbers representing the vertices of a quadrilateral $ABCD$ taken in order. If $z_1 - z_4 = z_2 - z_3$ and $\arg[(z_4 - z_1)/(z_2 - z_1)] = \pi/2$, then the quadrilateral is

a. rectangle b. rhombus c. square d. trapezium

76. If $\arg\left(\frac{z_1 - \frac{z}{|z|}}{\frac{z}{|z|}}\right) = \frac{\pi}{2}$ and $\left|\frac{z}{|z|} - z_1\right| = 3$ then $|z_1|$ equals to

a. $\sqrt{26}$ b. $\sqrt{10}$ c. $\sqrt{3}$ d. $2\sqrt{2}$

77. The points $z_1 = 3 + \sqrt{3}i$ and $z_2 = 2\sqrt{3} + 6i$ are given on a complex plane. The complex number lying on the bisector of the angle formed by the vectors z_1 and z_2 is

a. $z = \frac{(3+2\sqrt{3})}{2} + \frac{\sqrt{3}+2}{2}i$ **b.** $z = 5 + 5i$

c. $z = -1 - i$ **d.** none of these

78. Let C_1 and C_2 are concentric circles of radius 1 and 8/3, respectively, having centre at (3, 0) on the Argand plane. If the complex number z satisfies the inequality

$\log_{1/3}\left(\frac{|z-3|^2+2}{11|z-3|-2}\right) > 1$ then

a. z lies outside C_1 but inside C_2
b. z lies inside of both C_1 and C_2
c. z lies outside both of C_1 and C_2
d. none of these

79. If $|z-2-i| = |z|\left|\sin\left(\frac{\pi}{4} - \arg z\right)\right|$, then locus of z is

a. a pair of straight lines **b.** circle
c. parabola **d.** ellipse

80. If z is a complex number having least absolute value and $|z-2+2i| = 1$, then $z =$

a. $(2-1/\sqrt{2})(1-i)$ **b.** $(2-1/\sqrt{2})(1+i)$
c. $(2+1/\sqrt{2})(1-i)$ **d.** $(2+1/\sqrt{2})(1+i)$

81. If $z = 3/(2+\cos\theta + i\sin\theta)$, then locus of z is

a. a straight line
b. a circle having centre on y-axis
c. a parabola
d. a circle having centre on x-axis

82. If 'p' and 'q' are distinct prime numbers, then the number of distinct imaginary numbers which are p^{th} as well as q^{th} roots of unity are

a. min (p, q) **b.** max (p, q) **c.** 1 **d.** zero

83. If α is the n^{th} root of unity, then $1 + 2\alpha + 3\alpha^2 + \cdots$ to n terms equal to

a. $\frac{-n}{(1-\alpha)^2}$ **b.** $\frac{-n}{1-\alpha}$
c. $\frac{-2n}{1-\alpha}$ **d.** $\frac{-2n}{(1-\alpha)^2}$

84. Given z is a complex number with modulus 1. Then the equation $[(1+ia)/(1-ia)]^4 = z$ has

a. all roots real and distinct
b. two real and two imaginary
c. three roots real and one imaginary
d. one root real and three imaginary

85. Roots of the equations are $(z+1)^5 = (z-1)^5$ are

a. $\pm i\tan\left(\frac{\pi}{5}\right), \pm i\tan\left(\frac{2\pi}{5}\right)$ **b.** $\pm i\cot\left(\frac{\pi}{5}\right), \pm i\cot\left(\frac{2\pi}{5}\right)$

c. $\pm i\cot\left(\frac{\pi}{5}\right), \pm i\tan\left(\frac{2\pi}{5}\right)$ **d.** none of these

86. The value of z satisfying the equation $\log z + \log z^2 + \cdots + \log z^n = 0$ is

a. $\cos\frac{4m\pi}{n(n+1)} + i\sin\frac{4m\pi}{n(n+1)}$, $m = 1, 2, \ldots$

b. $\cos\frac{4m\pi}{n(n+1)} - i\sin\frac{4m\pi}{n(n+1)}$, $m = 1, 2, \ldots$

c. $\sin\frac{4m\pi}{n} + i\cos\frac{4m\pi}{n}$, $m = 1, 2, \ldots$

d. 0

87. If $n \in N > 1$ then sum of real part of roots of $z^n = (z+1)^n$ is equal to

a. $\frac{n}{2}$ **b.** $\frac{(n-1)}{2}$ **c.** $-\frac{n}{2}$ **d.** $\frac{(1-n)}{2}$

88. Which of the following represents a point in an Argand plane, equidistant from the roots of the equation $(z+1)^4 = 16z^4$?

a. (0, 0) **b.** $\left(-\frac{1}{3}, 0\right)$ **c.** $\left(\frac{1}{3}, 0\right)$ **d.** $\left(0, \frac{2}{\sqrt{5}}\right)$

89. $1, z_1, z_2, z_3, \ldots, z_{n-1}$ are the n^{th} roots of unity, then the value of $1/(3-z_1) + 1/(3-z_2) + \cdots + 1/(3-z_{n-1})$ is equal to

a. $\frac{n3^{n-1}}{3^n-1} + \frac{1}{2}$ **b.** $\frac{n\,3^{n-1}}{3^n-1} - 1$

c. $\frac{n\,3^{n-1}}{3^n-1} + 1$ **d.** none of these

Multiple Correct Answers Type

Solutions on page 2.66

Each question has 4 choices a, b, c and d, out of which *one or more* answers are correct.

1. If $z = \omega, \omega^2$, where ω is a non-real complex cube root of unity, are two vertices of an equilateral triangle in the Argand plane then the third vertex may be represented by

a. $z = 1$ **b.** $z = 0$ **c.** $z = -2$ **d.** $z = -1$

2. $P(z_1)$, $Q(z_2)$, $R(z_3)$ and $S(z_4)$ are four complex numbers representing the vertices of a rhombus taken in order on the complex plane, then which one of the following is/are correct?

a. $\frac{z_1 - z_4}{z_2 - z_3}$ is purely real

b. amp $\frac{z_1 - z_4}{z_2 - z_4}$ = amp $\frac{z_2 - z_4}{z_3 - z_4}$

c. $\frac{z_1 - z_3}{z_2 - z_4}$ is purely imaginary

d. it is not necessary that $|z_1 - z_3| \neq |z_2 - z_4|$

3. If $z^3 + (3+2i)z + (-1+ia) = 0$ has one real root, then the value of 'a' lies in the interval ($a \in R$)

a. (– 2, 1) **b.** (–1, 0) **c.** (0, 1) **d.** (–2, 3)

4. A rectangle of maximum area is inscribed in the circle $|z-3-4i| = 1$. If one vertex of the rectangle is $4+4i$, then another adjacent vertex of this rectangle can be

a. $2+4i$ **b.** $3+5i$ **c.** $3+3i$ **d.** $3-3i$

5. If $|z_1| = 15$ and $|z_2 - 3 - 4i| = 5$, then

a. $|z_1 - z_2|_{min} = 5$ b. $|z_1 - z_2|_{min} = 10$
c. $|z_1 - z_2|_{max} = 20$ d. $|z_1 - z_2|_{max} = 25$

6. If the points $A(z)$, $B(-z)$ and $C(1 - z)$ are the vertices of an equilateral triangle ABC, then

a. sum of possible z is 1/2
b. sum of possible z is 1
c. product of possible z is 1/4
d. product of possible z is 1/2

7. If $|(z-z_1)/(z-z_2)|=3$, where z_1 and z_2 are fixed complex numbers and z is a variable complex number, then 'z' lies on a

a. Circle with 'z_1' as its interior point
b. Circle with 'z_2' as its interior point
c. Circle with 'z_1' as its exterior point
d. Circle with 'z_2' as its exterior point

8. If $\arg(z + a) = \pi/6$ and $\arg(z - a) = 2\pi/3$ $(a \in R^+)$, then

a. $|z| = a$
b. $|z| = 2a$
c. $\arg(z) = \frac{\pi}{2}$
d. $\arg(z) = \frac{\pi}{3}$

9. Value(s) $(-i)^{1/3}$ is/are

a. $\frac{\sqrt{3}-i}{2}$ b. $\frac{\sqrt{3}+i}{2}$
c. $\frac{-\sqrt{3}-i}{2}$ d. $\frac{-\sqrt{3}+i}{2}$

10. If $1, z_1, z_2, z_3, \ldots, z_{n-1}$ be the n^{th} roots of unity and ω be a non-real complex cube root of unity, then the product $\prod_{r=1}^{n-1}(\omega - z_r)$ can be equal to

a. 0 b. 1 c. −1 d. $1 + \omega$

11. If the equation, $z^3 + (3 + i)z^2 - 3z - (m + i) = 0$, where $m \in R$, has at least one real root, then m can have the value equal to

a. 1 b. 2 c. 3 d. 5

12. Let $P(x)$ and $Q(x)$ be two polynomials. Suppose that $f(x) = P(x^3) + xQ(x^3)$ is divisible by $x^2 + x + 1$, then

a. $P(x)$ is divisible by $(x - 1)$ but $Q(x)$ is not divisible by $x - 1$
b. $Q(x)$ is divisible by $(x - 1)$ but $P(x)$ is not divisible by $x - 1$
c. Both $P(x)$ and $Q(x)$ are divisible by $x - 1$
d. $f(x)$ is divisible by $x - 1$

13. If amp $(z_1z_2) = 0$ and $|z_1| = |z_2| = 1$, then

a. $z_1 + z_2 = 0$ b. $z_1z_2 = 1$
c. $z_1 = \bar{z}_2$ d. none of these

14. If $|z - (1/z)| = 1$ then

a. $|z|_{max} = \frac{1+\sqrt{5}}{2}$ b. $|z|_{min} = \frac{\sqrt{5}-1}{2}$
c. $|z|_{max} = \frac{\sqrt{5}-2}{2}$ d. $|z|_{min} = \frac{\sqrt{5}-1}{\sqrt{2}}$

15. z_0 is a root of the equation $z^n \cos\theta_0 + z^{n-1}\cos\theta_1 + \cdots + z\cos\theta_{n-1} + \cos\theta_n = 2$, where $\theta_i \in R$, then

a. $|z_0| > 1$ b. $|z_0| > \frac{1}{2}$ c. $|z_0| > \frac{1}{4}$ d. $|z_0| > \frac{3}{2}$

16. If from a point P representing the complex number z_1 on the curve $|z| = 2$, two tangents are drawn from P to the curve $|z| = 1$, meeting at points $Q(z_2)$ and $R(z_3)$, then

a. complex number $(z_1 + z_2 + z_3)/3$ will be on the curve $|z| = 1$

b. $\left(\frac{4}{\bar{z}_1}+\frac{1}{\bar{z}_2}+\frac{1}{\bar{z}_3}\right)\left(\frac{4}{z_1}+\frac{1}{z_2}+\frac{1}{z_3}\right)=9$

c. $\arg\left(\frac{z_2}{z_3}\right)=\frac{2\pi}{3}$

d. orthocentre and circumcnetre of ΔPQR will coincide

17. If z_1, z_2 be two complex numbers $(z_1 \neq z_2)$ satisfying $|z_1^2 - z_2^2| = |\bar{z}_1^2 + \bar{z}_2^2 - 2\bar{z}_1\bar{z}_2|$, then

a. $\frac{z_1}{z_2}$ is purely imaginary b. $\frac{z_1}{z_2}$ is purely real
c. $|\arg z_1 - \arg z_2| = \pi$ d. $|\arg z_1 - \arg z_2| = \frac{\pi}{2}$

18. A complex number z is rotated in anticlockwise direction by an angle α and we get z' and if the same complex number z is rotated by an angle α in clockwise direction and we get z'' then

a. z', z, z'' are in G.P. b. z', z, z'' are in H.P.
c. $z' + z'' = 2z\cos\alpha$ d. $z'^2 + z''^2 = 2z^2\cos 2\alpha$

19. Let z be a complex number satisfying equation $z^p = \bar{z}^q$, where $p, q \in N$, then

a. if $p = q$, then number of solutions of equation will be infinite
b. if $p = q$, then number of solutions of equation will be finite
c. if $p \neq q$, then number of solutions of equation will be $p + q + 1$.
d. if $p \neq q$, then number of solutions of equation will be $p + q$

20. If $z_1 = 5 + 12i$ and $|z_2| = 4$ then

a. maximum $(|z_1 + iz_2|) = 17$

b. minimum $(|z_1 + (1 + i)z_2|) = 13 - 4\sqrt{2}$

c. minimum $\left|\frac{z_1}{z_2+\frac{4}{z_2}}\right|=\frac{13}{4}$

d. maximum $\left|\frac{z_1}{z_2+\frac{4}{z_2}}\right|=\frac{13}{3}$

21. If $p = a + b\omega + c\omega^2$, $q = b + c\omega + a\omega^2$ and $r = c + a\omega + b\omega^2$ where $a, b, c \neq 0$ and ω is the complex cube root of unity, then

a. If p, q, r lie on the circle $|z| = 2$, the triangle formed by these points is equilateral.

b. $p^2 + q^2 + r^2 = a^2 + b^2 + c^2$

c. $p^2 + q^2 + r^2 = 2(pq + qr + rp)$

d. none of these

22. z_1, z_2, z_3 and z'_1, z'_2, z'_3 are non-zero complex numbers such that $z_3 = (1 - \lambda)z_1 + \lambda z_2$ and $z'_3 = (1 - \mu)z'_1 + \mu z'_2$ then which of the following statements is/are true?

a. If $\lambda, \mu \in R - \{0\}$, then z_1, z_2 and z_3 are collinear and z'_1, z'_2, z'_3 are collinear separately.

b. If λ, μ are complex numbers, where $\lambda = \mu$ then triangles formed by points z_1, z_2, z_3 and z'_1, z'_2, z'_3 are similar.

c. If λ, μ are distinct complex numbers, then points z_1, z_2, z_3 and z'_1, z'_2, z'_3 are not connected by any well defined geometry.

d. If $0 < \lambda < 1$, then z_3 divides the line joining z_1 and z_2 internally and if $\mu > 1$ then z'_3 divides the line joining of z'_1, z'_2 externally.

23. If $|z - 3| = \min\{|z - 1|, |z - 5|\}$, then Re($z$) equals to

a. 2 **b.** $\frac{5}{2}$ **c.** $\frac{7}{2}$ **d.** 4

24. If n is a natural number ≥ 2, such that $z^n = (z + 1)^n$, then

a. roots of equation lie on a straight line parallel to y-axis

b. roots of equation lie on a straight line parallel to x-axis

c. sum of the real parts of the roots is $-[(n - 1)/2]$

d. none of these

25. If $|z - 1| = 1$, then

a. $\arg((z - 1 - i)/z)$ can be equal to $-\pi/4$

b. $(z - 2)/z$ is purely imaginary number

c. $(z - 2)/z$ is purely real number

d. if $\arg(z) = \theta$, where $z \neq 0$ and θ is acute, then $1 - 2/z = i \tan \theta$

26. If $z = x + iy$, then the equation $|(2z - i)/(z + 1)| = m$ represents a circle then m can be

a. 1/2 **b.** 1 **c.** 2 **d.** $3 < r < 2\sqrt{2}$

27. Given that the two curves $\arg(z) = \pi/6$ and $|z - 2\sqrt{3}i| = r$ intersect in two distinct points, then

a. $[r] \neq 2$ **b.** $0 < r < 3$ **c.** $r = 6$ **d.** $3 < r < 2\sqrt{3}$
([r] represents integral part of r)

28. If P and Q are represented by the complex numbers z_1 and z_2, such that $|1/z_2 + 1/z_1| = |1/z_2 - 1/z_1|$, then

a. ΔOPQ (where O is the origin) is equilateral

b. ΔOPQ is right angled

c. the circumcentre of ΔOPQ is $\frac{1}{2}(z_1 + z_2)$

d. the circiucentre of ΔOPQ is $\frac{1}{3}(z_1 + z_2)$

29. Given $z = f(x) + i\, g(x)$ where $f, g:(0, 1) \to (0, 1)$ are real valued functions. Then, which of the following does not hold good?

a. $z = \frac{1}{1 - ix} + i\left(\frac{1}{1 + ix}\right)$ **b.** $z = \frac{1}{1 + ix} + i\left(\frac{1}{1 - ix}\right)$

c. $z = \frac{1}{1 + ix} + i\left(\frac{1}{1 + ix}\right)$ **d.** $z = \frac{1}{1 - ix} + i\left(\frac{1}{1 - ix}\right)$

30. Given that the complex numbers which satisfy the equation $z\bar{z}^3 + \bar{z}z^3 = 350$ form a rectangle in the Argand plane with the length of its diagonal having an integral number of units, then

a. area of rectangle is 48 sq. units

b. if z_1, z_2, z_3, z_4 are vertices of rectangle then $z_1 + z_2 + z_3 + z_4 = 0$

c. rectangle is symmetrical about real axis

d. $\arg(z_1 - z_3) = \frac{\pi}{4}$ or $\frac{3\pi}{4}$

31. Equation of tangent drawn to circle $|z| = r$ at the point $A(z_0)$ is

a. $\text{Re}\left(\frac{z}{z_0}\right) = 1$ **b.** $z\bar{z}_0 + z_0\bar{z} = 2r^2$

c. $\text{Im}\left(\frac{z}{z_0}\right) = 1$ **d.** $\text{Im}\left(\frac{z_0}{z}\right) = 1$

32. z_1 and z_2 are the roots of the equation $z^2 - az + b = 0$, where $|z_1| = |z_2| = 1$ and a, b are non-zero complex numbers, then

a. $|a| \leq 1$ **b.** $|a| \leq 2$

c. $\arg(a^2) = \arg(b)$ **d.** $\arg a = \arg(b^2)$

33. Let z_1, z_2, z_3 be the three non-zero complex numbers such that $z_2 \neq 1$, $a = |z_1|$, $b = |z_2|$ and $c = |z_3|$. Let,

$$\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = 0$$

Then

a. $\arg\left(\frac{z_3}{z_2}\right) = \arg\left(\frac{z_3 - z_1}{z_2 - z_1}\right)^2$

b. orthocentre of triangle formed by z_1, z_2, z_3 is $z_1 + z_2 + z_3$

c. if triangle formed by z_1, z_2, z_3 is equilateral, then its area is $\frac{3\sqrt{3}}{2}|z_1|^2$

d. if triangle formed by z_1, z_2, z_3 is equilateral then $z_1 + z_2 + z_3 = 0$

34. Locus of complex number satisfying $\arg[(z - 5 + 4i)/(z + 3 - 2i)] = -\pi/4$ is the arc of a circle

a. whose radius is $5\sqrt{2}$

b. whose radius is 5

c. whose lengh (of arc) is $\frac{15\pi}{\sqrt{2}}$

d. whose centre is $-2 - 5i$

35. If α is a complex constant such that $\alpha z^2 + z + \bar{\alpha} = 0$ has a real root, then

a. $\alpha + \bar{\alpha} = 1$ **b.** $\alpha + \bar{\alpha} = 0$

c. $\alpha + \bar{\alpha} = -1$ **d.** the absolute value of the real root is 1

36. If $\sqrt{5-12i}+\sqrt{-5-12i}=z$, then principal value of arg z can be

a. $-\frac{\pi}{4}$ **b.** $\frac{\pi}{4}$ **c.** $\frac{3\pi}{4}$ **d.** $-\frac{3\pi}{4}$

Reasoning Type

Solutions on page 2.72

Each question has four choices a, b, c and d, out of which *only one* is correct. Each question contains STATEMENT 1 and STATEMENT 2.

a. Both the statements are TRUE and STATEMENT 2 is the correct explanation of STATEMENT 1.

b. Both the statements are TRUE but STATEMENT 2 is NOT the correct explanation of STATEMENT 1.

c. STATEMENT 1 is TRUE and STATEMENT 2 is FALSE.

d. STATEMENT 1 is FALSE and STATEMENT 2 is TRUE.

1. **Statement 1:** If $\arg(z_1z_2)=2\pi$, then both z_1 and z_2 are purely real (z_1 and z_2 have principal arguments).
   Statement 2: Principal argument of complex number lies in $(-\pi, \pi)$

2. **Statement 1:** If n is an odd integer greater than 3 but not a multiple of 3, then $(x+1)^n-x^n-1$ is divisible by x^3+x^2+x.
   Statement 2: If n is an odd integer greater than 3 but not a multiple of 3, we have $1+\omega^n+\omega^{2n}=3$.

3. **Statement 1:** If $z_1+z_2=a$ and $z_1 z_2=b$, where $a=\bar{a}$ and $b=\bar{b}$, then $\arg(z_1z_2)=0$.
   Statement 2: The sum and product of two complex numbers are real if and only if they are conjugate of each other.

4. **Statement 1:** If $x+(1/x)=1$ and $p=x^{4000}+(1/x^{4000})$ and q be the digit at unit place in the number $2^{2^n}+1$, $n\in N$ and $n>1$, then the value of $p+q=8$.
   Statement 2: If ω, ω^2 are the roots of $x+1/x=-1$, then $x^2+1/x^2=-1$, $x^3+(1/x^3)=2$.

5. **Statement 1:** Let z_1 and z_2 are two complex numbers such that $|z_1-z_2|=|z_1+z_2|$ then the orthocentre of ΔAOB is $[(z_1+z_2)/2]$. (where O is origin).
   Statement 2: In case of right angled triangle, orthocentre is that point at which the triangle is right angled.

6. **Statement 1:** Locus of z, satisfying the equation $|z-1|+|z-8|=5$ is an ellipse.
   Statement 2: Sum of focal distances of any point on ellipse is constant.

7. **Statement 1:** $|z_1-a|<a$, $|z_2-b|<b$, $|z_3-c|<c$, where a, b, c are positive real numbers, then $|z_1+z_2+z_3|$ is greater than $2|a+b+c|$.
   Statement 2: $|z_1\pm z_2|\le|z_1|+|z_2|$.

8. Let fourth roots of unity are z_1, z_2, z_3 and z_4 respectively.
   Statement 1: $z_1^2+z_2^2+z_3^2+z_4^2=0$.
   Statement 2: $z_1+z_2+z_3+z_4=0$.

9. **Statement 1:** If the equation $ax^2+bx+c=0$, $0<a<b<c$, has non-real complex roots z_1 and z_2, then $|z_1|>1$, $|z_2|>1$.
   Statement 2: Complex roots always occur in conjugate pairs.

10. **Statement 1:** If z_1, z_2 are the roots of the quadratic equation $az^2+bz+c=0$ such that $\text{Im}(z_1 z_2)\ne 0$, then at least one of a, b, c is imaginary.
    Statement 2: If quadratic equation having real coefficients has complex roots, then roots are always conjugate to each other.

11. **Statement 1:** The product of all values of $(\cos\alpha+i\sin\alpha)^{3/5}$ is $\cos 3\alpha+i\sin 3\alpha$.
    Statement 2: The product of fifth roots of unity is 1.

12. **Statement 1:** If $|z_1|=|z_2|=|z_3|$, $z_1+z_2+z_3=0$ and (z_1), $B(z_2)$, $C(z_3)$ are the vertices of ΔABC, then one of the values of $\arg\left((z_2+z_3-2z_1)/(z_3-z_2)\right)$ is $\pi/2$.
    Statement 2: In equilateral triangle orthocentre coincides with centroid.

13. **Statement 1:** Let z be a complex number, then the equation $z^4+z+2=0$ cannot have a root, such that $|z|<1$.
    Statement 2: $|z_1+z_2|\le|z_1|+|z_2|$

14. If $z_1\ne -z_2$ and $|z_1+z_2|=\left|(1/z_1)+(1/z_2)\right|$ then
    Statement 1: z_1z_2 is unimodular.
    Statement 2: z_1 and z_2 both are unimodular.

Linked Comprehension Type

Solutions on page 2.73

Based upon each paragraph, the relevant multiple choice questions have to be answered. Each question has four choices a, b, c and d, out of which *only one* is correct.

For Problems 1–4

Consider the complex numbers z_1 and z_2 satisfying the relation $|z_1+z_2|^2=|z_1|^2+|z_2|^2$.

1. Complex number $z_1\bar{z}_2$ is
   a. purely real **b.** purely imaginary
   c. zero **d.** none of these

2. Complex number z_1/z_2 is
   a. purely real **b.** purely imaginary
   c. zero **d.** none of these

3. One of the possible argument of complex number $i\,(z_1/z_2)$
   a. $\frac{\pi}{2}$ **b.** $-\frac{\pi}{2}$
   c. 0 **d.** none of these

4. Possible difference between the argument of z_1 and z_2 is
   a. 0 **b.** π
   c. $-\frac{\pi}{2}$ **d.** none of these

For Problems 5–8

Consider the complex numbers $z=(1-i\sin\theta)/(1+i\cos\theta)$.

5. The value of θ for which z is purely real are
   a. $n\pi-\frac{\pi}{4}$, $n\in I$ **b.** $n\pi+\frac{\pi}{4}$, $n\in I$
   c. $n\pi$, $n\in I$ **d.** none of these

6. The value of θ for which z is purely imaginary are

a. $n\pi - \frac{\pi}{4}, n \in I$ b. $n\pi + \frac{\pi}{4}, n \in I$
c. $n\pi,\ n \in I$ d. no real values of θ

7. The value of θ for which z is unimodular is given by

a. $n\pi \pm \frac{\pi}{6}, n \in I$ b. $n\pi \pm \frac{\pi}{3}, n \in I$
c. $n\pi \pm \frac{\pi}{4}, n \in I$ d. no real values of θ

8. If argument of z is $\pi/4$, then

a. $\theta = n\pi, n \in I$ only
b. $\theta = (2n + 1), n \in I$ only
c. both $\theta = n\pi$ and $\theta = (2n + 1)\frac{\pi}{2}, n \in$ I
d. none of these

For Problems 9–11

Consider a quadratic equation $az^2 + bz + c = 0$ where a, b, c are complex numbers.

9. The condition that the equation has one purely imaginary root is

a. $(c\bar{a} - a\bar{c})^2 = -(b\bar{c} + c\bar{b})(a\bar{b} + \bar{a}b)$
b. $(c\bar{a} + a\bar{c})^2 = (b\bar{c} + c\bar{b})(a\bar{b} + \bar{a}b)$
c. $(c\bar{a} - a\bar{c})^2 = (b\bar{c} - c\bar{b})(a\bar{b} - \bar{a}b)$
d. none of these

10. If equation has two purely imaginary roots, then which of the following is not true

a. $a\bar{b}$ is purely imaginary
b. $b\bar{c}$ is purely imaginary
c. $c\bar{a}$ is purely real
d. none of these

11. The condition that the equation has one purely real root is

a. $(c\bar{a} - a\bar{c})^2 = (b\bar{c} + c\bar{b})(a\bar{b} - \bar{a}b)$
b. $(c\bar{a} - a\bar{c})^2 = (b\bar{c} - c\bar{b})(a\bar{b} + \bar{a}b)$
c. $(c\bar{a} - a\bar{c})^2 = (b\bar{c} + c\bar{b})(a\bar{b} + \bar{a}b)$
d. $(c\bar{a} - a\bar{c})^2 = (b\bar{c} - c\bar{b})(a\bar{b} - \bar{a}b)$

For Problems 12–14

Consider the equation $az + b\bar{z} + c = 0$, where $a, b, c \in Z$.

12. If $|a| \neq |b|$, then z represents

a. circle   b. straight line
c. one point   d. ellipse

13. If $|a| = |b|$ and $\bar{a}c \neq b\bar{c}$, then z has

a. infinite solutions   b. no solutions
c. finite solutions   d. cannot say anything

14. If $|a| = |b| \neq 0$ and $\bar{a}c = b\bar{c}$, then $az + b\bar{z} + c = 0$ represents

a. an ellipse   b. a circle
c. a point   d. a straight line

For Problems 15–17

Let z be a complex number satisfying $z^2 + 2z\lambda + 1 = 0$, where λ is a parameter which can take any real value.

15. The roots of this equation lie on a certain circle if

a. $-1 < \lambda < 1$ b. $\lambda > 1$
c. $\lambda < 1$ d. none of these

16. One root lies inside the unit circle and one outside if

a. $-1 < \lambda < 1$ b. $\lambda > 1$
c. $\lambda < 1$ d. none of these

17. For every large value of λ, the roots are approximately

a. $-2\lambda, 1/\lambda$ b. $-\lambda, -1/\lambda$
c. $-2\lambda, -\frac{1}{2\lambda}$ d. none of these

For Problems 18–20

Consider the equation $az^2 + z + 1 = 0$ having purely imaginary root where $a = \cos\theta + i\sin\theta$, $i = \sqrt{-1}$ and function $f(x) = x^3 - 3x^2 + 3(1 + \cos\theta)x + 5$, then answer the following questions.

18. Which of the following is true about $f(x)$?

a. $f(x)$ decreases for $x \in [2n\pi, (2n + 1)\pi], n \in Z$
b. $f(x)$ decreases for $x \in \left[(2n-1)\frac{\pi}{2}, (2n+1)\frac{\pi}{2}\right], n \in Z$
c. $f(x)$ is non-monotonic function
d. $f(x)$ increases for $x \in R$.

19. Which of the following is true?

a. $f(x) = 0$ has three real distinct roots
b. $f(x) = 0$ has one positive real root
c. $f(x) = 0$ has one negative real root
d. $f(x) = 0$ has three but not distinct roots

20. Number of roots of the equation $\cos 2\theta = \cos\theta$ in $[0, 4\pi]$ are

a. 2   b. 3
c. 4   d. 6

For Problems 21–23

Complex numbers z satisfy the equation $|z - (4/z)| = 2$.

21. The difference between the least and the greatest moduli of complex numbers is

a. 2   b. 4
c. 1   d. 3

22. The value of arg (z_1/z_2), where z_1 and z_2 are complex numbers with the greatest and the least moduli can be

a. 2π b. π c. $\pi/2$ d. none of these

23. Locus of z if $|z - z_1| = |z - z_2|$, where z_1 and z_2 are complex numbers with the greatest and the least moduli is

a. line parallel to real axis
b. line parallel to imaginary axis
c. line having positive slope
d. line having negative slope

For Problems 24–26

Consider ΔABC in Argand plane. Let $A(0)$, $B(1)$ and $C(1 + i)$ be its vertices and M be the mid-point of CA. Let z be a variable complex number on the line BM. Let u be another variable complex number defined as $u = z^2 + 1$.

24. Locus of u is

a. parabola **b.** ellipse
c. hyperbola **d.** none of these

25. Axis of locus of u is

a. imaginary axis **b.** real axis
c. $z + \bar{z} = 2$ **d.** none of these

26. Directrix of locus of u is

a. imaginary axis **b.** $z - \bar{z} = 2i$
c. real axis **d.** none of these

For Problems 27–29

In an Argand plane z_1, z_2 and z_3 are, respectively, the vertices of an isosceles triangle ABC with $AC = BC$ and $\angle CAB = \theta$. If z_4 is the centre of triangle, then

27. the value of $AB \times AC/(IA)^2$ is

a. $\frac{(z_2 - z_1)(z_3 - z_1)}{(z_4 - z_1)^2}$ **b.** $\frac{(z_2 - z_1)(z_1 - z_3)}{(z_4 - z_1)^2}$

c. $\frac{(z_4 - z_1)}{(z_2 - z_1)(z_3 - z_1)}$ **d.** none of these

28. the value of $(z_4 - z_1)^2 (\cos\theta + 1) \sec\theta$ is

a. $\frac{(z_2 - z_1)(z_3 - z_1)}{(z_4 - z_1)}$ **b.** $(z_2 - z_1)(z_3 - z_1)$

c. $(z_2 - z_1)(z_3 - z_1)^2$ **d.** $\frac{(z_2 - z_1)(z_3 - z_1)}{(z_4 - z_1)^2}$

29. the value of $(z_2 - z_1)^2 \tan\theta \tan\theta/2$ is

a. $(z_1 + z_2 - 2z_3)$ **b.** $(z_1 + z_2 - z_3)(z_1 + z_2 - z_4)$
c. $-(z_1 + z_2 - 2z_3)(z_1 + z_2 - 2z_4)$ **d.** none of these

For Problems 30–32

$A(z_1)$, $B(z_2)$, $C(z_3)$ are the vertices of a triangle ABC inscribed in the circle $|z| = 2$. Internal angle bisector of the angle A meets the circumcircle again at $D(z_4)$.

30. Complex number representing point D is

a. $z_4 = \frac{1}{z_2} + \frac{1}{z_3}$ **b.** $\sqrt{\frac{z_2 + z_3}{z_1}}$

c. $\sqrt{\frac{z_2 z_3}{z_1}}$ **d.** $z_4 = \sqrt{z_2 z_3}$

31. $\arg [z_4/(z_2 - z_3)]$ is equal to

a. $\frac{\pi}{4}$ **b.** $\frac{\pi}{3}$
c. $\frac{\pi}{2}$ **d.** $\frac{2\pi}{3}$

32. For fixed positions of $B(z_2)$ and $C(z_3)$ all the bisectors (internal) of $\angle A$ will pass through a fixed point which is

a. H.M. of z_2 and z_3 **b.** A.M. of z_2 and z_3
c. G.M. of z_2 and z_3 **d.** none of these

Matrix-Match Type

Solutions on page 2.76

Each question contains statements given in two columns which have to be matched. Statements a, b, c, d in column I have to be matched with statements p, q, r, s in column II. If the correct match is a→p, a→s, b→r, c→p, c→q and d→s, then the correctly bubbled 4 × 4 matrix should be as follows:

	p	q	r	s
a	(p)	(q)	(r)	(s)
b	(p)	(q)	(r)	(s)
c	(p)	(q)	(r)	(s)
d	(p)	(q)	(r)	(s)

1.

Column I	Column II: possible argument of $z = a + ib$
a. $ab > 0$	**p.** $-\tan^{-1}\left\|\frac{b}{a}\right\|$
b. $ab < 0$	**q.** $\pi - \tan^{-1}\left\|\frac{b}{a}\right\|$
c. $a^2 + b^2 = 0$	**r.** $\tan^{-1}\frac{b}{a}$
d. $ab = 0$	**s.** $-\pi + \tan^{-1}\frac{b}{a}$
	t. not defined
	u. 0 or $\frac{\pi}{2}$

2.

Column I	Column II (one of the values of z)
a. $z^4 - 1 = 0$	**p.** $z = \cos\frac{\pi}{8} + i\sin\frac{\pi}{8}$
b. $z^4 + 1 = 0$	**q.** $z = \cos\frac{\pi}{8} - i\sin\frac{\pi}{8}$
c. $iz^4 + 1 = 0$	**r.** $z = \cos\frac{\pi}{4} + i\sin\frac{\pi}{4}$
d. $iz^4 - 1 = 0$	**s.** $z = \cos 0 + i\sin 0$

3.

Column I	Column II (Locus)
a. $\|z - 1\| = \|z - i\|$	**p.** pair of straight lines
b. $\|z + \bar{z}\| + \|z - \bar{z}\| = 2$	**q.** a line through the origin
c. $\|z + \bar{z}\| = \|z - \bar{z}\|$	**r.** circle
d. If $\|z\| = 1$, then $2/z$ lies on	**s.** square

4. Which of the condition/conditions in column II are satisfied by the quadrilateral formed by z_1, z_2, z_3, z_4 in order given in column I?

Column I	Column II
a. parallelogram	p. $z_1 - z_4 = z_2 - z_3$
b. rectangle	q. $\|z_1 - z_3\| = \|z_2 - z_4\|$
c. rhombus	r. $\frac{z_1 - z_2}{z_3 - z_4}$ is purely real
d. square	s. $\frac{z_1 - z_3}{z_2 - z_4}$ is purely imaginary
	t. $\frac{z_1 - z_2}{z_3 - z_2}$ is purely imaginary

5.

