PART FOUR

OSCILLATIONS AND WAVES

4.1 MECHANICAL OSCILLATIONS

4.1 (a) Given, $x = a\cos\left(\omega t - \frac{\pi}{4}\right)$

So, $v_x = \dot{x} = -a\,\omega\sin\left(\omega t - \frac{\pi}{4}\right)$ and $w_x = \ddot{x} = -a\,\omega^2\cos\left(\omega t - \frac{\pi}{4}\right)$ (1)

On-the basis of obtained expressions plots $x(t)$, $v_x(t)$ and $w_x(t)$ can be drawn as shown in the answersheet, (of the problem book).

(b) From Eqn (1)

$$v_x = -a\,\omega\sin\left(\omega t - \frac{\pi}{4}\right) \quad \text{So,} \quad v_x^2 = a^2\omega^2\sin^2\left(\omega t - \frac{\pi}{4}\right) \tag{2}$$

But from the law $x = a\cos(\omega t - \pi/4)$, so, $x^2 = a^2\cos^2(\omega t - \pi/4)$

or, $$\cos^2(\omega t - \pi/4) = x^2/a^2 \quad \text{or} \quad \sin^2(\omega t - \pi/4) = 1 - \frac{x^2}{a^2} \tag{3}$$

Using (3) in (2),

$$v_x^2 = a^2\omega^2\left(1 - \frac{x^2}{a^2}\right) \quad \text{or} \quad v_x^2 = \omega^2(a^2 - x^2) \tag{4}$$

Again from Eqn (4), $w_x = -a\,\omega^2\cos(\omega t - \pi/4) = -\omega^2 x$

4.2 (a) From the motion law of the particle

$$x = a\sin^2(\omega t - \pi/4) = \frac{a}{2}\left[1 - \cos\left(2\,\omega t - \frac{\pi}{2}\right)\right]$$

or, $$x - \frac{a}{2} = -\frac{a}{2}\cos\left(2\,\omega t - \frac{\pi}{2}\right) = -\frac{a}{2}\sin 2\,\omega t = \frac{a}{2}\sin(2\,\omega t + \pi)$$

i.e. $$x - \frac{a}{2} = \frac{a}{2}\sin(2\,\omega t + \pi). \tag{1}$$

Now compairing this equation with the general equation of harmonic oscillations : $X = A\sin(\omega_0 t + \alpha)$

Amplitude, $A = \frac{a}{2}$ and angular frequency, $\omega_0 = 2\,\omega$.

Thus the period of one full oscillation, $T = \frac{2\pi}{\omega_0} = \frac{\pi}{\omega}$

(b) Differentiating Eqn (1) w.r.t. time

$v_x = a\,\omega \cos(2\,\omega\, t + \pi)$ or $v_x^2 = a^2\,\omega^2 \cos^2(2\,\omega\, t + \pi) = a^2\,\omega^2\left[1 - \sin^2(2\,\omega\, t + \pi)\right]$ (2)

From Eqn (1) $\left(x - \frac{a}{2}\right)^2 = \frac{a^2}{4}\sin^2(2\,\omega\, t + \pi)$

or, $4\frac{x^2}{a^2} + 1 - \frac{4x}{a} = \sin^2(2\,\omega\, t + \pi)$ or $1 - \sin^2(2\,\omega\, t + \pi) = \frac{4\,x}{a}\left(1 - \frac{x}{a}\right)$ (3)

From Eqns (2) and (3), $v_x = a^2\,\omega^2\,\frac{4\,x}{a}\left(1 - \frac{x}{a}\right) = 4\,\omega^2\,x\,(a - x)$

Plot of $v_x(x)$ is as shown in the answersheet.

4.3 Let the general equation of S.H.M. be

$$x = a\cos(\omega\, t + \alpha) \qquad (1)$$

So,

$$v_x = -a\,\omega \sin(\omega\, t + \alpha) \qquad (2)$$

Let us assume that at $t = 0$, $x = x_0$ and $v_x = v_{x_0}$.

Thus from Eqns (1) and (2) for $t = 0$, $x_0 = a\cos\alpha$, and $v_{x_0} = -a\,\omega\sin\alpha$

Therefore $\tan\alpha = -\frac{v_{x_0}}{\omega\, x_0}$ and $a = \sqrt{x_0^2 + \left(\frac{v_{x_0}}{\omega}\right)^2} = 35.35$ cm

Under our assumption Eqns (1) and (2) give the sought x and v_x if

$$t = t = 2{\cdot}40\text{ s},\ a = \sqrt{x_0^2 + \left(v_{x_0}/\omega\right)^2} \text{ and } \alpha = \tan^{-1}\left(-\frac{v_x}{\omega\, x_0}\right) = -\frac{\pi}{4}$$

Putting all the given numerical values, we get :

$$x = -29 \text{ cm} \quad \text{and} \quad v_x = -81 \text{ cm/s}$$

4.4 From the Eqn, $v_x^2 = \omega^2(a^2 - x^2)$ (see Eqn. 4 of 4.1)

$$v_1^2 = \omega^2(a^2 - x_1^2) \quad \text{and} \quad v_2^2 = \omega^2(a^2 - x_2^2)$$

Solving these Eqns simultaneously, we get

$$\omega = \sqrt{(v_1^2 - v_2^2)/(x_2^2 - x_1^2)}\ ,\quad a = \sqrt{(v_1 x_2^2 - v_2^2 x_1^2)/(v_1^2 - v_2^2)}$$

4.5 (a) When a particle starts from an extreme position, it is useful to write the motion law as

$$x = a\cos\omega\, t \qquad (1)$$

(However x is the displacement from the equlibrium position)

It t_1 be the time to cover the distence $a/2$ then from (1)

$$a - \frac{a}{2} = \frac{a}{2} = a\cos\omega\, t_1 \quad \text{or} \quad \cos\omega\, t_1 = \frac{1}{2} = \cos\frac{\pi}{3}\ (\text{as } t_1 < T/4)$$

Thus

$$t_1 = \frac{\pi}{3\,\omega} = \frac{\pi}{3\,(2\,\pi/T)} = \frac{T}{6}$$

As $x = a\cos\omega t$, so, $v_x = -a\omega\sin\omega t$

Thus $v = |v_x| = -v_x = a\omega\sin\omega t$, for $t \le t_1 = T/6$

Hence sought mean velocity

$$<v> = \frac{\int v\,dt}{\int dt} = \int_0^{T/6} a\,(2\pi/T)\sin\omega t\,dt \Big/ T/6 = \frac{3a}{T} = 0{\cdot}5\text{ m/s}$$

(b) In this case, it is easier to write the motion law in the form :

$$x = a\sin\omega t \tag{2}$$

If t_2 be the time to cover the distance $a/2$, then from Eqn (2)

$$a/2 = a\sin\frac{2\pi}{T}t_2 \quad \text{or} \quad \sin\frac{2\pi}{T}t_2 = \frac{1}{2} = \sin\frac{\pi}{6} \quad (\text{as } t_2 < T/4)$$

Thus $$\frac{2\pi}{T}t_2 = \frac{\pi}{6} \quad \text{or}, \ t_2 = \frac{T}{12}$$

Differentiating Eqn (2) w.r.t time, we get

$$v_x = a\omega\cos\omega t = a\frac{2\pi}{T}\cos\frac{2\pi}{T}t$$

So, $$v = |v_x| = a\frac{2\pi}{T}\cos\frac{2\pi}{T}t, \ \text{for} \ t \le t_2 = T/12$$

Hence the sought mean velocity

$$<v> = \frac{\int v\,dt}{\int dt} = \frac{1}{(T/12)}\int_0^{T/12} a\frac{2\pi}{T}\cos\frac{2\pi}{T}t\,dt = \frac{6a}{T} = 1\text{ m/s}$$

4.6 (a) As $x = a\sin\omega t$ so, $v_x = a\omega\cos\omega t$

Thus $$<v_x> = \int v_x\,dt \Big/ \int dt = \frac{\int_0^{\frac{3}{8}T} a\omega\cos(2\pi/T)t\,dt}{\frac{3}{8}T} = \frac{2\sqrt{2}\,a\omega}{3\pi} \left(\text{using } T = \frac{2\pi}{\omega}\right)$$

(b) In accordance with the problem

$$\vec{v} = v_x\vec{i}, \ \text{so}, \ |<\vec{v}>| = |<v_x>|$$

Hence, using part (a), $$|<\vec{v}>| = \left|\frac{2\sqrt{2}\,a\omega}{3\pi}\right| = \frac{2\sqrt{2}\,a\omega}{3\pi}$$

(c) We have got, $v_x = a\omega\cos\omega t$

So, $$v = |v_x| = \begin{cases} a\omega\cos\omega t, & \text{for } t \le T/4 \\ -a\omega\cos\omega t, & \text{for } T/4 \le t \le \frac{3}{8}T \end{cases}$$

Hence, $$< v > = \frac{\int v\, dt}{\int dt} = \frac{\int_0^{T/4} a\,\omega \cos \omega\, t\, dt + \int_{T/4}^{3T/8} - a\,\omega \cos \omega\, t\, dt}{3\,T/8}$$

Using $\omega = 2\pi/T$, and on evaluating the integral we get

$$< v > = \frac{2\,(4 - \sqrt{2}\,)\, a\,\omega}{3\,\pi}$$

4.7 From the motion law, $x = a \cos \omega\, t$,, it is obvious that the time taken to cover the distance equal to the amplitude (a), starting from extreme position equals $T/4$.

Now one can write

$$t = n\frac{T}{4} + t_0\,, \quad \left(\text{where } \; t_0 < \frac{T}{4} \text{ and } n = 0\,,1\,,2\,,\ldots\right)$$

As the particle moves according to the law, $x = a \cos \omega\, t$,

so at $n = 1{,}3{,}5\ \ldots$ or for odd n values it passes through the mean positon and for even numbers of n it comes to an extreme position (if $t_0 = 0$).

Case (1) when n is an odd number :

In this case, from the equation

$x = \pm\, a \sin \omega\, t$, if the t is counted from $nT/4$ and the distance covered in the time interval to becomes,$s_1 = a \sin \omega\, t_0 = a \sin \omega \left(t - n\frac{T}{4}\right) = a \sin\left(\omega\, t - \frac{n\,\pi}{2}\right)$

Thus the sought distance covered for odd n is

$$s = n\,a + s_1 = n\,a + a \sin\left(\omega\, t - \frac{n\,\pi}{2}\right) = a\left[n + \sin\left(\omega\, t - \frac{n\,\pi}{2}\right)\right]$$

Case (2), when n is even, In this case from the equation

$x = a \cos \omega\, t$, the distance covered (s_2) in the interval t_0, is given by

$$a - s_2 = a \cos \omega\, t_0 = a \cos \omega\left(t - n\frac{T}{4}\right) = a \cos\left(\omega\, t - n\frac{\pi}{2}\right)$$

or, $$s_2 = a\left[1 - \cos\left(\omega\, t - \frac{n\,\pi}{2}\right)\right]$$

Hence the sought distance for n is even

$$s = n\,a + s_2 = n\,a + a\left[1 - \cos\left(\omega\, t - \frac{n\,\pi}{2}\right)\right] = a\left[n + 1 - \cos\left(\omega\, t - \frac{n\,\pi}{2}\right)\right]$$

In general

$$s = \begin{cases} a\left[n + 1 - \cos\left(\omega\, t - \frac{n\,\pi}{2}\right)\right], & n \text{ is even} \\ a\left[n + \sin\left(\omega\, t - \frac{n\,\pi}{2}\right)\right], & n \text{ is odd} \end{cases}$$

4.8 Obviously the motion law is of the from, $x = a \sin \omega t$. and $v_x = \omega a \cos \omega t$.
Comparing $v_x = \omega a \cos \omega t$ with $v_x = 35 \cos \pi t$, we get

$$\omega = \pi, \ a = \frac{35}{\pi}, \text{thus } T = \frac{2\pi}{\omega} = 2 \text{ and } T/4 = 0{\cdot}5 \text{ s}$$

Now we can write

$$t = 2{\cdot}8 \text{ s} = 5 \times \frac{T}{4} + 0{\cdot}3 \quad \left(\text{where } \frac{T}{4} = 0{\cdot}5 \text{ s}\right)$$

As $n = 5$ is odd, like (4·7), we have to basically find the distance covered by the particle starting from the extreme position in the time interval 0·3 s.

Thus from the Eqn.

$$x = a \cos \omega t = \frac{35}{\pi} \cos \pi (0{\cdot}3)$$

$$\frac{35}{\pi} - s_1 = \frac{35}{\pi} \cos \pi (0{\cdot}3) \quad \text{or} \quad s_1 = \frac{35}{\pi} \{1 - \cos 0{\cdot}3\pi\}$$

Hence the sought distance

$$s = 5 \times \frac{35}{\pi} + \frac{35}{\pi} \{1 - \cos 0{\cdot}3\pi\}$$

$$= \frac{35}{\pi} \{6 - \cos 0{\cdot}3\pi\} = \frac{35}{22} \times 7 (6 - \cos 54°) = 60 \text{ cm}$$

4.9 As the motion is periodic the particle repeatedly passes through any given region in the range $-a \le x \le a$. The probability that it lies in the range $(x, x + dx)$ is defined as the fraction $\frac{\Delta t}{t}$ (as $t \to \infty$) where Δt is the time that the particle lies in the range $(x, x + dx)$ out of the total time t. Because of periodicity this is

$$dP = \frac{dP}{dx} dx = \frac{dt}{T} = \frac{2\, dx}{v T}$$

where the factor 2 is needed to take account of the fact that the particle is in the range $(x, x + dx)$ during both up and down phases of its motion. Now in a harmonic oscillator.

$$v = \dot{x} = \omega a \cos \omega t = \omega \sqrt{a^2 - x^2}$$

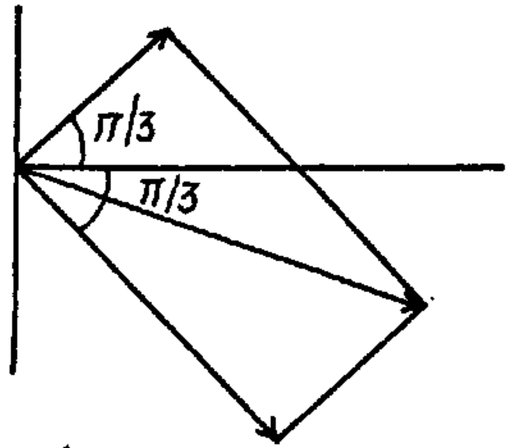
Thus since $\omega T = 2\pi$ (T is the time period)

We get $$dP = \frac{dP}{dx} dx = \frac{1}{\pi} \frac{dx}{\sqrt{a^2 - x^2}}$$

Note that $$\int_{-a}^{+a} \frac{dP}{dx} dx = 1$$

so $$\frac{dP}{dx} = \frac{1}{\pi} \frac{1}{\sqrt{a^2 - x^2}}$$ is properly normalized.

4.10 (a) We take a graph paper and choose an axis (X – axis) and an origin. Draw a vector of magnitude 3 inclined at an angle $\frac{\pi}{3}$ with the X-axis. Draw another vector of magnitude 8 inclined at an angle $-\frac{\pi}{3}$ (Since $\sin(\omega t + \pi/6) = \cos(\omega t - \pi/3)$) with the X – axis. The magnitude of the resultant of both these vectors (drawn from the origin) obtained using parallelogram law is the resultant, amplitude.



Clearly $\quad R^2 = 3^2 + 8^2 + 2 \cdot 3 \cdot 8 \cdot \cos\frac{2\pi}{3} = 9 + 64 - 48 \times \frac{1}{2}$

$$= 73 - 24 = 49$$

Thus $\quad R = 7$ units

(b) One can follow the same graphical method here but the result can be obtained more quickly by breaking into sines and cosines and adding :

Resultant $$x = \left(3 + \frac{5}{\sqrt{2}}\right)\cos\omega t + \left(6 - \frac{5}{\sqrt{2}}\right)\sin\omega t$$

$$= A\cos(\omega t + \alpha)$$

Then $$A^2 = \left(3 + \frac{5}{\sqrt{2}}\right)^2 + \left(6 - \frac{5}{\sqrt{2}}\right)^2 = 9 + 25 + \frac{30 - 60}{\sqrt{2}} + 36$$

$$= 70 - 15\sqrt{2} = 70 - 21 \cdot 2$$

So, $\quad A = 6 \cdot 985 \approx 7$ units

Note- In using graphical method convert all oscillations to either sines or cosines but do not use both.

4.11 Given, $x_1 = a\cos\omega t$ and $x_2 = a\cos 2\omega t$
so, the net displacement,

$$x = x_1 + x_2 = a\{\cos\omega t + \cos 2\omega t\} = a\{\cos\omega t + 2\cos^2\omega t - 1\}$$

and $$v_x = \dot{x} = a\{-\omega\sin\omega t - 4\omega\cos\omega t\sin\omega t\}$$

For $\dot{x}$ to be maximum,

$$\ddot{x} = a\omega^2\cos\omega t - 4a\omega^2\cos^2\omega t + 4a\omega^2\sin^2\omega t = 0$$

or, $\quad 8\cos^2\omega t + \cos\omega t - 4 = 0$, which is a quadratic equation for $\cos\omega t$.