Column I	Column II
a. If $\|z - 2i\| + \|z - 7i\| = k$, then locus of z is an ellipse if $k =$	p. 7
b. If $\|(2z - 3)/(3z - 2)\| = k$, then locus of z is a circle if 2/3 is a point inside circle and 3/2 is outside the circle if $k =$	q. 8
c. If $\|z - 3\| - \|z - 4i\| = k$, then locus of z is a hyperbola if k is	r. 2
d. If $\|z - (3 + 4i)\| = (k/50)\|a\bar{z} + \bar{a}z + b\|$, where $a = 3 + 4i$, then locus of z is a hyperbola with $k =$	s. 4
	t. 5

6.

Column I	Column II
a. The value of $\sum_{n=1}^{5}(x^n + 1/x^n)^2$ when $x^2 - x + 1 = 0$ is	p. 2
b. If $\left[\frac{1 + \cos\theta + i\sin\theta}{\sin\theta + i(1 + \cos\theta)}\right]^4 = \cos n\theta + i\sin n\theta$, then $n =$	q. 4
c. The adjacent vertices of a regular polygon of n sides having centre at origin are the points z and $\bar{z}$. If Im(z)/Re(z) = $\sqrt{2} - 1$, then the value of $n/4$ is	r. 9
d. $(1/50)\left\{\sum_{r=1}^{10}(r - \omega)(r - \omega^2)\right\}$ = (where ω is cube root of unity)	s. 8

Integer Type

Solutions on page 2.78

1. If $x = a + bi$ is a complex number such that $x^2 = 3 + 4i$ and $x^3 = 2 + 11i$ where $i = \sqrt{-1}$, then $(a + b)$ equal to.
2. If the complex numbers x and y satisfy $x^3 - y^3 = 98i$ and $x - y = 7i$ then $xy = a + ib$ where $a, b \in R$. The value of $(a + b)/3$ equals.
3. If $x = \omega - \omega^2 - 2$, then the value of $x^4 + 3x^3 + 2x^2 - 11x - 6$ is (where ω is cube root of unity).
4. Let $z = 9 + bi$ where b is non zero real and $i^2 = -1$. If the imaginary part of z^2 and z^3 are equal, then $b/3$ is.
5. Modulus of non zero complex number z, satisfying $\bar{z} + z = 0$ and $|z|^2 - 4zi = z^2$ is.
6. If the expression $(1 + ir)^3$ is of the form $s(1 + i)$ for some real 's' where 'r' is also real and, then the sum of all possible values of r is.
7. If complex number $z(z \neq 2)$ satisfies the equation $z^2 = 4z + |z|^2 + \frac{16}{|z|^3}$ then the value of $|z|^4$ is.
8. The complex number z satisfies $z + |z| = 2 + 8i$. The value of $(|z| - 8)$ is.
9. Let $|z| = 2$ and $w = \frac{z+1}{z-1}$ where $z, w \in C$ (where C is the set of complex numbers).

   Then product of least and greatest value of modulus of w is.
10. If $\left[\frac{1 + \cos\theta + i\sin\theta}{\sin\theta + i(1 + \cos\theta)}\right]^4 = \cos n\theta + i\sin n\theta$, then n is.
11. If z be a complex number satisfying $z^4 + z^3 + 2z^2 + z + 1 = 0$ then $|z|$ is equal to.
12. Let 1, w, w^2 be the cube root of unity. The least possible degree of a polynomial with real coefficients having roots $2w$, $(2 + 3w)$, $(2 + 3w^2)$, $(2 - w - w^2)$, is
13. If ω is the imaginary cube root of unity, then find the number of pairs of integers (a, b) such that $|a\omega + b| = 1$.
14. Suppose that z is a complex number that satisfies $|z - 2 - 2i| \leq 1$. The maximum value of $|2iz + 4|$ is equal to.
15. If $|z + 2 - i| = 5$ and maximum value of $|3z + 9 - 7i|$ is M then the value of $M/4$ is.
16. Let $Z_1 = (8 + i)\sin\theta + (7 + 4i)\cos\theta$ and $Z_2 = (1 + 8i)\sin\theta + (4 + 7i)\cos\theta$ are two complex numbers. If $Z_1 . Z_2 = a + ib$ where $a, b \in R$. If M is the greatest value of $(a + b)$ $\forall \theta \in R$, then the value of $M^{1/3}$ is.
17. Let $A = \{a \in R |$ the equation $(1 + 2i)x^3 - 2(3 + i)x^2 + (5 - 4i)x + 2a^2 = 0\}$ has at least one real root. Then the value of $\frac{\sum a^2}{2}$ is.
18. The minimum value of the expression $E = |z|^2 + |z - 3|^2 + |z - 6i|^2$ is m then the value of $m/5$ is.

Archives

Solutions on page 2.80

Subjective Type

1. Express $1/(1 - \cos\theta + 2i\sin\theta)$ in the form $x + iy$.

   (IIT-JEE, 1978)

2. If $x = a + b$, $y = a\beta + b\gamma$, $z = a\gamma + b\beta$ where γ and β are the complex cube roots of unity, show that $xyz = a^3 + b^3$ **(IIT-JEE, 1978)**

3. If $x + iy = \sqrt{(a+ib)/(c+id)}$, then prove that $(x^2 + y^2)^2 = (a^2 + b^2)/(c^2 + d^2)$. **(IIT-JEE, 1979)**

4. It is given that n is an odd integer greater than 3, but n is not a multiple of 3. Prove that $x^3 + x^2 + x$ is a factor of $(x + 1)^n - x^n - 1$. **(IIT-JEE, 1980)**

5. Find the real values of x and y for which of the following equation is satisfied:

$$\frac{(1+i)x-2i}{3+i} + \frac{(2-3i)y+i}{3-i} = i$$

(IIT-JEE, 1980)

6. Let the complex numbers z_1, z_2 and z_3 be the vertices of an equilateral triangle. Let z_0 be the circumcentre of the triangle. Then prove that $z_1^2 + z_2^2 + z_3^2 = 3z_0^2$. **(IIT-JEE, 1981)**

7. Prove that the complex numbers z_1, z_2 and the origin form an equilateral triangle only if $z_1^2 + z_2^2 - z_1z_2 = 0$.

8. Show that the area of the triangle on the Argand diagram formed by the complex numbers z, iz and $z + iz$ is $1/2|z|^2$. **(IIT-JEE, 1986)**

9. Complex numbers z_1, z_2, z_3 are the vertices A, B, C, respectively, of an isosceles right-angled triangle with right angle at C. Show that $(z_1 - z_2)^2 = 2(z_1 - z_3)(z_3 - z_2)$. **(IIT-JEE, 1986)**

10. Let $z_1 = 10 + 6i$ and $z_2 = 4 + 6i$. If z is any complex number such that the argument of $(z - z_1)/(z - z_2)$ is $\pi/4$, then prove that $|z - 7 - 9i| = 3\sqrt{2}$. **(IIT-JEE, 1990)**

11. If $iz^3 + z^2 - z + i = 0$, then show that $|z| = 1$. **(IIT-JEE, 1995)**

12. If $|z| \le 1$, $|w| \le 1$, then show that $|z - w|^2 \le (|z| - |w|)^2 + (\arg z - \arg w)^2$. **(IIT-JEE, 1995)**

13. Find all non-zero complex numbers z satisfying $\bar{z} = iz^2$. **(IIT-JEE, 1996)**

14. Let $\bar{b}z + b\bar{z} = c$, $b \neq 0$, be a line in the complex plane, where $\bar{b}$ is the complex conjugate of b. If a point z_1 is the reflection of a point z_2 through the line, then show that $c = \bar{z}_1 b + z_2\bar{b}$. **(IIT-JEE, 1997)**

15. Let z_1 and z_2 be roots of the equation $z^2 + pz + q = 0$, where the coefficients p and q may be complex numbers. Let A and B represent z_1 and z_2 in the complex plane. If $\angle AOB = \theta \neq 0$ and $OA = OB$, where O is the origin, prove that $p^2 = 4q\cos^2(\theta/2)$. **(IIT-JEE, 1997)**

16. For complex numbers z and w, prove that $|z|^2 w - |w|^2 z = z - w$ if and only if $z = w$ or $z\bar{w} = 1$. **(IIT-JEE, 1999)**

17. Let a complex number α, $\alpha \neq 1$, be a root of the equation $z^{p+q} - z^p - z^q + 1 = 0$, where p, q are distinct primes. Show that either $1 + \alpha + \alpha^2 + \cdots + \alpha^{p-1} = 0$ or $1 + \alpha + \alpha^2 + \cdots + \alpha^{q-1} = 0$, but not both together. **(IIT-JEE, 2002)**

18. If z_1 and z_2 are two complex numbers such that $|z_1| < 1 < |z_2|$, then prove that $|(1 - z_1\bar{z}_2)/(z_1 - z_2)| < 1$. **(IIT-JEE, 2003)**

19. Prove that there exists no complex number z such that $|z| < 1/3$ and $\sum_{r=1}^{n} a_r z^r = 1$ where $|a_r| < 2$. **(IIT-JEE, 2003)**

20. Find the centre and radius of the circle given by $|(z - \alpha)/(z - \beta)| = k$, $k \neq 1$, where $z = x + iy$, $\alpha = \alpha_1 + i\alpha_2$, $\beta = \beta_1 + i\beta_2$. **(IIT-JEE, 2004)**

21. If one of the vertices of the square circumscribing the circle $|z - 1| = \sqrt{2}$ is $2 + \sqrt{3}i$, find the other vertices of the square. **(IIT-JEE, 2005)**

22. The maximum value of $\left|Arg\left(\frac{1}{1-z}\right)\right|$ for $|z| = 1$, $z \neq 1$ is given by. (This question is part of matrix match question) **(IIT-JEE, 2011)**

23. The set $\text{Re}\left(\frac{2iz}{1-z^2}\right)$: z is a complex number, $|z| = 1$, $z = \pm 1\}$ is (This question is part of matrix match question) **(IIT-JEE, 2011)**

Objective Type

Fill in the blanks

1. If the expression $\frac{\left[\sin\left(\frac{x}{2}\right) + \cos\left(\frac{x}{2}\right) + i\tan(x)\right]}{\left[1 + 2i\sin\left(\frac{x}{2}\right)\right]}$ is real, then the set of all possible values of x is ______________. **(IIT-JEE, 1987)**

2. For any two complex numbers z_1, z_2 and any real numbers a and b, $|az_1 - bz_2|^2 + |bz_1 + az_2|^2 =$ ______________. **(IIT-JEE, 1988)**

3. If a, b, c are the numbers between 0 and 1 such that the points $z_1 = a + i$, $z_2 = 1 + bi$ and $z_3 = 0$ form an equilateral triangle, then $a =$ __________ and $b =$ ____________. **(IIT-JEE, 1989)**

4. $ABCD$ is a rhombus. Its diagonals AC and BD intersect at the point M and satisfy $BD = 2AC$. If the points D and M represent the complex numbers $1 + i$ and $2 - i$, respectively, then A represents the complex number __________ or __________. **(IIT-JEE, 1993)**

5. Suppose z_1, z_2, z_3 are the vertices of an equilateral triangle inscribed in the circle $|z| = 2$. If $z_1 = 1 + i\sqrt{3}$ then z_2 = ____________, z_3 = ____________. **(IIT-JEE, 1994)**

6. The value of the expression $1 \times (2 - \omega) \times (2 - \omega^2) + 2 \times (3 - \omega) \times (3 - \omega^2) + \cdots + (n - 1) \times (n - \omega) \times (n - \omega^2)$, where ω is an imaginary cube root of unity, is ____________. **(IIT-JEE, 1996)**

True or false

1. For complex number $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, we write $z_1 \cap z_2$, if $x_1 \le x_2$ and $y_1 \le y_2$. Then for all complex numbers z with $1 \cap z$, we have $((1 - z)/(1 + z)) \cap 0$. **(IIT-JEE, 1984)**
2. If the complex numbers z_1, z_2 and z_3 represent the vertices of an equilateral triangle such that $|z_1| = |z_2| = |z_3|$, then $z_1 + z_2 + z_3 = 0$. **(IIT-JEE, 1984)**
3. If three complex numbers are in A.P. then they lie on a circle in the complex plane. **(IIT-JEE, 1985)**
4. The cube roots of unity when represented on Argand diagram form the vertices of an equilateral triangle. **(IIT-JEE, 1988)**

Multiple choice questions with one correct answer

1. If the cube roots of unity are $1, \omega, \omega^2$, then the roots of the equation $(x - 1)^3 + 8 = 0$ are
   a. $-1, 1 + 2\omega, 1 + 2\omega^2$ **b.** $-1, 1 - 2\omega, 1 - 2\omega^2$
   c. $-1, -1, -1$ **d.** none of these
   (IIT-JEE, 1979)

2. The smallest positive integer n for which $[(1 + i)/(1 - i)]^n = 1$ is
   a. $n = 8$ **b.** $n = 16$
   c. $n = 12$ **d.** none of these
   (IIT-JEE, 1980)

3. The complex numbers $z = x + iy$ which satisfy the equation $|(z - 5i)/(z + 5i)| = 1$ lie on
   a. the x-axis
   b. the straight line $y = 5$
   c. a circle passing through the origin
   d. none of these **(IIT-JEE, 1981)**

4. If $z = [(\sqrt{3}/2) + i/2]^5 + [(\sqrt{3}/2) - i/2]^5$, then
   a. $\text{Re}(z) = 0$ **b.** $\text{Im}(z) = 0$
   c. $\text{Re}(z) > 0, \text{Im}(z) > 0$ **d.** $\text{Re}(z) > 0, \text{Im}(z) < 0$
   (IIT-JEE, 1982)

5. The inequality $|z - 4| < |z - 2|$ represents the region given by
   a. $\text{Re}(z) \ge 0$ **b.** $\text{Re}(z) < 0$
   c. $\text{Re}(z) > 0$ **d.** none of these
   (IIT-JEE, 1982)

6. If $z = x + iy$ and $\omega = (1 - iz)/(z - i)$, then $|\omega| = 1$ implies that, in the complex plane
   a. z lies on the imaginary axis **b.** z lies on the real axis
   c. z lies on the unit circle **d.** none of these
   (IIT-JEE, 1983)

7. The points z_1, z_2, z_3, z_4 in the complex plane are the vertices of a parallelogram taken in order if and only if
   a. $z_1 + z_4 = z_2 + z_3$ **b.** $z_1 + z_3 = z_2 + z_4$
   c. $z_1 + z_2 = z_3 + z_4$ **d.** none of these
   (IIT-JEE, 1983)

8. The complex numbers $\sin x + i \cos 2x$ and $\cos x - i \sin 2x$ are conjugate to each other for
   a. $x = n\pi$ **b.** $x = 0$
   c. $x = (n + 1/2)\pi$ **d.** no value of x
   (IIT-JEE, 1988)

9. If $\omega\ (\neq 1)$ is a cube root of unity and $(1 + \omega)^7 = A + B\omega$ then A and B are respectively
   a. 0, 1 **b.** 1, 1
   c. 1, 0 **d.** −1, 1 **(IIT-JEE, 1995)**

10. Let z and ω be two non-zero complex numbers such that $|z| = |\omega|$ and $\arg z = \pi - \arg \omega$, then z equals
    a. ω **b.** $-\omega$
    c. $\bar{\omega}$ **d.** $-\bar{\omega}$ **(IIT-JEE, 1995)**

11. Let z and ω be two complex numbers such that $|z| \le 1$, $|\omega| \le 1$ and $|z - i\omega| = |z - i\bar{\omega}| = 2$ then z equals
    a. 1 or i **b.** i or $-i$ **c.** 1 or −1 **d.** i or −1

12. For positive integers n_1, n_2 the value of the expression $(1+i)^{n_1} + (1+i^3)^{n_1} + (1+i^5)^{n_2} + (1+i^7)^{n_2}$, where $i = \sqrt{-1}$ is a real number if and only if
    a. $n_1 = n_2 + 1$ **b.** $n_1 = n_2 - 1$
    c. $n_1 = n_2$ **d.** $n_1 > 0, n_2 > 0$
    (IIT-JEE, 1996)

13. If $i = \sqrt{-1}$, then $4 + 5[(-1/2) + i\sqrt{3}/2]^{334} + 3[(-1/2) + (i\sqrt{3}/2)]^{365}$ is equal to
    a. $1 - i\sqrt{3}$ **b.** $-1 + i\sqrt{3}$
    c. $i\sqrt{3}$ **d.** $-i\sqrt{3}$ **(IIT-JEE, 1999)**

14. If $\arg(z) < 0$, then $\arg(-z) - \arg(z) =$
    a. π **b.** $-\pi$
    c. $-\dfrac{\pi}{2}$ **d.** $\dfrac{\pi}{2}$ **(IIT-JEE, 2000)**

15. If z_1, z_2 and z_3 are complex numbers such that $|z_1| = |z_2| = |z_3| = |(1/z_1) + (1/z_2) + (1/z_2)| = 1$, then $|z_1 + z_2 + z_3|$ is
    a. equal to 1 **b.** less than 1
    c. greater than 3 **d.** equal to 3

16. Let z_1 and z_2 be n^{th} roots of unity which subtend a right angle at the origin. Then n must be of the form

a. $4k + 1$ **b.** $4k + 2$
c. $4k + 3$ **d.** $4k$ **(IIT-JEE, 2001)**

17. The complex numbers z_1, z_2 and z_3 satisfying $[(z_1 - z_3)/(z_2 - z_3)] = [(1 - i\sqrt{3})/2]$ are the vertices of a triangle which is
a. of area zero **b.** right-angled isosceles
c. equilateral **d.** obtuse-angled isosceles

18. For all complex numbers z_1, z_2 satisfying $|z_1| = 12$ and $|z_2 - 3 - 4i| = 5$, the minimum value of $|z_1 - z_2|$ is
a. 0 **b.** 2 **c.** 7 **d.** 17

19. If $|z| = 1$ and $\omega = (z - 1)/(z + 1)$ (where $z \neq -1$), then Re(ω) is
a. 0 **b.** $\frac{1}{|z+1|^2}$
c. $\left|\frac{z}{z+1}\right| \frac{1}{|z+1|^2}$ **d.** $\frac{\sqrt{2}}{|z+1|^2}$
(IIT-JEE, 2003)

20. If ω ($\neq 1$) be a cube root of unity and $(1 + \omega^2)^n = (1 + \omega^4)^n$, then the least positive value of n is
a. 2 **b.** 3 **c.** 5 **d.** 6

21. The locus of z which lies in shaded region (excluding the boundaries) is best represented by

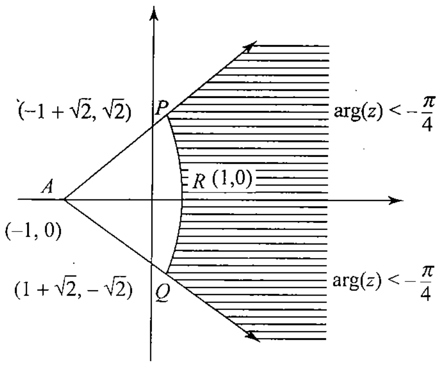


Fig. 2.47

a. z: $|z + 1| > 2$ and $|\arg(z + 1)| < \pi/4$
b. z: $|z - 1| > 2$ and $|\arg(z - 1)| < \pi/4$
c. z: $|z + 1| < 2$ and $|\arg(z + 1)| < \pi/2$
d. z: $|z - 1| < 2$ and $|\arg(z + 1)| < \pi/2$ **(IIT-JEE, 2005)**

22. a, b, c are integers, not all simultaneously equal and ω is cube root of unity ($\omega \neq 1$), then minimum value of $|a + b\omega + c\omega^2|$ is
a. 0 **b.** 1 **c.** $\frac{\sqrt{3}}{2}$ **d.** $\frac{1}{2}$
(IIT-JEE, 2005)

23. If a, b, c and u, v, w are complex numbers representing the vertices of two triangles such that $c = (1 - r)a + rb$ and $w = (1 - r)u + rv$, where r is a complex number, then the two triangles
a. have the same area **b.** are similar
c. are congruent **d.** none of these
(IIT-JEE, 1985)

24. If z_1 and z_2 are two non-zero complex numbers such that $|z_1 + z_2| = |z_1| + |z_2|$, then arg z_1 – arg z_2 is equal to
a. $-\pi$ **b.** $-\frac{\pi}{2}$ **c.** 0
d. $\frac{\pi}{2}$ **e.** π **(IIT-JEE, 1987)**

25. The value of $\sum_{k=1}^{6}(\sin(2\pi k/7) - i\cos(2\pi k/7))$ is
a. –1 **b.** 0 **c.** $-i$
d. i **e.** none **(IIT-JEE, 1987)**

26. If ω is an imaginary cube root of unity, then $(1 + \omega - \omega^2)^7$ equals
a. 128ω **b.** -128ω
c. $128\omega^2$ **d.** $-128\omega^2$ **(IIT-JEE, 1998)**

27. The value of the sum $\sum_{n=1}^{13}(i^n + i^{n+1})$, where $i = \sqrt{-1}$, equals
a. i **b.** $i - 1$
c. $-i$ **d.** 0 **(IIT-JEE, 1998)**

28. If $\begin{vmatrix} 6i & -3i & 1 \\ 4 & 3i & -1 \\ 20 & 3 & i \end{vmatrix} = x + iy$, then
a. $x = 3, y = 0$ **b.** $x = 1, y = 3$
c. $x = 0, y = 3$ **d.** $x = 0, y = 0$ **(IIT-JEE, 1998)**

29. Let $\omega = (-1/2) + i(\sqrt{3}/2)$. Then the value of the determinant $\begin{vmatrix} 1 & 1 & 1 \\ 1 & -1-\omega^2 & \omega^2 \\ 1 & \omega^2 & \omega^4 \end{vmatrix}$ is
a. 3ω **b.** $3\omega(\omega - 1)$
c. $3\omega^2$ **d.** $3\omega(1 - \omega)$ **(IIT-JEE, 2002)**

30. If $(w - \bar{w}z)/(1 - z)$ is purely real where $w = \alpha + i\beta, \beta \neq 0$ and $z \neq 1$, then the set of the values of z is
a. $\{z : |z| = 1\}$ **b.** $\{z : z = \bar{z}\}$
c. $\{z : z \neq 1\}$ **d.** $\{z : |z| = 1, z \neq 1\}$
(IIT-JEE, 2006)

31. A man walks a distance of 3 units from the origin towards the north-east (N 45° E) direction. Form there, he walks a distance of 4 units towards the north-west (N 45° W) direction to reach a point P, Then the position of P in the Argand plane is
a. $3e^{i\pi/4} + 4i$ **b.** $(3 - 4i)e^{i\pi/4}$
c. $(4 + 3i)e^{i\pi/4}$ **d.** $(3 + 4i)e^{i\pi/4}$ **(IIT-JEE, 2007)**

32. If $|z| = 1$ and $z \neq = 1$, then all the values of $z/(1 - z^2)$ lie on
a. a line not passing through the origin
b. $|z| = \sqrt{2}$
c. the x-axis
d. the y-axis **(IIT-JEE, 2007)**

33. A particle P starts from the point $z_0 = 1 + 2i$, where $i = \sqrt{-1}$. It moves first horizontally away form origin by 5 units and then vertically away form origin by 3 units to reach a point z_1. From z_1 the particle moves $\sqrt{2}$ units in the direction of the vector $i + j$ and then it moves through an angle $\pi/2$ in anticlockwise direction on a circle with centre at origin to reach a point z_2. Then point z_2 is given by

a. $6 + 7i$ **b.** $-7 + 6i$

c. $7 + 6i$ **d.** $-6 + 7i$

34. Let $z = x + iy$ be a complex number where x and y are integers. Then the area of the rectangle whose vertices are the roots of the equation $\bar{z}z^3 + z\bar{z}^3 = 350$ is

a. 48 **b.** 32

c. 40 **d.** 80 **(IIT-JEE, 2009)**

Multiple choice questions with one or more than one correct answer

1. If $z_1 = a + ib$ and $z_2 = c + id$ are complex numbers such that $|z_1| = |z_2| = 1$ and Re $(z_1\bar{z}_2) = 0$, then the pair of complex numbers $\omega_1 = a + ic$ and $\omega_2 = b + id$ satisfies

a. $|\omega_1| = 1$ **b.** $|\omega_2| = 1$

c. $\text{Re}(\omega_1\bar{\omega}_2) = 0$ **d.** $\omega_1\bar{\omega}_2 = 0$

2. Let z_1 and z_2 be complex numbers such that $z_1 \neq z_2$ and $|z_1| = |z_2|$. If z_1 has positive real part and z_2 has negative imaginary part, then $(z_1 + z_2)/(z_1 - z_2)$ may be

a. zero **b.** real and positive

c. real and negative **d.** purely imaginary

(IIT-JEE, 1986)

3. Let z_1 and z_2 be two distinct complex numbers and let $z = (1 - t)\, z_1 + tz_2$ for some real number t with $0 < t < 1$. If arg (w) denotes the principal argument of a non–zero complex number w, then

a. $|z - z_1| + |z - z_2| = |z_1 - z_2|$ **b.** $(z - z_1) = (z - z_2)$

c. $\begin{vmatrix} z - z_1 & \bar{z} - \bar{z}_1 \\ z_2 - z_1 & \bar{z}_2 - \bar{z}_1 \end{vmatrix} = 0$ **d.** arg $(z - z_1) = $ arg $(z_2 - z_1)$

(IIT-JEE, 2010)

Comprehension

For Problems 1–3

Let A, B, C be three sets of complex numbers as defined below:

$A = \{z: \text{Im}\, z \geq 1\}$

$B = \{z: |z - 2 - i| = 3\}$

$C = \{z: \text{Re}((1 - i)z) = \sqrt{2}\}$ **(IIT-JEE, 2008)**

1. The number of elements in the set $A \cap B \cap C$ is

a. 0 **b.** 1 **c.** 2 **d.** ∞

2. Let z be any point in $A \cap B \cap C$. Then, $|z + 1 - i|^2 + |z - 5 - i|^2$ lies between

a. 25 and 29 **b.** 30 and 34

c. 35 and 39 **d.** 40 and 44

3. Let z be any point in $A \cap B \cap C$ and let w be any point satisfying $|w - 2 - i| < 3$. Then, $|z| - |w| + 3$ lies between

a. -6 and 3 **b.** -3 and 6

c. -6 and 6 **d.** -3 and 9

Matrix-match type

1. Match the statements in column-I with those I column-II [Note: Here z takes the values in the complex plane and Im(z) and Re(z) denote, respectively, the imaginary part and the real part of z]

Column I	**Column II:**
a. The set of points z satisfying $\|z - i\|z\|\| - \|z + i\|z\|\| = 0$ is contained in or equal to	**p.** an ellipse with eccentricity 4/5
b. The set of points z satisfying $\|z + 4\| + \|z - 4\| = 10$ is contained in or equal to	**q.** the set of points z satisfying Im $z = 0$
c. If $\|\omega\| = 2$, then the set of points $z = \omega - (1/\omega)$ is contained in or equal to	**r.** the set of points z satisfying $\|\text{Im}\, z\| \leq 1$
d. If $\|\omega\| = 1$, then the set of points $z = \omega + 1/\omega$ is contained in or equal to	**s.** the set of points z satisfying $\|\text{Re}\, z\| \leq 1$
	t. the set of points z satisfying $\|z\| \leq 3$

(IIT-JEE, 2010)

Integer type

1. Let ω be the complex number $\cos\dfrac{2\pi}{3} + i\sin\dfrac{2\pi}{3}$. Then the number of distinct complex numbers z satisfying

$$\begin{vmatrix} z+1 & \omega & \omega^2 \\ \omega & z+\omega^2 & 1 \\ \omega^2 & 1 & z+\omega \end{vmatrix} = 0$$ is equal to. **(IIT-JEE 2010)**

2. If z is any complex number satisfying $|z - 3 - 2i| \leq 2$, then the minimum value of $|2z - 6 + 5i|$ is. **(IIT-JEE 2011)**

3. Let $\omega = e^{i\pi/3}$, and a, b, c, x, y, z be non–zero complex numbers such that

$a + b + c = x$

$a + b\omega + c\omega^2 = y$

$a + b\omega^2 + c\omega = z$

Then the value of $\dfrac{|x|^2 + |y|^2 + |z|^2}{|a|^2 + |b|^2 + |c|^2}$ is. **(IIT-JEE 2011)**

ANSWERS AND SOLUTIONS

Subjective Type

1. Given that

$$\left|\frac{z}{|\bar{z}|} - \bar{z}\right| = 1 + |z|$$

Putting $z = re^{i\theta} \Rightarrow \bar{z} = re^{-i\theta}$, we have

$\Rightarrow \left|\frac{z}{|\bar{z}|} - \bar{z}\right| = |e^{i\theta} - re^{-i\theta}| = 1 + r$

$\Rightarrow (1-r)^2 \cos^2\theta + (1+r)^2 \sin^2\theta = (1+r)^2$

$\Rightarrow (1-r)^2 \cos^2\theta - (1+r)^2 \cos^2\theta = 0$

$\Rightarrow \cos^2\theta = 0 \Rightarrow \text{Re}(z) = 0$

Hence, z is a purely imaginary number.

2.

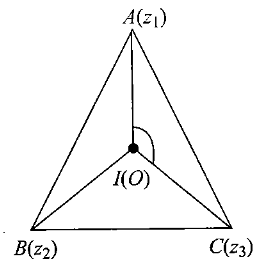


Fig. 2.48

Let z_1, z_2 and z_3 represent the vertices A, B and C, respectively, of triangle ABC. Now, the triangle is isosceles.

$\therefore \angle B = \angle C$

$\angle AIC = \frac{\pi}{2} + \frac{B}{2}$

$\angle BIA = \frac{\pi}{2} + \frac{C}{2}$

$\Rightarrow \frac{z_1}{z_3} = \left|\frac{z_1}{z_3}\right| e^{i\left(\frac{\pi}{2}+\frac{B}{2}\right)}$ (1)

and

$\frac{z_2}{z_1} = \left|\frac{z_2}{z_1}\right| e^{i\left(\frac{\pi}{2}+\frac{C}{2}\right)}$ (2)

On dividing Eq. (1) by Eq. (2), we get

$$\frac{z_1^2}{z_2 z_3} = \frac{|z_1|^2}{|z_2||z_3|} e^{i\left(\frac{B-C}{2}\right)}$$

$\Rightarrow \frac{z_1^2}{z_2 z_3} = \frac{|z_1|^2}{|z_2||z_3|}$ = positive real number $(\because \angle B = \angle C)$

$\Rightarrow z_1^2 = kz_2z_3$ (where $k \in R^+$)

Hence z_2, z_1 and kz_3 are in G.P.

3. Here $z - 1 = e^{i\theta}$ so that

$z = 1 + \cos\theta + i \sin\theta$

$\Rightarrow z = 2\cos^2\frac{\theta}{2} + i\, 2 \sin\frac{\theta}{2} \cos\frac{\theta}{2}$

$\Rightarrow z = 2\cos\frac{\theta}{2} (e^{i\theta/2})$

Hence

$$\tan\left(\frac{\arg(z-1)}{2}\right) - \frac{2i}{z} = \tan\frac{\theta}{2} - \frac{i}{\cos\theta/2} e^{-i\theta/2}$$

$$\Rightarrow \tan\frac{\theta}{2} - i\frac{\left(\cos\frac{\theta}{2} - i\sin\frac{\theta}{2}\right)}{\cos\frac{\theta}{2}} = -i$$

4.

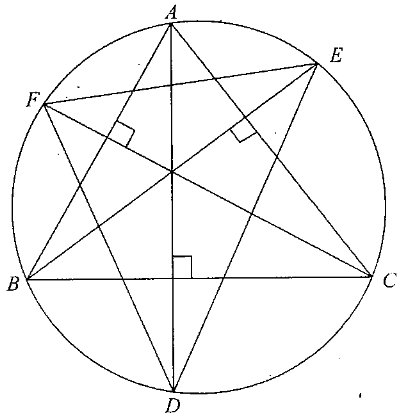


Fig. 2.49

$\angle FDA = \angle FCA = 90° - A$

$\angle ADE = \angle ABF = 90° - A$

$\Rightarrow \angle FDE = 180° - 2A = 2\pi - 2A$

Simpliarly $\angle DFE = 2\pi - 2C$ and $\angle DEF = 2\pi - 2B$

The angles of ΔDEF are $\pi - 2A$, $\pi - 2B$ and $\pi - 2C$, respectively.

Also it is given that $(z_3 - z_1)/(z_2 - z_1)$ is purely real. Hence,

$$\arg\left(\frac{z_3 - z_1}{z_2 - z_1}\right) = 0 \text{ or } \pi$$

$\Rightarrow \pi - 2A = 0$ or π

$\Rightarrow A = \frac{\pi}{2}$ or 0 (not permissible)

Hence triangle ABC is right angled at A.

5.

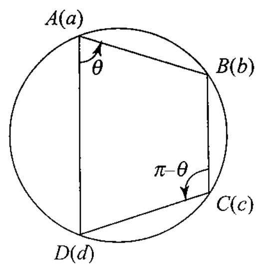


Fig. 2.50

Let complex number representing point 'D' is d and $\angle DAB = \theta$.

So, $\angle BCD = \pi - \theta$ (A, B, C, D are concyclic), Now, applying rotation formula on A and C, we get

$$\frac{b-a}{d-a} = \frac{AB}{AD} e^{i\theta} \text{ and } \frac{d-c}{b-c} = \frac{CD}{CB} e^{i(\pi-\theta)}$$

Multiplying these two, we get

$$\left(\frac{b-a}{d-a}\right)\left(\frac{d-c}{b-c}\right) = \frac{AB \times CD}{AD \times CB} e^{i\pi}$$

$\Rightarrow \frac{d(b-a) - c(b-a)}{d(b-c) - a(b-c)} = -1 \quad \left(\because \frac{AD}{AB} = \frac{CD}{CB}\right)$

$\Rightarrow\ d = \dfrac{2ac - b(a+c)}{a+c-2b}$

6. $i^{2i+3} \ln\left(\dfrac{i^3x^2+2x+i}{ix^2+2x+i^3}\right) = i^{2i}i^3 \ln\left(\dfrac{2x+i(1-x^2)}{2x-i(1-x^2)}\right)$

Let $2x = r\cos\theta$ and $1 - x^2 = r\sin\theta$. Then, above expression becomes

$(i)^{2i}(-i)\ln\left(\dfrac{r(\cos\theta+i\sin\theta)}{r(\cos\theta-i\sin\theta)}\right)$

$= (e^{i\pi/2})^{2i}(-i)\ln(e^{2i\theta})$

$= e^{-\pi}(-i)(2i\theta) = \dfrac{1}{e^{\pi}}(2\theta)$

Now,

$\theta = \tan^{-1}\left(\dfrac{1-x^2}{2x}\right) = \cot^{-1}\left(\dfrac{2x}{1-x^2}\right)$ $\left[\because \dfrac{1-x^2}{2x} > 0 \text{ when } x \in (0,1)\right]$

$= \dfrac{\pi}{2} - \tan^{-1}\left(\dfrac{2x}{1-x^2}\right) = \dfrac{\pi}{2} - 2\tan^{-1}x$ $[\because x \in (0,1)]$

So,

$\dfrac{1}{e^{\pi}}(2\theta) = \dfrac{1}{e^{\pi}}(\pi - 4\tan^{-1}x)$

7. Let α be the real and $i\beta$ be the imaginary roots of the given equation. Then

$\alpha + i\beta = -a \Rightarrow \alpha - i\beta = -\bar{a}$

$\Rightarrow\ 2\alpha = -(a+\bar{a})$ and $2i\beta = -(a-\bar{a})$

$\therefore\ 4i\alpha\beta = a^2 - \bar{a}^2 \Rightarrow 4b = a^2 - \bar{a}^2$

Alternative solution:

If one root is real and the other is imaginary, their product will be imaginary $\Rightarrow b$ is purely imaginary. Let $b = ik$, so that the equation $x^2 + ax + ik = 0$ has one purely real root. Let it be α. Then,

$\alpha^2 + a\alpha + ik = 0 \Rightarrow \alpha^2 + \bar{a}\alpha - ik = 0$

Hence,

$\dfrac{\alpha^2}{-ika - i\bar{a}k} = \dfrac{\alpha}{ik+ik} = \dfrac{1}{\bar{a}-a}$

$\Rightarrow\ \alpha^2 = \dfrac{ik(a+\bar{a})}{a-\bar{a}}$ and $\alpha = \dfrac{-2ik}{a-\bar{a}}$

$\therefore\ \dfrac{ik(a+\bar{a})}{a-\bar{a}} = \dfrac{-4k^2}{(a-\bar{a})^2} \Rightarrow a^2 - \bar{a}^2 = 4ik = 4b$

8. Put $z = a + ib$. Then,

$a^2 + b^2 - 2ai + 2b + 2c + 2ci = 0$

$\Rightarrow\ (a^2 + b^2 + 2b + 2c) + (2c - 2a)i = 0$

$\Rightarrow\ a^2 + b^2 + 2b + 2c = 0$ and $2c - 2a = 0$

$\Rightarrow\ a = c$

Now,

$b^2 + 2b + (c^2 + 2c) = 0$

$\Rightarrow\ b = -1 \pm \sqrt{1 - 2c - c^2}$

Since b is real,

$1 - 2c - c^2 \geq 0$

$\Rightarrow\ c \in [-1-\sqrt{2}, -1+\sqrt{2}]$

$\Rightarrow\ z = c + i(-1 \pm \sqrt{1-2c-c^2})$, where $c \in [-1-\sqrt{2}, -1+\sqrt{2}]$

9. $az^2 + z + 1 = 0$ (1)

Taking conjugate of both sides,

$\overline{az^2 + z + 1} = \bar{0}$

$\Rightarrow\ \bar{a}(\bar{z})^2 + \bar{z} + \bar{1} = 0$

$\bar{a}z^2 - z + 1 = 0$ (2)

(since $\bar{z} = -z$ as z is purely imaginary)

Eliminating z from both the equations, we get

$(\bar{a} - a)^2 + 2(a + \bar{a}) = 0$

Let,

$a = \cos\theta + i\sin\theta$ $(\because |a| = 1)$

$\Rightarrow\ (-2i\sin\theta)^2 + 2(2\cos\theta) = 0$

$\Rightarrow\ \cos\theta = \dfrac{-1 \pm \sqrt{1+4}}{2}$

$= \dfrac{\sqrt{5}-1}{2}$

Hence, $a = \cos\theta + i\sin\theta$, where $\theta = \cos^{-1}\left(\dfrac{\sqrt{5}-1}{2}\right)$.

10. a. Let

$z = |z|(\cos\theta + i\sin\theta)$

$\Rightarrow\ \left|\dfrac{z}{|z|} - 1\right| = |\cos\theta + i\sin\theta - 1|$

$= \sqrt{(\cos\theta - 1)^2 + \sin^2\theta}$

$= \sqrt{2 - 2\cos\theta}$

$= \sqrt{4\sin^2\dfrac{\theta}{2}}$

$= 2\left|\sin\dfrac{\theta}{2}\right|$

$\leq 2\left|\dfrac{\theta}{2}\right|$ $(\because |\sin\theta| \leq |\theta|)$

$= |\theta| = |\arg z|$ (1)

b. $|z - 1| = |z - |z| + |z| - 1|$

$\leq |z - |z|| + ||z| - 1|$

$= |z|\left|\dfrac{z}{|z|} - 1\right| + ||z| - 1|$

$\leq |z||\arg z| + ||z| - 1|$ [From (1)]

11. We have to prove that

$|\mathrm{Im}(z^n)| \leq n|\mathrm{Im}(z)||z|^{n-1}$

$\Rightarrow\ \left|\dfrac{z^n - \bar{z}^n}{2i}\right| \leq n\left|\dfrac{z - \bar{z}}{2i}\right||z|^{n-1}$

$\Rightarrow\ \left|\dfrac{z^n - \bar{z}^n}{z - \bar{z}}\right| \leq n|z|^{n-1}$

Now,

$\left|\dfrac{z^n - \bar{z}^n}{z - \bar{z}}\right| = \left|z^{n-1} + z^{n-2}\bar{z} + z^{n-3}\bar{z}^2 + \cdots + \bar{z}^n\right|$

$\leq \left|z^{n-1}\right| + \left|z^{n-2}\bar{z}\right| + \left|z^{n-3}\bar{z}^2\right| + \cdots + \left|\bar{z}^n\right|$

$$= |z|^{n-1} + |z|^{n-2}|\bar{z}| + |z|^{n-3}|\bar{z}|^2 + \cdots + |\bar{z}|^n$$

$$= |z|^{n-1} + |z|^{n-1} + |z|^{n-1} + \cdots + |z|^{n-1} \quad [\because |\bar{z}| = |z|]$$

$$= n|z|^{n-1}$$

Hence proved.

12. Given $OA = 1$ and $|z| = 1$.

$\therefore \quad OP = |z - 0| = |z| = 1 \Rightarrow OP = OA$

$OP_0 = |z_0 - 0| = |z_0|$

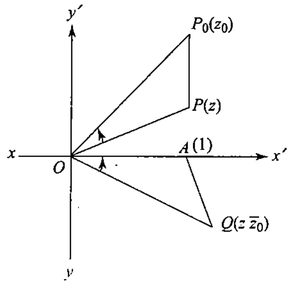


Fig. 2.51

$OQ = |z\bar{z}_0 - 0| = |z\bar{z}_0| = |z|\,|z| = |z_0|$

Also,

$$\angle P_0OA = \arg\left(\frac{z_0 - 0}{z - 0}\right) = \arg\left(\frac{z_0}{z}\right)$$

$$= \arg\left(\frac{\bar{z}z_0}{z\bar{z}}\right)$$

$$= \arg\left(\frac{\bar{z}z_0}{1}\right) = -\arg\left(\overline{\bar{z}z_0}\right)$$

$$= -\arg(z\,\bar{z}_0) = \arg\left(\frac{1}{z\bar{z}_0}\right)$$

$$= \arg\left(\frac{1 - 0}{z\bar{z}_0 - 0}\right) = \angle AOQ$$

Thus, the triangles POP_0 and AOQ are congruent. Hence,

$PP_0 = AQ \Rightarrow |z - z_0| = |z\bar{z}_0 - 1|$

13. We have

$$az^3 + bz^2 + \bar{b}z + \bar{a} = 0 \qquad (1)$$

Taking conjugate of both sides,

$$\bar{a}\bar{z}^3 + \bar{b}\bar{z}^2 + b\bar{z} + a = 0$$

Dividing this equation by $\bar{z}^3$ and writing the terms in reverse order, we get

$$\frac{a}{\bar{z}^3} + \frac{b}{\bar{z}^2} + \frac{\bar{b}}{\bar{z}} + \bar{a} = 0 \quad (\text{for } \bar{z} \neq 0) \qquad (2)$$

Since Eqs. (1) and (2) are identical,

$$z^3 = \frac{1}{\bar{z}^3}$$

$\Rightarrow |z|^6 = 1 \Rightarrow |z| = 1$

Hence, α is a root of the given equation, such that $|\alpha| = 1$.

14. $z = t^2 - 1 + \sqrt{t^4 - t^2}$

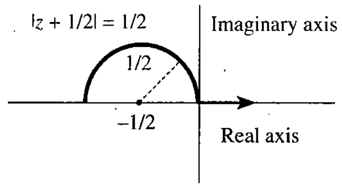


Fig. 2.52

Case I: $t^2 < 1$

$$z = t^2 - 1 + i\sqrt{t^2 - t^4}$$

$$\Rightarrow \quad z + \frac{1}{2} = \left(t^2 - \frac{1}{2}\right) + i\sqrt{\frac{1}{4} - \left(t^2 - \frac{1}{2}\right)^2}$$

$$\Rightarrow \quad \left|z + \frac{1}{2}\right|^2 = \left(t^2 - \frac{1}{2}\right)^2 + \frac{1}{4} - \left(t^2 - \frac{1}{2}\right)^2 \Rightarrow \left|z + \frac{1}{2}\right| = \frac{1}{2}$$

In this case z will lie on the upper half of circle of radius 1/2, centred at –1/2, in the Argand plane.

Case II: $t^2 \geq 1 \Rightarrow t \in (-\infty, -1] \cup [1, \infty)$

Clearly, 'z' is purely real. Also 'z' is an even function of t. Hence,

$$\frac{dz}{dt} = 2t + \frac{(4t^3 - 2t)}{2\sqrt{t^4 - t^2}}$$

$$= 2t + \frac{2t(t^2 - 1)}{2\sqrt{t^2(t^2 - 1)}}$$

$$> 0 \ \forall \ t \in (1, \infty)$$

Hence, 'z' will attain all values in the interval $[0, \infty)$ (as z is equal to zero at $t = 1$). In this case z will lie on the positive real axis in the Argand plane $\forall \ t \in (-\infty, -1) \cup (1, \infty)$.

15.

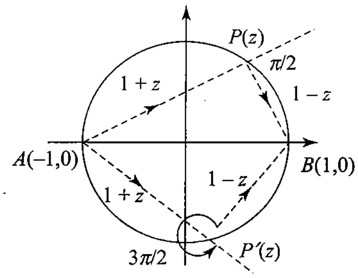


Fig. 2.53

Since $|z| = 1$, z lies on a unit circle having centre at the origin.

$$\arg\left(\frac{1 + z}{1 - z}\right) = +\frac{\pi}{2} \text{ or } +\frac{3\pi}{2}$$

$$\Rightarrow \quad \frac{1 + z}{1 - z} = ke^{i\pi/2} \text{ or } ke^{i3\pi/2}$$

where k is a real parameter and its value depends upon the position of z. Let,

$$\alpha = \sqrt{\frac{1 + z}{1 - z}}$$

$$= \sqrt{k}\ e^{i\pi/4} \text{ or } \sqrt{k}\ e^{i3\pi/4}$$

Therefore, α lies on one of the two perpendicular lines.

16. From figure, it is clear that $|z - i| < |z + i|$ (as z lies above the real axis). Hence,

$$|\alpha| = \frac{|z-i|}{|z+i|} < 1$$

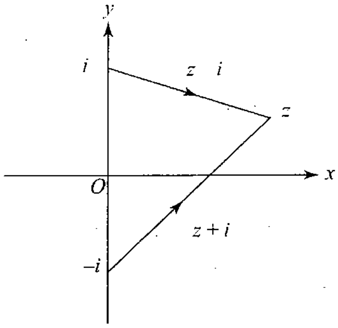


Fig. 2.54

Therefore, α lies within the unit circle which has centre at the origin. Now, if z is travelling on the real axis Im(z) = 0, Re(z) varies from $-\infty$ to $+\infty$. Let,

$$z = x + i0$$

$$\Rightarrow \quad \alpha = \frac{x-i}{x+i} \Rightarrow |\alpha| = \frac{|x-i|}{|x+i|} = 1 \text{ (as } |x-i| = |x+i| \; \forall x \in R$$

Hence, α moves on the unit circle which has centre at the origin

17. Suppose $x^2 + ax + b = 0$ has roots x_1 and x_2. Then,

$$\therefore \quad x_1 + x_2 = -a \qquad (1)$$

and

$$x_1 x_2 = b \qquad (2)$$

From (2),

$$|x_1|\,|x_2| = |b|$$

$$\Rightarrow \quad |b| = 1 \qquad (3)$$

Also,

$$|-a| = |x_1 + x_2|$$

$$\Rightarrow \quad |a| \le |x_1| + |x_2|$$

or

$$|a| \le 2 \qquad (4)$$

Now suppose $y^2 + |a|y + |b| = 0$ has roots y_1 and y_2. Then,

$$y_1, y_2 = \frac{-|a| \pm \sqrt{|a|^2 - 4|b|}}{2}$$

$$= \frac{-|a| \pm \left(\sqrt{4 - |a|^2}\right)i}{2}$$

$$\Rightarrow \quad |y_1|, |y_2| = \frac{\sqrt{|a|^2 + 4 - |a|^2}}{2} = 1$$

Hence, $|y_1| = |y_2| = 1$.

18.

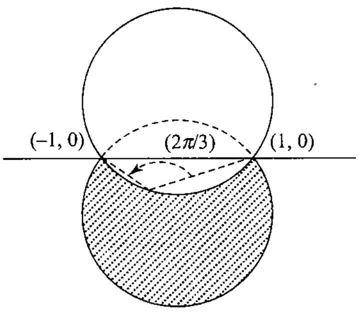


Fig. 2.55

Let us take arg $[(z+1)/(z-1)] = 2\pi/3$. Clearly, z lies on the minor arc of the circle passing through (1, 0) and (–1, 0). Similarly, arg $[(z+1)/(z-1)] = \pi/3$ means that 'a' is lying on the major arc of the circle passing through (1, 0) and (–1, 0). Now, if we take any point in the region included between the two arcs, say $P_1(z_1)$, we get

$$\frac{\pi}{3} \le \arg\left(\frac{z+1}{z-1}\right) \le \frac{2\pi}{3}$$

Thus $\pi/3 \le \arg[(z+1)/(z-1)] \le 2\pi/3$ represents the shaded region [excluding the points (1, 0) and (– 1, 0)].

19.

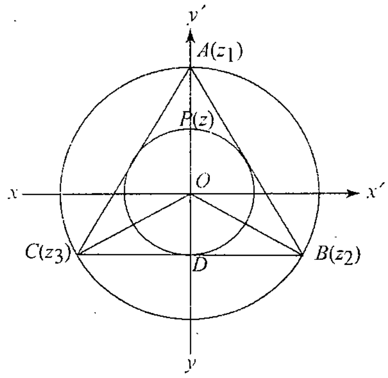


Fig. 2.56

Let,

$$z_1 = \frac{2}{\sqrt{3}} e^{i\frac{\pi}{2}}, \; z_2 = \frac{2}{\sqrt{3}} e^{-i\frac{\pi}{6}}, \; z_3 = \frac{2}{\sqrt{3}} e^{-i\frac{5}{6}\pi}$$

Clearly, the points lie on the circle $|z| = 2/\sqrt{3}$. And ΔABC is equilateral and its centroid coincides with circumcentre. Hence,

$$z_1 + z_2 + z_3 = 0 \text{ and } \bar{z}_1 + \bar{z}_2 + \bar{z}_3 = 0$$

Since triangle is equilateral, inradius $r = OD = 1/\sqrt{3}$. Theequation of incircle is

$$|z| = 1/\sqrt{3}$$

Let $P(z)$ be any point on the incircle. Now,

$$AP^2 = |z - z_1|^2 = |z|^2 + |z_1|^2 - (z\bar{z}_1 + \bar{z}z_1)$$

Similarly,

$$BP^2 = |z|^2 + |z_2|^2 - (z\bar{z}_2 + \bar{z}z_2)$$

$$CP^2 = |z|^2 + |z_3|^2 - (z\bar{z}_3 + \bar{z}z_3)$$

$$\therefore \quad AP^2 + BP^2 + CP^2$$

$$= 3|z|^2 + |z_1|^2 + |z_2|^2 + |z_3|^2 - z(\bar{z}_1 + \bar{z}_2 + \bar{z}_3) - \bar{z}(z_1 + z_2 + z_3)$$

$$= 3 \times \frac{4}{3} + \frac{1}{3} + \frac{1}{3} + \frac{1}{3} - z(0) - \bar{z}(0)$$

$$= 5$$

20. Given equation of line is

$$a\bar{z} + \bar{a}z + b = 0, \; \forall \, b \in R$$

Let PQ be the segment intercepted between the axes. For real intercept Z_R,

$$z = \bar{z}$$

$$\Rightarrow \quad Z_R(a + \bar{a}) + b = 0$$

$$\Rightarrow \quad Z_R = \frac{-b}{(a + \bar{a})}$$

For imaginary intercept Z_I,

$$z + \bar{z} = 0$$

$\Rightarrow \quad Z_I(a-\bar{a})+b=0$

$\Rightarrow \quad Z_I=-\dfrac{b}{\bar{a}-a}$

Mid-point is

$$z=\frac{Z_R+Z_I}{2}$$

$$=\frac{-b}{2}\left[\frac{1}{\bar{a}+a}+\frac{1}{\bar{a}-a}\right]$$

$$=\frac{\bar{a}b}{(a+\bar{a})(a-\bar{a})}$$

$$=\frac{\bar{a}b}{a^2-(\bar{a})^2}$$

$\Rightarrow \quad \dfrac{z[a^2-(\bar{a})^2]}{\bar{a}}=b=b\text{ (= real)}$

$\Rightarrow \quad z\dfrac{a^2-(\bar{a})^2}{\bar{a}}=\bar{z}\dfrac{(\bar{a})^2-(a)^2}{a}$

$\Rightarrow \quad az+\overline{az}=0$

Objective Type

1. b. Verify by selecting particular values of a and b.

Let $a=-9$ and $b=4$. Then,

$$\sqrt{a}\sqrt{b}=\sqrt{-9}\sqrt{4}=(3i)(2)=6i$$

From option (a), we have

$$-\sqrt{|a|b}=-\sqrt{|-9|\times 4}=-\sqrt{36}=-6$$

From option (b), we have

$$\sqrt{|a|b}\,i=\sqrt{|-9|\times 4}\;i=6i$$

2. c. $x=9^{\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\cdots}=9^{\frac{\frac{1}{3}}{1-\frac{1}{3}}}=9^{\frac{1}{2}}=3$

$$y=4^{\frac{1}{3}-\frac{1}{9}+\frac{1}{27}+\cdots}=4^{\frac{\frac{1}{3}}{1+\frac{1}{3}}}=4^{\frac{1}{4}}=\sqrt{2}$$

$$z=\sum_{r=1}^{\infty}(1+i)^{-r}=\frac{1}{1+i}+\frac{1}{(1+i)^2}+\frac{1}{(1+i)^3}+\cdots$$

$$=\frac{\frac{1}{1+i}}{1-\frac{1}{1+i}}=\frac{1}{i}=-i$$

Let $a=x+yz=3-\sqrt{2}\,i$ (fourth quadrant). Then,

$$\arg\alpha=-\tan^{-1}\left(\frac{\sqrt{2}}{3}\right)$$

3. d. $10z^2-3iz-k=0$

$\Rightarrow \quad z=\dfrac{3i\pm\sqrt{-9+40k}}{20}$

Now, $D=-9+40k$. If $k=1$, then $D=31$. So (a) is false.

If k is a negative real number, then D is a negative real number. So (d) is true.

If $k=i$, then $D=-9+40i=16+40i-25=(4+5i)^2$, and the roots are $(1/5)+(2/5)i$ and $-(1/5-1/10)i$. So (c) is false.

If $k=0$ (which is a complex number), then the roots are 0 and $(3/10)i$. So (b) is false.

4. d. Let $z=x+iy$, so that $\bar{z}=x-iy$.

$\therefore \quad z^2+\bar{z}=0$

$\Rightarrow \quad (x^2-y^2+x)+i(2xy-y)=0$

Equating real and imaginary parts, we get

$$x^2-y^2+x=0 \qquad (1)$$

and

$$2xy-y=0\Rightarrow y=0\text{ or }x=\frac{1}{2}$$

If $y=0$, then (1) gives $x^2+x=0\Rightarrow x=0$ or $x=-1$

If $x=1/2$, then from (1),

$$y^2=\frac{1}{4}+\frac{1}{2}=\frac{3}{4}\Rightarrow y=\pm\frac{\sqrt{3}}{2}$$

Hence, there are four solutions in all.

5. c. Given that $a^2+b^2=1$. Therefore,

$$\frac{1+b+ia}{1+b-ia}=\frac{(1+b+ia)(1+b+ia)}{(1+b-ia)(1+b+ia)}$$

$$=\frac{(1+b)^2-a^2+2ia(1+b)}{1+b^2+2b+a^2}$$

$$=\frac{(1-a^2)+2b+b^2+2ia(1+b)}{2(1+b)}$$

$$=\frac{2b^2+2b+2ia(1+b)}{2(1+b)}$$

$$=b+ia$$

6. b. Let,

$$\sin\frac{\pi}{8}+i\cos\frac{\pi}{8}=z$$

$$\Rightarrow \left[\frac{1+\sin\frac{\pi}{8}+i\cos\frac{\pi}{8}}{1+\sin\frac{\pi}{8}-i\cos\frac{\pi}{8}}\right]^8$$

$$=\left(\frac{1+z}{1+\frac{1}{z}}\right)^8$$

$$=z^8$$

$$=\left(\sin\frac{\pi}{8}+i\cos\frac{\pi}{8}\right)^8$$

$$=\left(\cos\left(\frac{\pi}{2}-\frac{\pi}{8}\right)+i\sin\left(\frac{\pi}{2}-\frac{\pi}{8}\right)\right)^8$$

$$=\left(\cos\frac{3\pi}{8}+i\sin\frac{3\pi}{8}\right)^8$$

$$=\cos 3\pi=-1$$

7. a. Let $z_1=a+ib$ and $z_2=c-id$, where $a>0$ and $d>0$. Then,

$$|z_1|=|z_2|\Rightarrow a^2+b^2=c^2+d^2 \qquad (1)$$

Now,

$$\frac{z_1+z_2}{z_1-z_2}=\frac{(a+ib)+(c-id)}{(a+ib)-(c-id)}$$

$$=\frac{\left[(a+c)+i(b-d)\right]\left[(a-c)-i(b+d)\right]}{\left[(a-c)+i(b+d)\right]\left[(a-c)-i(b+d)\right]}$$

$$=\frac{(a^2+b^2)-(c^2+d^2)-2(ad+bc)i}{a^2+c^2-2ac+b^2+d^2+2bd}$$

$$=\frac{-(ad+bc)i}{a^2+b^2-ac+bd} \quad \text{[Using (1)]}$$

Hence, $(z_1+z_2)/(z_1-z_2)$ is purely imaginary. However, if $ad + bc = 0$, then $(z_1+z_2)/(z_1-z_2)$ will be equal to zero. According to the conditions of the equation, we can have $ad + bc = 0$.

8. a. We have,

$$\arg\left(\frac{z_1}{z_2}\right)=\pi$$

$\Rightarrow$ $\arg(z_1) - \arg(z_2) = \pi$

$\Rightarrow$ $\arg(z_1) = \arg(z_2) + \pi$

Let $\arg(z_2) = \theta$. Then $\arg(z_1) = \pi + \theta$.

$\therefore$ $z_1 = |z_1|[\cos(\pi + \theta) + i\sin(\pi + \theta)]$
$= |z_1|(-\cos\theta - i\sin\theta)$

and

$z_2 = |z_2|(\cos\theta + i\sin\theta)$
$= |z_1|(\cos\theta + i\sin\theta)$ $(\because |z_1| = |z_2|)$
$= -z_1$

$\Rightarrow$ $z_1 + z_2 = 0$

9. c. Let
$a = \cos\alpha + i\sin\alpha$
$b = \cos\beta + i\sin\beta$
$c = \cos\gamma + i\sin\gamma$

Then,
$a + 2b + 3c = (\cos\alpha + 2\cos\beta + 3\cos\gamma) + i(\sin\alpha + 2\sin\beta + 3\sin\gamma) = 0$

$\Rightarrow$ $a^3 + 8b^3 + 27c^3 = 18abc$

$\Rightarrow$ $\cos 3\alpha + 8\cos 3\beta + 27\cos 3\gamma = 18\cos(\alpha + \beta + \gamma)$

and

$\sin 3\alpha + 8\sin 3\beta + 27\sin 3\gamma = 18\sin(\alpha + \beta + \gamma)$

10. d.

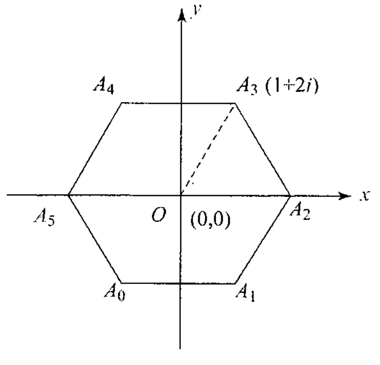


Fig. 2.57

Let the vertices be $z_0, z_1, \ldots, z_5$ w.r.t. centre O at origin and $|z_0| = \sqrt{5}$.

Now ΔOA_2A_3 is equilateral $\Rightarrow$ $OA_2 = OA_3 = A_2A_3 = \sqrt{5}$
$= |z_0|\,|\cos\theta + i\sin\theta - 1|$

Perimeter $= 6\sqrt{5}$.

11. a. $$\frac{1+iz}{1-iz}=\frac{1+i(b+ic)/(1+a)}{1-i(b+ic)/(1+a)}$$

$$=\frac{1+a-c+ib}{1+a+c-ib}$$

$$=\frac{(1+a-c+ib)(1+a+c+ib)}{(1+a+c)^2+b^2}$$

$$=\frac{1+2a+a^2-b^2-c^2+2ib+2iab}{1+a^2+c^2+b^2+2ac+2(a+c)}$$

$$=\frac{2a+2a^2+2ib+2iab}{2+2ac+2(a+c)} \quad (\because a^2+b^2+c^2=1)$$

$$=\frac{a+a^2+ib+iab}{1+ac+(a+c)}$$

$$=\frac{a(a+1)+ib(a+1)}{(a+1)(c+1)}$$

$$=\frac{a+ib}{c+1}$$

12. b. If z_1, z_2, z_3 are three complex numbers, then

$$A=\begin{vmatrix}\arg z_1 & \arg z_2 & \arg z_3\\ \arg z_2 & \arg z_3 & \arg z_1\\ \arg z_3 & \arg z_1 & \arg z_2\end{vmatrix}$$

$$\Rightarrow A=(\arg z_1+\arg z_2+\arg z_3)\begin{vmatrix}1 & \arg z_2 & \arg z_3\\ 1 & \arg z_3 & \arg z_1\\ 1 & \arg z_1 & \arg z_2\end{vmatrix}$$

(Using $C_1 \to C_1 + C_2 + C_3$)

$$\Rightarrow A=\arg(z_1z_2z_3)\begin{vmatrix}1 & \arg z_2 & \arg z_3\\ 1 & \arg z_3 & \arg z_1\\ 1 & \arg z_1 & \arg z_2\end{vmatrix}$$

Hence, A is divisible by $\arg(z_1z_2z_3)$.

13. c. $\bar{z} + i\bar{w} = 0$

$\Rightarrow$ $z - iw = 0$

$\Rightarrow$ $z = iw$ (1)

$\arg zw = \pi$

$\Rightarrow$ $\arg z + \arg w = \pi$

$\Rightarrow$ $\arg z + \arg \dfrac{z}{i} = \pi$ [Using (1)]

$\Rightarrow$ $\arg z + \arg z - \arg i = \pi$

$\Rightarrow$ $2\arg z - \dfrac{\pi}{2} = \pi$

$\Rightarrow$ $2\arg z = \dfrac{3\pi}{2}$

$\Rightarrow$ $\arg z = \dfrac{3\pi}{4}$

14. c. We have,

$|z_1 - z_2|^2 = |z_1|^2 + |z_2|^2 - 2|z_1|\,|z_2| \cos(\theta_1 - \theta_2)$

where $\theta_1 = \arg(z_1)$ and $\theta_2 = \arg(z_2)$. Given,

$\arg(z_1 - z_2) = 0$

$\Rightarrow \quad |z_1 - z_2|^2 = |z_1|^2 + |z_2|^2 - 2|z_1|\,|z_2|$

$= (|z_1| - |z_2|)^2$

$\Rightarrow \quad |z_1 - z_2| = ||z_1| - |z_2||$

15. a. $\alpha = \dfrac{z - \bar{w}}{k^2 + z\bar{w}} \Rightarrow \bar{\alpha} = \dfrac{\bar{z} - w}{k^2 + \bar{z}w}$

But $z\bar{z} = w\bar{w} = k^2$. Hence,

$$\Rightarrow \quad \bar{\alpha} = \frac{\dfrac{k^2}{z} - \dfrac{k^2}{\bar{w}}}{k^2 + \dfrac{k^2}{z}\dfrac{k^2}{\bar{w}}} = \frac{\bar{w} - z}{z\bar{w} + k^2} = -\alpha$$

$\Rightarrow \quad \alpha + \bar{\alpha} = 0$

$\Rightarrow \quad \mathrm{Re}(\alpha) = 0$

16. a. Here $x = 4\cos\theta$, $y = 4\sin\theta$.

$\therefore \quad ||x| - |y||$

$= |4|\cos\theta| - 4|\sin\theta||$

$= 4||\cos\theta| - |\sin\theta||$

$= 4\sqrt{1 - 2|\cos\theta||\sin\theta|}$

$= 4\sqrt{1 - |\sin 2\theta|}$

Hence, the range is [0, 4].

17. c. $|k + z^2| = |z^2| - k = |z^2| + |k|$

$\Rightarrow$ k, z^2 and $0 + i0$ are collinear

$\Rightarrow \quad \arg(z^2) = \arg(k)$

$\Rightarrow \quad 2\arg(z) = \pi$

$\Rightarrow \quad \arg(z) = \dfrac{\pi}{2}$

18. b. The given equation is

$|z|^n = (z^2 + z)\,|z|^{n-2} + 1$

$\Rightarrow$ $z^2 + z$ is real

$\Rightarrow \quad z^2 + z = \bar{z}^2 + \bar{z}$

$\Rightarrow \quad (z - \bar{z})(z + \bar{z} + 1) = 0$

$\Rightarrow \quad z = \bar{z} = x$ as $z + \bar{z} + 1 \neq 0$ $(x \neq -1/2)$

Hence, the given equation reduces to

$x^n = x^n + x|x|^{n-2} + 1$

$\Rightarrow \quad x|x|^{n-2} = -1$

$\Rightarrow \quad x = -1$

So number of solutions is 1.

19. c. Observing carefully the system of equations, we find

$\dfrac{1+i}{2i} = \dfrac{1-i}{2} = \dfrac{1}{1+i}$

Hence, there are infinite number of solutions.

20. d. Given,

$$z^3 + \frac{3(\bar{z})^2}{|z|} = 0$$

Let,

$z = re^{i\theta}$

$\Rightarrow \quad r^3 e^{i3\theta} + 3re^{-i2\theta} = 0$

Since 'r' cannot be zero, so

$re^{i5\theta} = -3$

which will hold for $r = 3$ and five distinct values of 'θ'. Thus there are five solutions.

21. c.
$$\frac{(1+i)^5 (1+\sqrt{3}i)^2}{-2i(-\sqrt{3}+i)} = \frac{(\sqrt{2})^5 \left(\dfrac{1}{\sqrt{2}} + \dfrac{i}{\sqrt{2}}\right)^5 2^2 \left(\dfrac{1}{2} + \dfrac{\sqrt{3}}{2}i\right)^2}{2i2\left(\dfrac{\sqrt{3}}{2} - \dfrac{i}{2}\right)}$$

$\Rightarrow \quad \text{Argument} = \dfrac{5\pi}{4} + \dfrac{2\pi}{3} - \dfrac{\pi}{2} + \dfrac{\pi}{6} = \dfrac{19\pi}{12}$

Therefore, the principal argument is $-5\pi/12$.

22. d. Let $f(x) = x^6 + 4x^5 + 3x^4 + 2x^3 + x + 1$. Hence,

$f(\omega) = \omega^6 + 4\omega^5 + 3\omega^4 + 2\omega^3 + \omega + 1$

$= 1 + 4\omega^2 + 3\omega + 2 + \omega + 1$

$= 4(\omega^2 + \omega + 1)$

$= 0$

Hence $f(x)$ is divisible by $x - \omega$. Then $f(x)$ is also divisible by $x - \omega^2$ (as complex roots occur in conjugate pairs).

$f(-\omega) = (-\omega)^6 + 4(-\omega)^5 + 3(-\omega)^4 + 2(-\omega)^3 + (-\omega) + 1$

$= \omega^6 - 4\omega^5 + 3\omega^4 - 2\omega^3 - \omega + 1$

$= 1 - 4\omega^2 + 3\omega - 2 - \omega + 1$

$\neq 0$

23. c. We have,

$(\cos\theta + i\sin\theta)(\cos 2\theta + i\sin 2\theta) \cdots \times (\cos n\theta + i\sin n\theta) = 1$

$\Rightarrow \quad \cos(\theta + 2\theta + 3\theta + \cdots + n\theta) + i\sin(\theta + 2\theta + \cdots + n\theta) = 1$

$\Rightarrow \quad \cos\left(\dfrac{n(n+1)}{2}\theta\right) + i\sin\left(\dfrac{n(n+1)}{2}\theta\right) = 1$

$\Rightarrow \quad \cos\left(\dfrac{n(n+1)}{2}\theta\right) = 1$ and $\sin\left(\dfrac{n(n+1)}{2}\theta\right) = 0$

$\Rightarrow \quad \dfrac{n(n+1)}{2}\theta = 2m\pi \Rightarrow \theta = \dfrac{4m\pi}{n(n+1)}$, where $m \in Z$

24. c. $z = (1 + i\sqrt{3})^{100} = 2^{100}\left(\cos\dfrac{\pi}{3} + i\sin\dfrac{\pi}{3}\right)^{100}$

$= 2^{100}\left(\cos\dfrac{100\pi}{3} + i\sin\dfrac{100\pi}{3}\right)$

$= 2^{100}\left(-\cos\dfrac{\pi}{3} - i\sin\dfrac{\pi}{3}\right)$

$= 2^{100}\left(-\dfrac{1}{2} - \dfrac{i\sqrt{3}}{2}\right)$

$\Rightarrow \quad \dfrac{\mathrm{Re}(z)}{\mathrm{Im}(z)} = \dfrac{-1/2}{-\sqrt{3}/2} = \dfrac{1}{\sqrt{3}}$

25. a. $i = \cos\dfrac{\pi}{2} + i\sin\dfrac{\pi}{2} = e^{\frac{i\pi}{2}}$

$\Rightarrow \quad i^i = \left(e^{\frac{i\pi}{2}}\right)^i = e^{-\frac{\pi}{2}}$

$\Rightarrow \quad z = (i)^{(i)^i} = i^{e^{-\frac{\pi}{2}}}$

$\Rightarrow \quad |z| = 1$

26. d. $z = i\log(2-\sqrt{3})$

$\Rightarrow \quad e^{iz} = e^{i^2\log(2-\sqrt{3})} = e^{-\log(2-\sqrt{3})}$

$\Rightarrow \quad e^{iz} = e^{\log(2-\sqrt{3})^{-1}} = e^{\log(2+\sqrt{3})} = (2+\sqrt{3})$

$\Rightarrow \quad \cos z = \dfrac{e^{iz}+e^{-iz}}{2} = \dfrac{(2+\sqrt{3})+(2-\sqrt{3})}{2} = 2$

27. b. Let xi be the root where $x \neq 0$ and $x \in R$

$x^4 - a_1x^3i - a_2x^2 + a_3xi + a_4 = 0$

$\Rightarrow \quad x^4 - a_2x^2 + a_4 = 0$ (1)

and

$a_1x^3 - a_3x = 0$ (2)

From Eq. (2),

$a_1x^2 - a_3 = 0$

$\Rightarrow \quad x^2 = a_3/a_1$ (as $x \neq 0$)

Putting the value of x^2 in Eq. (1), we get

$\dfrac{a_3^2}{a_1^2} - \dfrac{a_2a_3}{a_1} + a_4 = 0$

$\Rightarrow \quad a_3^2 + a_4a_1^2 = a_1a_2a_3$

$\Rightarrow \quad \dfrac{a_3}{a_1a_2} + \dfrac{a_1a_4}{a_2a_3} = 1$ (dividing by $a_1a_2a_3$)

28. b. Let $A = x + iy$. Given,

$|A| = 1 \Rightarrow x^2 + y^2 = 1$

and

$|A+1| = 1 \Rightarrow (x+1)^2 + y^2 = 1$

$\Rightarrow \quad x = -\dfrac{1}{2}$ and $y = \pm\dfrac{\sqrt{3}}{2}$

$\Rightarrow \quad A = \omega$ or ω^2

$\Rightarrow \quad (\omega)^n = (1+\omega)^n = (-\omega^2)^n$

Therefore, n must be even and divisible by 3.