Solving for acceptable value

$$\cos\omega t = 0 \cdot 644$$

thus $$\sin\omega t = 0 \cdot 765$$

and $$v_{max} = |v_{x_{max}}| = +a\omega[0 \cdot 765 + 4 \times 0 \cdot 765 \times 0 \cdot 644] = +2 \cdot 73\ a\omega$$

4.12 We write :

$$a \cos 2{\cdot}1\, t \cos 50{\cdot}0\, t = \frac{a}{2}\left\{\cos 52{\cdot}1\, t + \cos 47{\cdot}9\, t\right\}$$

Thus the angular frequencies of constituent oscillations are

$$52{\cdot}1\ \text{s}^{-1} \text{ and } 47{\cdot}9\ \text{s}^{-1}$$

To get the beat period note that the variable amplitude $a \cos 2{\cdot}1\, t$ becomes maximum (positive or negative), when

$$2{\cdot}1\, t = n\,\pi$$

Thus the interval between two maxima is

$$\frac{\pi}{2{\cdot}1} = 1{\cdot}5 \text{ s nearly.}$$

4.13 If the frequency of A with respect to K' is ν_0 and K' oscillates with frequency $\bar{\nu}$ with respect to K, the beat frequency of the point A in the K-frame will be ν when

$$\bar{\nu} = \nu_0 \pm \nu$$

In the present case $\bar{\nu} = 20$ or 24. This means

$$\nu_0 = 22. \quad \& \quad \nu = 2$$

Thus beats of $2\,\nu = 4$ will be heard when $\bar{\nu} = 26$ or 18.

4.14 (a) From the Eqn : $x = a \sin \omega t$

$$\sin^2 \omega t = x^2/a^2 \quad \text{or} \quad \cos^2 \omega t = 1 - \frac{x^2}{a^2} \qquad (1)$$

And from the equation : $y = b \cos \omega t$

$$\cos^2 \omega t = y^2/b^2 \qquad (2)$$

From Eqns (1) and (2), we get :

$$1 - \frac{x^2}{a^2} = \frac{y^2}{b^2} \quad \text{or} \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

which is the standard equation of the ellipse shown in the figure.
we observe that,

at $\quad t = 0,\ x = 0$ and $y = b$

and at $\quad t = \dfrac{\pi}{2\,\omega},\ x = +a$ and $y = 0$

Thus we observe that at $t = 0$, the point is at point 1 (Fig.) and at the following moments, the co-ordinate y diminishes and x becomes positive. Consequently the motion is clockwise.

(b) As $x = a \sin \omega t$ and $y = b \cos \omega t$

So we may write $\vec{r} = a \sin \omega t\ \vec{i} + b \cos \omega t\ \vec{j}$

Thus $\ddot{\mathbf{r}} = \vec{w} = -\,\omega^2\, \vec{r}$

4.15 (a) From the Eqn. : $x = a \sin \omega t$, we have

$$\cos \omega t = \sqrt{1 - (x^2/a^2)}$$

and from the Eqn. :$y = a \sin 2 \omega t$

$$y = 2 a \sin \omega t \cos \omega t = 2 x \sqrt{1 - (x^2/a^2)} \quad \text{or} \quad y^2 = 4 x^2 \left(1 - \frac{x^2}{a^2}\right)$$

(b) From the Eqn. : $x = a \sin \omega t$;

$$\sin^2 \omega t = x^2/a^2$$

From $y = a \cos 2 \omega t$

$$y = a (1 - 2 \sin^2 \omega t) = a \left(1 - 2 \frac{x^2}{a^2}\right)$$

For the plots see the plots of answersheet of the problem book.

4.16 As $U(x) = U_0 (1 - \cos a x)$

So, $$F_x = -\frac{dU}{dx} = -U_0 a \sin a x \quad (1)$$

or, $F_x = - U_0 a a x$ (because for small angle of oscillations $\sin a x \approx a x$)

or, $$F_x = - U_0 a^2 x \quad (1)$$

But we know $F_x = - m \omega_0^2 x$, for small oscillation

Thus $$\omega_0^2 = \frac{U_0 a^2}{m} \quad \text{or} \quad \omega_0 = a \sqrt{\frac{U_0}{m}}$$

Hence the sought time period

$$T = \frac{2\pi}{\omega_0} = \frac{2\pi}{a} \sqrt{\frac{m}{U_0}} = 2\pi \sqrt{\frac{m}{a^2 U_0}}$$

4.17 If $U(x) = \frac{a}{x^2} - \frac{b}{x}$

then the equilibrium position is $x = x_0$ when $U'(x_0) = 0$

or $$-\frac{2a}{x_0^3} + \frac{b}{x_0^2} = 0 \Rightarrow x_0 = \frac{2a}{b}.$$

Now write : $$x = x_0 + y$$

Then $$U(x) = \frac{a}{x_0^2} - \frac{b}{x_0} + (x - x_0) U'(x_0) + \frac{1}{2} (x - x_0)^2 U''(x_0)$$

But $$U''(x_0) = \frac{6a}{x_0^4} - \frac{2b}{x_0^3} = (2a/b)^{-3} (3b - 2b) = b^4/8a^3$$

So finally : $$U(x) = U(x_0) + \frac{1}{2}\left(\frac{b^4}{8a^3}\right) y^2 + \dots$$

We neglect remaining terms for small oscillations and compare with the P.E. for a harmonic, oscillator :

$$\frac{1}{2} m \omega^2 y^2 = \frac{1}{2}\left(\frac{b^4}{8a^3}\right) y^2 , \text{ so } \omega = \frac{b^2}{\sqrt{8a^3 m}}$$

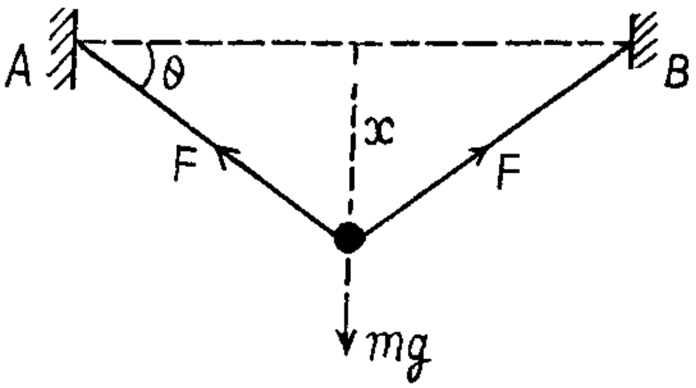
Thus $$T = 2\pi \frac{\sqrt{8 m a^3}}{b^2}$$

Note : Equilibrium position is generally a minimum of the potential energy. Then $U'(x_0) = 0$, $U''(x_0) > 0$. The equilibrium position can in principle be a maximum but then $U''(x_0) < 0$ and the frequency of oscillations about this equilibrium position will be imaginary.

The answer given in the book is incorrect both numerically and dimensionally.

4.18 Let us locate and depict the forces acting on the ball at the position when it is at a distance x down from the undeformed position of the string.
At this position, the unbalanced downward force on the ball

$$= mg - 2F\sin\theta$$



By Newton's law, $m\ddot{x} = mg - 2F\sin\theta$

$$= mg - 2F\theta \text{ (when } \theta \text{is small)}$$

$$= mg - 2F\frac{x}{l/2} = mg - \frac{4F}{l}x$$

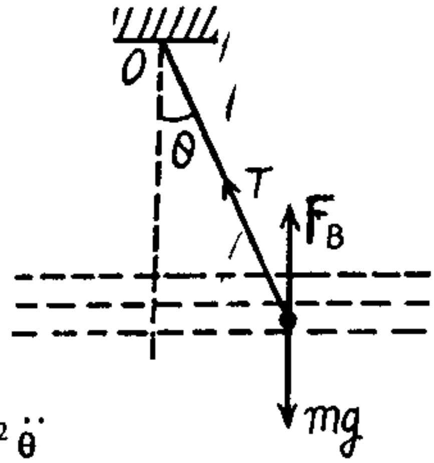
Thus $$\ddot{x} = g - \frac{4F}{ml}x = -\frac{4F}{ml}\left(x - \frac{mgl}{4F}\right)$$

putting $x' = x - \frac{mgl}{T}$, we get

$$\ddot{x}' = -\frac{4T}{ml}x'$$

Thus $T = \pi\sqrt{\frac{ml}{F}} = 0{\cdot}2$ s

4.19 Let us depict the forces acting on the oscillating ball at an arbitraty angular position θ. (Fig.), relative to equilibrium position where F_B is the force of buoyancy. For the ball from the equation :



$N_Z = I\beta_Z$, (where we have taken the positive sense of Z axis in the direction of angular velocity i.e. $\dot{\theta}$ of the ball and passes through the point of suspension of the pendulum O), we get :

$$-mgl\sin\theta + F_B l\sin\theta = ml^2\ddot{\theta} \tag{1}$$

Using $m = \frac{4}{3}\pi r^3 \sigma$, $F_B = \frac{4}{3}\pi r^3 \rho$ and $\sin\theta \approx \theta$ for small θ, in Eqn (1), we get :

$$\ddot{\theta} = -\frac{g}{l}\left(1-\frac{\rho}{\sigma}\right)\theta$$

Thus the sought time period

$$T = 2\pi\frac{1}{\sqrt{\frac{g}{l}\left(1-\frac{\rho}{\sigma}\right)}} = 2\alpha\sqrt{\frac{l/g}{1-\frac{1}{\eta}}}$$

Hence $$T = 2\alpha\sqrt{\frac{\eta l}{g(\eta-1)}} = 1{\cdot}1\text{s}$$

4.20 Obviously for small β the ball execute part of S.H.M. Due to the perfectly elastic collision the velocity of ball simply reversed. As the ball is in S.H.M. ($|\theta| < \alpha$ on the left)its motion law in differential from can be written as

$$\ddot{\theta} = -\frac{g}{l}\theta = -\omega_0^2\theta \qquad (1)$$

If we assume that the ball is released from the extreme position, $\theta = \beta$ at $t = 0$, the solution of differential equation would be taken in the form

$$\theta = \beta\cos\omega_0 t = \beta\cos\sqrt{\frac{g}{l}}\,t \qquad (2)$$

If t' be the time taken by the ball to go from the extreme position $\theta = \beta$ to the wall i.e. $\theta = -\alpha$, then Eqn. (2) can be rewritten as

$$-\alpha = \beta\cos\sqrt{\frac{g}{l}}\,t'$$

or $$t' = \sqrt{\frac{l}{g}}\cos^{-1}\left(-\frac{\alpha}{\beta}\right) = \sqrt{\frac{l}{g}}\left(\pi-\cos^{-1}\frac{\alpha}{\beta}\right)$$

Thus the sought time $T = 2t' = 2\sqrt{\frac{l}{g}}\left(\pi-\cos^{-1}\frac{\alpha}{\beta}\right)$

$$= 2\sqrt{\frac{l}{g}}\left(\frac{\pi}{2}+\sin^{-1}\frac{\alpha}{\beta}\right), \quad [\text{because } \sin^{-1}x+\cos^{-1}x = \pi/2]$$

4.21 Let the downward acceleration of the elevator car has continued for time t', then the sought time

$t = \sqrt{\frac{2h}{w}} + t'$, where obviously $\sqrt{\frac{2h}{w}}$ is the time of upward acceleration of the elevator.

One should note that if the point of suspension of a mathematical pendulum moves with an acceleration $\vec{w}$, then the time period of the pendulum becomes

$$2\pi\sqrt{\frac{l}{|\vec{g}-\vec{w}|}} \qquad (\text{see } 4.30)$$

In this problem the time period of the pendulum while it is moving upward with acceleration w becomes

$2\pi\sqrt{\dfrac{l}{g+w}}$ and its time period while the elevator moves downward with the same magnitude of acceleration becomes

$$2\pi\sqrt{\frac{l}{g-w}}$$

As the time of upward acceleration equals $\sqrt{\dfrac{2h}{w}}$, the total number of oscillations during this time equals

$$\frac{\sqrt{2h/w}}{2\pi\sqrt{l/(g+w)}}$$

Thus the indicated time = $\dfrac{\sqrt{2h/w}}{2\pi\sqrt{l/(g+w)}}\cdot 2\pi\sqrt{l/g} = \sqrt{2h/w}\,\sqrt{(g+w)/g}$

Similarly the indicated time for the time interval t'

$$= \frac{t'}{2\pi\sqrt{l/(g-w)}}\,2\pi\sqrt{l/g} = t'\sqrt{(g-w)/g}$$

we demand that

$$\sqrt{2h/w}\,\sqrt{(g+w)/g} + t'\sqrt{(g-w)/g} = \sqrt{2h/w} + t'$$

or, $$t' = \sqrt{2h/w}\,\frac{\sqrt{g+w}-\sqrt{g}}{\sqrt{g}-\sqrt{g-w}}$$

Hence the sought time

$$t = \sqrt{\frac{2h}{w}} + t' = \sqrt{\frac{2h}{w}}\,\frac{\sqrt{g+w}-\sqrt{g-w}}{\sqrt{g}-\sqrt{g-w}}$$

$$= \sqrt{\frac{2h}{w}}\,\frac{\sqrt{1+\beta}-\sqrt{1-\beta}}{1-\sqrt{1-\beta}}, \text{ where } \beta = w/g$$

4.22 If the hydromoter were in equlibrium or floating, its weight will be balanced by the buoyancy force acting on it by the fluid. During its small oscillation, let us locate the hydrometer when it is at a vertically downward distance x from its equilibrium position. Obviously the net unbalanced force on the hydrometer is the excess buoyancy force directed upward and equals $\pi r^2 x \rho g$. Hence for the hydrometer.

$$m\ddot{x} = -\pi r^2 \rho g x$$

or, $$\ddot{x} = -\frac{\pi r^2 \rho g}{m}x$$

Hence the sought time period

$$T = 2\alpha\sqrt{\frac{m}{\pi r^2 \rho g}} = 2{\cdot}5\text{ s}.$$

4.23 At first let us calculate the stiffness κ_1 and κ_2 of both the parts of the spring. If we subject the original spring of stiffness κ having the natural length l_0 (say), under the deforming forces $F - F$ (say) to elongate the spring by the amount x, then

$$F = \kappa x \tag{1}$$

Therefore the elongation per unit length of the spring is x/l_0 . Now let us subject one of the parts of the spring of natural length $\eta\, l_0$ under the same deforming forces $F - F$. Then the elongation of the spring will be

$$\frac{x}{l_0}\eta\, l_0 = \eta\, x$$

Thus
$$F = \kappa_1\,(\eta\, x) \tag{2}$$

Hence from Eqns (1) and (2)

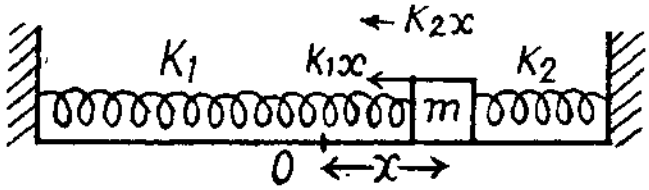
$$\kappa = \eta\,\kappa_1 \quad \text{or} \quad \kappa_1 = \kappa/\eta \tag{3}$$

Similarly
$$\kappa_2 = \frac{\kappa}{1-\eta}$$

The position of the block m when both the parts of the spring are non-deformed, is its equilibrium position O. Let us displace the block m towards right or in positive x axis by the small distance x. Let us depict the forces acting on the block when it is at a distance x from its equilibrium position (Fig.). From the second law of motion in projection form i.e. $F_x = m\, w_x$

$$-\kappa_1 x - \kappa_2 x = m\ddot{x}$$

or,
$$-\left(\frac{\kappa}{\eta} + \frac{\kappa}{1-\eta}\right)x = m\ddot{x}$$



Thus
$$\ddot{x} = -\frac{\kappa}{m}\frac{1}{\eta\,(m)}x$$

Hence the sought time period

$$T = 2\pi\sqrt{\eta\,(1-\eta)\,m/\kappa} = 0{\cdot}13\text{ s}$$

4.24 Similar to the Soln of 4.23, the net unbalanced force on the block m when it is at a small horizontal distance x from the equilibrium position becomes $(\kappa_1 + \kappa_2)\,x$.

From $F_x = m\, w_x$ for the block :

$$-(\kappa_1 + \kappa_2)\,x = m\ddot{x}$$

Thus
$$\ddot{x} = -\left(\frac{\kappa_1 + \kappa_2}{m}\right)x$$

Hence the sought time period $T = 2\pi\sqrt{\dfrac{m}{\kappa_1 + \kappa_2}}$

Alternate : Let us set the block m in motion to perform small oscillation. Let us locate the block when it is at a distance x from its equilibrium position.

As the spring force is restoring conservative force and deformation of both the springs are same, so from the conservation of mechanical energy of oscillation of the spring-block system :

$$\frac{1}{2}m\left(\frac{dx}{dt}\right)^2 + \frac{1}{2}\kappa_1 x^2 + \frac{1}{2}\kappa_2 x^2 = \text{Constant}$$

Differentiating with respect to time

$$\frac{1}{2}m\,2\dot{x}\ddot{x} + \frac{1}{2}(\kappa_1+\kappa_2)\,2x\dot{x} = 0$$

or, $$\ddot{x} = -\frac{(\kappa_1+\kappa_2)}{m}x$$

Hence the sought time period $T = 2\pi\sqrt{\dfrac{m}{\kappa_1+\kappa_2}}$

4.25 During the vertical oscillation let us locate the block at a vertical down distance x from its equilibrium position. At this moment if x_1 and x_2 are the additional or further elongation of the upper & lower springs relative to the equilibrium position, then the net unbalanced force on the block will be $\kappa_2 x_2$ directed in upward direction. Hence

$$-\kappa_2 x_2 = m\ddot{x} \qquad (1)$$

We also have $$x = x_1 + x_2 \qquad (2)$$

As the springs are massless and initially the net force on the spring is also zero so for the spring

$$\kappa_1 x_1 = \kappa_2 x_2 \qquad (3)$$

Solving the Eqns (1), (2) and (3) simultaneously, we get

$$-\frac{\kappa_1\kappa_2}{\kappa_1+\kappa_2}x = m\ddot{x}$$

Thus $$\ddot{x} = -\frac{(\kappa_1\kappa_2/\kappa_1+\kappa_2)}{m}x$$

Hence the sought time period $T = 2\pi\sqrt{m\dfrac{(\kappa_1\kappa_2)}{\kappa_1\kappa_2}}$

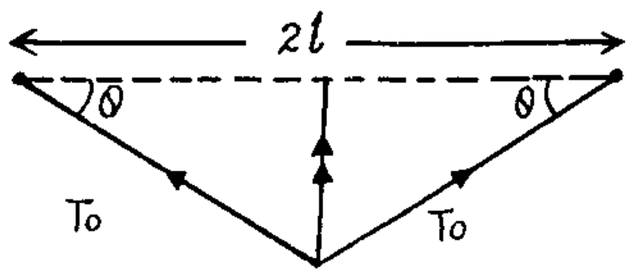
4.26 The force F, acting on the weight deflected from the position of equilibrium is $2\,T_0\sin\theta$.