29. c. Let $z = \cos x + i\sin x$, $x \in [0, 2\pi)$. Then,

$1 = \left|\dfrac{z}{\bar{z}} + \dfrac{\bar{z}}{z}\right|$

$= \dfrac{|z^2 + \bar{z}^2|}{|z|^2}$

$= |\cos 2x + i\sin 2x + \cos 2x - i\sin 2x|$

$= 2|\cos 2x|$

$\Rightarrow \quad \cos 2x = \pm 1/2$

Now,

$\cos 2x = 1/2$

$\Rightarrow \quad x_1 = \dfrac{\pi}{6}, x_2 = \dfrac{5\pi}{6}, x_3 = \dfrac{7\pi}{6}, x_4 = \dfrac{11\pi}{6}$

$\cos 2x = -\dfrac{1}{2}$

$\Rightarrow \quad x_5 = \dfrac{\pi}{3}, x_6 = \dfrac{2\pi}{3}, x_7 = \dfrac{4\pi}{3}, x_8 = \dfrac{5\pi}{3}$

30. a. $u^2 - 2u + 2 = 0 \Rightarrow u = 1 \pm i$

$\Rightarrow \quad \dfrac{(x+\alpha)^n - (x+\beta)^n}{\alpha - \beta}$

$= \dfrac{[(\cot\theta - 1) + (1+i)]^n - [(\cot\theta - 1) + (1-i)]^n}{2i}$

$(\because \cot\theta - 1 = x)$

$= \dfrac{(\cos\theta + i\sin\theta)^n - (\cos\theta - i\sin\theta)^n}{\sin^n\theta \, 2i}$

$= \dfrac{2i\sin n\theta}{\sin^n\theta \, 2i}$

$= \dfrac{\sin n\theta}{\sin^n\theta}$

31. b. $f(z) = g(z)(z-i)(z+i) + az + b$; $a, b \in C$

Given,

$f(i) = i \Rightarrow ai + b = i$ (1)

and

$f(-i) = 1 + i$

$\Rightarrow \quad a(-i) + b = 1 + i$ (2)

From (1) and (2), we have

$a = \dfrac{i}{2}, b = \dfrac{1}{2} + i$

Hence, the required remainder is $az + b = (1/2)iz + (1/2) + i$.

32. c. We have,

$z_1(z_1^2 - 3z_2^2) = 2$ (1)

$z_2(3z_1^2 - z_2^2) = 11$ (2)

Multiplying (2) by i and adding it to (1), we get

$z_1^3 - 3z_2^2z_1 + i(3z_1^2z_2 - z_2^3) = 2 + 11i$

$\Rightarrow \quad (z_1 + iz_2)^3 = 2 + 11i$ (3)

Multiplying (2) by i and subtracting it from (1), we get

$z_1^3 - 3z_2^2z_1 - i(3z_1^2z_2 - z_2^3) = 2 - 11i$

$\Rightarrow \quad (z_1 - iz_2)^3 = 2 - 11i$ (4)

Multiplying (3) and (4), we get

$(z_1^2 + z_2^2)^3 = 2^2 - 121i^2 = 4 + 121 = 125$

$\Rightarrow \quad z_1^2 + z_2^2 = 5$

33. a. Assuming $\arg z_1 = \theta$ and $\arg z_2 = \theta + \alpha$,

$\dfrac{az_1}{bz_2} + \dfrac{bz_2}{az_1} = \dfrac{a|z_1|e^{i\theta}}{b|z_2|e^{i(\theta+\alpha)}} + \dfrac{b|z_2|e^{i(\theta+\alpha)}}{a|z_1|e^{i\theta}}$

$= e^{-i\alpha} + e^{i\alpha} = 2\cos\alpha$

Hence, the point lies on the line segment [– 2, 2] of the real axis.

34. d. $x^2 + x + 1 = 0 \Rightarrow x = \omega$ or ω^2

Let $x = \omega$. Then,

$x + \dfrac{1}{x} = \omega + \dfrac{1}{\omega} = \omega + \omega^2 = -1$

$$x^2+\frac{1}{x^2}=\omega^2+\frac{1}{\omega^2}=\omega^2+\omega=-1$$

$$x^3+\frac{1}{x^3}=\omega^3+\frac{1}{\omega^3}=2$$

$$x^4+\frac{1}{x^4}=\omega^4+\frac{1}{\omega^4}=\omega+\omega^2=-1, \text{etc.}$$

$$\therefore \left(x+\frac{1}{x}\right)^2+\left(x^2+\frac{1}{x^2}\right)^2+\left(x^3+\frac{1}{x^3}\right)^2+\cdots+\left(x^{27}+\frac{1}{x^{27}}\right)^2$$

$$=\left[\left(x+\frac{1}{x}\right)^2+\left(x^2+\frac{1}{x^2}\right)^2+\left(x^4+\frac{1}{x^4}\right)^2+\cdots+\left(x^{26}+\frac{1}{x^{26}}\right)^2\right]$$

$$+\left[\left(x^3+\frac{1}{x^3}\right)^2+\left(x^6+\frac{1}{x^6}\right)^2+\left(x^9+\frac{1}{x^9}\right)\right.$$

$$\left.\cdots+\left(x^{27}+\frac{1}{x^{27}}\right)^2\right]$$

$= 18 + 9(2^2) = 54$

35. c. $E=\sum_{r=1}^{n}(ar+b)\omega^{r-1}$

$=(a+b)+(2a+b)\omega+(3a+b)\omega^2+\cdots+(na+b)\omega^{n-1}$

$=b\underbrace{(1+\omega+\omega^2+\cdots+\omega^{n-1})}_{\text{zero}}+a(1+2\omega+3\omega^2+\cdots+n\omega^{n-1})$

Now,

$S=1+2\omega+3\omega^2+\cdots+n\omega^{n-1}$

$S\omega=\omega+2\omega^2+\cdots+(n-1)\omega^{n-1}+n\omega^n$

$\therefore \quad S(1-\omega)=\underbrace{1+\omega+\omega^2+\cdots+\omega^{n-1}}_{\text{zero}}-n\omega^n=-n \ (\because \omega^n=1)$

$\Rightarrow \quad S=\dfrac{n}{\omega-1}$

$\Rightarrow \quad E=\dfrac{na}{\omega-1}$

36. b. Taking cube roots of both sides, we get

$z+ab=a(1)^{1/3}=a, a\omega, a\omega^2$

where

$$\omega=-\frac{1}{2}+i\frac{\sqrt{3}}{2}, \omega^2=-\frac{1}{2}-i\frac{\sqrt{3}}{2}$$

$\therefore \quad z_1=a-ab, z_2=a\omega-ab, z_3=a\omega^2-ab$

$|z_1-z_2|=|a(1-\omega)|$

$$=|a|\left|1-\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)\right|$$

$$=|a|\left|\frac{3}{2}-i\frac{\sqrt{3}}{2}\right|$$

$$=|a|\left(\frac{9}{4}+\frac{3}{4}\right)^{1/2}=\sqrt{3}|a|$$

Similarly,

$|z_2-z_3|=|z_3-z_1|=\sqrt{3}|a|$

37. a. We have,

$z^3+2z^2+2z+1=0$

$\Rightarrow \quad (z^3+1)+2z(z+1)=0$

$\Rightarrow \quad (z+1)(z^2+z+1)=0$

$\Rightarrow \quad z=-1, \omega, \omega^2$

Since $z=-1$ does not satisfy $z^{1985}+z^{100}+1=0$ while $z=\omega, \omega^2$ satisfy it, hence sum is $\omega+\omega^2=-1$.

38. b. Let $z_1=|z_1|(\cos\theta_1+i\sin\theta_1)$. Now,

$$\left|\frac{z_1}{z_2}\right|=1\Rightarrow |z_1|=|z_2|$$

Also,

$\arg(z_1z_2)=0\Rightarrow \arg(z_1)+\arg(z_2)=0$

$\Rightarrow \quad \arg(z_2)=-\theta_1$

$\Rightarrow \quad z_2=|z_2|(\cos(-\theta_1)+i\sin(-\theta_1))$

$=|z_1|(\cos\theta_1-i\sin\theta_1)=\bar{z}_1$

$\Rightarrow \quad \bar{z}_2=\overline{(\bar{z}_1)}=z_1$

$\Rightarrow \quad |z_2|^2=z_1z_2$

39. d. $(x-3)^3+1=0$

$\Rightarrow \quad \left(\dfrac{x-3}{-1}\right)^3=1$

$\Rightarrow \quad \dfrac{x-3}{-1}=1, \omega, \omega^2$

$\Rightarrow \quad x=2, 3-\omega, 3-\omega^2$

Hence, the um of complex roots is $6-(\omega+\omega)^2=6+1=7$.

40. b. $x=\sqrt[3]{-1}$

$\Rightarrow \quad x^3=-1$

$\Rightarrow \quad (-x)^3=1$

$\Rightarrow \quad -x=1, \omega, \omega^2$

$\Rightarrow \quad x=-1, -\omega, -\omega^2$

$$=-1, \frac{1+\sqrt{3}i}{2}, \frac{1-\sqrt{3}i}{2}$$

$$=-1, \frac{-\sqrt{3}+i}{2i}, \frac{\sqrt{3}+i}{2i}$$

$$=-1, \frac{-\sqrt{3}+\sqrt{-1}}{\sqrt{-4}}, \frac{\sqrt{3}+\sqrt{-1}}{\sqrt{-4}}$$

41. d. $|\omega z-1-\omega^2|=a$

$\Rightarrow \quad |z+1|=a\Rightarrow |z-1+2|=a$

$\Rightarrow \quad |z-1|+2\geq a\Rightarrow 0\leq a\leq 4$

42. b. $|z^2-3|\geq |z|^2-3$

$\Rightarrow \quad 3|z|\geq |z|^2-3$

$\Rightarrow \quad |z|^2-3|z|-3\leq 0$

$\Rightarrow \quad 0<|z|\leq \dfrac{3+\sqrt{21}}{2}$

43. b. $|2z-1|=|z-2|$

$\Rightarrow \quad |2z-1|^2=|z-2|^2$

$\Rightarrow \quad (2z-1)(2\bar{z}-1)=(z-2)(\bar{z}-2)$

$\Rightarrow\ 4z\bar{z} - 2\bar{z} - 2z + 1 = z\bar{z} - 2\bar{z} - 2z + 4$

$\Rightarrow\ 3|z|^2 = 3$

$\Rightarrow\ |z| = 1$

Again,

$$|z_1 + z_2| = |z_1 - \alpha + z_2 - \beta + \alpha + \beta|$$
$$\le |z_1 - \alpha| + |z_2 - \beta| + |\alpha + \beta|$$
$$< \alpha + \beta + |\alpha + \beta|$$
$$= 2|\alpha + \beta| \quad [\because \alpha, \beta > 0]$$

$\therefore\ \left|\frac{z_1 + z_2}{\alpha + \beta}\right| < 2$

$\Rightarrow\ \left|\frac{z_1 + z_2}{\alpha + \beta}\right| < 2\,|z|$

44. a. $a_0 z^n + a_1 z^{n-1} + a_2 z^{n-2} + \cdots + a_{n-1} z + a_n = 3$

$\Rightarrow\ |3| = |a_0 z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n|$

$\Rightarrow\ 3 \le |a_0|\,|z|^n + |a_1|\,|z|^{n-1} + \cdots + |a_{n-1}|\,|z| + |a_n|$

$\Rightarrow\ 3 < 2(|z|^n + |z|^{n-1} + \cdots + |z| + 1)$

$\Rightarrow\ 1 + |z| + |z|^2 + \cdots + |z|^n > \frac{3}{2}$

If $|z| \ge 1$, the inequality is clearly satisfied. For $|z| < 1$, we must have,

$$\frac{1 - |z|^{n+1}}{1 - |z|} > \frac{3}{2}$$

$\Rightarrow\ 2 - 2|z|^{n+1} > 3 - 3|z|$

$\Rightarrow\ 2|z|^{n+1} < 3|z| - 1$

$\Rightarrow\ 3|z| - 1 > 0$

$\Rightarrow\ |z| > \frac{1}{3}$

45. a. $8iz^3 + 12z^2 - 18z + 27i = 0$

$\Rightarrow\ 4iz^2(2z - 3i) - 9(2z - 3i) = 0$

$\Rightarrow\ (2z - 3i)(4iz^2 - 9) = 0$

$\Rightarrow\ z = \frac{3i}{2}$ and $z^2 = \frac{9}{4i}$

$\Rightarrow\ |z| = \frac{3}{2}$ and $|z^2| = \frac{9}{4}$

$\Rightarrow\ |z| = \frac{3}{2}$

46. a. $|z^2 + 2z\cos\alpha| \le |z^2| + |2z\cos\alpha|$

$$= |z|^2 + 2|z|\,|\cos\alpha|$$
$$\le |z|^2 + 2|z|$$
$$< (\sqrt{2} - 1)^2 + 2(\sqrt{2} - 1) = 1$$

47. b. $\left|z + \frac{1}{z}\right| \ge \left||z| - \frac{1}{|z|}\right|$

Hence the least value occurs when $|z| = 3$.

$\therefore\ \left|z + \frac{1}{z}\right|_{\text{least}} = 3 - \frac{1}{3} = \frac{8}{3}$

48. c. $\left|\sum_{r=1}^{n} z_r\right| \le \sum_{r=1}^{n} |z_r| \le \sum_{r=1}^{n} |z_r - r| + \sum_{r=1}^{n} r \le 2\sum_{r=1}^{n} r$

49. d. Let $z = x + iy$. Then,

$|z^2 - 1| = |z|^2 + 1$

$\Rightarrow\ |(x^2 - y^2 - 1) + 2ixy| = x^2 + y^2 + 1$

$\Rightarrow\ (x^2 - y^2 - 1)^2 + 4x^2y^2 = (x^2 + y^2 + 1)^2$

$\Rightarrow\ x = 0$

Hence, z lies on imaginary axis.

50. a. $|z| = 1$, let $\alpha = -1 + 3z$

$\Rightarrow\ \alpha + 1 = 3z$

$\Rightarrow\ |\alpha + 1| = 3|z| = 3$

Hence, 'α' lies on a circle centred at -1 and radius equal to 3.

51. b. Let $z = x + iy$. Then,

$x = \lambda + 3$ and $y = -\sqrt{5 - \lambda^2}$

$\Rightarrow\ (x - 3)^2 = \lambda^2$ (1)

and

$y^2 = 5 - \lambda^2$ (2)

From (1) and (2),

$(x - 3)^2 = 5 - y^2 \Rightarrow (x - 3)^2 + y^2 = 5$

Obviously it is a semicircle as $y < 0$. Hence part of the circle lies below the x-axis.

52. c.

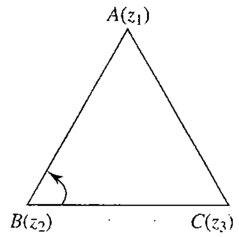


Fig. 2.58

$$\frac{z_1 - z_2}{z_3 - z_2} = \frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}} = e^{i\pi/4}$$

$\angle CBA = \frac{\pi}{4}$

Also,

$|z_1 - z_2| = |z_3 - z_2|$

Hence, ΔABC is isosceles.

53. c. Given that

$|z_1 - i| = |z_2 - i| = |z_3 - i|$

Hence, z_1, z_2, z_3 lie on the circle whose centre is i. Also given that the triangle is equilateral. Hence centroid and cirucmcentre coincides.

$\therefore\ \frac{z_1 + z_2 + z_3}{3} = i$

$\Rightarrow\ |z_1 + z_2 + z_3| = 3$

54. d.

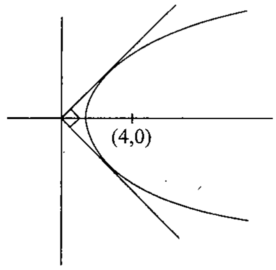


Fig. 2.59

$|z-4| = \text{Re}(z)$

$\Rightarrow \quad \sqrt{(x-4)^2+y^2} = x$

$\Rightarrow \quad x^2 - 8x + 16 + y^2 = x^2$

$\Rightarrow \quad y^2 = 8(x-2)$

The given relation represents the part of the parabola with focus (4, 0) lying above x-axis and the imaginary axis as the directirx. The two tangents from directrix are at right angle. Hence greatest positive argument of z is $\pi/4$.

55. d.

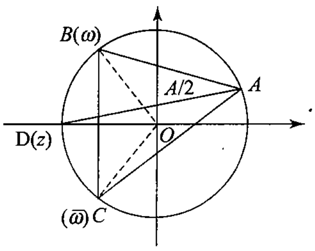


Fig. 2.60

Clearly,

$\angle DOB = \angle COD = A$

$\Rightarrow \quad z = \omega e^{iA}$ and $\bar{\omega} = ze^{iA}$ (Applying rotation about O)

$\Rightarrow \quad z^2 = \omega\bar{\omega} = 1$

$\Rightarrow \quad z = -1$ (As A and D are on opposite sides of BC)

56. b.

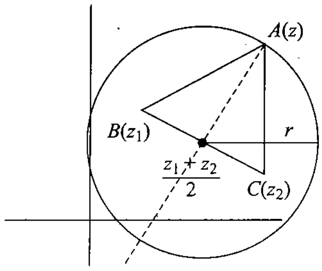


Fig. 2.61

By the given conditions, the area of the triangle ABC is given by $(1/2)|z_1 - z_2|r$.

57. d. The given equation is written as

$$\arg(z-(1+i)) = \begin{cases} \dfrac{3\pi}{4}, & \text{when } x \le 2 \\ \dfrac{-\pi}{4}, & \text{when } x > 2 \end{cases}$$

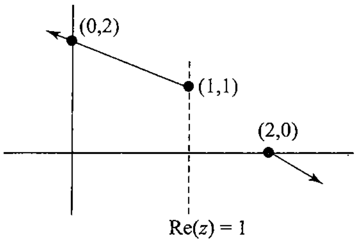


Fig. 2.62

Therefore, the locus is a set of two rays.

58. a.

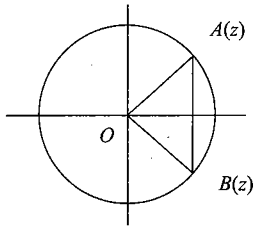


Fig. 2.63

$|z-\bar{z}|$ is the length AB while $|z|(\arg z - \arg\bar{z})$ is arc length AB.

$\therefore \quad |z-\bar{z}| \le |z| (\arg z - \arg\bar{z})$

59. c.

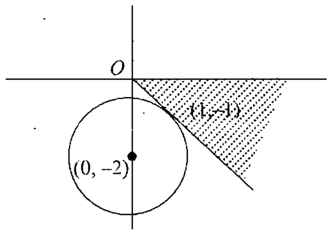


Fig. 2.64

$kz/(k+1)$ represents any point lying on the line joining origin and z.
Given,

$$\left|\frac{kz}{k+1} + 2i\right| > \sqrt{2}$$

Hence, $kz/(k+1)$ should lie outside the circle $|z+2i| > \sqrt{2}$. So, z should lie in the shaded region.

$\therefore \quad -\dfrac{\pi}{4} < \arg(z) < 0$

60. b. $2\left|z-\dfrac{1}{2}\right| = |z-1|$

$\therefore \quad \dfrac{|z-1|}{\left|z-\dfrac{1}{2}\right|} = 2$

So, locus of z is a circle.

61. d. $|z_2 + iz_1| = |z_1| + |z_2|$

$\Rightarrow \quad iz_1$, $0 + i0$ and z_2 are collinear

$\Rightarrow \quad \arg(iz_1) = \arg(z_2)$

$\Rightarrow \quad \arg(z_2) - \arg(z_1) = \dfrac{\pi}{2}$

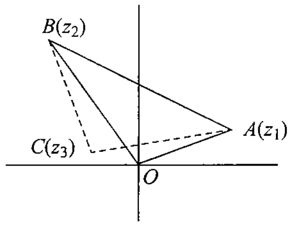


Fig. 2.65

Let,

$$z_3 = \frac{z_2 - iz_1}{1-i}$$

$\Rightarrow \quad (1-i)z_3 = z_2 - iz_1$

$\Rightarrow \quad z_2 - z_3 = i(z_1 - z_3)$

$\therefore \quad \angle ACB = \dfrac{\pi}{2}$

and

$|z_1 - z_3| = |z_2 - z_3|$

$\Rightarrow \quad AC = BC$

$\because \quad AB^2 = AC^2 + BC^2$

$\Rightarrow \quad AC = \dfrac{5}{\sqrt{2}} \qquad (\because AB = 5)$

Therefore area of ΔABC is $(1/2)AC \times BC = AC^2/2 = 25/4$ sq. units

62. a.

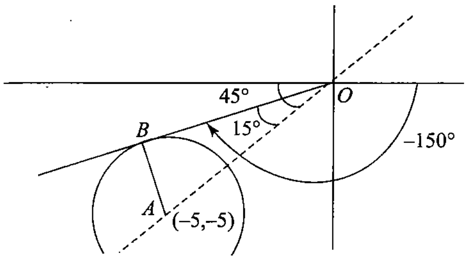


Fig. 2.66

$|2z + 10 + 10i| \le 5\sqrt{3} - 5$

$\Rightarrow \quad |z + 5 + 5i| \le \dfrac{5(\sqrt{3}-1)}{2}$

Point B has least principal argument. Now,

$AB = \dfrac{5(\sqrt{3}-1)}{2}$

$OA = 5\sqrt{2}$

$\angle AOB = \dfrac{\pi}{12}$

$\therefore \quad \arg(z) = -\dfrac{5\pi}{6}$

63. c. $z = \dfrac{at+b}{t-c} \Rightarrow t = \dfrac{b+cz}{z-a}$

Now,

$|t| = 1$

$\Rightarrow \quad \left|\dfrac{b+cz}{z-a}\right| = 1$

$\Rightarrow \quad \left|\dfrac{z+\frac{b}{c}}{z-a}\right| = \dfrac{1}{|c|} \quad (\neq 1 \text{ as } |c| \neq |t|)$

$\Rightarrow$ locus of z is a circle

64. a. Let $z = x + iy$. Then,

$|z - 3 - i| = |z - 9 - i|$

$\Rightarrow \quad \sqrt{(x-3)^2 + (y-1)^2} = \sqrt{(x-9)^2 + (y-1)^2}$

$\Rightarrow \quad x = 6$

$|z - 3 + 3i| = 3$

$\Rightarrow \quad \sqrt{(x-3)^2 + (y+3)^2} = 3$

For $x = 6$, $y = -3$.

$\therefore \quad z = 6 - 3i$

65. a. Given,

$z_k = 1 + a + a^2 + \cdots + a^{k-1} = \dfrac{1-a^k}{1-a}$

$\Rightarrow \quad z_k - \dfrac{1}{1-a} = -\dfrac{a^k}{1-a}$

$\Rightarrow \quad \left|z_k - \dfrac{1}{1-a}\right| = \dfrac{|a|^k}{|1-a|} < \dfrac{1}{|1-a|} \qquad [\because |a| < 1]$

Hence, z_k lies within the circle.

$\therefore \quad \left|z - \dfrac{1}{1-a}\right| = \dfrac{1}{|1-a|}$

66. b. $2z^2 + 2z + \lambda = 0$

Let the roots be z_1, z_2. Then,

$z_1 + z_2 = -1$ and $z_1 z_2 = \dfrac{\lambda}{2}$

$0, z_1, z_2$ form an equilateral triangle.

$\therefore \quad z_1^2 + z_2^2 = z_1 z_2$

$\Rightarrow \quad (z_1 + z_2)^2 = 3z_1 z_2$

$\Rightarrow \quad 1 = 3\dfrac{\lambda}{2}$

$\Rightarrow \quad \lambda = \dfrac{2}{3}$

67. a. $x + iy = 1 - t + i\sqrt{t^2 + t + 2}$

$\Rightarrow \quad x = 1 - t,\ y = \sqrt{t^2 + t + 2}$

Eliminating t,

$y^2 = t^2 + t + 2 = (1 - x)^2 + 1 - x + 2 = \left(x - \dfrac{3}{2}\right)^2 + \dfrac{7}{4}$

$\Rightarrow \quad y^2 - \left(x - \dfrac{3}{2}\right)^2 = \dfrac{7}{4}$, which is a hyperbola

68. c. $z^2 + z|z| + |z|^2 = 0 \Rightarrow \left(\dfrac{z}{|z|}\right)^2 + \dfrac{z}{|z|} + 1 = 0$

$\Rightarrow \quad \dfrac{z}{|z|} = \omega, \omega^2 \Rightarrow z = \omega|z|$ or $z = \omega^2|z|$

$\Rightarrow \quad x + iy = |z|\left(\dfrac{-1}{2} + \dfrac{i\sqrt{3}}{2}\right)$ or $x + iy = |z|\left(\dfrac{-1}{2} - \dfrac{i\sqrt{3}}{2}\right)$

$\Rightarrow \quad x = -\dfrac{1}{2}|z|,\ y = |z|\dfrac{\sqrt{3}}{2}$ or $x = -\dfrac{|z|}{2},\ y = -\dfrac{|z|\sqrt{3}}{2}$

$\Rightarrow \quad y + \sqrt{3}x = 0$ or $y - \sqrt{3}x = 0 \Rightarrow y^2 - 3x^2 = 0$

69. c. $S_1 = \Sigma z_1 = -3a$, $S_2 = \Sigma z_1 z_2 = 3b$

Since the triangle is equilateral, we have

$\Sigma z_1^2 = \Sigma z_1 z_2$

$\Rightarrow \quad (\Sigma z_1)^2 - 2\Sigma z_1 z_2 = \Sigma z_1 z_2$

$\Rightarrow \quad (\Sigma z_1)^2 = 3\Sigma z_1 z_2$

$\Rightarrow \quad (-3a)^2 = 3(3b)$

$\Rightarrow \quad 9a^2 = 9b$

$\Rightarrow \quad a^2 = b$

70. b.

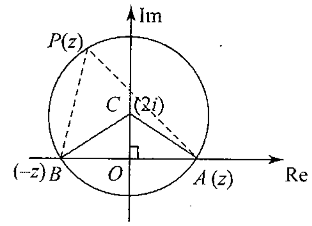


Fig. 2.67

$CA = CB = 2\sqrt{2},\ OC = 2$

$\Rightarrow \quad OA = OB = 2$

$\Rightarrow \quad A \equiv 2 + 0i,\ B = -2 + 0i$

Clearly,

$\angle BCA = \pi/2$

$\Rightarrow \quad \angle BPA = \pi/4$

$$\Rightarrow \quad \arg\left(\frac{z-2}{z+2}\right) = \frac{\pi}{4}$$

71. a.

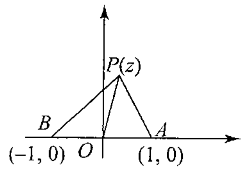


Fig. 2.68

When $|z-1| < |z+1|$ (or $x > 0$)

$|z| = |z-1|$

$\Rightarrow \quad x^2 + y^2 = (x-1)^2 + y^2$

$\Rightarrow \quad x = 1/2$

$\Rightarrow \quad z + \bar{z} = 1$

When $|z-1| > |z+1|$ (or $x < 0$),

$|z| = |z+1|$

$\Rightarrow \quad x^2 + y^2 = (x+1)^2 + y^2$

$\Rightarrow \quad x = -1/2$

$\Rightarrow \quad z + \bar{z} = -1$

72. b. Let $z = x + iy$. Then,

$$\text{Re}\left(\frac{1}{z}\right) = k$$

$$\Rightarrow \quad \text{Re}\left(\frac{1}{x+iy}\right) = k$$

$$\Rightarrow \quad \text{Re}\left(\frac{x}{x^2+y^2} - \frac{iy}{x^2+y^2}\right) = k$$

$$\Rightarrow \quad \frac{x}{x^2+y^2} = k$$

$$\Rightarrow \quad x^2 + y^2 - \frac{1}{k}x = 0$$

which is a circle.

73. b.

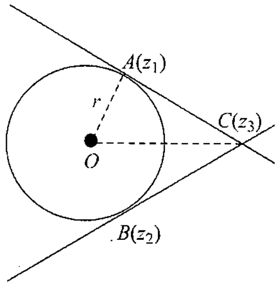


Fig. 2.69

As ΔOAC is a right-angled triangle with right angle at A, so

$|z_1|^2 + |z_3 - z_1|^2 = |z_3|^2$

$\Rightarrow \quad 2|z_1|^2 - \bar{z}_3 z_1 - \bar{z}_1 z_3 = 0$

$$\Rightarrow \quad 2\bar{z}_1 - \bar{z}_3 - \frac{\bar{z}_1}{z_1} z_3 = 0 \qquad (1)$$

Similarly,

$$2\bar{z}_2 - \bar{z}_3 - \frac{\bar{z}_2}{z_2} z_3 = 0 \qquad (2)$$

Subtracting (2) from (1), we get

$$2(\bar{z}_2 - \bar{z}_1) = z_3\left(\frac{\bar{z}_1}{z_1} - \frac{\bar{z}_2}{z_2}\right)$$

$$\Rightarrow \quad \frac{2r^2(z_1 - z_2)}{z_1 z_2} = z_3 r^2\left(\frac{z_2^2 - z_1^2}{z_1^2 z_2^2}\right) \qquad [\because |z_1|^2 = |z_2|^2 = r^2]$$

$$\Rightarrow \quad z_3 = \frac{2z_1 z_2}{z_2 + z_1}$$

74. a. $|z_1| = |z_2| = |z_3| = 1$

Hence, the circumcentre of triangle is origin. Also, centroid $(z_1 + z_2 + z_3)/3 = 0$, which coincides with the circumcentre. So the triangle is equilateral. Since radius is 1, length of side is $a = \sqrt{3}$. Therefore, the area of the triangle is $(\sqrt{3}/4)a^2 = (3\sqrt{3}/4)$.

75. a.

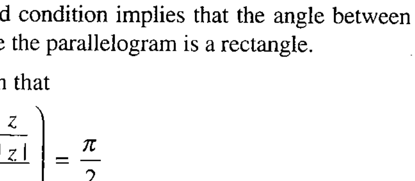


Fig. 2.70

The first condition implies that $(z_1 + z_3)/2 = (z_2 + z_4)/2$, i.e., diagonals AC and BD bisect each other. Hence, quadrilateral is a parallelogram. The second condition implies that the angle between AD and AB is 90°. Hence the parallelogram is a rectangle.

76. b. Given that

$$\arg\left(\frac{z_1 - \frac{z}{|z|}}{\frac{z}{|z|}}\right) = \frac{\pi}{2}$$

and

$$\left|\frac{z}{|z|} - z_1\right| = 3$$

from which we can establish the following geometry.

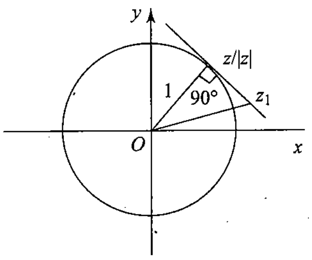


Fig. 2.71

From the diagram,

$$\left|\frac{z}{|z|} - z_1\right| = 3, |z_1| = \sqrt{9+1} = \sqrt{10}$$

77. b. Note that $z_1 = 3 + \sqrt{3}i$ lies on the line $y = (1/\sqrt{3})x$ and $z_2 = 2\sqrt{3} + 6i$ lies on the line $y = \sqrt{3}x$.

Hence $z = 5 + 5i$ will only lie on the bisector of z_1 and z_2, i.e., $y = x$.

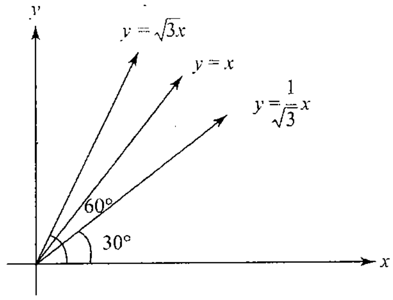


Fig. 2.72

78. a. $\log_{1/3}\left(\frac{|z-3|^2+2}{11|z-3|-2}\right) > 1$

$\Rightarrow \quad \frac{|z-3|^2+2}{11|z-3|-2} < \frac{1}{3}$

$\Rightarrow \quad (3t-8)(t-1) < 0$ (where $|z-3| = t$)

$\Rightarrow \quad 1 < |z-3| < 8/3$

Hence, z lies between the two concentric circles.

79. c. We have,

$$|(x-2) + i(y-1)| = |z| \left|\frac{1}{\sqrt{2}}\cos\theta - \frac{1}{\sqrt{2}}\sin\theta\right|$$

where $\theta = \arg z$.

$$\sqrt{(x-2)^2 + (y-1)^2} = \frac{1}{\sqrt{2}}|x-y|,$$

which is a parabola

80. a. We have,

$|z - 2 + 2i| = 1$

$\Rightarrow \quad |z - (2-2i)| = 1$

Hence, z lies on a circle having centre at (2, –2) and radius 1. It is evident from the figure that the required complex number z is given by the point P. We find that OP makes an angle $\pi/4$ with OX and

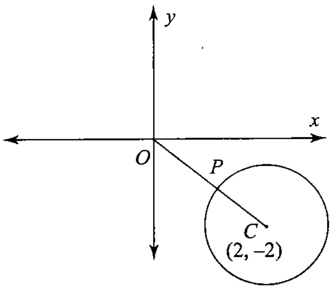


Fig. 2.73

$$OP = OC - CP = \sqrt{2^2+2^2} - 1 = 2\sqrt{2} - 1$$

So, coordinates of P are $[(2\sqrt{2}-1)\cos(\pi/4),$ $-(2\sqrt{2}-1)\sin(\pi/4)]$, i.e., $((2-1/\sqrt{2}), -(2-1/\sqrt{2}))$. Hence,

$$z = \left(2 - \frac{1}{\sqrt{2}}\right) + \left\{-\left(2-\frac{1}{\sqrt{2}}\right)\right\}i = \left(2-\frac{1}{\sqrt{2}}\right)(1-i)$$

81. d. Given,

$$z = \frac{3}{2+\cos\theta + i\sin\theta}$$

$\Rightarrow \quad \cos\theta + i\sin\theta = \frac{3}{z} - 2 = \frac{3-2z}{z}$

$\Rightarrow \quad 1 = \frac{|3-2z|}{|z|}$ [taking modulus]

$\Rightarrow \quad \frac{\left|z - \frac{3}{2}\right|}{|z|} = \frac{1}{2}$

Hence, locus of z is a circle.

82. d. We have,

$e^{\frac{2\pi ri}{p}} = e^{\frac{2\pi m}{q}}$

$r = 0, 1, \ldots, p-1$

$m = 0, 1, \ldots, q-1$

This is possible iff $r = m = 0$ but for $r = m = 0$ we get 1 which is not an imaginary number.