Since the angle θ is small, the net restoring force, $F = 2\,T_0\dfrac{x}{l}$

or, $F = kx$, where $k = \dfrac{2\,T_0}{l}$

So, by using the formula,

$$\omega_0 = \sqrt{\frac{k}{m}}, \quad \omega_0 = \sqrt{\frac{2\,T_0}{m\,l}}$$



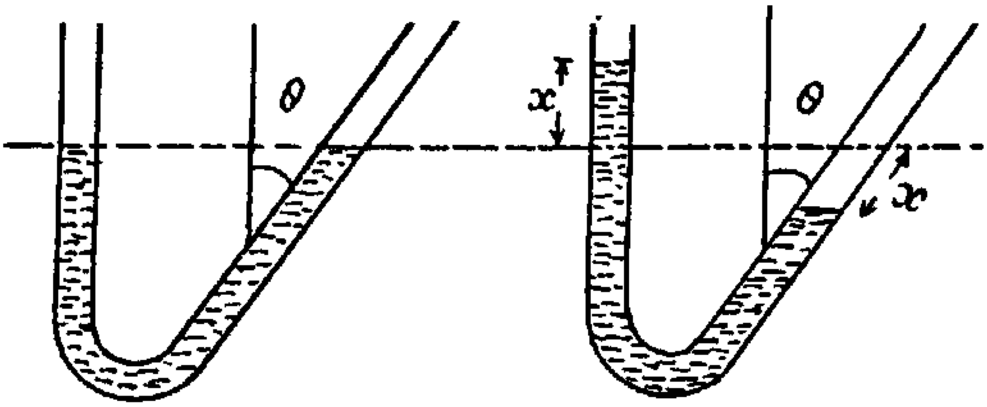
4.27 If the mercury rises in the left arm by x it must fall by a slanting length equal to x in the other arm. Total pressure difference in the two arms will then be

$$\rho g x + \rho g x\cos\theta = \rho g x(1+\cos\theta)$$

This will give rise to a restoring force

$$-\rho g S x(1+\cos\theta)$$

This must equal mass times acceleration which can be obtained from work energy principle.



The K.E. of the mercury in the tube is clearly : $\frac{1}{2} m \dot{x}^2$

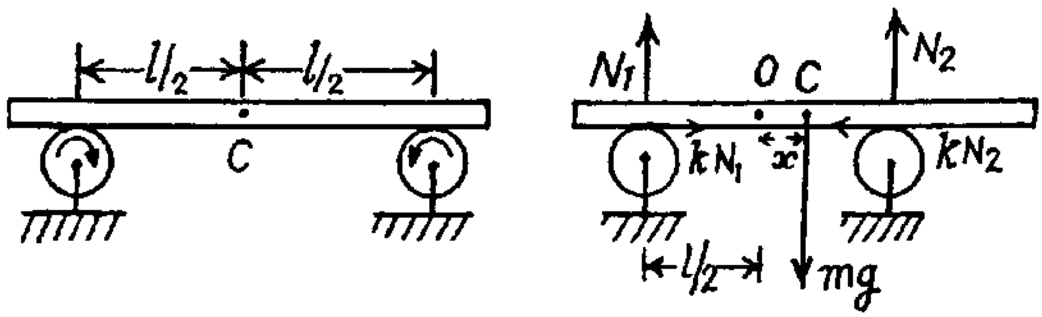
So mass times acceleration must be : $m \ddot{x}$

Hence $$m \ddot{x} + \rho g S (1 + \cos\theta) x = 0$$

This is S.H.M. with a time period

$$T = 2\pi \sqrt{\frac{m}{\rho g S (1 + \cos\theta)}}.$$

4.28 In the equilibrium position the C.M. of the rod lies nid way between the two rotating wheels. Let us displace the rod horizontally by some small distance and then release it. Let us depict the forces acting on the rod when its C.M. is at distance x from its equilibrium position (Fig.). Since there is no net vertical force acting on the rod, Newton's second law gives :



$$N_1 + N_2 = mg \tag{1}$$

For the translational motion of the rod from the Eqn. : $F_x = m\, w_{cx}$

$$k N_1 - k N_2 = m \ddot{x} \tag{2}$$

As the rod experiences no net torque about an axis perpendicular to the plane of the Fig. through the C.M. of the rod.

$$N_1 \left(\frac{l + x}{2} \right) = N_2 \left(\frac{l - x}{2} \right) \tag{3}$$

Solving Eqns. (1), (2) and (3) simultaneously we get

$$\ddot{x} = - k \frac{2g}{l} x$$

Hence the sought time period

$$T = 2\pi \sqrt{\frac{l}{2kg}} = \pi \sqrt{\frac{2l}{kg}} = 1 \cdot 5 \text{ s}$$

4.29 (a) The only force acting on the ball is the gravitational force $\vec{F}$, of magnitude $\gamma \frac{4}{3} \pi \rho m r$, where γ is the gravitational constant ρ, the density of the Earth and r is the distance of the body from the centre of the Earth.

But, $g = \gamma \frac{4\pi}{3} \rho R$, so the expression for $\vec{F}$ can be written as,

$\vec{F} = -m g \frac{\vec{r}}{R}$, here R is the radius of the Earth and the equation of motion in projection form has the form, or, $m \ddot{x} + \frac{m g}{R} x = 0$

(b) The equation, obtained above has the form of an equation of S.H.M. having the time period,

$$T = 2\pi \sqrt{\frac{R}{g}},$$

Hence the body will reach the other end of the shaft in the time,

$$t = \frac{T}{2} = \pi \sqrt{\frac{R}{g}} = 42 \text{ min}.$$

(c) From the conditions of S.H.M., the speed of the body at the centre of the Earth will be maximum, having the magnitude,

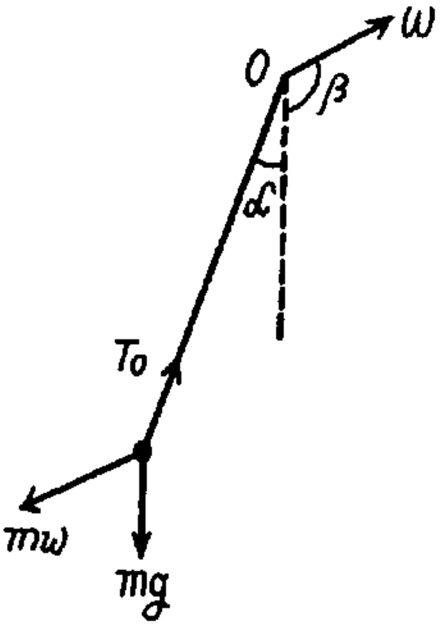
$$v = R\omega = R\sqrt{g/R} = \sqrt{gR} = 7{\cdot}9 \text{ km/s}.$$

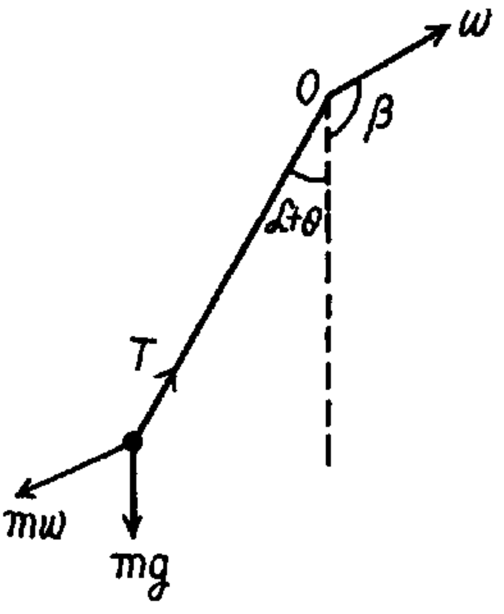
4.30 In the frame of point of suspension the mathematical pendulum of mass m (say) will oscillate. In this frame, the body m will experience the inertial force $m(-\vec{w})$ in addition to the real forces during its oscillations. Therefore in equilibrium position m is deviated by some angle say α. In equilibrium position

$$T_0 \cos\alpha = m g + m w \cos(\pi - \beta) \quad \text{and} \quad T_0 \sin\alpha = m w \sin(\pi - \beta)$$

So, from these two Eqns

$$\left.\begin{aligned} \tan\alpha &= \frac{g - w\cos\beta}{w\sin\beta} \\ \text{and } \cos\alpha &= \sqrt{\frac{m^2 w^2 \sin^2\beta + (m g - m w\cos\beta)^2}{m g - m w \cos\beta}} \end{aligned}\right] \quad (1)$$





Let us displace the bob m from its equilibrium position by some small angle and then release it. Now locate the ball at an angular position $(\alpha + \theta)$ from vertical as shown in the figure.

From the Eqn. : $N_{0z} = I\beta_z$

$$-m g l \sin(\alpha+\theta) - m w \cos(\pi-\beta)\, l \sin(\alpha+\theta) + m w \sin(\pi-\beta)\, l \cos(\alpha+\theta) = m l^2 \ddot{\theta}$$

or, $-g(\sin\alpha\cos\theta + \cos\alpha\sin\theta) - w\cos(\pi-\beta)(\sin\alpha\cos\theta+\cos\alpha\sin\theta) + w\sin\beta$

$$(\cos\alpha\cos\theta - \sin\alpha\sin\theta) = l\ddot{\theta}$$

But for small θ, $\sin\theta \approx \theta$ $\cos\theta \approx 1$

So, $$-g(\sin\alpha+\cos\alpha\,\theta) - w\cos(\pi-\beta)(\sin\alpha+\cos\alpha\,\theta) + w\sin\beta(\cos\alpha - \sin\alpha\,\theta) = l\ddot{\theta}$$

or, $$(\tan\alpha+\theta)(w\cos\beta - g) + w\sin\beta(1-\tan\alpha\,\theta) = \frac{l}{\cos\alpha}\ddot{\theta} \qquad (2)$$

Solving Eqns (1) and (2) simultaneously we get

$$-(g^2 - 2wg\cos\beta + w^2)\theta = l\sqrt{g^2+w^2-2wg\cos\beta}\;\ddot{\theta}$$

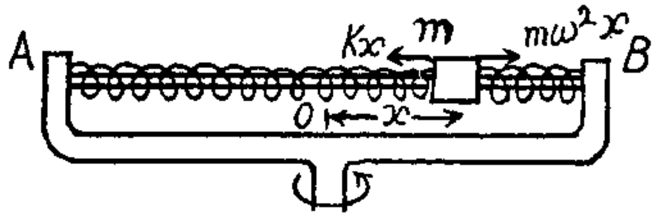
Thus $$\ddot{\theta} = -\frac{|\vec{g}-\vec{w}|}{l}\theta$$

Hence the sought time period $T = \dfrac{2\pi}{\omega_0} = 2\pi\sqrt{\dfrac{l}{|\vec{g}-\vec{w}|}}$

4.31 Obviously the sleeve performs small oscillations in the frame of rotating rod. In the rod's frame let us depict the forces acting on the sleeve along the length of the rod while the sleeve is at a small distance x towards right from its equilibrium position. The free body diagram of block does not contain Coriolis force, because it is perpendicualr to the length of the rod. From $F_x = m w_x$ for the sleeve in the frame of rod

$$-\kappa x + m\omega^2 x = m\ddot{x}$$

or, $$\ddot{x} = -\left(\frac{\kappa}{m} - \omega^2\right)x \qquad (1)$$



Thus the sought time period

$$T = \frac{2\pi}{\sqrt{\frac{\kappa}{m}-\omega^2}} = 0{\cdot}7\,\text{s}$$

It is obvious from Eqn (1) that the sleeve will not perform small oscillations if $\omega \geq \sqrt{\frac{\kappa}{m}}$ 10 rod/s.

4.32 When the bar is about to start sliding along the plank, it experiences the maximum restoring force which is being provided by the limiting friction,

Thus

$$kN = m\omega_0^2 a \qquad \text{or,} \quad kmg = m\omega_0^2 a$$

or, $$k = \frac{\omega_0^2 a}{g} = \frac{a}{g}\left(\frac{2\pi}{T}\right)^2 = 4\,s.$$

4.33 The natural angular frequency of a mathematical pendulum equals $\omega_0 = \sqrt{g/l}$

(a) We have the solution of S.H.M. equation in angular form :

$$\theta = \theta_m \cos(\omega_0 t + \alpha)$$

If at the initial moment i.e. at $t = 0$, $\theta = \theta_m$ than $\alpha = 0$.

Thus the above equation takes the form

$$\theta = \theta_m \cos \omega_0 t$$

$$= \theta_m \cos \sqrt{\frac{g}{l}}\, t = 3° \cos \sqrt{\frac{9{\cdot}8}{0{\cdot}8}}\, t$$

Thus $$\theta = 3° \cos 3{\cdot}5\ t$$

(b) The S.H.M. equation in angular form :

$$\theta = \theta_m \sin(\omega_0 t + \alpha)$$

If at the initial moment $t = 0$, $\theta = 0$, then $\alpha = 0$ .Then the above equation takes the form

$$\theta = \theta_m \sin \omega_0 t$$

Let v_0 be the velocity of the lower end of pendulam at $\theta = 0$, then from conserved of mechanical energy of oscillaton

$$E_{mean} = E_{extreme} \quad \text{or} \quad T_{mean} = U_{extrem}$$

or, $$\frac{1}{2} m v_0^2 = m g l (1 - \cos \theta_m)$$

Thus

$$\theta_m = \cos^{-1}\left(1 - \frac{v_0^2}{2gl}\right) = \cos^{-1}\left[1 - \frac{(0 \cdot 22)^2}{2 \times 9 \cdot 8 \times 0 \cdot 8}\right] = 4 \cdot 5°$$

Thus the sought equation becomes

$$\theta = \theta_m \sin \omega_0 t = 4{\cdot}5° \sin 3{\cdot}5\, t$$

(c) Let θ_0 and v_0 be the angular deviation and linear velocity at $t = 0$.

As the mechanical energy of oscillation of the mathematical pendulum is conservation

$$\frac{1}{2} m v_0^2 + m g l (1 - \cos \theta_0) = m g l (1 - \cos \theta_m)$$

or, $$\frac{v_0^2}{2} = g l (\cos \theta_0 - \cos \theta_m)$$

Thus $$\theta_m = \cos^{-1}\left\{\cos \theta_0 - \frac{v_0^2}{2gl}\right\} = \cos^{-1}\left\{\cos 3° - \frac{(0{\cdot}22)^2}{2 \times 9{\cdot}8 \times 0{\cdot}8}\right\} = 5{\cdot}4°$$

Then from $\theta = 5.4° \sin(3.5t + \alpha)$, we see that $\sin\alpha = \frac{3}{5.4}$ and $\cos\alpha < 0$ because the velovity is directed towards the centre. Thus $\alpha = \frac{\pi}{2} + 1.0$ radians and we get the answer.

4.34 While the body A is at its upper extreme position, the spring is obviously elongated by the amount

$$\left(a - \frac{m_1 g}{\kappa}\right).$$

If we indicate y-axis in vertically downward direction, Newton's second law of motion in projection form i.e. $F_y = m\,w_y$ for body A gives :

$$m_1 g + \kappa\left(a - \frac{m_1 g}{\kappa}\right) = m_1\omega^2 a \quad \text{or, } \kappa\left(a - \frac{m_1 g}{\kappa}\right) = m_1(\omega^2 a - g) \qquad (1)$$

(Because at any extreme position the magnitude of acceleration of an oscillating body equals $\omega^2 a$ and is restoring in nature.)

If N be the normal force exerted by the floor on the body B, while the body A is at its upper extreme position, from Newton's second law for body B

$$N + \kappa\left(a - \frac{m_1 g}{\kappa}\right) = m_2 g$$

or, $$N = m_2 g - \kappa\left(a - \frac{m_1 g}{\kappa}\right) = m_2 g - m_1(\omega^2 a - g) \text{ (using Eqn. 1)}$$

Hence$N = (m_1 + m_2) g - m_1\omega^2 a$

When the body A is at its lower extreme position, the spring is compresed by the distance $\left(a + \frac{m_1 g}{\kappa}\right)$.

From Newton's second law in projeciton form i.e. $F_y = m\,w_y$ for body A at this state:

$$m_1 g - \kappa\left(a + \frac{m_1 g}{\kappa}\right) = m_1(-\omega^2 a) \quad \text{or, } \kappa\left(a + \frac{m_1 g}{\kappa}\right) = m_1(g + \omega^2 a) \qquad (3)$$

In this case if N' be the normal force exerted by the floor on the body B, From Newton's second law

for body B we get: $N' = \kappa\left(a + \frac{m_1 g}{\kappa}\right) + m_2 g = m_1(g + \omega^2 a) + m_2 g$ (using Eqn. 3)

Hence $$N' = (m_1 + m_2) g + m_1\omega^2 a$$

From Newton's third law the magnitude of sought forces are N' and N, respectively.

4.35 (a) For the block from Newton's second law in projection form $F_y = m\,w_y$

$$N - mg = m\ddot{y} \qquad (1)$$

But from $$y = a(1 - \cos\omega t)$$

We get $\ddot{y} = \omega^2 a \cos \omega t$ (2

From Eqns (1) and (2)

$$N = m g \left(1 + \frac{\omega^2 a}{g} \cos \omega t \right) \quad (3)$$

From Newtons's third law the force by which the body m exerts on the block is directed vertically downward and equls $N = m g \left(1 + \frac{\omega^2 a}{g} \cos \omega t \right)$

(b) When the body m starts, falling behind the plank or loosing contact, $N = 0$, (because the normal reaction is the contact force). Thus from Eqn. (3)

$$m g \left(1 + \frac{\omega^2 a}{g} \cos \omega t \right) = 0 \quad \text{for some } t.$$

Hence $a_{\min} = g/\omega^2 = 8$ cm.