83. b. Let,

$S = 1 + 2a + 3a^2 + \cdots + na^{n-1}$

$\Rightarrow \quad \alpha S = \alpha + 2\alpha^2 + 3\alpha^3 + \cdots + (n-1)\alpha^{n-1} + n\alpha^n$

On subtracting, we get

$S(1-\alpha) = 1 + [\alpha + \alpha^2 + \alpha^{n-1}] - n\alpha^n$

$$= 1 + \frac{\alpha(1-\alpha^{n-1})}{1-\alpha} - n\alpha^n$$

$\Rightarrow \quad S = \frac{1}{1-\alpha} + \frac{\alpha - \alpha^n}{(1-\alpha)^2} - \frac{n\alpha^n}{1-\alpha}$ $\quad [\because \alpha^n = 1]$

$$= \frac{1}{1-\alpha} + \frac{\alpha-1}{(1-\alpha)^2} - \frac{n}{1-\alpha} = -\frac{n}{1-\alpha}$$

84. a. $\left(\frac{1+ia}{1-ia}\right)^4 = z$

$\Rightarrow \quad \left|\frac{1+ia}{1-ia}\right|^4 = |z|$

$\Rightarrow \quad \left|\frac{a-i}{a+i}\right|^4 = 1$

$\Rightarrow \quad |a-i| = |a+i|$

Therefore, a lies on the perpendicular bisector of i and $-i$, which is real axis. Hence all the roots are real.

85. b. For $z \neq 1$, the given equation can be written as

$$\left(\frac{z+1}{z-1}\right)^5 = 1$$

$\Rightarrow \quad \frac{z+1}{z-1} = e^{2k\pi i/5}$

where $k = -2, -1, 1, 2$.

If we denote this value of z by z_k, then

$$z_k = \frac{e^{2k\pi i/5}+1}{e^{2k\pi i/5}-1}$$

$$= \frac{e^{k\pi i/5}+e^{-k\pi i/5}}{e^{k\pi i/5}-e^{-k\pi i/5}}$$

$$= -i\cot\left(\frac{k\pi}{5}\right),\ k=-2,-1,1,2$$

Therefore, roots of the equation are $\pm i\cot(\pi/5), \pm i\cot(2\pi/5)$.

86. a. We have,

$\log z + \log z^2 + \log z^3 + \cdots + \log z^n = 0$

$\Rightarrow \log(z\,z^2\,z^3\cdots z^n) = 0$

$\Rightarrow \log\left(z^{\frac{n(n+1)}{2}}\right) = 0$

$\Rightarrow z^{\frac{n(n+1)}{2}} = 1$

$\Rightarrow z = 1^{\frac{2}{n(n+1)}}$

$= (\cos 0° + i\sin 0°)^{\frac{2}{n(n+1)}}$

$= (\cos 2m\pi + i\sin 2m\pi)^{\frac{2}{n(n+1)}}$, $m = 0, 1, 2, 3, \ldots$

$= \cos\frac{4m\pi}{n(n+1)} + i\sin\frac{4m\pi}{n(n+1)}$, $m = 0, 1, 2, \ldots$

87. d. The equation $z^n = (z+1)^n$ will have exactly $n-1$ roots. We have,

$\left(\frac{z+1}{z}\right)^n = 1$

$\Rightarrow \left|\frac{z+1}{z}\right| = 1$

$\Rightarrow |z+1| = |z|$

Therefore, 'z' lies on the right bisector of the segment connecting the points (0, 0) and (–1, 0). Thus Re(z) = –1/2. Hence, roots are collinear and will have their real parts equal to –1/2.

Hence sum of the real parts of roots is $(-1/2)(n-1)$.

88. c. $\left(\frac{z+1}{z}\right)^4 = 16$

$\Rightarrow \frac{z+1}{z} = \pm 2, \pm 2i$

The roots are $1, -1/3, (-1/5-(2/5)i)$ and $(-1/5+(2/5)i)$.

Note that (–1/3, 0) and (1, 0) are equidistant from (1/3, /0) and since it lies on the perpendicular bisector of AB, it will be equidistant from A and B also.

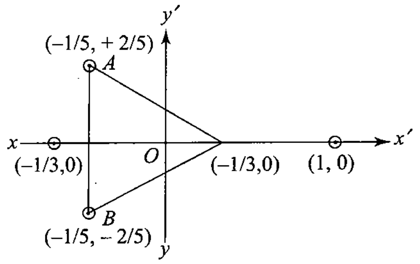


Fig. 2.74

89. d. $(z^n-1) = (z-1)(z-z_1)(z-z_2)\ldots(z-z_{n-1})$ (1)

Differentiating w.r.t. x, and then dividing by (1), we have

$$\frac{nz^{n-1}}{z^n-1} = \frac{1}{z-1}+\frac{1}{z-z_1}+\frac{1}{z-z_2}+\cdots+\frac{1}{z-z_{n-1}}$$

Putting $z = 3$, we get

$$\frac{n3^{n-1}}{3^n-1} = \frac{1}{2}+\frac{1}{3-z_1}+\frac{1}{3-z_2}+\cdots+\frac{1}{3-z_{n-1}}$$

$$\Rightarrow \frac{1}{3-z_1}+\frac{1}{3-z_2}+\cdots+\frac{1}{3-z_{n-1}} = \frac{n3^{n-1}}{3^n-1}-\frac{1}{2}$$

Multiple Correct Answers Type

1. a, c.

Clearly, we have to find it for real z. Let $z = x$. Then,

$|x-w| = |x-w^2| = |w-w^2|$

$\Rightarrow \left(x+\frac{1}{2}\right)^2+\frac{3}{4} = \left|\frac{-1+\sqrt{3}i}{2}-\frac{-1-\sqrt{3}i}{2}\right|^2 = 3 \Rightarrow x+\frac{1}{2} = \pm\frac{3}{2}$

$\Rightarrow x = 1, -2$

2. a, b, c, d.

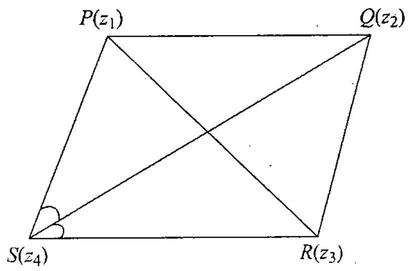


Fig. 2.75

a. $PS \parallel QR \Rightarrow \arg\left(\frac{z_1-z_4}{z_2-z_3}\right) = 0 \Rightarrow \frac{z_1-z_4}{z_2-z_3}$ is purely real

b. Since diagonal bisect the angle

$\Rightarrow \text{amp}\left(\frac{z_1-z_4}{z_2-z_4}\right) = \text{amp}\left(\frac{z_2-z_4}{z_3-z_4}\right)$

c. Diagonals of rhombus are perpendicular. Hence, $(z_1-z_3)/(z_2-z_4)$ is purely imaginary.

d. Diagonals of rhombus are not equal. Hence, $|z_1-z_3| \neq |z_2-z_4|$.

3. a, b, d.

Let $z = \alpha$ be a real root. Then,

$\alpha^3 + (3+2i)\alpha + (-1+i\alpha) = 0$

$\Rightarrow (\alpha^3+3\alpha-1) + i(\alpha+2\alpha) = 0$

$\Rightarrow \alpha^3+3\alpha-1 = 0$ and $\alpha = -a/2$

$\Rightarrow -\frac{a^3}{8}-\frac{3a}{2}-1 = 0$

$\Rightarrow a^3+12a+8 = 0$

Let $f(a) = a^3+12a+8$.

$\therefore f(-1)<0, f(0)>0, f(-2)<0, f(1)>0$ and $f(3)>0$

Hence, $a \in (-1, 0)$ or $a \in (-2, 1)$ or $a \in (-2, 3)$.

4. b, c.

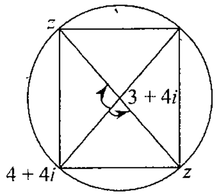


Fig. 2.76

Clearly, the inscribed rectangle is a square. Let the adjacent vertex be z. Then,

$$\frac{3+4i-(z)}{(3+4i)-(4+4i)} = e^{\pm i\pi/2} \text{ (by rotation about center)}$$

$\Rightarrow \quad 3 + 4i - z = \pm i(-1)$

$\Rightarrow \quad z = 3 + 4i \pm i = 3 + 5i$ or $3 + 3i$

5. a, d.

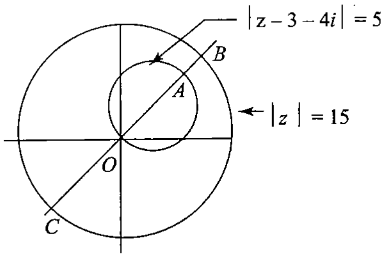


Fig. 2.77

We have,

$|z_1| = 15, |z_2 - 3 - 4i| = 5$

Minimum value of $|z_1 - z_2|$ is $AB = OB - OA = 15 - 10 = 5$. Maximum value of $|z_1 - z_2|$ is $CA = OA + OC = 10 + 15 = 25$.

6. a, c.

Triangle ABC is equilateral. Hence,

$z^2 + (-z)^2 + (1-z)^2 = z(-z) + z(1-z) + (-z)(1-z)$

$\Rightarrow \quad 3z^2 - 2z + 1 = -z^2$

$\Rightarrow \quad 4z^2 - 2z + 1 = 0$

sum of roots = 2

and product of roots is = $\frac{1}{4}$

7. b, c.

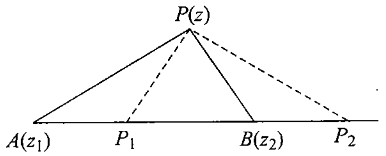


Fig. 2.78

Let internal and external bisectors of $\angle APB$ meet the line joining A and B at P_1 and P_2, respectively. Hence,

$AP_1 : P_1B \equiv PA : PB \equiv 3:1$ (internal division)

$AP_2 : P_2B \equiv PA : PB \equiv 3:1$ (external division)

Thus, P_1 and P_2 are fixed points. Also,

$\angle P_1PP_2 = \frac{\pi}{2}$

Thus 'P' lies on a circle having P_1P_2 as its diameter. Clearly, $B(z_2)$ lies inside this circle.

8. a, d.

Refer the figure, z lies on the point of intersection of the rays from A and B. $\angle ACB$ is a right angle and OBC is an equilateral triangle. Hence,

$$OC = a \Rightarrow z = a\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$$

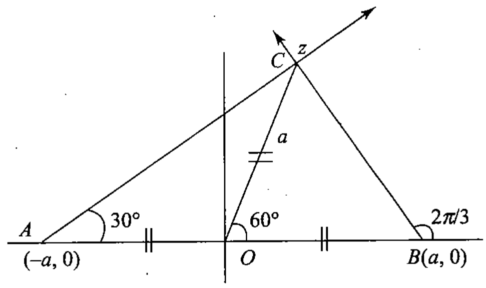


Fig. 2.79

9. a, c.

$(-i)^{1/3} = (i^3)^{1/3} = i, i\omega, i\omega^2$

where

$$\omega = \frac{-1+\sqrt{3}i}{2}$$

Hence roots are i, $(-\sqrt{3}-i)/2, (\sqrt{3}-i)/2$.

10. a, b, d.

$x^n - 1 = (x-1)(x-z_1)(x-z_2)\cdots(x-z_{n-1})$

$$\Rightarrow \quad \frac{x^n-1}{x-1} = (x-z_1)(x-z_2)\cdots(x-z_{n-1})$$

Putting $x = \omega$, we have

$$\prod_{r=1}^{n-1}(\omega - z_r) = \frac{\omega^n - 1}{\omega - 1} = \begin{cases} 0, & \text{if } n = 3k,\ k \in Z \\ 1, & \text{if } n = 3k+1,\ k \in Z \\ 1+\omega, & \text{if } n = 3k+2,\ k \in Z \end{cases}$$

11. a, d.

If α is a real root, then

$\alpha^3 + (3+i)\alpha^2 - 3\alpha - (m+i) = 0$

$\therefore \quad \alpha^3 + 3\alpha^2 - 3\alpha - m = 0$ and $\alpha^2 - 1 = 0$

$\Rightarrow \quad \alpha = 1$ or -1

$\alpha = 1 \Rightarrow m = 1$

$\alpha = -1 \Rightarrow m = 5$

12. c, d.

We have,

$x^2 + x + 1 = (x-\omega)(x-\omega^2)$

Since $f(x)$ is divisible by $x^2 + x + 1$, $f(\omega) = 0, f(\omega^2) = 0$, so

$P(\omega^3) + \omega Q(\omega^3) = 0 \Rightarrow P(1) + \omega Q(1) = 0$ (1)

$P(\omega^6) + \omega^2 Q(\omega^6) = 0 \Rightarrow P(1) + \omega^2 Q(1) = 0$ (2)

Solving (1) and (2), we obtain

$P(1) = 0$ and $Q(1) = 0$

Therefore, both $P(x)$ and $Q(x)$ are divisible by $x - 1$. Hence, $P(x^3)$ and $Q(x^3)$ are divisible by $x^3 - 1$ and so by $x - 1$ Since $f(x) = P(x^3) + xQ(x^3)$, we get $f(x)$ is divisible by $x - 1$.

13. b, c.

$\text{amp}(z_1z_2) = 0 \Rightarrow \text{amp } z_1 + \text{amp } z_2 = 0$

$\therefore \quad \text{amp } z_1 = -\text{amp } z_2 = \text{amp } \bar{z}_2$

Since $|z_1| = |z_2|$, we get $|z_1| = |\bar{z}_2|$. So, $z_1 = \bar{z}_2$. Also, $z_1z_2 = \bar{z}_2z_2 = |z_2|^2 = 1$ because $|z_2| = 1$.

14. a, b.

$$\left|z-\frac{1}{z}\right|=1$$

$\Rightarrow \quad 1 \ge \left||z|-\frac{1}{|z|}\right|$

$\Rightarrow \quad -1 \le |z|-\frac{1}{|z|} \le 1$

$\Rightarrow \quad -|z| \le |z|^2-1 \le |z|$

From $|z|^2-1 \ge -|z|$, we get

$|z|^2+|z|-1 \ge 0$

$\Rightarrow \quad |z| \ge \frac{-1+\sqrt{5}}{2}$ (1)

From $|z|^2-1 \le |z|$, we get

$|z|^2-|z|-1 \le 0$

$\Rightarrow \quad \frac{1-\sqrt{5}}{2} \le |z| \le \frac{1+\sqrt{5}}{2}$ (2)

From (1) and (2), we get

$\Rightarrow \quad \frac{-1+\sqrt{5}}{2} \le |z| \le \frac{1+\sqrt{5}}{2}$

$\Rightarrow \quad |z|_{min}=\frac{\sqrt{5}-1}{2},\ |z|_{max}=\frac{1+\sqrt{5}}{2}$

15. a, b, c.

$z^n \cos\theta_0 + z^{n-1}\cos\theta_1 + \cdots + z\cos\theta_{n-1} + \cos\theta_n = 2$

$\Rightarrow \quad 2 = |z_0^n \cos\theta_0 + z_0^{n-1}\cos\theta_1 + \cdots + z_0\cos\theta_{n-1} + \cos\theta_n|$

$\Rightarrow \quad 2 \le |z_0|^n |\cos\theta_0| + |z|^{n-1}|\cos\theta_1| + \cdots + |z_0||\cos\theta_{n-1}| + |\cos\theta_n|$

$\Rightarrow \quad 2 \le |z_0|^n + |z_0|^{n-1} + |z_0|^{n-2} + \cdots + |z_0| + 1$

which is clearly satisfied for $|z_0| \ge 1$. If $|z_0| < 1$, then

$2 < 1 + |z_0| + |z_0|^2 + \cdots + |z|^n + \cdots \infty$

$\Rightarrow \quad 2 < \frac{1}{1-|z_0|}$

$\Rightarrow \quad |z_0| > \frac{1}{2}$

16. a, b, c, d.

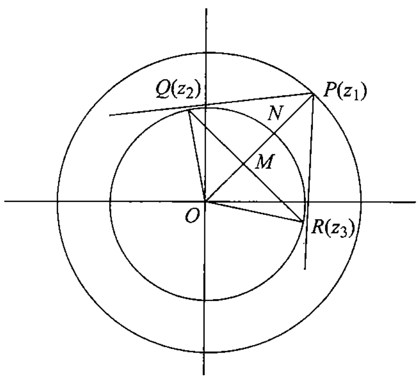


Fig. 2.80

Since $OQ = 1$ and $OP = 2$, so $\sin(\angle OPQ) = 1/2$ and hence $\angle QPR = \pi/3$. Then $\angle PQR$ is equilateral. Also, $OM \perp QR$. Then from $\angle OMQ$, $OM = 1/2$. Hence $MN = 1/2$. Then centroid of $\angle PQR$ lies on $|z| = 1$.

As PQR is an equilateral triangle, so orthocentre, circumcentre and centroid will coincide. Now,

$$\left|\frac{z_1+z_2+z_3}{3}\right| = 1$$

$\Rightarrow \quad |z_1+z_2+z_3|^2 = 9$

$\Rightarrow \quad (z_1+z_2+z_3)(\bar{z}_1+\bar{z}_2+\bar{z}_3) = 9$

$\Rightarrow \quad \left(\frac{4}{\bar{z}_1}+\frac{1}{\bar{z}_2}+\frac{1}{\bar{z}_3}\right)\left(\frac{4}{z_1}+\frac{1}{z_2}+\frac{1}{z_3}\right) = 9$

and

$\angle QOR = 120°$

17. a, d. $\left|z_1^2 - z_2^2\right| = \left|\bar{z}_1^2 + \bar{z}_1^2 - 2\bar{z}_1\bar{z}_2\right|$

$\Rightarrow \quad |z_1 - z_2||z_1 + z_2| = |\bar{z}_1 - \bar{z}_2|^2$

$\Rightarrow \quad |z_1 + z_2| = |\bar{z}_1 - \bar{z}_2|$

$|z_1 + z_2| = |z_1 - z_2|$

$\Rightarrow \quad \left|\frac{z_1}{z_2}+1\right| = \left|\frac{z_1}{z_2}-1\right|$

$\Rightarrow \quad \frac{z_1}{z_2}$ lies on $\perp$ bisector of 1 and -1

$\Rightarrow \quad \frac{z_1}{z_2}$ lies on imaginary axis

$\Rightarrow \quad \frac{z_1}{z_2}$ is purely imaginary

$\Rightarrow \quad \arg\left(\frac{z_1}{z_2}\right) = \pm\frac{\pi}{2}$

$|\arg(z_1) - \arg(z_2)| = \frac{\pi}{2}$

18. a, c, d.

$z' = ze^{i\alpha}$ (1)

$z'' = ze^{-i\alpha}$ (2)

$\therefore \quad z'z'' = z^2$

$\Rightarrow \quad z', z, z''$ are in G.P.

Also,

$$\left(\frac{z'}{z}\right)^2 + \left(\frac{z''}{z}\right)^2 = 2\cos 2\alpha$$

$\Rightarrow \quad z'^2 + z''^2 = 2z^2\cos 2\alpha$

$\Rightarrow \quad z'^2 + z''^2 = 2z^2(2\cos^2\alpha - 1)$

$\Rightarrow \quad z'^2 + z''^2 + 2z^2 = 4z^2\cos^2\alpha$

$\Rightarrow \quad z'^2 + z''^2 + 2z'z'' = 4z^2\cos^2\alpha$

$\Rightarrow \quad (z' + z'')^2 = 4z^2\cos^2\alpha$

$\Rightarrow \quad z' + z'' = 2z\cos\alpha$

19. a, c. If $p = q$, then equation becomes $z^p = \bar{z}^q$ and it has infinite number of solutions because any $z \in R$ will satisfy it. If $p \neq q$, let $p > q$, then $z^p = \bar{z}^q$.

$\therefore \quad |z|^p = |z|^q$

$\Rightarrow \quad |z|^p (|z|^{p-q} - 1) = 0$

$\Rightarrow \quad |z| = 0$ or $|z| = 1$

$|z| = 0 \Rightarrow z = 0 + i0$

$|z| = 1 \Rightarrow z = e^{i\theta}$

$\Rightarrow \quad e^{(p+q)\theta i} = 1$

$\Rightarrow \quad z = 1^{1/(p+q)}$

Therefore, the number of solutions is $p + q + 1$.

20. a, b, d.

$z_1 = 5 + 12i, |z_2| = 4$

$|z_1 + iz_2| \le |z_1| + |z_2| = 13 + 4 = 17$

$\therefore \quad |z_1 + (1 + i) z_2| \ge ||z_1| - |1 + i| |z_2||$

$= 13 - 4\sqrt{2}$

$\therefore \quad \min (|z_1 + (1 + i)z_2|) = 13 - 4\sqrt{2}$

$$\left|z_2 + \frac{4}{z_2}\right| \le |z_2| + \frac{4}{|z_2|} = 4 + 1 = 5$$

$$\left|z_2 + \frac{4}{z_2}\right| \ge |z_2| - \frac{4}{|z_2|} = 4 - 1 = 3$$

$$\therefore \quad \max\left|\frac{z_1}{z_2 + \frac{4}{z_2}}\right| = \frac{13}{3} \text{ and } \min\left|\frac{z_1}{z_2 + \frac{4}{z_2}}\right| = \frac{13}{5}$$

21. a, c.

$p + q + r = a + b\omega + c\omega^2$
$\quad + b + c\omega + a\omega^2$
$\quad + c + a\omega + b\omega^2$

$\therefore \quad p + q + r = (a + b + c)(1 + \omega + \omega^2) = 0 \qquad (1)$

p, q, r lie on the circle $|z| = 2$, whose circumcentre is origin. Also, $(p + q + r)/3 = 0$. Hence the cenroid coincides with circumcentre. So, the triangle is equilateral. Now,

$(p + q + r)^2 = 0.$

$$\Rightarrow \quad p^2 + q^2 + r^2 = -2pqr\left[\frac{1}{p} + \frac{1}{q} + \frac{1}{r}\right]$$

$$= -2pqr\left[\frac{1}{a+b\omega+c\omega^2} + \frac{1}{b+c\omega+a\omega^2} + \frac{1}{c+a\omega+b\omega^2}\right]$$

$$= -2pqr\left[\frac{1}{\omega^2(a\omega+b\omega^2+c)} + \frac{1}{\omega(b\omega^2+c+a\omega)} + \frac{1}{c+a\omega+b\omega^2}\right]$$

$$= \frac{-2pqr}{a\omega + b\omega^2 + c}\left[\frac{1}{\omega^2} + \frac{1}{\omega} + \frac{1}{1}\right] = 0 \qquad (2)$$

Hence,

$p^2 + q^2 + r^2 = 2(pq + qr + rp)$

22. a, b, c, d.

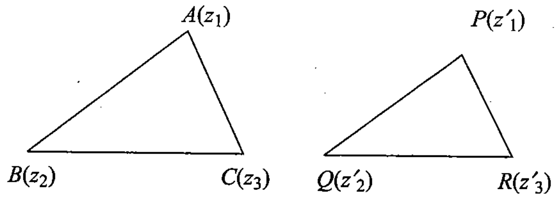


Fig. 2.81

$$z_3 = (1 - \lambda)z_1 + z_2 = \frac{(1-\lambda)z_1 + \lambda z_2}{1 - \lambda + \lambda}$$

Hence, z_3 divides the line joining $A(z_1)$ and $B(z_2)$ in the ratio $\lambda:(1-\lambda)$. That means the given points are collinear. Also, the ratio $\lambda/(1-\lambda) > 0$ (or $0 < \lambda < 1$) if z_3 divides the line joining z_1 and z_2 internally and $\mu/(1-\mu) < 0$ (or $\mu < 0$ or $\mu > 1$) if z_3' divides the line joining z_1', z_2' externally.

When λ, μ are complex numbers, where $\lambda = \mu$, we have $z_3 = (1 - \lambda)z_1 + \lambda z_2$ and $z_3' = (1 - \lambda)z_1' + \lambda z_2'$. Comparing the value of λ, we have

$$\frac{z_3 - z_1}{z_2 - z_1} = \frac{z_3' - z_1'}{z_2' - z_1'}$$

$$\Rightarrow \quad \left|\frac{z_3 - z_1}{z_2 - z_1}\right| = \left|\frac{z_3' - z_1'}{z_2' - z_1'}\right| \text{ and } \arg\left(\frac{z_3 - z_1}{z_2 - z_1}\right) = \arg\left(\frac{z_3' - z_1'}{z_2' - z_1'}\right)$$

$$\Rightarrow \quad \frac{AC}{AB} = \frac{PR}{PQ} \text{ and } \angle BAC = \angle QPR$$

Hence, triangles ABC and PQR are similar.

23. a, d.

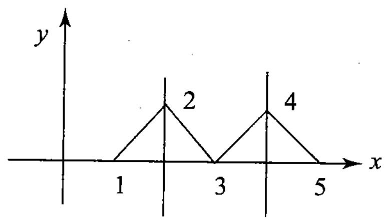


Fig. 2.82

24. a, c.

Given,

$z^n = (z + 1)^n \Rightarrow |z^n| = |(z + 1)^n|$

$\therefore \quad |z|^n = |z + 1|^n \Rightarrow |z| = |z + 1|$

$\Rightarrow \quad |z|^2 = |z + 1|^2$

$\Rightarrow \quad x^2 + y^2 = (x + 1)^2 + y^2$, where $z = x + iy$

$\Rightarrow \quad x = -\frac{1}{2}$

Hence, z lies on the line $x = -1/2$. Hence sum of real parts of the roots is $-(n-1)/2$ (since equation has $n - 1$ roots).

25. a, b, d.

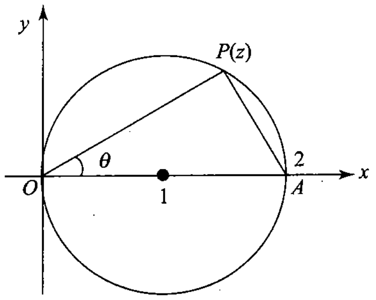


Fig. 2.83

Since arg $((z - 1 - i)/z)$ is the angle subtended by the chord joining the points O and $1 + i$ at the circumference of the circle $|z - 1| = 1$, so it is equal to $-\pi/4$. The line joining the points $z = 0$ and $z = 2 + 0i$ is the diameter.

$$\arg \frac{z-2}{z} = \pm\frac{\pi}{2}$$

$\Rightarrow$ $\dfrac{z-2}{z-0}$ is purely imaginary

We have,

$$\angle OPA = \frac{\pi}{2}$$

$$\Rightarrow \arg\left(\frac{2-z}{0-z}\right) = \frac{\pi}{2} \Rightarrow \frac{z-2}{z} = \frac{AP}{OP}i$$

Now in ΔOAP,

$$\tan\theta = \frac{AP}{OP}$$

Thus.

$$\frac{z-2}{z} = i\tan\theta$$

26. a, b, d.

$$\left|\frac{2z-i}{z+1}\right| = m \Rightarrow \left|z - \frac{i}{2}\right| = \frac{m}{2}|z+1|$$

This shows that the given equation will represent a circle, if $m/2 \neq 1$, i.e., $m \neq 2$.

27. a, d.

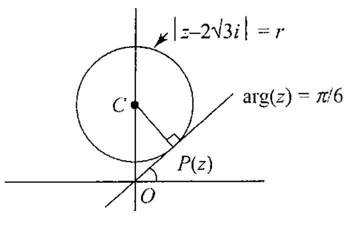


Fig. 2.84

$$CP = r,\ OC = 2\sqrt{3},\ \angle COP = \pi/3$$

$$\Rightarrow CP = OC\sin\frac{\pi}{3} = 2\sqrt{3}\,\frac{\sqrt{3}}{2} = 3$$

Thus, when $r = 3$, the circle touches the line. Hence, for two distinct points of intersection $3 < r < 2\sqrt{3}$.

28. b, c.

We have,

$$\left|\frac{1}{z_2} + \frac{1}{z_1}\right| = \left|\frac{1}{z_2} - \frac{1}{z_1}\right| \Rightarrow |z_1 + z_2| = |z_1 - z_2|$$

Squaring both sides, we have

$$|z_1|^2 + |z_2|^2 + 2(z_1\bar{z}_2 + \bar{z}_1 z_2) = |z_1|^2 + |z_2|^2 - 2(z_1\bar{z}_2 + \bar{z}_1 z_2)$$

$$\Rightarrow 4(z_1\bar{z}_2 + \bar{z}_1 z_2) = 0$$

$$\Rightarrow \frac{z_1}{z_2} + \frac{\bar{z}_1}{\bar{z}_2} = 0$$

$$\Rightarrow \arg\left(\frac{z_1}{z_2}\right) = \frac{\pi}{2} = \arg\left(\frac{z_1 - 0}{z_2 - 0}\right)$$

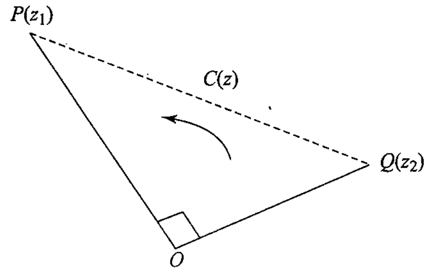


Fig. 2.85

That is angle between z_2, O and z_1 is a right angle, taken in order, as shown in the above diagram. Now, the circumcentre of the above diagram will lie on the line PQ as diameter and is represented by C which is the centre of PQ, such that $z = (z_1 + z_2)/2$, where z is the affix of circumcentre.

29. a, c, d.

Choice (a) on simplification gives

$$z = \frac{1+x}{1+x^2} + i\frac{1+x}{1+x^2}$$

For $x = 0.5$, $f(0.5) > 1$ which is out of range, Hence, (a) is not correct. From choice (b),

$$z = \frac{1-x}{1+x^2} + i\frac{1-x}{1+x^2}$$

$f(x)$ and $g(x) \in (0, 1)$ if $x \in (0, 1)$. Hence, (b) is correct. From choice (c),

$$z = \frac{1+x}{1+x^2} + \frac{1-x}{1+x^2}i$$

Hence, (c) is not correct. From choice (d),

$$z = \frac{1-x}{1+x^2} + \frac{1+x}{1+x^2}i$$

Hence, (d) is not correct.

30. a, b, c.

Let $z = x + iy$ where x, y satisfy the given equation. Hence,

$$(x^2 + y^2)(x^2 - y^2) = 175$$

$\Rightarrow$ $x^2 + y^2 = 25$ and $x^2 - y^2 = 7$ (as all other possibilities will give non-integral solutions)

Hence, possible values of z will be $4 + 3i$, $4 - 3i$, $-4 + 3i$ and $-4 - 3i$. Clearly, it will form a rectangle having length of the diagonal 10.

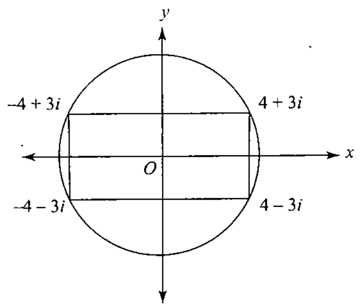


Fig. 2.86

From the diagram, options (a), (b), (c) are correct.

31. a, b.

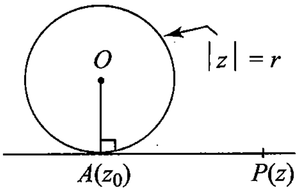


Fig. 2.87

$$\angle OAP = \frac{\pi}{2}$$

$$\Rightarrow \quad \frac{z - z_0}{z_0} \text{ is purely imaginary}$$

$$\Rightarrow \quad \frac{z - z_0}{z_0} + \frac{\bar{z} - \bar{z}_0}{\bar{z}_0} = 0$$

$$\Rightarrow \frac{z}{z_0} + \frac{\bar{z}}{\bar{z}_0} = 2 \qquad (1)$$

$$\Rightarrow \quad \text{Re}\left(\frac{z}{z_0}\right) = 1$$

From (1),

$$z\bar{z}_0 + z_0\bar{z} = 2|z_0|^2 = 2r^2$$

32. b., c. z_1 and z_2 are the roots of the equation $z^2 - az + b = 0$. Hence,

$z_1 + z_2 = a, z_1z_2 = b$

Now,

$|z_1 + z_2| \le |z_1| + |z_2|$

$\Rightarrow \quad |z_1 + z_2| = |a| \le 1 + 1 = 2 \qquad (\because |z_1| = |z_2| = 1)$

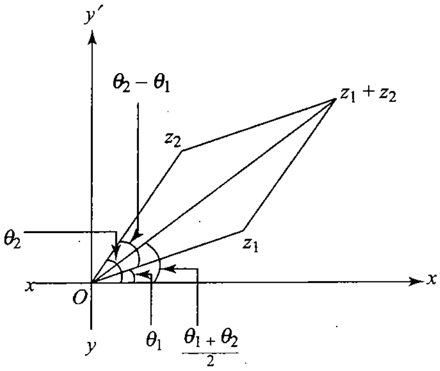


Fig. 2.88

$$\Rightarrow \quad \arg(a) = \frac{1}{2}[\arg(z_2) + \arg(z_1)]$$

Also,

$\arg(b) = \arg(z_1z_2) = \arg(z_1) + \arg(z_2)$

$\Rightarrow \quad 2\arg(a) = \arg(b)$

33. a, b, d. Given,

$$\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = 0$$

$\Rightarrow \quad a^3 + b^3 + c^3 - 3abc = 0$

$\Rightarrow \quad (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca) = 0$

$\Rightarrow \quad \frac{1}{2}(a+b+c)[(a-b)^2 + (b-c)^2 + (c-a)^2] = 0$

$\Rightarrow \quad (a - b)^2 + (b - c)^2 + (c - a)^2 = 0$

$\Rightarrow \quad a = b = c \qquad [\because a + b + c \ne 0, \because z_1 \ne 0, \therefore |z_1| = a \ne 0 \text{ etc.}]$

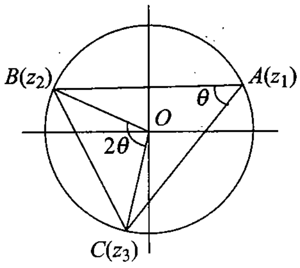


Fig. 2.89

Hence, $OA = OB = OC$, where O is the origin and A, B, C are the points representing z_1, z_2 and z_3, respectively. Therefore, O is circumcentre of $\angle ABC$. Now,

$$\arg\left(\frac{z_3}{z_2}\right) = \angle BOC \qquad \text{(i)}$$

$$= 2\angle BAC = 2\arg\left(\frac{z_3 - z_1}{z_2 - z_1}\right) \qquad \text{(ii)}$$

$$= \arg\left(\frac{z_3 - z_1}{z_2 - z_1}\right)^2 \qquad [\because \angle BOC = 2\angle BAC]$$

Hence,

$$\arg\left(\frac{z_3}{z_2}\right) = \arg\left(\frac{z_3 - z_1}{z_2 - z_1}\right)^2$$

Also, centroid is $(z_1 + z_2 + z_3)/3$. Since $HG:GO \equiv 2:1$ (where H is orthocentre and G is centroid), then orthocentre is $z_1 + z_2 + z_3$ (by section formula). When triangle is equilateral centroid coincides with circumcentre; hence $z_1 + z_2 + z_3 = 0$.

Also, the area for equilateral triangle is $(\sqrt{3}/4)L^2$, where L is length of side. Since radius is $|z_1|$, $L = \sqrt{3}|z_1|$, hence area is $(3\sqrt{3}/4)|z_1|^2$.

34. a, c, d.

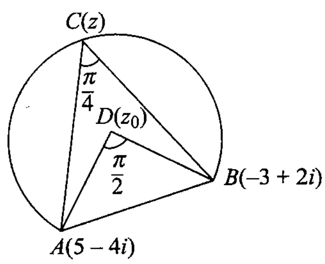


Fig. 2.90

$$\frac{z_0-(-3+2i)}{z_0-(5-4i)}=\frac{BD}{AD}e^{i\pi/2}=i$$

$\Rightarrow$ $z_0+3-2i=iz_0-5i-4$

$\Rightarrow$ $z_0=-2-5i$

$\Rightarrow$ Radius $AD=|5-4i-(-2-5i)|$

$=|7+i|$

$=\sqrt{50}=5\sqrt{2}$

Length of arc $=\frac{3}{4}$(perimeter of circle)

$=\frac{3}{4}(2\pi\times5\sqrt{2})$

$=\frac{15\pi}{\sqrt{2}}$

35. a, c, d.