(c) We observe that the motion takes place about the mean position $y = a$. At the initial instant $y = 0$. As shown in (b) the normal reaction vanishes at a height (g/ω^2) above the position of equilibrium and the body flies off as a free body. The speed of the body at a distance (g/ω^2) from the equilibrium position is $\omega \sqrt{a^2 - (g/\omega^2)^2}$, so that the condition of the problem gives

$$\frac{[\omega \sqrt{a^2 - (g/\omega^2)^2}\,]^2}{2g} + \frac{g}{\omega^2} + a = h$$

Hence solving the resulting quadratic equation and taking the positive roof,

$$a = -\frac{g}{\omega^2} + \sqrt{\frac{2hg}{\omega^2}} = 20 \text{ cm.}$$

4.36 (a) Let $y(t)$ = displacement of the body from the end of the unstreched position of the spring (not the equilibrium position). Then

$$m \ddot{y} = -\kappa y + m g$$

This equation has the solution of the form

$$y = A + B \cos(\omega t + \alpha)$$

if $-m \omega^2 B \cos(\omega t + \alpha) = -\kappa [A + B \cos(\omega t + \alpha)] + m g$

Then $\omega^2 = \frac{\kappa}{m}$ and $A = \frac{m g}{\kappa}$

we have $y = 0$ and $\dot{y} = 0$ at $t = 0$. So

$$-\omega B \sin \alpha = 0$$

$$A + B \cos \alpha = 0$$

Since $B > 0$ and $A > 0$ we must have $\alpha = \pi$

$$B = A = \frac{m g}{\kappa}$$

and $$y = \frac{mg}{\kappa}(1 - \cos \omega t)$$

(b) Tension in the spring is

$$T = \kappa y = mg(1 - \cos \omega t)$$

so $$T_{max} = 2mg, \; T_{min} = 0$$

4.37 In accordance with the problem

$$\vec{F} = -\alpha m \vec{r}$$

So, $$m(\ddot{x}\vec{i} + \ddot{y}\vec{j}) = -\alpha m(x\vec{i} + y\vec{j})$$

Thus $$\ddot{x} = -\alpha x \text{ and } \ddot{y} = -\alpha y$$

Hence the solution of the differential equation

$$\ddot{x} = -\alpha x \quad \text{becomes } x = a\cos(\omega_0 t + \delta), \text{ where } \omega_0^2 = \alpha \qquad (1)$$

So, $$\dot{x} = -a\omega_0 \sin(\omega_0 t + \alpha) \qquad (2)$$

From the initial conditions of the problem, $v_x = 0$ and $x = r_0$ at $t = 0$

So from Eqn. (2) $\alpha = 0$, and Eqn takes the form

$$x = r_0 \cos \omega_0 t \quad \text{so, } \cos \omega_0 t = x/r_0 \qquad (3)$$

One of the solution of the other differential Eqn $\ddot{y} = -\alpha y$, becomes

$$y = a' \sin(\omega_0 t + \delta'), \text{ where } \omega_0^2 = \alpha \qquad (4)$$

From the initial condition, $y = 0$ at $t = 0$, so $\delta' = 0$ and Eqn (4) becomes :

$y = a' \sin \omega_0 t$(5)

Differentiating w.r.t. time we get

$$\dot{y} = a'\omega_0 \cos \omega_0 t \qquad (6)$$

But from the initial condition of the problem, $\dot{y} = v_0$ at $t = 0$,

So, from Eqn (6) $v_0 = a'\omega_0$ or, $a' = v_0/\omega_0$

Using it in Eqn (5), we get

$$y = \frac{v_0}{\omega_0}\sin \omega_0 t \quad \text{or} \quad \sin \omega_0 t = \frac{\omega_0 y}{v_0} \qquad (7)$$

Squaring and adding Eqns (3) and (7) we get :

$$\sin^2 \omega_0 t + \cos^2 \omega_0 t = \frac{\omega_0^2 y^2}{v_0^2} + \frac{x^2}{r_0^2}$$

or, $$\left(\frac{x}{r_0}\right)^2 + \alpha\left(\frac{y}{v_0}\right)^2 = 1 \left(\text{as } \alpha, = \omega_0^2\right)$$

4.38 (a) As the elevator car is a translating non-inertial frame, therefore the body m will experience an inertial force mw directed downward in addition to the real forces in the elevator's frame. From the Newton's second law in projection form

$F_y = m w_y$ for the body in the frame of elevator car:

$$-\kappa\left(\frac{mg}{\kappa} + y\right) + mg + mw = m\ddot{y} \qquad (A)$$

(Because the initial elongation in the spring is $m\,g/\kappa$)

so, $$m\ddot{y} = -\kappa y + m w = -\kappa\left(y - \frac{m w}{\kappa}\right)$$

or, $$\frac{d^2}{dt^2}\left(y - \frac{m w}{\kappa}\right) = -\frac{\kappa}{m}\left(y - \frac{m w}{\kappa}\right) \tag{1}$$

Eqn. (1) shows that the motion of the body m is S.H.M. and its solution becomes

$$y - \frac{m w}{\kappa} = a \sin\left(\sqrt{\frac{\kappa}{m}}\, t + \alpha\right) \tag{2}$$

Differentiating Eqn (2) w.r.t. time

$$\dot{y} = a\sqrt{\frac{\kappa}{m}} \cos\left(\sqrt{\frac{\kappa}{m}}\, t + \alpha\right) \tag{3}$$

Using the initial condition $y(0) = 0$ in Eqn (2), we get :

$$a \sin\alpha = -\frac{m w}{\kappa}$$

and using the other initial condition $\dot{y}(0) = 0$ in Eqn (3)

we get $$a\sqrt{\frac{\kappa}{m}} \cos\alpha = 0$$

Thus $$\alpha = -\alpha/2 \text{ and } a = \frac{m w}{\kappa}$$

Hence using these values in Eqn (2), we get

$$y = \frac{m w}{\kappa}\left(1 - \cos\sqrt{\frac{\kappa}{m}}\, t\right)$$

(b) Proceed up to Eqn.(1). The solution of this differential Eqn be of the form :

$$y - \frac{m w}{\kappa} = a \sin\left(\sqrt{\frac{\kappa}{m}}\, t + \delta\right)$$

or, $$y - \frac{\alpha t}{\kappa/m} = a \sin\left(\sqrt{\frac{\kappa}{m}}\, t + \delta\right)$$

or, $$y - \frac{\alpha t}{\omega_0^2} = a \sin(\omega_0 t + \delta) \left(\text{wher } \omega_0 = \sqrt{\frac{\kappa}{m}}\right) \tag{4}$$

From the initial condition that at$t = 0$, $y(0) = 0$, so $0 = a \sin\delta$ or $\delta = 0$

Thus Eqn.(4) takes the from : $y - \frac{\alpha t}{\omega_0^2} = a \sin\omega_0 t$ (5)

Differentiating Eqn. (5) we get : $\dot{y} - \frac{\alpha}{\omega_0^2} = a\,\omega_0 \cos\omega t$ (6)

But from the other initial condition $\dot{y}(0) = 0$ at $t = 0$.

So, from Eqn.(6) $\quad -\frac{\alpha}{\omega_0^2} = a\,\omega_0$ or $a = -\alpha/\omega_0^3$

Putting the value of a in Eqn. (5), we get the sought $y(t)$. i.e.

$$y - \frac{\alpha t}{\omega_0^2} = -\frac{\alpha}{\omega_0^3}\sin\omega_0 t \quad \text{or} \quad y = \frac{\alpha}{\omega_0^3}(\omega_0 t - \sin\omega_0 t)$$

4.39 There is an important difference between a rubber cord or steel coire and a spring. A spring can be pulled or compressed and in both cases, obey's Hooke's law. But a rubber cord becomes loose when one tries to compress it and does not then obey Hooke's law. Thus if we suspend a body by a rubber cord it stretches by a distance mg/κ in reaching the equilibrium configuration. If we further strech it by a distance Δh it will execute harmonic oscillations when released if $\Delta h \le mg/\kappa$ because only in this case will the cord remain taut and obey Hooke's law.