Let $z=c$ be a real root. Then,

$ac^2+c+\bar{a}=0$ (1)

Putting $a=p+iq$, we have

$(p+iq)c^2+c+p-iq=0$

$\Rightarrow$ $pc^2+c+p=0$ and $qc^2-q=0\Rightarrow c=\pm1$ $(\because q\neq0)$

$\therefore$ $(1)\Rightarrow a\pm1+\bar{a}=0$

Also,

$|c|=1$

36. a, b, c, d.

$\sqrt{5-12i}=\sqrt{(3-2i)^2}=\pm(3-2i)$

$\sqrt{-5-12i}=\sqrt{(2-3i)^2}=\pm(2-3i)$

$\Rightarrow$ $z=\sqrt{5-12i}+\sqrt{-5-12i}$

$=-1-i,-5+5i,5-5i,1+i$

Therefore, principal values of arg z are $-3\pi/4$, $3\pi/4$, $-\pi/4$, $\pi/4$.

Reasoning Type

1. a. $\arg(z_1z_2)=2\pi\Rightarrow\arg(z_1)+\arg(z_2)=2\pi\Rightarrow\arg(z_1)=\arg(z_2)=\pi$, as principal arguments are from $-\pi$ to π.

Hence both the complex numbers are purely real. Hence both the statements are true and statement 2 is correct explanation of statement 1.

2. c. $x^3+x^2+x=x(x^2+x+1)=x(x-\omega)(x-\omega^2)$

Now $f(x)=(x+1)^n-x^n-1$ is divisible by x^3+x^2+x. Then $f(0)=0, f(\omega)=0\, f(\omega^2)=0$. Now,

$f(0)=(0+1)^n-0^n-1=0$

$f(\omega)=(\omega+1)^n-\omega^n-1=(-\omega^2)^n-\omega^n-1=-(\omega^{2n}+\omega^n+1)$
$=0$ (as n is not a multiple of 3)

Similarly, we have $f(\omega^2)=0$.

Hence statement 1 is correct but statement 2 is false.

3. a. First, let the two complex numbers be conjugate of each other. Let complex numbers be $z_1=x+iy$ and $z_2=x-iy$. Then, $z_1+z_2=(x+iy)+(x-ix)=2x$, which is real and $z_1z_2=(x+iy)(x-iy)=x^2-i^2y^2=x^2+y^2$, which is real.

Conversely, let z_1 and z_2 be two complex numbers such that their sum z_1+z_2 and product z_1z_2 both are real. Let $z_1=x_1+iy_1$ and $z_2=x_2+iy_2$. Then,

$z_1+z_2=(x_1+x_2)+i(y_1+y_2)$ and $z_1z_2=(x_1xs_2-y_1y_2)+i(x_1y_2+x_2y_1)$

Now, z_1+z_2 and z_1z_2 are real. Hence,

$b_1+b_2=0$ and $a_1b_2+a_2b_1=0$ $[\because z$ is real $\Rightarrow \text{Im}(z)=0]$

$\Rightarrow$ $b_2=-b_1$ and $a_1b_2+a_2b_1=0$

$=-b_1$ and $-a_1b_1+a_2b_1=0$

$=-b_1$ and $(a_2-a_1)\,b_1=0$

$=-b_1$ and $a_2-a_1=0$

$=-b_1$ and $a_2=a_1$

$\Rightarrow$ $z_2=a_2+ib_2=a_1-ib_1$

$=\bar{z}_1$

Hence, z_1 and z_2 are conjugate of each other. Hence, statement 2 is true.

Also in statement 1, $a=\bar{a}$ and $b=\bar{b}$, then a and b are real. Thus, z_1+z_2 and z_1z_2 are real. So,

$z_2=\bar{z}_2$

$\Rightarrow$ $\arg(z_1z_2)=\arg(z_1\bar{z}_1)=\arg(|z_1|^2)=0$

Hence, statement 1 is correct and statement 2 is correct explanation of statement 1.

4 d. $x+\frac{1}{x}=1$

$\Rightarrow$ $x^2-x+1=0$

$\therefore$ $x=-\omega,-\omega^2$

Now for $x=-\omega$,

$p=\omega^{4000}+\frac{1}{\omega^{4000}}=\omega+\frac{1}{\omega}=-1$

Similarly for $x=-\omega^2$, $P=-1$. For $n>1$,

$2^n=4k$

$\therefore$ $2^{2^n}=2^{4k}=(16)^k=$ a number with last digit 6

$\Rightarrow$ $q=6+1=7$

Hence, $p+q=-1+7=6$.

5. d.

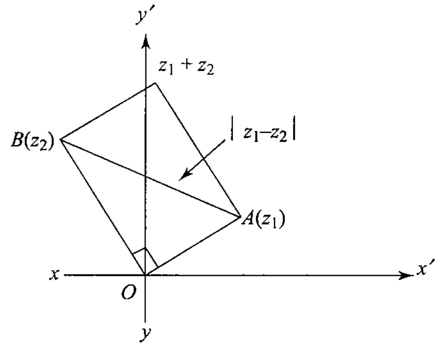


Fig. 2.91

From the diagram when $|z_1-z_2|=|z_1+z_2|$, OAB is right-angled triangle. Hence orthocentre is 0.

6. d. Statement 2 is true as it is the definition of an ellipse. Statement 1 is false as distance between 1 and 8 is 7 but $|z-1|+|z-8| = 5 < 7$. Hence no such z exists.

7. d. $|z_1 + z_2 + z_3| = |z_1 - a + z_2 - b + z_3 - c + (a+b+c)|$

$$\le |z_1 - a| + |z_2 - b| + |z_3 - c| + |a+b+c|$$
$$\le 2|a+b+c|$$

Hence, $|z_1 + z_2 + z_3|$ is less than $2|a+b+c|$.

8. b. Fourth roots of unity are $-1, 1, -i$ and i.

$\therefore \quad z_1^2 + z_2^2 + z_3^2 + z_4^2 = 0$ and $z_1 + z_2 + z_3 + z_4 = 0$

9. a. If roots of $ax^2 + bx + c = 0$, $0 < a < b < c$, are non-real, then they will be the conjugate of each other. Hence,

$$z_2 = \bar{z}_1 \Rightarrow |z_1| = |z_2|$$

Now,

$$z_1 z_2 = \frac{c}{a} > 1 \Rightarrow |z_1|^2 > 1$$

$\Rightarrow \quad |z_1| > 1$

$\Rightarrow \quad |z_2| > 1$

10. a. We have,

$$az^2 + bz + c = 0 \qquad (1)$$

and z_1, z_2 [roots of (1)] are such that $\text{Im}\,(z_1 z_2) \ne 0$. So, z_1 and z_2 are not conjugate of each other. That is complex roots of (1) are not conjugate of each other, which implies that coefficients a, b, c cannot all be real. Hence, at least one of a, b, c is imaginary.

11. b. We have,

Let $x = (\cos\theta + i\sin\theta)^{3/5}$

$\Rightarrow \quad x^5 = (\cos\theta + i\sin\theta)^3$

$\Rightarrow \quad x^5 - (\cos 3\theta + i\sin 3\theta) = 0$

$\Rightarrow \quad$ Product of roots $= \cos 3\theta + i\sin 3\theta$

Also product of roots of the equation $x^5 - 1 = 0$ is 1. Hence statement 2 is true. But it is not correct explanation of statement 1.

12. b. Since $|z_1| = |z_2| = |z_3|$, circumcentre of Δ is origin

Also $\dfrac{z_1 + z_2 + z_3}{3} = 0$

Centroid concide with circumcentre

$\Rightarrow \quad \Delta$ is equilateral

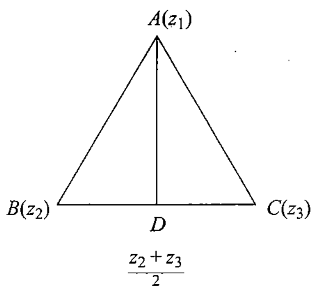


Fig. 2.92

$$\arg\left(\frac{z_2 + z_3 - 2z_1}{z_3 - z_2}\right) = \arg\left(2\left(\frac{\frac{z_2+z_3}{2} - z_1}{z_3 - z_2}\right)\right)$$

$$= \arg\left(\frac{\frac{z_2+z_3}{2} - z_1}{z_3 - z_2}\right)$$

$(z_2 + z_3)/2$ is the mid-point of side BC. Clearly, line joining A and mid-point of BC will be perpendicular to side BC. Thus,

$$\arg\left(\frac{\frac{z_2+z_3}{2} - z_1}{z_3 - z_2}\right) = \frac{\pi}{2}$$

Hence, statement 2 is also true. However, it does not explain statement 1.

13. a. Suppose there exists a complex number z which satisfies the given equation and is such that $|z| < 1$. Then,

$$z^4 + z + 2 = 0 \Rightarrow -2 = z^4 + z \Rightarrow |-2| = |z^4 + z|$$

$\Rightarrow \quad 2 \le |z^4| + |z| \Rightarrow 2 < 2$, because $|z| < 1$

But $2 < 2$ is not possible. Hence given equation cannot have a root z such that $|z| < 1$.

14. c. $|z_1 + z_2| = \left|\dfrac{z_1 + z_2}{z_1 z_2}\right|$

$\Rightarrow \quad |z_1 + z_2|\left(1 - \dfrac{1}{|z_1 z_2|}\right) = 0$

$\Rightarrow \quad |z_1 z_2| = 1$

Hence, statement 1 is true. However, it is not necessary that $|z_1| = |z_2| = 1$. Hence, statement 2 is false.

Linked Comprehension Type

For Problems 1–4

1. b, 2. b, 3. c, 4. c.

Sol. Given that

$$|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2$$

$\Rightarrow \quad |z_1|^2 + |z_2|^2 + z_1\bar{z}_2 + \bar{z}_1 z_2 = |z_1|^2 + |z_2|^2$

$\Rightarrow \quad z_1\bar{z}_2 + \bar{z}_1 z_2 = 0 \qquad (1)$

$\Rightarrow \quad \dfrac{z_1}{z_2} + \dfrac{\bar{z}_1}{\bar{z}_2} = 0 \qquad$ (dividing by $z_2\,\bar{z}_2$)

$\Rightarrow \quad \dfrac{z_1}{z_2} + \overline{\left(\dfrac{z_1}{z_2}\right)} = 0 \qquad (2)$

From (1), $z_1\,\bar{z}_2$ is purely imaginary. From (2), z_1/z_2 is purely imaginary. Hence,

$$\arg\left(\frac{z_1}{z_2}\right) = \pm\frac{\pi}{2} \Rightarrow \arg(z_1) - \arg(z_2) = \pm\frac{\pi}{2}$$

Also, $i(z_1/z_2)$ is purely real. Hence its possible arguments are 0 and π.

For Problems 5–8

5. a, 6. d, 7. c, 8. d.

Sol.

$$z = \frac{1 - i\sin\theta}{1 + i\cos\theta} = \frac{(1 - i\sin\theta)(1 - i\cos\theta)}{(1 + i\cos\theta)(1 - i\cos\theta)}$$

$$= \frac{(1 - \sin\theta\cos\theta) - i(\cos\theta + \sin\theta)}{(1 + \cos^2\theta)}$$

If z is purely real, then

$$\cos\theta + \sin\theta = 0$$

or

$$\tan\theta = -1$$

$$\Rightarrow \quad \theta = n\pi - \frac{\pi}{4}, n \in I$$

If z is purely imaginary, $1 - \sin\theta\cos\theta = 0$ or $\sin\theta\cos\theta = 1$, which is not possible.

$$|z| = \left|\frac{1 - i\sin\theta}{1 + i\cos\theta}\right| = \frac{\sqrt{1+\sin^2\theta}}{\sqrt{1+\cos^2\theta}}$$

If $|z| = 1$, then

$$\cos^2\theta = \sin^2\theta \Rightarrow \tan^2\theta = 1 \Rightarrow \theta = n\pi \pm \frac{\pi}{4}, n \in I$$

We have,

$$\arg(z) = \tan^{-1}\left(\frac{-(\cos\theta + \sin\theta)}{(1 - \sin\theta\cos\theta)}\right)$$

Now,

$$\arg(z) = \pi/4$$

$$\Rightarrow \quad \frac{-(\cos\theta + \sin\theta)}{(1 - \sin\theta\cos\theta)} = 1$$

$$\Rightarrow \quad \cos^2\theta + \sin^2\theta + 2\sin\theta\cos\theta$$

$$= 1 + \sin^2\theta\cos^2\theta - 2\sin\theta\cos\theta$$

$$\Rightarrow \quad 1 + 4\sin\theta\cos\theta = 1 + \sin^2\theta\cos^2\theta$$

$$\Rightarrow \quad \sin^2\theta\cos^2\theta - 4\sin\theta\cos\theta = 0$$

$$\Rightarrow \quad \sin\theta\cos\theta(\sin\theta\cos\theta - 4) = 0$$

$$\Rightarrow \quad \sin\theta\cos\theta = 0 \qquad (\because \sin\theta\cos\theta = 4 \text{ is not possible})$$

$$\Rightarrow \quad \theta = (2n+1)\pi \text{ or } \theta = (4n-1)\pi/2, n \in I$$

$$(\because -\cos\theta - \sin\theta > 0)$$

For Problems 9–11

9. a, 10. d, 11. d.

Sol.

9. Let z_1 (purely imaginary) be a root of the given equation. Then,

$$z_1 = -\bar{z}_1$$

and

$$az_1^2 + bz_1 + c = 0 \qquad (1)$$

$$\Rightarrow \quad \overline{az_1^2 + bz_1 + c} = 0$$

$$\Rightarrow \quad \bar{a}\bar{z}_1^2 + \bar{b}\bar{z}_1 + \bar{c} = 0$$

$$\Rightarrow \quad \bar{a}z_1^2 - \bar{b}z_1 + \bar{c} = 0 \qquad (\text{as } \bar{z}_1 = -z_1) \qquad (2)$$

Now Eqs. (1) and (2) must have one common root.

$$\therefore \quad (c\bar{a} - a\bar{c})^2 = (b\bar{c} + c\bar{b})(-a\bar{b} - \bar{a}b)$$

Let z_1 and z_2 be two purely imaginary roots. Then,

$$\bar{z}_1 = -z_1, \bar{z}_2 = -z_2$$

Now,

$$az^2 + bz + c = 0 \qquad (1)$$

$$\Rightarrow \quad \overline{az^2 + bz + c} = \bar{0}$$

$$\Rightarrow \quad \bar{a}\bar{z}^2 + \bar{b}\bar{z} + \bar{c} = 0$$

$$\Rightarrow \quad \bar{a}z^2 - \bar{b}z + \bar{c} = 0 \qquad (2)$$

Equations (1) and (2) must be identical as their roots are same.

$$\therefore \quad \frac{a}{\bar{a}} = -\frac{b}{\bar{b}} = \frac{c}{\bar{c}}$$

$$\Rightarrow \quad a\bar{c} = \bar{a}c, a\bar{b} + \bar{a}b = 0 \text{ and } b\bar{c} + \bar{b}c = 0$$

Hence, $a\bar{c}$ is purely real and $a\bar{b}$ and $b\bar{c}$ are purely imaginary.

Let z_1 (purely real) be a root of the given equation. Then,

$$z_1 = \bar{z}_1$$

and

$$az_1^2 + bz_1 + c = 0 \qquad (1)$$

$$\Rightarrow \quad \overline{az_1^2 + bz_1 + c} = \bar{0}$$

$$\Rightarrow \quad \bar{a}\bar{z}_1^2 + \bar{b}\bar{z}_1 + \bar{c} = 0$$

$$\Rightarrow \quad \bar{a}z_1^2 + \bar{b}z_1 + \bar{c} = 0 \qquad (2)$$

Now (1) and (2) must have one common root. Hence,

$$(c\bar{a} - a\bar{c})^2 = (b\bar{c} - c\bar{b})(a\bar{b} - \bar{a}b)$$

For Problems 12–14

12. c, 13. b, 14. d.

Sol. $az + b\bar{z} + c = 0 \qquad (1)$

$$\Rightarrow \quad \bar{a}\bar{z} + \bar{b}z + \bar{c} = 0 \qquad (2)$$

Eliminating $\bar{z}$ from (1) and (2), we get

$$z = \frac{c\bar{a} - b\bar{c}}{|b|^2 - |a|^2}$$

If $|a| \neq |b|$, then z represents one point on the Argand plane. If $|a| = |b|$ and $\bar{a}c \neq b\bar{c}$, then no such z exists. Adding (1) and (2),

$$(\bar{a} + b)\bar{z} + (a + \bar{b})z + (c + \bar{c}) = 0$$

This is of the form $A\bar{z} + \bar{A}z + B = 0$, where $B = c + \bar{c}$ is real.

Hence locus of z is a straight line.

For Problems 15–17

15. a, 16. b, 17. c.

$$z = -\lambda \pm \sqrt{\lambda^2 - 1}$$

Case I:

When $-1 < \lambda < 1$, we have

$$\lambda^2 < 1 \Rightarrow \lambda^2 - 1 < 0$$

$$z = -\lambda \pm i\sqrt{1-\lambda^2} \text{ or } x = -\lambda, y = \pm\sqrt{1-\lambda^2}$$

$$\Rightarrow \quad y^2 = 1 - x^2 \text{ or } x^2 + y^2 = 1$$

Case II:

$$\lambda > 1 \Rightarrow \lambda^2 - 1 > 0$$

$$z = -\lambda \pm \sqrt{\lambda^2 - 1} \text{ or } x = -\lambda \pm \sqrt{\lambda^2 - 1}, y = 0$$

Roots are $(-\lambda + \sqrt{\lambda^2-1}, 0)$, $(-\lambda - \sqrt{\lambda^2-1}, 0)$. One root lies inside the unit circle and the other root lies outside the unit circle.

Case III:

When λ is very large, then

$$z = -\lambda - \sqrt{\lambda^2 - 1} \approx -2\lambda$$

$$z = -\lambda + \sqrt{\lambda^2 - 1} = \frac{(-\lambda + \sqrt{\lambda^2-1})(-\lambda - \sqrt{\lambda^2-1})}{(-\lambda - \sqrt{\lambda^2-1})}$$

$$= \frac{1}{-\lambda - \sqrt{\lambda^2-1}} = -\frac{1}{2\lambda}$$

For Problems 18–20

18. d, 19. c, 20. c.

Sol. We have,

$$az^2 + z + 1 = 0 \quad (1)$$

$$\Rightarrow \overline{az^2 + z + 1} = 0 \quad \text{(taking conjugate of both sides)}$$

$$\Rightarrow \bar{a}z^2 - z + 1 = 0 \quad (2)$$

[since z is purely imaginary $\bar{z} = -z$]

Eliminating z from (1) and (2) by cross-multiplication rule,

$$(\bar{a} - a)^2 + 2(a + \bar{a}) = 0 \Rightarrow \left(\frac{\bar{a} - a}{2}\right)^2 + \frac{a + \bar{a}}{2} = 0$$

$$\Rightarrow -\left(\frac{a - \bar{a}}{2i}\right)^2 + \left(\frac{a + \bar{a}}{2}\right) = 0 \Rightarrow -\sin^2\theta + \cos\theta = 0$$

$$\Rightarrow \cos\theta = \sin^2\theta \quad (3)$$

Now,

$$f(x) = x^3 - 3x^2 + 3(1 + \cos\theta)x + 5$$

$$f'(x) = 3x^2 - 6x + 3(1 + \cos\theta)$$

Its discriminant is

$$36 - 36(1 + \cos\theta) = -36\cos\theta = -36\sin^2\theta < 0$$

$$\Rightarrow f'(x) > 0 \;\forall\, x \in R$$

Hence, $f(x)$ is increasing $\forall\, x \in R$. Also, $f(0) = 5$, then $f(x) = 0$ has one negative root. Now,

$$\cos 2\theta = \cos\theta \Rightarrow 1 - 2\sin^2\theta = \cos\theta$$

$$\Rightarrow 1 - 2\cos\theta = \cos\theta$$

$$\Rightarrow \cos\theta = 1/3$$

which has four roots for $\theta \in [0, 4\pi]$.

For Problems 21–23

21. a, 22. b, 23. b.

Sol.

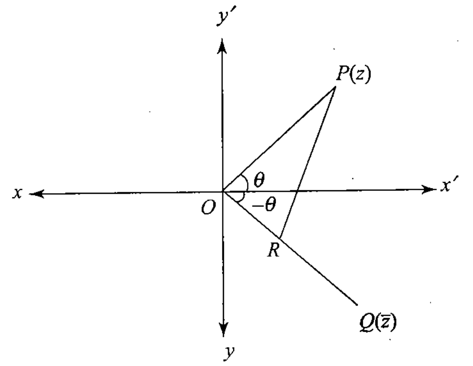


Fig. 2.93

We have,

$$\left| |z| - \left|\frac{4}{z}\right| \right| \le \left| z - \frac{4}{z} \right| = 2$$

$$\Rightarrow -2 \le |z| - \frac{4}{|z|} \le 2$$

$$\Rightarrow |z|^2 + 2|z| - 4 \ge 0 \text{ and } |z|^2 - 2|z| - 4 \le 0$$

$$\Rightarrow (|z| + 1)^2 - 5 \ge 0 \text{ and } (|z| - 1)^2 \le 5$$

$$\Rightarrow (|z| + 1 + \sqrt{5})(|z| + 1 - \sqrt{5}) \ge 0$$

$$\text{and } (|z| - 1 + \sqrt{5}) \times (|z| - 1 - \sqrt{5}) \le 0$$

$$\Rightarrow |z| \le -\sqrt{5} - 1 \text{ or } |z| \ge \sqrt{5} - 1 \text{ and } \sqrt{5} - 1 \le |z| \le \sqrt{5} + 1$$

$$\Rightarrow \sqrt{5} - 1 \le |z| \le \sqrt{5} + 1$$

Hence, the least modulus is $\sqrt{5} - 1$ and the greatest modulus is $\sqrt{5} + 1$. Also,

$$|z| = \sqrt{5} + 1 \Rightarrow \frac{4}{|z|} = \sqrt{5} - 1$$

Now,

$$\frac{4}{z} = \frac{4\bar{z}}{|z|^2}$$

Hence, $4/z$ lies in the direction of $\bar{z}$.

$$\left| z - \frac{4}{z} \right| = PR = 2 \text{ (given)}$$

We have,

$$OP = \sqrt{5} + 1 \text{ and } OR = \sqrt{5} - 1$$

$$\Rightarrow \cos 2\theta = \frac{OP^2 + OR^2 - PR^2}{2\,OP \cdot OR}$$

$$= \frac{(\sqrt{5} + 1)^2 + (\sqrt{5} - 1)^2 - 4}{2(5 - 1)} = 1$$

$$\Rightarrow 2\theta = 0, 2\pi$$

$$\Rightarrow \theta = 0, \pi$$

$\Rightarrow$ z is purely real

$$\Rightarrow z = \pm(\sqrt{5} + 1)$$

Similarly for $|z| = \sqrt{5} - 1$, we have $z = \pm(\sqrt{5} - 1)$.

For Problems 24–26

24. a, 25. c, 26. b.

Sol. $BM \equiv y - 0 = -1(x - 1)$

$$x + y = 1$$

$$\therefore \sqrt{u - 1} = t + i(1 - t)$$

$$u = 2t + 2it(1 - t)$$

$$x = 2t \text{ and } y = 2t(1 - t)$$

$$y = x(1 - x/2)$$

$$2y = 2x - x^2$$

$$\Rightarrow (x - 1)^2 = -2\left(y - \frac{1}{2}\right)$$

which is a parabola. Its axis is $x = 1$, i.e., $z + \bar{z} = 2$ and directrix is $y = 1$, i.e., $z - \bar{z} = 2i$.

For Problems 27–29

27. a, 28. b, 29. c.

Sol. 27.

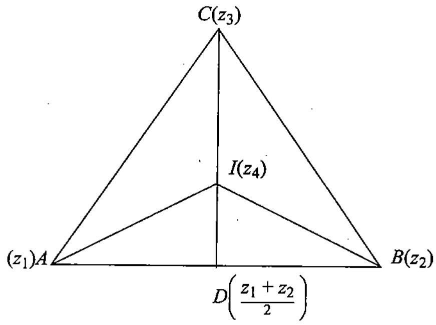


Fig. 2.94

$$\frac{AB \times AC}{(IA)^2} = \frac{AB}{IA} \times \frac{AC}{IA}$$

$$\angle IAB = \frac{\theta}{2},\ \angle IAC = \frac{\theta}{2}$$

$$\frac{z_2 - z_1}{z_4 - z_1} = \frac{|z_2 - z_1|}{|z_4 - z_1|} e^{-\frac{i\theta}{2}}$$

and

$$\frac{z_3 - z_1}{z_4 - z_1} = \frac{|z_3 - z_1|}{|z_4 - z_1|} e^{\frac{i\theta}{2}}$$

Multiplying,

$$\frac{z_2 - z_1}{z_4 - z_1} \frac{z_3 - z_1}{z_4 - z_1} = \frac{|z_2 - z_1|}{|z_4 - z_1|} \frac{|z_3 - z_1|}{|z_4 - z_1|}$$

$$\Rightarrow \quad \frac{(z_2 - z_1)(z_3 - z_1)}{(z_4 - z_1)^2} = \frac{AB \times AC}{IA^2} \qquad (1)$$

28. From (1),

$$\frac{(z_2 - z_1)(z_3 - z_1)}{(z_4 - z_1)^2} = 2\left(\frac{AD}{IA}\right)^2 \left(\frac{AC}{AD}\right) \qquad (\because AB = 2\,AD)$$

$$\Rightarrow \quad (z_2 - z_1)(z_3 - z_1) = (z_4 - z_1)^2\, 2\cos^2\frac{\theta}{2} \sec\theta$$

$$= (z_4 - z_1)^2 (\cos\theta + 1)\sec\theta$$

29. Keeping in mind that $\tan\theta = CD/AD$ and $\tan\theta/2 = ID/BD$, We have,

$$\frac{z_3 - \frac{z_1 + z_2}{2}}{z_1 - \frac{z_1 + z_2}{2}} = \frac{\left|z_3 - \frac{z_1 + z_2}{2}\right|}{\left|z_1 - \frac{z_1 + z_2}{2}\right|} e^{-i\frac{\pi}{2}}$$

$$\Rightarrow \quad \frac{2z_3 - z_1 - z_2}{z_1 - z_2} = \frac{CD}{AD} e^{-i\frac{\pi}{2}} \qquad (1)$$

and

$$\frac{z_4 - \frac{z_1 + z_2}{2}}{z_2 - \frac{z_1 + z_2}{2}} = \frac{\left|z_4 - \frac{z_1 + z_2}{2}\right|}{\left|z_2 - \frac{z_1 + z_2}{2}\right|} e^{i\frac{\pi}{2}}$$

$$\Rightarrow \quad \frac{2z_4 - z_1 - z_2}{z_2 - z_1} = \frac{ID}{BD} e^{i\frac{\pi}{2}} \qquad (2)$$

Multiplying (1) and (2), we have

$$\frac{2z_3 - z_1 - z_2}{z_1 - z_2} \frac{2z_4 - z_1 - z_2}{z_2 - z_1} = \frac{CD}{AD}\frac{ID}{BD} = \tan\theta \tan\frac{\theta}{2}$$

$$\Rightarrow \quad (z_2 - z_1)^2 \tan\theta \tan\frac{\theta}{2} = -(z_1 + z_2 - 2z_3)(z_1 + z_2 - 2z_4)$$

For Problems 30–32

30. d, 31. c, 32. c.

Sol.

30. $\angle BOD = 2\angle BAD = A$

$\angle COD = 2\angle CAD = A$

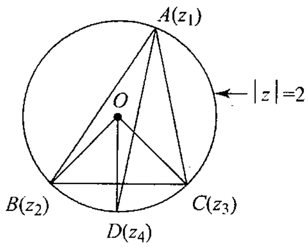


Fig. 2.95

$$\frac{z_4}{z_2} = e^{iA},\ \frac{z_3}{z_4} = e^{iA} \qquad \text{(From rotation about the point 'O')}$$

$$\Rightarrow \quad z_4^2 = z_2 z_3$$

31. Clearly, OD bisects $\angle BAC$ of isosceles triangle BOC.

Thus angle between segments OD and BC is $\pi/2$.

$$\therefore \quad \arg\left(\frac{z_4}{z_2 - z_3}\right) = \frac{\pi}{2}$$

32. See theory.

Matrix-Match Type

1.a → r, s; b → p, q; r → t; d → u, t.

a. If $ab > 0$, then either a, b both are positive or both a, b are negative. Hence $z = a + ib$ lies in either first or third quadrant, then argument of z is $\tan^{-1} b/a$ or $-\pi + \tan^{-1} b/a$.

b. If $ab < 0$, then a and b have opposite signs, then z lies in either second or fourth quadrant, then argument of z is $-\tan^{-1} b/a$ or $\pi - \tan^{-1} |b/a|$ or $\tan^{-1} b/a$.

c. If $a^2 + b^2 = 0$, then $a = b = 0$, so $z = 0 + i0$ whose argument is not defined.

d. If $ab = 0$, then either $a = 0$ or $b = 0$ or both are 0, then argument is 0 or $\pi/2$ or not defined.

2. a → s; b → r; c → p; d → q.

a. $z^4 - 1 = 0 \Rightarrow z^4 = 1 = \cos 0 + i \sin 0 \Rightarrow z = (\cos 0 + i \sin 0)^{1/4}$

$= \cos 0 + i \sin 0$

b. $z^4 + 1 = 0 \Rightarrow z^4 = -1 = \cos\pi + i \sin\pi \Rightarrow z = (\cos\pi + i \sin\pi)^{1/4}$

$$= \cos\frac{\pi}{4} + i\sin\frac{\pi}{4}$$

c. $iz^4 + 1 = 0 \Rightarrow z^4 = i = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} \Rightarrow z = \left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)^{1/4}$

$$= \cos\frac{\pi}{8} + i\sin\frac{\pi}{8}$$

d. $iz^4 - 1 = 0 \Rightarrow z^4 = -i = \cos\frac{\pi}{2} - i\sin\frac{\pi}{2}$

$$\Rightarrow z = \left(\cos\frac{\pi}{2} - i\sin\frac{\pi}{2}\right)^{1/4} = \cos\frac{\pi}{8} - i\sin\frac{\pi}{8}$$

3. a → q; b → s; c → p; d → r.

a. $|z - 1| = |z - i|$

Hence it lies on the perpendicular bisector of the line joining (1, 0) and (0, 1) which is a straight line passing through the origin.

b. $|z + \bar{z}| + |z - \bar{z}| = 2$

$\Rightarrow |x| + |y| = 1$

Hence, z lies on a square.

c. Let $z = x + iy$. Then,

$|z + \bar{z}| = |z - \bar{z}|$

$\Rightarrow |2x| = |2iy|$

$\Rightarrow |x| = |y|$

$\Rightarrow x = \pm y$

Hence, the locus of z is a pair of straight lines.

d. Let $Z = 2/z$. Then,

$$|Z| = \left|\frac{2}{z}\right| = \frac{2}{|z|} = \frac{2}{1} = 2$$

This shows that Z lies on a circle with centre at the origin and radius 2 units.

4. a → p, r; b → p, q, r, t; c → p, r, s; d → p, q, r, s, t.

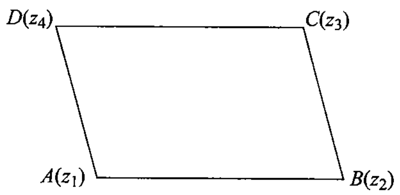


Fig. 2.96

In parallelogram, the mid-points of diagonals coincide

$$\frac{z_1 + z_3}{2} = \frac{z_2 + z_4}{2}$$

$$\Rightarrow z_1 - z_4 = z_2 - z_3$$

Also in parallelogram, $AB \| CD$. Hence,

$$\arg\left(\frac{z_1 - z_2}{z_3 - z_4}\right) = 0$$

$\Rightarrow \frac{z_1 - z_2}{z_3 - z_4}$ is purely real

In rectangle, adjacent sides are perpendicular. Hence,

$$\arg\left(\frac{z_1 - z_2}{z_3 - z_2}\right) = \frac{\pi}{2}$$

$\Rightarrow \frac{z_1 - z_2}{z_3 - z_2}$ is purely imaginary

Also in rectangle,

$AC = BD \Rightarrow |z_1 - z_3| = |z_2 - z_4|$

In rhombus,

$AC \perp BD \Rightarrow \frac{z_1 - z_3}{z_2 - z_4}$ is purely imaginary

5. a → p, q; b → p, q, r, s, t; c → r, s; d → p, q.

a. $|z - 2i| + |z - 7i| = k$ is ellipse if $k > |7i - 2i|$ or $k > 5$

b. $\left|\frac{2z-3}{3z-2}\right| = k \Rightarrow \left|\frac{z-\frac{3}{2}}{z-\frac{2}{3}}\right| = \frac{3k}{2} \Rightarrow 3k/2 > 1 \Rightarrow k > 2/3$

c. $|z - 3| - |z - 4i| = k$ is hyperbola, if $k < |3 - 4i| \Rightarrow 0 < k < 5$

d. $|z - (3 + 4i)| = \frac{k}{50}|a\bar{z} + \bar{a}z + b|$

$\Rightarrow |z - (3 + 4i)| = \frac{k}{5}\frac{|a\bar{z} + \bar{a}z + b|}{2|3+4i|}$

This is hyperbola if $k/5 > 1 \Rightarrow k > 5$.

6. a → s; b → q; c → p; d → r.

a. $x^2 - x + 1 = 0$

$$\Rightarrow x = \frac{1 \pm i\sqrt{3}}{2}$$

$$= -\omega, -\omega^2$$

$$\Rightarrow \left(x^n + \frac{1}{x^n}\right)^2 = (-1)^{2n}\left(\omega^n + \frac{1}{\omega^n}\right)^2$$

$$= (\omega^n + \omega^{2n})^2 \quad (1)$$

$$\because \frac{1}{\omega^n} = \frac{\omega^{2n}}{\omega^{3n}} = \omega^{2n}$$

Now,

$$1 + \omega^n + \omega^{2n} = \frac{1-\omega^{3n}}{1-\omega^n} = 0 \text{ for } n \neq 3p$$

$\therefore \omega^n + \omega^{2n} = -1$ for $n \neq 3p$

$= 2$ for $n = 3p$

$$\therefore \sum_{n=1}^{5}\left(x^n + \frac{1}{x^n}\right)^2 = 8$$

b. In the expression,

$$\left[\frac{1+\cos\theta+i\sin\theta}{\sin\theta+i(1+\cos\theta)}\right]^4$$

numerator is

$$1+\cos\theta+i\sin\theta=2\cos\frac{\theta}{2}\left[\cos\frac{\theta}{2}+i\sin\frac{\theta}{2}\right]$$

$$=2\cos\frac{\theta}{2}\,e^{i\theta/2}$$

and denominator is

$-i^2\sin\theta+i(1+\cos\theta)=i$ [conjugate of numerator]

$$=i\,2\cos\frac{\theta}{2}\,e^{-\theta/2}$$

$$\therefore\quad E=\left(\frac{N^r}{D^r}\right)=\left[\frac{1}{i}\frac{e^{i\theta/2}}{e^{-i\theta/2}}\right]^4=\frac{1}{i^4}e^{4i\theta}$$

$$=\cos 4\theta+i\sin 4\theta$$

$\therefore\quad n=4$

c. We know that if $z=re^{i\theta}$, then $\bar{z}=re^{-i\theta}$.

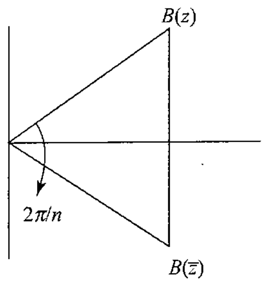


Fig. 2.97

$$\therefore\quad \frac{\text{Im}(z)}{\text{Re}(z)}=\frac{r\sin\theta}{r\cos\theta}=\tan\theta=\tan\frac{\pi}{n}=\sqrt{2}-1$$

$$\Rightarrow\quad \tan\frac{\pi}{n}=\tan\frac{\pi}{8}\Rightarrow n=8$$

d. $\displaystyle\sum_{r=1}^{10}(r-\omega)(r-\omega^2)=\sum_{r=1}^{10}(r^2+r+1)$

$$=\Sigma r^2+\Sigma r+10$$

$$=\frac{10\times 11\times 21}{6}+\frac{10\times 11}{2}+10$$

$$=450$$

$$\Rightarrow\quad \frac{1}{50}\left\{\sum_{r=1}^{10}(r-\omega)(r-\omega^2)\right\}=9$$

Integer Type

1.(3) $\displaystyle x=\frac{x^3}{x^2}=\frac{2+11i}{3+4i}\times\frac{3-4i}{3-4i}=\frac{50+25i}{25}=2+i$

2.(7) We have

$x^3-y^3=98i$

$\Rightarrow (x-y)^3+3xy(x-y)=98i$

$\Rightarrow -343i+3(a+ib)(7i)=98i$

$\Rightarrow -343+3(a+bi)7=98$

$\Rightarrow a+ib=21$

$\Rightarrow a=21$ and $b=0$

$\Rightarrow a+b=21$

3.(1) We have $x=\omega-\omega^2-2$ or $x+2=\omega-\omega^2$

Squaring, $x^2+4x+4=\omega^2+\omega^4-2\omega^3=\omega^2+\omega-2=-3$

$\Rightarrow x^2+4x+7=0.$

Dividing $x^4+3x^3+2x^2-11x-6$ by x^2+4x+7, we get

$x^4+3x^3+2x^2-11x-6=(x^2+4x+7)(x^2-x-1)+1$

$=(0)(x^2-x-1)+1=0+1=1$

4.(5) $z^2=81-b^2+18bi$

$z^3=729+243bi-27b^2-b^3i$

$z^2=z^3\Rightarrow 243b-b^3=18b$ and $243-b^2=18\Rightarrow b=15$

5.(2) $\bar{z}+z=0$

$\Rightarrow \bar{z}=-z$ (1)

Now $|z|^2-4zi=z^2$

$\Rightarrow -z^2-4zi=z^2$ (from (1))

$\Rightarrow 2z=-4i$

$\Rightarrow z=-2i$

$\Rightarrow |z|=2$

6.(3) $(1+ri)^3=s(1+i)$

$\Rightarrow 1+3ri+3r^2i^2+r^3i^3=s(1+i)$

$\Rightarrow 1-3r^2+i(3r-r^3)=s+si$

$\Rightarrow 1-3r^2=s=3r-r^3$

Hence, $1-3r^2=3r-r^3$

$\Rightarrow r^3-3r^2-3r+1=0$

$\Rightarrow$ sum of three roots is 3.

7.(4) We have $\displaystyle |z|^2+\frac{16}{|z|^3}=z^2-4z=\bar{z}^2-4\bar{z}$

$\Rightarrow (z-\bar{z})(z+\bar{z}-4)=0$

$\Rightarrow z=\bar{z}=x\ (x\neq 2)$

So, $\displaystyle x^2=4x+x^2+\frac{16}{|x|^3}$

$\displaystyle\Rightarrow x=\frac{-4}{|x|^3}\quad\Rightarrow\quad x=-\sqrt{2}$

$\therefore\ z=-\sqrt{2}$

$\therefore\ |z|^4=4$

8.(9) Let $z=a+bi$.

$\Rightarrow |z|^2=a^2+b^2.$

Now $z+|z|=2+8i$

$\Rightarrow a+bi+\sqrt{a^2+b^2}=2+8i$

$\Rightarrow a+\sqrt{a^2+b^2}=2,\ b=8$

$\Rightarrow a+\sqrt{a^2+64}=2$

$\Rightarrow a^2+64=(2-a)^2=a^2-4a+4,$

$\Rightarrow 4a=-60,\ a=-15.$

Thus, $a^2+b^2=225+64=289$

$\therefore\ |z|=\sqrt{a^2+b^2}=\sqrt{289}=17$

9.(1) Let $z=a+ib$

Given $|z|=2\Rightarrow a^2+b^2=4\Rightarrow a,b\in[-2,2]$

Now $w = \dfrac{(a+1)+ib}{(a-1)+ib}$;

$\Rightarrow |w| = \sqrt{\dfrac{(a+1)^2+b^2}{(a-1)^2+b^2}}$

$= \sqrt{\dfrac{a^2+b^2+2a+1}{a^2+b^2-2a+1}} = \sqrt{\dfrac{5+2a}{5-2a}}$

$|w|_{max} = \sqrt{\dfrac{5+4}{1}} = 3$ (when $a = 2$)

$|w|_{min} = \sqrt{\dfrac{5-4}{9}} = \dfrac{1}{3}$ (when $a = -2$)

Hence, required product is 1.

10.(4) $= \left[\dfrac{1+\cos\theta+i\sin\theta}{\sin\theta+i(1+\cos\theta)}\right]^4$

$= i^4\left[\dfrac{1+\cos\theta+i\sin\theta}{i\sin\theta+i^2(1+\cos\theta)}\right]^4$

$= \left[\dfrac{2\cos^2\frac{\theta}{2}+i2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\cos^2\frac{\theta}{2}-i2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}\right]^4$

$= \left[\dfrac{\cos\frac{\theta}{2}+i\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}-i\sin\frac{\theta}{2}}\right]^4$

$= \left[\left(\cos\frac{\theta}{2}+i\sin\frac{\theta}{2}\right)^2\right]^4$

$= \cos 8\frac{\theta}{2}+i\sin 8\frac{\theta}{2} = \cos 4\theta + i\sin 4\theta \Rightarrow n = 4$

11.(1) $z^4+z^3+z^2+z^2+z+1=0$

$\Rightarrow (z^2(z^2+z+1)+(z^2+z+1)=0$

$\Rightarrow (z^2+z+1)(z^2+1)=0$

$\therefore\ z = i, -i, \omega, \omega^2$. For each, $|z| = 1$

12.(5) Roots are 2ω, $(2+3\omega)$, $(2+3\omega^2)$, $(2-\omega-\omega^2)$

$2+3\omega$ and $2+3\omega^2$ are conjugate to each other.

2ω is complex root, then other root must be $2\omega^2$ (as complex roots occur in conjugate pair)

$2-\omega-\omega^2 = 2-(-1) = 3$ which is real.

Hence least degree of the polynomial is 5.

13.(6) We have $|a\omega+b| = 1$

$\Rightarrow |a\omega+b|^2 = 1$

$\Rightarrow (a\bar{\omega}+b) = 1$

$\Rightarrow a^2+ab(\omega+\bar{\omega})+b^2 = 1$

$\Rightarrow a^2-ab+b^2 = 1$

$\Rightarrow (a-b)^2+ab = 1$ (1)

when $(a-b)^2 = 0$ and $ab = 1$ then (1, 1); (–1, –1)

when $(a-b)^2 = 1$ and $ab = 0$ then (0, 1); (1, 0); (0, –1); (–1, 0)

Hence there are 6 ordered pairs

14.(3) $|z-2-2i| \le 1$

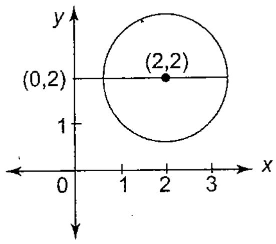


Fig. 2.98

denotes the region inside a circle with centre (2, 2) and radius is 1

$|2iz+4| = |2i(z-2i)|$

$= |2i||z-2i|$

$= 2|z-2i|$

$|z-2i|$ = distance of z from $P(0, 2)$

Hence, maximum value is 3.

15.(5) $|3z+9-7i| = |(3z+6-3i)+(3-4i)|$

$\le |3z+6-3i|+|3-4i|$

$= 3|z+2-i|+5$

$= 20$

16.(5) $Z_1 = (8\sin\theta+7\cos\theta)+i(\sin\theta+4\cos\theta)$

$Z_2 = (\sin\theta+4\cos\theta)+i(8\sin\theta+4\cos\theta)$

Hence, $Z_1 = x+iy$ and $Z_2 = y+ix$

where $x = (8\sin\theta+7\cos\theta)$ and $y = (\sin\theta+4\cos\theta)$

$Z_1\cdot Z_2 = (xy-xy)+i(x^2+y^2) = i(x^2+y^2) = a+ib$

$\Rightarrow a = 0; b = x^2+y^2$

Now, $x^2+y^2 = (8\sin\theta+7\cos\theta)^2+(\sin\theta+4\cos\theta)^2$

$= 65\sin^2\theta+65\cos^2\theta+120\sin\theta.\cos\theta$

$= 65+60\sin 2\theta$

$\Rightarrow |Z_1\cdot Z_2|_{max} = 125$

17.(9) $A \equiv (1+2i)x^3-2(3+i)x^2+(5-4i)x+2a^2 = 0$

Let the real root of equation be α

Then $(1+2i)\alpha^3-2(3+i)\alpha^2+(5-4i)\alpha+2a^2 = 0$

equating imaginary part zero, we get

$2\alpha^3-2\alpha^2-4\alpha = 0$

$\Rightarrow$ or $\alpha(\alpha^2-\alpha-2) = 0$

$\Rightarrow \alpha = 0$ or $\alpha = -1, 2$

Now equating real part zero

$\alpha^3-6\alpha^2+5\alpha+2a^2 = 0$

$\alpha = 0 \Rightarrow a = 0$

$\alpha = -1 \Rightarrow a = \pm\sqrt{6}$

$\alpha = 2 \Rightarrow a = \pm\sqrt{3}$

$\Rightarrow \sum a^2 = (0)^2+(+\sqrt{6})^2+(-\sqrt{6})^2+(+\sqrt{3})^2+(-\sqrt{3})^2 = 18$

18.(6) Let $z = x+iy$

$\therefore\ E = z\bar{z}+(z-3)(\bar{z}-3)+(z-6i)(\bar{z}+6i)$

$= 3z\bar{z}-3(z+\bar{z})+9+6(z-\bar{z})i+36$

$= 3(x^2+y^2)-6x-12y+45$

$= 3[x^2+y^2-2x-4y+15]$

$= 3[(x-1)^2+(y-2)^2+10]$

$\therefore\ E_{min} = 30$ when $x = 1$ and $y = 2$

Archives

Subjective Type

1. $\frac{1}{1-\cos\theta+2i\sin\theta}=\frac{1}{2\sin^2\theta/2+4i\sin\theta/2\ \cos\theta/2}$

$$=\frac{1}{2\sin\theta/2}\left[\frac{\sin\theta/2-2i\cos\theta/2}{(\sin\theta/2+2i\cos\theta/2)(\sin\theta/2-2i\cos\theta/2)}\right]$$

$$=\frac{1}{2\sin\theta/2}\left[\frac{\sin\theta/2-2i\cos\theta/2}{\sin^2\theta/2+4\cos^2\theta/2}\right]$$

$$=\frac{1}{2\sin\theta/2}\left[\frac{2\sin\theta/2-4i\cos\theta/2}{1-\cos\theta+4+4\cos\theta}\right]$$

$$=\frac{1}{\sin\theta/2}\left[\frac{\sin\theta/2-2i\cos\theta/2}{5+3\cos\theta}\right]$$

$$=\left(\frac{1}{5+3\cos\theta}\right)+\left(\frac{-2\cot\theta/2}{5+3\cos\theta}\right)i$$

2. As β and γ are the complex cube roots of unity, therefore let $\beta=\omega$ and $\gamma=\omega^2$ so that $\omega+\omega^2+1=0$ and $\omega^3=1$. Then,

$xyz=(a+b)(a\omega+b\omega^2)(a\omega^2+b\omega)$
$=(a+b)(a^2\omega^3+ab\omega^4+ab\omega^2+b^2\omega^3)$
$=(a+b)(a^2+ab\omega+ab\omega^2+b^2)$ (Using $\omega^3=1$)
$=(a+b)(a^2+ab(\omega+\omega^2)+b^2)$
$=(a+b)(a^2-ab+b^2)$ (Using $\omega+\omega^2=-1$)
$=a^3+b^3$

3. Given,

$$x+iy=\sqrt{\frac{a+ib}{c+id}}$$

$\Rightarrow (x+iy)^2=\frac{a+ib}{c+id}$ (1)

$\Rightarrow \left|(x+iy)^2\right|=\left|\frac{a+ib}{c+id}\right|$

$\Rightarrow |x+iy|^4=\left|\frac{a+ib}{c+id}\right|^2$

$\Rightarrow (x^2+y^2)^2=\frac{a^2+b^2}{c^2+d^2}$

4. Given that n is an odd integer >3 and n is not a multiple of 3.
Let,
$p(x)=(x+1)^n-x^n-1$
and
$q(x)=x^3+x^2+x$
$=x(x^2+x+1)$
$=x(x-\omega)(x-\omega^2)$

where ω and ω^2 are cube roots of unity. Clearly, 0, ω, ω^2 are zeros of the polynomial $q(x)$. Now,

$p(0)=1^n-0-1=0$

Hence, 0 is a zero of $p(x)$.

$p(\omega)=(\omega+1)^n-\omega^n-1$
$=(-\omega^2)^n-\omega^n-1$
$=-(\omega^{2n}+\omega^n+1)$ [$\because n$ is odd]
$=0$ [$\because \omega^n+\omega^{2n}+1=0$ if $n\neq 3m$]

Therefore, ω is a zero of $p(x)$. Also,

$p(\omega^2)=(\omega^2+1)^n-(\omega^2)^n-1$
$=(-\omega)^n-\omega^{2n}-1$
$=-\omega^n-\omega^{2n}-1$
$=-(1+\omega^n+\omega^{2n})$
$=0$ [for $n\neq 3m$]

Hence, ω^2 is a zero of $p(x)$.

Since 0, ω, ω^2 are zeros of $p(x)$, hence x, $x-\omega$, $x-\omega^2$ are factors of $p(x)$. Hence, $x(x-\omega)(x-\omega^2)$ is a factor of $p(x)$, i.e, x^3+x^2+x is a factor of $p(x)$.

5. $\frac{(1+i)x-2i}{3+i}+\frac{(2-3i)y+i}{3-i}=i$

$\Rightarrow (4+2i)x-6i-2+(9-7i)y+3i-1=10i$
$\Rightarrow (4x+9y-3)+(2x-7y-3)i=10i$
$\Rightarrow 4x+9y-3=0$ and $2x-7y-3=10$

On solving, we get $x=3$, $y=-1$.

6. $A(z_1)$, $B(z_2)$, $C(z_3)$ are the vertices of an equilateral triangle. Hence,

$$z_1^2+z_2^2+z_3^2=z_1z_2+z_2z_3+z_3z_1$$

Now,

$$(z_1+z_2+z_3)^2=z_1^2+z_2^2+z_3^2+2(z_1z_2+z_2z_3+z_3z_1)$$
$$=3(z_1^2+z_2^2+z_3^2)$$

We also have,

$z_0=\frac{z_1+z_2+z_3}{3}$ (as centroid will coincide with circumcentre)

$\Rightarrow z_1^2+z_2^2+z_3^2=3z_0^2$

7. We know that if z_1, z_2, z_3 form an equilateral triangle, then

$$z_1^2+z_2^2+z_3^2=z_1z_2+z_2z_3+z_3z_1$$

Putting $z_3=0$, we get

$z_1^2+z_2^2=z_1z_2$

$\Rightarrow z_1^2+z_2^2-z_1z_2=0$

8.

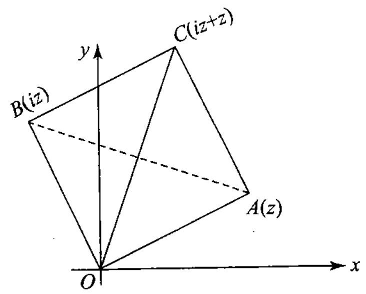


Fig. 2.99

Let the vertices of the triangle be $A(z)$, $B(iz)$, $C(z + iz)$. We know that iz is obtained by rotating OA through an angle 90°. Also point $z + iz$ can be obtained by completing the parallelogram two of whose adjacent sides are OA and OB. From Argand diagram, it is clear that

Area of ΔABC = Area of ΔOAB

$$= \frac{1}{2} \times OA \times OB \quad [\because \text{ it is right angled at point } O]$$

$$= \frac{1}{2}|z| \times |iz|$$

$$= \frac{1}{2}|z|^2$$

9.

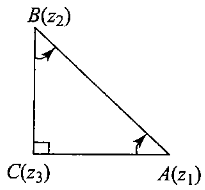


Fig. 2.100

Applying rotation about point C,

$$\frac{z_2 - z_3}{z_1 - z_3} = e^{i\pi/2} \quad (1)$$

Applying rotation about point B,

$$\frac{z_1 - z_2}{z_3 - z_2} = \sqrt{2}e^{i\pi/4} \quad (2)$$

Applying rotation about point A,

$$\frac{z_2 - z_1}{z_3 - z_1} = \sqrt{2}\, e^{-i\pi/4} \quad (3)$$

Multiplying (2) and (3), we get

$$\frac{(z_1 - z_2)(z_2 - z_1)}{(z_3 - z_2)(z_3 - z_1)} = 2$$

$$(z_1 - z_2)^2 = -2(z_3 - z_2)(z_3 - z_1)$$

$$= 2(z_1 - z_3)(z_3 - z_2)$$

10.

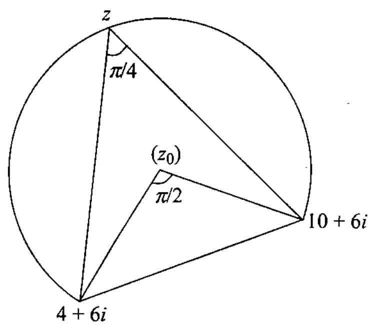


Fig. 2.101

$$\arg\left(\frac{z - z_1}{z - z_2}\right) = \frac{\pi}{4}$$

Locus of z is major arc whose centre is at z_0. Applying rotation at z_0, we have

$$\frac{z_0 - (10+6i)}{z_0 - (4+6i)} = \frac{|z_0 - (10+6i)|}{|z_0 - (4+6i)|} e^{i\frac{\pi}{2}}$$

$$\Rightarrow \quad \frac{z_0 - (10+6i)}{z_0 - (4+6i)} = i$$

$$\Rightarrow \quad z_0 - 10 - 6i = iz_0 - 4i + 6$$

$$\Rightarrow \quad z_0 = 7 + 9i$$

Thus centre is at $7 + 9i$ and z is any point on the arc.

Hence, $|z - (7 + 9i)| = |10 + 6i - (7 + 9i)| = 3\sqrt{2}$.

11. Dividing throughout by i, we get

$$z^3 - iz^2 + iz + 1 = 0$$

$\Rightarrow \quad z^2 (z - i) + i(z - i) = 0$ as $1 = -i^2$

$\Rightarrow \quad (z - i)(z^2 + i) = 0$

$\Rightarrow \quad z = i$ or $z^2 = -i$

$\Rightarrow \quad |z| = |i| = 1$ or $|z^2| = |z|^2 = |-i| = 1$

$\Rightarrow \quad |z| = 1$

Hence, in either case $|z| = 1$.

12. Let $z = |z|e^{i\alpha}$ and $w = |w|e^{i\beta}$. Now,

$$|z - w|^2 = |z|^2 + |w|^2 - z\bar{w} - \bar{z}w$$

$$= (|z| - |w|)^2 + 2|z||w| - |z||w|e^{i(\alpha-\beta)} - |z||w|e^{-i(\alpha-\beta)}$$

$$= (|z| - |w|)^2 + |z||w|(2 - 2\cos(\alpha - \beta))$$

$$\leq (|z| - |w|)^2 + 4\sin^2\left(\frac{\alpha - \beta}{2}\right)^2 \quad (\because |z| \leq 1, |w| \leq 1)$$

$$\leq (|z| - |w|)^2 + 4\left(\frac{\alpha - \beta}{2}\right)^2 \quad [\because \sin\theta < \theta \text{ for } \theta \in (0, \pi/2)]$$

$$= (|z| - |w|)^2 + 4(\alpha - \beta)^2$$

$$= (|z| - |w|)^2 + (\arg z - \arg w)^2$$

13. Let $z = x + iy$. Then,

$$\bar{z} = iz^2$$

$\Rightarrow \quad x - iy = i(x^2 - y^2 + 2ixy)$

$\Rightarrow \quad x - iy = i(x^2 - y^2) - 2xy$

$\Rightarrow \quad x(1 + 2y) = 0 \quad (1)$

and

$$x^2 - y^2 + y = 0 \quad (2)$$

From (1), $x = 0$ or $y = -1/2$. From (2), when $x = 0$, $y = 0, 1$ and when $y = -1/2$, $x = \pm(\sqrt{3}/2)$. For non-zero complex number z,

$$z = i, \frac{\sqrt{3}}{2} - \frac{i}{2}, -\frac{\sqrt{3}}{2} - \frac{i}{2}$$

14. Given that z_1 is the reflection of z_2 through the line

$$b\bar{z} + \bar{b}z = c \quad (1)$$

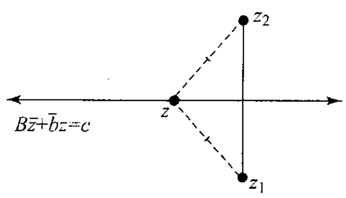


Fig. 2.102

Therefore, for any arbitrary point z on the line, we must have

$$|z - z_1| = |z - z_2|$$

$\Rightarrow \quad |z - z_1|^2 = |z - z_2|^2$

$\Rightarrow \quad |z|^2 + |z_1|^2 - z\bar{z}_1 - \bar{z}z_1 = |z|^2 + |z_2|^2 - z\bar{z}_2 - \bar{z}z_2$

$\Rightarrow \quad (\bar{z}_2 - \bar{z}_1)z + (z_2 - z_1)\bar{z} = |z_2|^2 - |z_1|^2 \qquad (1)$

Comparing (1) with (2), we have

$b = z_2 - z_1$ and $c = |z_2|^2 - |z_1|^2$

$\Rightarrow \quad \bar{z}_1 b + z_2 \bar{b} = \bar{z}_1(z_2 - z_1) + z_2(\bar{z}_2 - \bar{z}_1) = |z_2|^2 - |z_1|^2 = c$

15.

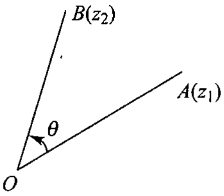


Fig. 2.103

Let z_1 and z_2 be roots of the equation $z^2 + pz + q = 0$. Then,

$z_1 + z_2 = -p, \ z_1 z_2 = q$

Also,

$$\frac{z_2}{z_1} = e^{i\theta} \Rightarrow z_2 = z_1 e^{i\theta}$$

$\Rightarrow \quad z_1(1 + e^{i\theta}) = -p, \ z_1^2 e^{i\theta} = q$

$$z_1^2 = q\, e^{-i\theta} = \frac{p^2}{(1 + e^{i\theta})^2}$$

$\Rightarrow \quad p^2 = qe^{-i\theta}(1 + e^{2i\theta} + 2e^{i\theta})$

$= q(e^{-i\theta} + e^{i\theta} + 2)$

$= q(2\cos\theta + 2)$

$= 4q \cos^2 \dfrac{\theta}{2}$

16. Given that z and ω are two complex numbers. To prove $|z|^2 w - |w|^2 z = z - w \Leftrightarrow z = w$ or $z\bar{w} = 1$

First let us consider

$|z|^2 w - |w|^2 z = z - w \qquad (1)$

$\Rightarrow \quad z(1 + |w|^2) = w(1 + |z|^2)$

$$\Rightarrow \quad \frac{z}{w} = \frac{1 + |z|^2}{1 + |w|^2} = \text{a real number}$$

$$\Rightarrow \quad \overline{\left(\frac{z}{w}\right)} = \frac{z}{w} \Rightarrow \frac{\bar{z}}{\bar{w}} = \frac{z}{w}$$

$\Rightarrow \bar{z}w = z\bar{w} \qquad (2)$

Again from Eq. (1),

$z\bar{z}w - w\bar{w}z = z - w$

$z(\bar{z}w - 1) - w(\bar{w}z - 1) = 0$

$z(z\bar{w} - 1) - w(z\bar{w} - 1) = 0 \qquad$ [Using Eq. (2)]

$\Rightarrow \quad (z\bar{w} - 1)(z - w) = 0$

$\Rightarrow \quad z\bar{w} = 1$ or $z = w$

Conversely if $z = w$, then L.H.S. of (1) is $|w|^2 w - |w|^2 w = 0$ and R.H.S. of (1) is $w - w = 0$. Therefore, Eq. (1) holds. Also, if $\bar{w}z = 1$, then $w\bar{z} = 1$. L.H.S. of (1) is $z\bar{z}w - w\bar{w}z = zz\bar{w} - ww\bar{z}$ = R.H.S. Hence proved.

17. $z^{p+q} - z^p - z^q + 1 = 0$

$\Rightarrow \quad (z^p - 1)(z^q - 1) = 0$

$\Rightarrow \quad z = (1)^{1/p}$ or $(1)^{1/q} \qquad (1)$

where p and q are distinct prime numbers. Hence both the equations will have distinct roots and as $z \neq 01$, both will be simultaneously zero for any value of z given by Eq. (1). Also,

$$1 + \alpha + \alpha^2 + \cdots + \alpha^{p-1} = \frac{1 - \alpha^p}{1 - \alpha} \qquad (\alpha \neq 1)$$

or

$$1 + \alpha + \alpha^2 + \cdots + \alpha^q = \frac{1 - \alpha^q}{1 - \alpha} \qquad (\alpha \neq 1)$$

Because of (1), either $\alpha^p = 1$ or $\alpha^q = 1$ but not both simultaneously as p and q are distinct primes.

18. Given that $|z_1| < 1 < |z_2|$. Now,

$$\left|\frac{1 - z_1\bar{z}_2}{z_1 - z_2}\right| < 1$$

$\Rightarrow \quad |1 - z_1\bar{z}_2| < |z_1 - z_2|$

$\Rightarrow \quad |1 - z_1\bar{z}_2|^2 < |z_1 - z_2|^2$

$\Rightarrow \quad (1 - z_1\bar{z}_2)\left(\overline{1 - z_1\bar{z}_2}\right) < (z_1 - z_2)\left(\overline{z_1 - z_2}\right)$

$\Rightarrow \quad (1 - z_1\bar{z}_2)(1 - \bar{z}_1 z_2) < (z_1 - z_2)(\bar{z}_1 - \bar{z}_2)$

$\Rightarrow \quad 1 - z_1\bar{z}_2 - \bar{z}_1 z_2 + z_1\bar{z}_1 z_2\bar{z}_2 < z_1\bar{z}_1 - z_1\bar{z}_2 - \bar{z}_1 z_2 + z_2\bar{z}_2$

$\Rightarrow \quad 1 + |z_1|^2 |z_2|^2 < |z_1|^2 + |z_2|^2$

$\Rightarrow \quad (1 - |z_1|^2)(1 - |z_2|^2) < 0$

which is obviously true as

$|z_1| < 1 < |z_2|$

$\Rightarrow \quad |z_1|^2 < 1 < |z_2|^2$

$\Rightarrow \quad (1 - |z_1|^2) > 0$ and $(1 - |z_2|^2) < 0$

19. $\displaystyle\sum_{r=1}^{n} a_r z^r = 1$ (where $|a_r| < 2$)

$\Rightarrow \quad a_1 z + a_2 z^2 + a_3 z^3 + \cdots + a_n z^n = 1$

$\Rightarrow \quad |a_1 z + a_2 z^2 + a_3 z^3 + \cdots + a_n z^n| = 1 \qquad (1)$

$\Rightarrow \quad 1 = |a_1 z + a_2 z^2 + a_3 z^3 + \cdots + a_n z^n|$

$\leq |a_1 z| + |a_2 z^2| + \cdots + |a_n z^n|$

$= |a_1||z| + |a_2||z^2| + |a_3||z^3| + \cdots + |a_n||z^n|$

$< 2\,[|z| + |z|^2 + |z|^3 + \cdots + |z|^n] \qquad (\because |a_r| < 2, \forall\, r$ and $|z^n| = |z|^n)$

$$= 2\left[\frac{|z|(1 - |z|^n)}{1 - |z|}\right]$$

$$= 2\left[\frac{|z| - |z|^{n+1}}{1 - |z|}\right]$$

$\Rightarrow \quad 2[|z| - |z|^{n+1}] > 1 - |z| \qquad (\because 1 - |z| > 0$ as $|z| < 1/3)$

$\Rightarrow \quad \frac{3}{2}|z| > \frac{1}{2} + |z|^{n+1}$

$\Rightarrow \quad |z| > \frac{1}{3} + \frac{2}{3}|z|^{n+1}$

$\Rightarrow \quad |z| > \frac{1}{3}$

which is a contradiction. Hence, there exists no such complex number.

20.

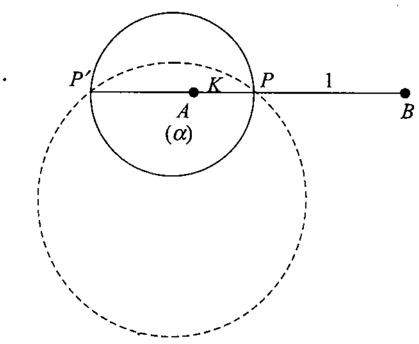


Fig. 2.104

$$\left|\frac{z-\alpha}{z-\beta}\right| = k$$

$\Rightarrow \quad |z-\alpha| = k\,|z-\beta|$

Let points A, B and P represent complex numbers α, β and z, respectively. Then,

$$|z-\alpha| = k\,|z-\beta|$$

Therefore, z is the complex number whose distance from A is k times its distance from B, i.e.,

$$PA = k\,PB$$

Hence, P divides AB in the ratio $k{:}1$ internally or externally (at P'). Then,

$$P \equiv \left(\frac{k\beta+\alpha}{k+1}\right) \text{ and } P' \equiv \left(\frac{k\beta-\alpha}{k-1}\right)$$

Now through PP' there can pass a number of circles, but with given data we can find radius and centre of that circle for which PP' is diameter. Hence the centre is the mid-point of PP', and is given by

$$\frac{\left(\frac{k\beta+\alpha}{k+1} + \frac{k\beta-\alpha}{k-1}\right)}{2}$$

$$= \frac{k^2\beta + k\alpha - k\beta - \alpha + k^2\beta - k\alpha + k\beta - \alpha}{2(k^2-1)}$$

$$= \frac{k^2\beta - \alpha}{k^2-1}$$

$$= \frac{\alpha - k^2\beta}{1-k^2}$$

$$\text{Radius} = \frac{1}{2}|PP'|$$

$$= \frac{1}{2}\left|\frac{k\beta+\alpha}{k+1} - \frac{k\beta-\alpha}{k-1}\right|$$

$$= \frac{1}{2}\left|\frac{k^2\beta + k\alpha - k\beta - \alpha - k^2\beta + k\alpha - k\beta + \alpha}{k^2-1}\right|$$

$$= \frac{k\,|\alpha-\beta|}{|1-k^2|}$$

21. The given circle is $|z-1| = \sqrt{2}$ where $z_0 = 1$ is the centre and $\sqrt{2}$ is radius of the circle. z_1 is one of the vertices of the square inscribed in the given circle.

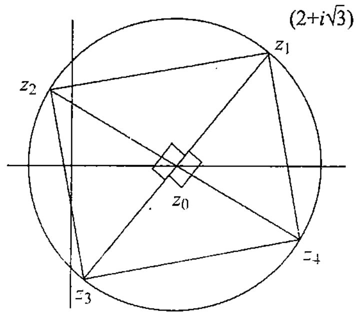


Fig. 2.105

Clearly, z_2 can be obtained by rotating z_1 by an angle of 90° in anticlockwise sense about centre z_0. Thus,

$$z_2 - z_0 = (z_1 - z_0)\,e^{i\pi/2}$$

$\Rightarrow \quad z_2 - 1 = (2 + i\sqrt{3} - 1)i$

$\Rightarrow \quad z_2 = i - \sqrt{3} + 1$

$\Rightarrow \quad z_2 = (1-\sqrt{3}) + i$

Now z_0 is mid-point of z_1 and z_3 and z_2 and z_4

$\therefore \quad \frac{z_1+z_3}{2} = z_0 \Rightarrow \frac{2+i\sqrt{3}+z_3}{2} = 1$

$\Rightarrow \quad z_3 = -i\sqrt{3}$

and

$$\frac{z_2+z_4}{2} = z_0 \Rightarrow \frac{\left(1-\sqrt{3}\right)+i+z_4}{2} = 1$$

$\Rightarrow \quad z_4 = (\sqrt{3}+1) - i$

22. Let $u = \frac{1}{1-z} \Rightarrow z = 1 - \frac{1}{u}$

$$|z| = 1 \Rightarrow = \left|1 - \frac{1}{u}\right| = 1$$

$\Rightarrow \quad |u-1| = |u|$

$\therefore$ locus of u is perpendicular bisector of line segment joining 0 and 1

$\Rightarrow$ maximum arg u approaches $\frac{\pi}{2}$ but will not attain.

23. $z = \frac{2i(x+iy)}{1-(x+iy)^2} = \frac{2i\,(x+iy)}{1-(x^2-y^2+2ixy)}$

Using $1 - x^2 = y^2$

$$z = \frac{2ix - 2y}{2y^2 - 2ixy} = -\frac{1}{y}.$$

$\because \; -1 \le y \le 1 \Rightarrow -\frac{1}{y} \le -1 \text{ or } -\frac{1}{y} \ge 1.$

Objective Type

Fill in the blanks

1. Let,

$$z=\frac{\sin\frac{x}{2}+\cos\frac{x}{2}+i\tan x}{1+2i\sin\frac{x}{2}}$$

$$=\frac{\left(\sin\frac{x}{2}+\cos\frac{x}{2}+i\tan x\right)\left(1-2i\sin\frac{x}{2}\right)}{\left(1+2i\sin\frac{x}{2}\right)\left(1-2i\sin\frac{x}{2}\right)}$$

$$=\frac{\left(\sin\frac{x}{2}+\cos\frac{x}{2}+2\sin\frac{x}{2}\tan x\right)+i\left(\tan x-2\sin^2\frac{x}{2}-2\sin\frac{x}{2}\cos\frac{x}{2}\right)}{1+4\sin^2\frac{x}{2}}$$

Now,

Im$(z)=0$ (as z is real)

$\Rightarrow\ \tan x-2\sin\frac{x}{2}\left(\sin\frac{x}{2}+\cos\frac{x}{2}\right)=0$

$\Rightarrow\ \frac{\sin x}{\cos x}-2\sin^2(x/2)-2\sin(x/2)\cos(x/2)=0$

$\Rightarrow\ \frac{\sin x}{\cos x}-(1-\cos x)-\sin x=0$

$\Rightarrow\ \sin x\left[\frac{1}{\cos x}-1\right]-[1-\cos x]=0$

$\Rightarrow\ (1-\cos x)\left[\frac{\sin x}{\cos x}-1\right]=0$

$\Rightarrow\ \cos x=1\Rightarrow x=2n\pi$ or $\tan x=1\Rightarrow x=n\pi+\pi/4,\ n\in Z$

2. $|az_1-bz_2|^2+|bz_1+az_2|^2$

$=a^2|z_1|^2+b^2|z_2|^2-2ab\,\text{Re}(z_1\bar{z}_2)+b^2|z_1|^2+a^2|z_2|^2+2ab\times\text{Re}(z_1\bar{z}_2)$

$=(a^2+b^2)(|z_1|^2+|z_2|^2)$

3. As $z_1=a+i$, $z_2=1+bi$ and $z_3=0$ from an equilateral triangle, therefore

$|z_1-z_3|=|z_2-z_3|=|z_1-z_2|$

$\Rightarrow\ |a+i|=|1+bi|=|(a-1)+i(1-b)|$

$\Rightarrow\ a^2+1=1+b^2=(a-1)^2+(1-b)^2$

$\Rightarrow\ a^2=b^2=a^2+b^2-2a-2b+1$

$\Rightarrow\ a=b\qquad(\because a,b>0\ \therefore a\neq-b)$ (1)

and

$b^2-2a-2b+1=0$ (2)

Solving $a^2-2a-2a+1=0$, we get

$a^2-4a+1=0$

$\Rightarrow\ a=\frac{4\pm2\sqrt{3}}{2}=2\pm\sqrt{3}$

But $0<a,b<1$.

$\therefore\ a=2-\sqrt{3}$ and $b=2-\sqrt{3}$

4.

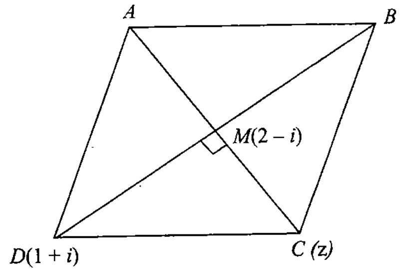


Fig. 2.106

Rotating DM about M by an angle 90°, we have

$$\frac{z-(2-i)}{(1+i)-(2-i)}=\frac{|z-(2-i)|}{|(1+i)-(2-1)|}e^{\pm i\frac{\pi}{2}}$$

$\Rightarrow\ \frac{z-(2-i)}{-1+2i}=\pm\frac{i}{2}$

$\Rightarrow\ 2z=(-i-2)+(4-2i)$ or $(i+2)+(4-2i)$

$\Rightarrow\ z=1-\frac{3}{2}i$ or $3-\frac{i}{2}$.

5. Let z_1, z_2, z_3 be the vertices A, B and C, respectively, of equilateral ΔABC, inscribed in a circle $|z|=2$ with centre (0, 0) and radius = 2. Given $z_1=1+i\sqrt{3}$.

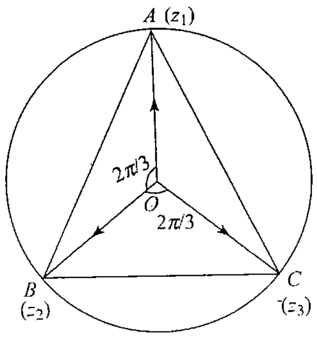


Fig. 2.107

Rotating OA about O by an angle $2\pi/3$, we have

$$\frac{z-0}{1+i\sqrt{3}-0}=\frac{|z-0|}{|1+i\sqrt{3}-0|}e^{\pm i\frac{2\pi}{3}}$$

$\Rightarrow\ z=(1+i\sqrt{3})\left(\cos\frac{2\pi}{3}\pm i\sin\frac{2\pi}{3}\right)$

$\Rightarrow\ z=(1+i\sqrt{3})\left(-\frac{1}{2}\pm i\frac{\sqrt{3}}{2}\right)$

$\Rightarrow\ z=-\frac{(1+i\sqrt{3})^2}{2}$ or $-\frac{(1+i\sqrt{3})(1-i\sqrt{3})}{2}$

$\Rightarrow\ z=1-i\sqrt{3}$ or -2

6. $S=1(2-\omega)(2-\omega^2)+2(3-\omega)(3-\omega^2)+\cdots+(n-1)(n-\omega)(n-\omega^2)$

Here,

$T_n=(n-1)(n-\omega)(n-\omega^2)$

$=n^3-1$

$$S=\sum_{n=2}^{n}(n^3-1)$$

$$= \sum_{n=1}^{n}\left(n^3 - 1\right)$$

$$= \left[\left(\frac{n(n+1)}{2}\right)^2 - n\right]$$

$$= \frac{n^2(n^2 + 2n + 1) - 4n}{4}$$

$$= \frac{1}{4}n(n^3 + 2n^2 + n - 4)$$

$$= \frac{1}{2}n[n-1]\left[n^2 + 3n + 4\right]$$

True or false

1. Let $z = x + iy$. Then

$1 \cap z \Rightarrow 1 \le x$ and $0 \le y$ (by definition)

$$\frac{1-z}{1+z} = \frac{1-(x+iy)}{1+(x+iy)}$$

$$= \frac{(1-x)-iy}{(1+x)+iy} \times \frac{(1+x)-iy}{(1+x)-iy}$$

$$= \frac{1-x^2-y^2}{(1+x)^2+y^2} - \frac{iy(1-x+1+x)}{(1+x)^2+y^2}$$

$$= \frac{1-x^2-y^2}{(1+x)^2+y^2} - \frac{2iy}{(1+x)^2+y^2}$$

Now,

$$\frac{1-z}{1+z} \le 0 \Rightarrow \frac{1-x^2-y^2}{(1+x)^2+y^2} \le 0 \text{ and } \frac{-2y}{(1+x)^2+y^2} \le 0$$

$\Rightarrow$ $1 - x^2 - y^2 \le 0$ and $-2y \le 0$

$\Rightarrow$ $x^2 + y^2 \ge 1$ and $y \ge 0$

which is true as $x > 1$ and $y > 0$. Therefore, the given statement is true, $\forall z \in C$.

2. As $|z_1| = |z_2| = |z_3|$, therefore, z_1, z_2, z_3 are equidistant from origin.

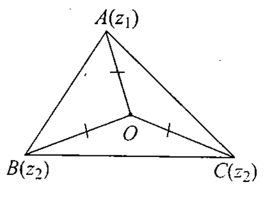


Fig. 2.108

Hence O is circumcentre of ΔABC. But according to the question, ΔABC is equilateral and we know that in an equilateral triangle circumcnetre and centroid coincide. Hence, centroid of ΔABC is 0. Hence,

$$\frac{z_1 + z_2 + z_3}{3} = 0$$

$\Rightarrow$ $z_1 + z_2 + z_3 = 0$

Therefore, the statement is true.

3. If z_1, z_2, z_3 are in A.P., then $(z_1 + z_3)/2 = z_2$. So, z_2 is mid-point of line joining z_1 and z_3. Hence, z_1, z_2, z_3 lie on a straight line. Hence, given statement is false.

4. See the theory of cube roots of unity. The given statement is true.

Multiple choice questions with one correct answer

1. b. $\left(\frac{x-1}{-2}\right)^3 = 1$

$\Rightarrow$ $\frac{x-1}{-2} = 1, \omega, \omega^2$

$\Rightarrow$ $x = -1, 1 - 2\omega, 1 - 2\omega^2$

2. d. $\frac{1+i}{1-i} = \frac{(1+i)^2}{(1-i)(1+i)} = \frac{1-1+2i}{2} = i$

Now $i^n = 1$. Hence, the smallest positive integral value of n should be 4.

3. a. We know that $|z - z_1| = |z - z_2|$. Then locus of z is the line, which is a perpendicular bisector of line segment joining z_1 and z_2. Hence,

$z = x + iy$

$\Rightarrow$ $|z - 5i| = |z + 5i|$

Therefore, z remains equidistant from $z_1 = 5i$ and $z_2 = -5i$. Hence, z lies on perpendicular bisector of line segment joining z_1 and z_2, which is clearly the real axis or $y = 0$.

Alternative solution:

$$\left|\frac{z-5i}{z+5i}\right| = 1$$

$\Rightarrow$ $|x + iy - 5i| = |x + iy + 5i|$

$\Rightarrow$ $|x + (y-5)i| = |x + (y+5)i|$

$\Rightarrow$ $x^2 + (y-5)^2 = x^2 + (y+5)^2$

$\Rightarrow$ $x^2 + y^2 - 10y + 25 = x^2 + y^2 + 10y + 25$

$\Rightarrow$ $20y = 0$

$\Rightarrow$ $y = 0$

4. b. $z = \left(\frac{\sqrt{3}}{2} + \frac{i}{2}\right)^5 + \left(\frac{\sqrt{3}}{2} - \frac{i}{2}\right)^5$

$$= \left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)^5 + \left(\cos\frac{\pi}{6} - i\sin\frac{\pi}{6}\right)^5$$

$$= \left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right) + \left(\cos\frac{5\pi}{6} - i\sin\frac{5\pi}{6}\right)$$

$$= 2\cos\frac{5\pi}{6}$$

$$= -\sqrt{3}$$

$\Rightarrow$ $\text{Re}(z) < 0$ and $\text{Im}(z) = 0$

Alternative solution:

$z = \bar{z}_1 + \bar{z}_1$

where

$$\left(\frac{\sqrt{3}}{2} + \frac{i}{2}\right)^5$$

$\Rightarrow$ z is real

$\Rightarrow$ $\text{Im}(z) = 0$

5. d. $|z-4|<|z-2|$

$\Rightarrow |(x-4)+iy|<|(x-2)+iy|$

$\Rightarrow (x-4)^2+y^2<(x-2)^2+y^2$

$\Rightarrow -8x+16<-4x+4$

$\Rightarrow 4x-12>0$

$\Rightarrow x>3$

$\Rightarrow \text{Re}(z)>3$

6. b. $|\omega|=1$

$\Rightarrow \left|\dfrac{1-iz}{z-i}\right|=1$

$\Rightarrow |1-iz|=|z-i|$

$\Rightarrow |-i|\,|z+i|=|z-i|$

$\Rightarrow |z+i|=|z-i|$

Hence, z is equidistant from (0, –1) and (0, 1). So, z lies on perpendicular bisector of (0, –1) and (0, 1). i.e., x-axis, and $y=0$. Therefore, z lies on real axis.

7. b. If vertices of a parallelogram are z_1, z_2, z_3, z_4, then as diagonals bisect each other as given,

$$\frac{z_1+z_3}{2}=\frac{z_2+z_4}{2}$$

$\Rightarrow z_1+z_3=z_2+z_4$

8. d. Let $z_1=\sin x+i\cos 2x$; $z_2=\cos x-i\sin 2x$. Then

$\bar{z}_1=z_2$

$\Rightarrow \sin x-i\cos 2x=\cos x-i\sin 2x$

$\sin x=\cos x$ and $\cos 2x=\sin 2x$

$\Rightarrow \tan x=1$ and $\tan 2x=1$

$\Rightarrow x=\dfrac{\pi}{4}$ and $x=\dfrac{\pi}{8}$

which is not possible. Hence, there is no value of x.

9. b. $(1+\omega)^7=A+B\omega$

$\Rightarrow (-\omega^2)^7=A+B\omega \quad (\because 1+\omega+\omega^2=0)$

$\Rightarrow -\omega^{14}=A+B\omega$

$\Rightarrow -\omega^2=A+B\omega \quad (\because \omega^3=1)$

$\Rightarrow 1+\omega=A+B\omega$

$\Rightarrow A=1, B=1$

10. d. We have,

$|z|=|\omega|$ and $\arg z=\pi-\arg\omega$

Let $\omega=re^{i\theta}$. Then

$z=re^{i(\pi-\theta)}$

$\Rightarrow z=re^{i\pi}e^{-i\theta}=(re^{-i\theta})(\cos\pi+i\sin\pi)$

$=\bar{\omega}(-1)=-\bar{\omega}$

11. c. We have,

$2=|z-i\omega|\le|z|+|\omega| \quad (\because |z_1+z_2|\le|z_1|+|z_2|)$

$\therefore |z|+|\omega|\ge 2$ (i)

But given that $|z|\le 1$ and $|\omega|\le 1$. Hence,

$\Rightarrow |z|+|\omega|\le 2$ (ii)

From (i) and (ii),

$|z|=|\omega|=1$

Also,

$|z+i\omega|=|z-i\bar{\omega}|$

$\Rightarrow |z-(-i\omega)|=|z-i\bar{\omega}|$

Hence, z lies on perpendicular bisector of the line segment joining $(-i\omega)$ and $(i\bar{\omega})$, which is a real axis, as $(-i\omega)$ and $(i\bar{\omega})$ are conjugate to each other. For z, $\text{Im}(z)=0$. If $z=x$, then

$|z|\le 1\Rightarrow x^2\le 1$

$\Rightarrow -1\le x\le 1$

12. d. $(1+i)^{n_1}+(1+i^3)^{n_1}+(1+i^5)^{n_2}+(1+i^7)^{n_2}$

$=\left[(1+i)^{n_1}+(1-i)^{n_1}\right]+\left[(1+i)^{n_2}+(1-i)^{n_2}\right]$

$=\left[(1+i)^{n_1}+\overline{(1+i)^{n_1}}\right]+\left[(1+i)^{n_2}+\overline{(1+i)^{n_2}}\right]$

$=$ [purely real number] + [purely real number]

Hence, n_1 and n_2 are any integers.

13. c. $E=4+5(\omega)^{334}+3(\omega)^{365}$

$=4+5\omega+3\omega^2$

$=1+2\omega+3(1+\omega+\omega^2)$

$=1+(-1+i\sqrt{3})$

$=i\sqrt{3}$

14. a. $\arg(-z)-\arg(z)=\arg\left(\dfrac{-z}{z}\right)=\arg(-1)=\pi$

15. a. $|z_1|=|z_2|=|z_3|$ (given)

Now,

$|z_1|=1\Rightarrow |z_1|^2=1\Rightarrow z_1\bar{z}_1=1$

Similarly,

$z_2\bar{z}_2=1, z_3\bar{z}_3=1$

Now,

$\left|\dfrac{1}{z_1}+\dfrac{1}{z_2}+\dfrac{1}{z_3}\right|=1$

$\Rightarrow |\bar{z}_1+\bar{z}_2+\bar{z}_3|=1$

$\Rightarrow \left|\overline{z_1+z_2+z_3}\right|=1$

$\Rightarrow |z_1+z_2+z_3|=1$

16. d. Let,

$z=(1)^{1/n}=(\cos 2k\pi+i\sin 2k\pi)^{1/n}$

$=\cos\dfrac{2k\pi}{n}+i\sin\dfrac{2k\pi}{n}$, $k=0, 1, 2, \ldots, n-1$

Let

$z_1=\cos\left(\dfrac{2k_1\pi}{n}\right)+i\sin\left(\dfrac{2k_1\pi}{n}\right)$

and

$z_2=\cos\left(\dfrac{2k_2\pi}{n}\right)+i\sin\left(\dfrac{2k_2\pi}{n}\right)$

be the two values of z such that they subtend angle of 90° at origin. Then,

$\Rightarrow \dfrac{2k_1\pi}{n}-\dfrac{2k_2\pi}{n}=\pm\dfrac{\pi}{2}\Rightarrow 4(k_1-k_2)=\pm n$

As k_1 and k_2 are integers and $k_1\ne k_2$, therefore $n=4m, m\in Z$.

17. c. $\dfrac{z_1-z_3}{z_2-z_3}=\dfrac{1-i\sqrt{3}}{2}$

$\Rightarrow \arg\left(\dfrac{z_1-z_3}{z_2-z_3}\right)=\arg\left(\dfrac{1-i\sqrt{3}}{2}\right)$

Hence, the angle between z_1-z_3 and z_2-z_3 is 60°. Also,

$\left|\dfrac{z_1-z_3}{z_2-z_3}\right|=\left|\dfrac{1-i\sqrt{3}}{2}\right|$

$$\Rightarrow \quad \left|\frac{z_1 - z_3}{z_2 - z_3}\right| = 1$$

$\Rightarrow \quad |z_1 - z_3| = |z_2 - z_3|$

Hence, the triangle with vertices z_1, z_2 and z_3 is isosceles with vertical angle 60°. Hence rest of the two angles should also be 60° each. Therefore, the required triangle is an equilateral triangle.

18. b. $|z_1| = 12$. Therefore, z_1 lies on a circle with centre (0, 0) and radius 12 units. As $|z_2 - 3 - 4i| = 5$, so z_2 lies on a circle with centre (3, 4) and radius 5 units.

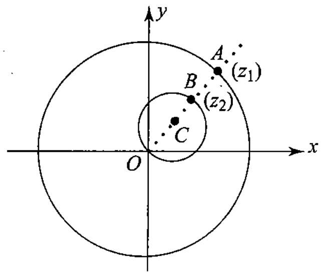


Fig. 2.109

From the above figure it is clear that $|z_1 - z_2|$, i.e., distance between z_1 and z_2 will be minimum when they lie at A and B, respectively, i.e., on diagram as shown. Then $|z_1 - z_2| = AB = OA - OB = 12 - 2(5) = 2$. As it is the minimum value, we must have $|z_1 - z_2| \geq 2$.

19. a. $\omega = \dfrac{z-1}{z+1}$

$\Rightarrow \quad z = \dfrac{1+\omega}{1-\omega}$

Now,

$|z| = 1$

$$\Rightarrow \quad \left|\frac{1+\omega}{1-\omega}\right| = 1$$

$\Rightarrow \quad |\omega + 1| = |\omega - 1|$

Therefore, ω is equidistant from (1, 0) and (–1, 0) and hence must lie on perpendicular bisector of line segment joining (1, 0) and (–1, 0), i.e., imaginary axis. Hence, ω is purely imaginary, i.e., $\text{Re}(\omega) = 0$.

20. b. $(1 + \omega^2)^n = (1 + \omega^4)^n$

$\Rightarrow \quad (-\omega)^n = (1 + \omega)^n = (-\omega^2)^n$

$\Rightarrow \quad \omega^n = 1$

Hence, the least positive value of n is 3.

21. a. Here we observe that

$PA = AQ = AR = 2$

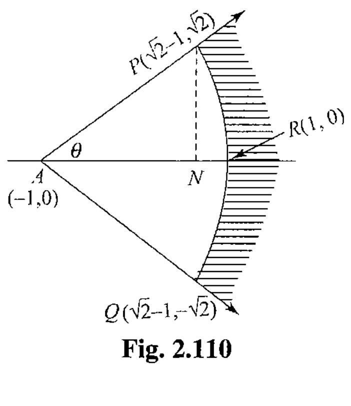


Fig. 2.110

Therefore, PRQ is an arc of a circle with centre at A and radius 2. Shaded region is outer (exterior) part of the sector $APRQA$.

Hence, for any point x on arc PRQ, we should have

$|z - (-1)| = 2$

and for shaded region,

$|z + 1| > 2$ (1)

Also,

$$\tan\theta = \frac{PN}{AN} = \frac{\sqrt{2}}{(\sqrt{2}-1)-(-1)} = \frac{\sqrt{2}}{\sqrt{2}} = 1$$

$\Rightarrow \quad \theta = \pi/4$

and by symmetry, $\arg(z + 1)$ varies from $-\pi/4$ to $\pi/4$ as it moves from Q to P on arc QRP. Hence, for shaded region, we also have

$-\pi/4 < \arg(z + 1) < \pi/4$

or

$|\arg(z + 1)| < \pi/4$ (2)

Combining (i) and (ii), we find that (a) is the correct option.

22. b. Given that a, b, c are integers not all equal, ω is cube root of unity $\neq 1$. Then

$$|a + b\omega + c\omega^2| = \left|a + b\left(\frac{-1+i\sqrt{3}}{2}\right) + c\left(\frac{-1-i\sqrt{3}}{2}\right)\right|$$

$$= \left|\left(\frac{2a-b-c}{2}\right) + i\left(\frac{b\sqrt{3}-c\sqrt{3}}{2}\right)\right|$$

$$= \frac{1}{2}\sqrt{(2a-b-c)^2 + 3(b-c)^2}$$

$$= \frac{1}{2}\sqrt{4a^2 + b^2 + c^2 - 4ab + 2bc - 4ac + 3b^2 + 3c^2 - 6bc}$$

$$= \sqrt{a^2 + b^2 + c^2 - ab - bc - ca}$$

$$= \sqrt{\frac{1}{2}[(a-b)^2 + (b-c)^2 + (c-a)^2]}$$

R.H.S. will be minmum when $a = b = c$, but we cannot take $a = b = c$ as per the question. Hence, the minimum value is obtained when any two are zero and third is a minimum magnitude integer, i.e., 1. Thus $b = c = 0$, $a = 1$ gives us the minimum value of 1.

23. b. If a, b, c and u, v, w are complex numbers representing the vertices of two triangles such that they are similar, then

$$\begin{vmatrix} a & u & 1 \\ b & v & 1 \\ c & w & 1 \end{vmatrix} = 0$$

or

$$\frac{a-c}{a-b} = \frac{u-w}{u-v} = r$$

24. c. Let $z_1 = |z_1| (\cos\theta_1 + i\sin\theta_1)$ and $z_2 = |z_2| (\cos\theta_2 + i\sin\theta_2)$.

Also,

$|z_1 + z_2| = |z_1| + |z_2|$

$\Rightarrow \quad |z_1 + z_2|^2 = (|z_1| + |z_2|)^2$

$\Rightarrow \quad |z_1|^2 + |z_2|^2 + 2\text{Re}(z_1\bar{z}_2) = |z_1|^2 + |z_2|^2 + 2|z_1||z_2|$

$\Rightarrow \quad \text{Re}(z_1\bar{z}_2) = 2|z_1||z_2|$

$\Rightarrow \quad 2|z_1||z_2|\cos(\theta_1 - \theta_2) = 2|z_1||z_2|$

$\Longrightarrow \cos(\theta_1 - \theta_2) = 1$

$\Longrightarrow \theta_1 - \theta_2 = 0$

$\Longrightarrow \arg z_1 - \arg z_2 = 0$

25. d. Let $z = \cos(2\pi/7) + i\sin(2\pi/7)$. Then by De Moivre's theorem, we have

$$z^k = \cos\frac{2\pi k}{7} + i\sin\frac{2\pi k}{7}$$

Now,

$$\sum_{k=1}^{6}\left(\sin\frac{2\pi k}{7} - i\cos\frac{2\pi k}{7}\right) = \sum_{k=1}^{6}(-i)\left(\cos\frac{2\pi k}{7} + i\sin\frac{2k\pi}{7}\right)$$

$$= (-i)\sum_{k=1}^{6} z^k$$

$$= -i\,\frac{z(1-z^6)}{1-z}$$

$$= -i\left(\frac{z-z^7}{1-z}\right)$$

$$= (-i)\left(\frac{z-1}{1-z}\right)$$

[Using $z^7 = \cos 2\pi + i\sin 2\pi = 1$]

$$= (-i)\left(\frac{1-z}{1-z}\right)$$

$$= i$$

26. d. We have,

$$(1+\omega-\omega^2)^7 = (-\omega^2-\omega^2)^7$$
$$= (-2)^7(\omega^2)^7$$
$$= -128\omega^{14}$$
$$= -128\omega^2$$

27. b. $\displaystyle\sum_{i=1}^{13}\left(i^n + i^{n+1}\right) = \sum_{i=1}^{13} i^n(1+i)$

$$= (1+i)\sum_{i=1}^{13} i^n$$

$$= i(1+i)\,\frac{\left(1-i^{13}\right)}{1-i}$$

$$= i - 1 \text{ as } i^{13} = i$$

28. d. Taking $-3i$ common from C_2, we get

$$-3i\begin{vmatrix} 6i & 1 & 1 \\ 4 & -1 & -1 \\ 20 & i & i \end{vmatrix} = 0 \qquad (\because C_2 \equiv C_3)$$

$\Rightarrow x = 0, y = 0$

29. b. Operating $R_1 \to R_1 + R_2 + R_3$, we get

$$\begin{vmatrix} 3 & 0 & 0 \\ 1 & -1-\omega^2 & \omega^2 \\ 1 & \omega^2 & \omega^4 \end{vmatrix} = 3[-\omega^4 - \omega^6 - \omega^4]$$

$$= 3(-1-2\omega)$$
$$= 3(\omega^2 - \omega)$$
$$= 3\omega(\omega - 1)$$

30. d. Since $(w - \bar{w}z)/(1-z)$ is purely real, therefore

$$\overline{\left(\frac{w-\bar{w}z}{1-z}\right)} = \left(\frac{w-\bar{w}z}{1-z}\right)$$

$$\Rightarrow \frac{\bar{w}-w\bar{z}}{1-\bar{z}} = \frac{w-\bar{w}z}{1-z}$$

$$\Rightarrow \bar{w} - \bar{w}z - w\bar{z} + wz\bar{z} = w - w\bar{z} - \bar{w}z + \bar{w}z\bar{z}$$

$$\Rightarrow w - \bar{w} = (w - \bar{w})|z|^2$$

$$\Rightarrow |z|^2 = 1 \qquad (\because w = \alpha + i\beta \text{ and } \beta \neq 0)$$

$$\Rightarrow |z| = 1$$

Also given $z \neq 1$. Therefore, the required set is $\{z : |z| = 1, z \neq 1\}$.

31. d. $\overline{OP} = \overline{OA} + \overline{AP}$

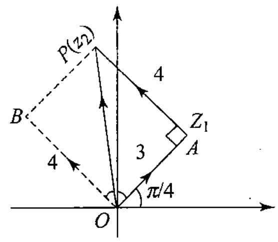


Fig. 2.111

Rotating OA by an angle 45° in anticlockwise direction to get OP, we have

$$\frac{z_2 - 0}{z_1 - 0} = \frac{|z_2|}{|z_1|}e^{i\theta} \text{ (where } \tan\theta = 4/3)$$

$$\Rightarrow \frac{z_2 - 0}{3e^{i\pi/4}} = \frac{5}{3}(\cos\theta + i\sin\theta)$$

$$\Rightarrow \frac{z_2 - 0}{e^{i\pi/4}} = 5\left(\frac{3}{5} + i\frac{4}{5}\right)$$

$$\Rightarrow z_2 = (3+4i)e^{i\pi/4}$$

32. d. Given $|z| = 1$ and $z \neq \pm 1$. To find locus of $\omega = z/(1-z^2)$. We have,

$$\omega = \frac{z}{1-z^2} = \frac{z}{z\bar{z} - z^2} \qquad (\because |z| = 1 \Rightarrow |z|^2 = 1 \Rightarrow z\bar{z} = 1)$$

$$= \frac{1}{\bar{z} - z}$$

which is a purely imaginary number. Therfore, ω must lie on y-axis.

33. d.

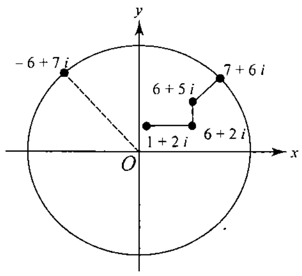


Fig. 2.112

$z_0 \equiv (1 + 2i)$

$z_1 \equiv (6 + 5i)$

$z_2 \equiv (-6 + 7i)$

34. a. $z\bar{z}(\bar{z}^2 + z^2) = 350$

Puting $z = x + iy$, we have

$(x^2 + y^2)(x^2 - y^2) = 175$

$(x^2 + y^2)(x^2 - y^2) = 5 \times 5 \times 7$

$x^2 + y^2 = 25$

and

$x^2 - y^2 = 7$

(as other combinations give non-integral values of x and y)

$\therefore \quad x = \pm 4, y = \pm 3 \ (x, y \in I)$

Hence, area is $8 \times 6 = 48$ sq. units.

Multiple choice questions with one or more than one correct answer

1. a, b, c.

We have,

$|z_1| = |z_2| = 1 \Rightarrow a^2 + b^2 = c^2 + d^2 = 1 \qquad (1)$

and

$\text{Re}(z_1 \bar{z}_2) = 0 \Rightarrow \text{Re}\{(a + ib)(c - id)\} = 0 \Rightarrow ac + bd = 0 \qquad (2)$

Now from (1) and (2),

$$a^2 + b^2 = 1 \Rightarrow a^2 + \frac{a^2c^2}{d^2} = 1 \Rightarrow a^2 = d^2 \qquad (3)$$

Also,

$$c^2 + d^2 = 1 \Rightarrow c^2 + \frac{a^2c^2}{b^2} = 1 \Rightarrow b^2 = c^2 \qquad (4)$$

$|\omega_1| = \sqrt{a^2 + c^2} = \sqrt{a^2 + b^2} = 1$ [From (1) and (4)]

and

$|\omega_2| = \sqrt{b^2 + d^2} = \sqrt{a^2 + b^2} = 1$ [From (1) and (4)]

Further,

$\text{Re}(\omega_1\bar{\omega}_2) = \text{Re}\{(a + ic)(b - id)\}$

$= ab + cd$

$= ab + \left(-\dfrac{ac^2}{b}\right)$ [From (2)]

$= \dfrac{ab^2 - ac^2}{b} = 0$ [From (4)]

Also,

$$\text{Im}(\omega_1\bar{\omega}_2)\ bc - ad = bc - a\left(-\frac{ac}{b}\right) = \frac{(a^2 + b^2)c}{b} = \frac{c}{b} = \pm 1 \neq 0$$

$\therefore \quad |\omega_1| = 1, |\omega_2| = 1$ and $\text{Re}(\omega_1\bar{\omega}_2) = 0$

2. a, d. Let $z_1 = a + ib$, $a > 0$ and $b \in R$; $z_2 = c + id$, $d < 0$, $c \in R$.

Given,

$|z_1| = |z_2|$

$\Rightarrow \quad a^2 + b^2 = c^2 + d^2$

$\Rightarrow \quad a^2 - c^2 = d^2 - b^2 \qquad (1)$

Now,

$$\frac{z_1 + z_2}{z_1 - z_2} = \frac{(a + c) + i(b + d)}{(a - c) + i(b - d)}$$

$$= \frac{[(a^2 - c^2) + (b^2 - d^2)] + i[(a - c)(b + d) - (a + c)(b - d)]}{(a - c)^2 + (b - d)^2}$$

which is a purely imaginary number or zero in case $a + c = b + d = 0$.

3. a, c, d.

Given $z = (1 - t)z_1 + tz_2$

$\Rightarrow \quad z = \dfrac{(1 - t)z_1 + tz_2}{(1 - t) + t}$

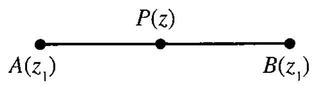
$\Rightarrow$ z divides the line segment joining z_1 and z_2 in ration $(1 - t) : t$ internally as $0 < t < 1$

$\Rightarrow$ z, z_1, and z_2 are collinear.

$\Rightarrow$ $\arg(z - z_1) = \arg(z_2 - z_1)$

$\Rightarrow \quad \dfrac{z - z_1}{z_2 - z_1} = \dfrac{\bar{z} - \bar{z}_1}{\bar{z}_2 - \bar{z}_1}$

$\Rightarrow \quad \begin{vmatrix} z - z_1 & \bar{z} - \bar{z}_1 \\ z_2 - z_1 & \bar{z}_2 - \bar{z}_1 \end{vmatrix} = 0$



AP + PB = AB

$\Rightarrow \quad |z - z_1| + |z - z_2| = |z_1 - z_2|$

Comprehension

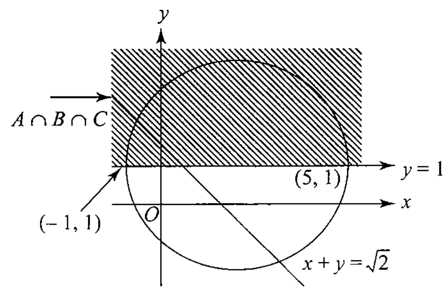


Fig. 2.113

1. b. A is the set of points on and above the line $y = 1$ in the Argand plane. B is the set of points on the circle $(x - 2)^2 + (y - 1)^2 = 9$ and

$C = \text{Re}(1 - i)z = \text{Re}((1 - i)(x + iy))$

$\Rightarrow \quad x + y = \sqrt{2}$

Hence $A \cap B \cap C$ has only one point of intersection.

2. c. The points $(-1, 1)$ and $(5, 1)$ are the extremities of a diameter of the given circle. Hence,

$|z + 1 - i|^2 + |z - 5 - i|^2 = 36$

3. d. $||z| - |w|| < |z - w|$ and $|z - w|$ is the distance between z and w. Here, z is fixed. Hence distance between z and w would be maximum for diametrically opposite points. Therefore,

$|z - w| < 6$

$\Rightarrow \quad -6 < |z| - |w| < 6$

$\Rightarrow \quad -3 < |z| - |w| + 3 < 9$

Matrix-match type

a→q

$$\left|\frac{z}{|z|} - i\right| = \left|\frac{z}{|z|} + i\right|, z \neq 0$$

$\frac{z}{|z|}$ is unimodular complex number

and lies on perpendicular bisector of i and $-i$

$\Rightarrow \frac{z}{|z|} = \pm 1 \Rightarrow z = \pm 1 |z|$

$\Rightarrow z$ is real number $\Rightarrow \text{Im}(z) = 0$.

b→p

$|z+4| + |z-4| = 10$

z lies on an ellipse whose focus are (4,0) and (–4,0) and length of major axis is 10

$\Rightarrow 2ae = 8$ and $2a = 10 \Rightarrow e = 4/5$

$|\text{Re}(z)| \le 5$.

c→p, t

$|\omega| = 2 \Rightarrow w = 2(\cos\theta + i\sin\theta)$

$\Rightarrow z = x + iy = 2(\cos\theta + i\sin\theta) - \frac{1}{2}(\cos\theta - i\sin\theta)$

$= \frac{3}{2}\cos\theta + i\frac{5}{2}\sin\theta \Rightarrow \frac{x^2}{(3/2)^2} + \frac{y^2}{(5/2)^2} = 1$

$\Rightarrow e^2 = 1 - \frac{9/4}{25/4} = 1 - \frac{9}{25} = \frac{16}{25} \Rightarrow e = \frac{4}{5}$

d→q, t

$|\omega| = 1 \Rightarrow x + iy = \cos\theta + i\sin\theta + \cos\theta - i\sin\theta$

$x + iy = 2\cos\theta$

$|\text{Re}(z)| \le 1$, $\text{Im}(z) = 0$.

Integer type

1. (1)

$\omega = e^{i2\pi/3}$

$$\begin{vmatrix} z+1 & \omega & \omega^2 \\ \omega & z+\omega^2 & 1 \\ \omega^2 & 1 & z+\omega \end{vmatrix} = 0$$

Applying $(C_1 \to C_1 + C_2 + C_3)$

$$\Rightarrow z\begin{vmatrix} 1 & \omega & \omega^2 \\ 1 & z+\omega^2 & 1 \\ 1 & 1 & z+\omega \end{vmatrix} = 0$$

$\Rightarrow z^3 = 0$

$z = 0$ is only solution.

2.(5) $|z - 3 - 2i| \le 2$

$\Rightarrow z$ lies on or inside the circle radius 2 and centre (3, 2)

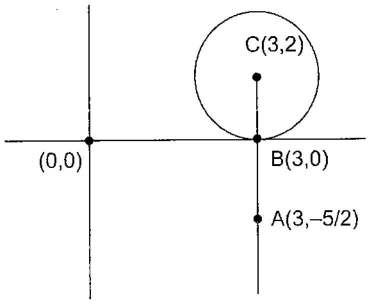


Fig. 2.114

$|2z - 6 + 5i|_{\text{min.}}$

$= 2|z - 3 + (5/2)i|_{\text{min.}}$

= 2(minimum distance of any point on the circle to the point (3, – 5/2)

= 2(5/2) = 5

3.(0) The expression may not attain integral value for all a, b, c

If we consider $a = b = c$, then

$y = a(1 + \omega + \omega^2) = a(1 + i\sqrt{3})$

$z = a(1 + \omega^2 + \omega) = a(1 + i\sqrt{3})$

$\therefore |x|^2 + |y|^2 + |z|^2 = 9|a|^2 + 4|a|^2 + 4|a|^2 = 17|a|^2$

$\therefore \frac{|x|^2 + |y|^2 + |z|^2}{|a|^2 + |b|^2 + |c|^2} = \frac{17}{3}$

Note: *However if $\omega = e^{i(2\pi/3)}$, then the value of the expression = 3.*