Thus
$$\Delta h_{max} = mg/\kappa$$

The energy of oscillation in this case is

$$\frac{1}{2}\kappa(\Delta h_{max})^2 = \frac{1}{2}\frac{m^2g^2}{\kappa}$$

4.40 As the pan is of negligible mass, there is no loss of kinetic energy even though the collision is inelastic. The mechanical energy of the body m in the field generated by the joint action of both the gravity force and the elastic force is conserved i.e. $\Delta E = 0$. During the motion of the body m from the initial to the final (position of maximum compression of the spring) position $\Delta T = 0$, and therefore $\Delta U = \Delta U_{gr} + \Delta U_{sp} = 0$

or
$$-mg(h+x) + \frac{1}{2}\kappa x^2 = 0$$

On solving the quadratic equation :

$$x = \frac{mg}{\kappa} \pm \sqrt{\frac{m^2g^2}{\kappa^2} + \frac{2mgh}{\kappa}}$$

As minus sign is not acceptable

$$x = \frac{mg}{\kappa} + \sqrt{\frac{m^2g^2}{\kappa^2} + \frac{2mgh}{\kappa}}$$

If the body m were at rest on the spring, the corresponding position of m will be its equilibrium position and at this position the resultant force on the body m will be zero. Therefore the equilibrium compression Δx (say) due to the body m will be given by

$$\kappa\,\Delta x = mg \quad \text{or} \quad \Delta x = mg/\kappa$$

Therefore seperation between the equilibrium position and one of the extreme position i.e. the sought amplitude

$$a = x - \Delta x = \sqrt{\frac{m^2g^2}{\kappa^2} + \frac{2mgh}{\kappa}}$$

The mechanical energy of oscillation which is conserved equals $E = U_{extreme}$,because at the extreme position kinetic energy becomes zero.

Although the weight of body m is a conservative force , it is not restoring in this problem, hence $U_{extreme}$ is only concerned with the spring force. Therefore

$$E = U_{extreme} = \frac{1}{2}\kappa a^2 = m g h + \frac{m^2 g^2}{2\kappa}$$

4.41 Unlike the previous (4.40) problem the kinetic energy of body m decreases due to the perfectly inelastic collision with the pan.Obviously the body m comes to strike the pan with velocity $v_0 = \sqrt{2gh}$ .If v be the common velocity of the " body $m +$ pan " system due to the collision then from the conservation of linear momentum

$$m v_0 = (M+m) v$$

or
$$v = \frac{m v_0}{(M+m)} = \frac{m\sqrt{2gh}}{(M+m)} \quad 1)$$

At the moment the body m strikes the pan, the spring is compressed due to the weight of the pan by the amount Mg/κ . If l be the further compression of the spring due to the velocity acquired by the "pan – body m " system, then from the conservation of mechanical energy of the said system in the field generatad by the joint action of both the gravity and spring forces

$$\frac{1}{2}(M+m)v^2 + (M+m)g l = \frac{1}{2}\kappa\left(\frac{Mg}{\kappa}+l\right)^2 - \frac{1}{2}\kappa\left(\frac{Mg}{\kappa}\right)^2$$

or, $\frac{1}{2}(M+m)\frac{m^2 2gh}{(M+m)} + (M+m)gl = \frac{1}{2}\kappa\left(\frac{Mg}{\kappa}\right)^2 + \frac{1}{2}\kappa l^2 + Mgl - \frac{1}{2}\kappa\left(\frac{Mg}{\kappa}\right)^2$ (Using 1)

or,
$$\frac{1}{2}\kappa l^2 - mgl - \frac{m^2 gh}{(m+M)} = 0$$

Thus
$$l = \frac{mg \pm \sqrt{m^2 g^2 + \frac{2\kappa g h m^2}{M+m}}}{\kappa}$$

As minus sign is not acceptable

$$l = \frac{mg}{\kappa} + \frac{1}{\kappa}\sqrt{m^2 g^2 + \frac{2\kappa m^2 g h}{(M+m)}}$$

If the oscillating "pan + body m" system were at rest it correspond to their equilbrium position i.e. the spring were compressed by $\frac{(M+m)g}{\kappa}$ therefore the amplitude of oscillation

$$a = l - \frac{mg}{\kappa} = \frac{mg}{\kappa}\sqrt{1 + \frac{2h\kappa}{mg}}$$

The mechanical energy of oscillation which is only conserved with the restoring forces becomes $E = U_{extreme} = \frac{1}{2}\kappa a^2$ (Because spring force is the only restoring force not the weight of the body)

Alternately $$E = T_{mean} = \frac{1}{2}(M+m)a^2\omega^2$$

thus $$E = \frac{1}{2}(M+m)a^2\left(\frac{\kappa}{M+m}\right) = \frac{1}{2}\kappa a^2$$

4.42 We have $\vec{F} = a(\dot{y}\vec{i} - \dot{x}\vec{j})$

or, $$m(\ddot{x}\vec{i} + \ddot{y}\vec{j}) = a(\dot{y}\vec{i} - \dot{x}\vec{j})$$

So, $$m\ddot{x} = a\dot{y} \text{ and } m\ddot{y} = -a\dot{x} \quad (1)$$

From the initial condition,at $t = 0$, $\dot{x} = 0$ and $y = 0$

So, integrating Eqn,$m\ddot{x} = a\dot{y}$

we get $$m\dot{x} = ay \text{ or } \dot{x} = \frac{a}{m}y \quad (2)$$

Using Eqn (2) in the Eqn $m\ddot{y} = -a\dot{x}$, we get

$$m\ddot{y} = -\frac{a^2}{m}y \text{ or } \ddot{y} = -\left(\frac{a}{m}\right)^2 y \quad (3)$$

one of the solution of differential Eqn (3) is

$$y = A\sin(\omega_0 t + \alpha), \text{ where } \omega_0 = a/m.$$

As at $t = 0$, $y = 0$, so the solution takes the form $y = A\sin\omega_0 t$

On differentiating w.r.t. time $\dot{y} = A\,\omega_0\cos\omega_0 t$

From the initial condition of the problem, at $t = 0$, $\dot{y} = v_0$

So, $$v_0 = A\omega_0 \text{ or } A = v_0/\omega_0$$

Thus $$y = (v_0/\omega_0)\sin\omega_0 t \quad (4)$$

Thus from (2) $\dot{x} = v_0\sin\omega_0 t$ so integrating

$$x = B - \frac{v_0}{\omega_0}\cos\omega_0 t \quad (5)$$

On using $$x = 0 \text{ at } t = 0,\ B = \frac{v_0}{\omega_0}$$

Hence finally $$x = \frac{v_0}{\omega_0}(1 - \cos\omega_0 t) \quad (6)$$

Hence from Eqns (4) and (6) we get

$$[x - (v_0/\omega_0)]^2 + y^2 = (v_0/\omega_0)^2$$

which is the equation of a circle of radius (v_0/ω_0) with the centre at the point $x_0 = v_0/\omega_0$, $y_0 = 0$

4.43 If water has frozen, the system consisting of the light rod and the frozen water in the hollow sphere constitute a compound (physical) pendulum to a very good approximation because we can take the whole system to be rigid. For such systems the time period is given by

$T_1 = 2\pi\sqrt{\frac{l}{g}}\sqrt{1+\frac{k^2}{l^2}}$ where $k^2 = \frac{2}{5}R^2$ is the radius of gyration of the sphere.

The situation is different when water is unfrozen. When dissipative forces (viscosity) are neglected, we are dealing with ideal fluids. Such fluids instantaneously respond to (unbalanced) internal stresses. Suppose the sphere with liquid water actually executes small rigid oscillations. Then the portion of the fluid above the centre of the sphere will have a greater acceleration than the portion below the centre because the linear acceleration of any element is in this case, equal to angular acceleration of the element multiplied by the distance of the element from the centre of suspension (Recall that we are considering small oscillations). Then, as is obvious in a frame moving with the centre of mass, there will appear an unbalanced couple (not negated by any pseudoforces) which will cause the fluid to move rotationally so as to destroy differences in acceleration. Thus for this case of ideal fluids the pendulum must move in such a way that the elements of the fluid all undergo the same acceleration. This implies that we have a simple (mathematical) pendulum with the time period :

$$T_0 = 2\pi\sqrt{\frac{l}{g}}$$

Thus
$$T_1 = T_0\sqrt{1+\frac{2}{5}\left(\frac{R}{l}\right)^2}$$

(One expects that a liquid with very small viscosity will have a time period close T_0 while one with high viscosity will have a time period closer to T_1.)

4.44 Let us locate the rod at the position when it makes an angle θ from the vertical. In this problem both, the gravity and spring forces are restoring conservative forces, thus from the conservation of mechanical energy of oscillation of the oscillating system :

$$\frac{1}{2}\frac{m l^2}{3}(\dot{\theta})^2 + m g\frac{l}{2}(1-\cos\theta) + \frac{1}{2}\kappa(l\theta)^2 = \text{constant}$$

Differentiating w.r.t. time, we get :

$$\frac{1}{2}\frac{m l^2}{3}2\dot{\theta}\ddot{\theta} + \frac{m g l}{2}\sin\theta\,\dot{\theta} + \frac{1}{2}\kappa l^2 2\theta\dot{\theta} = 0$$

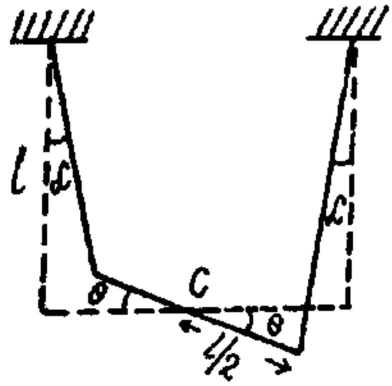
Thus for very small θ

$$\ddot{\theta} = -\frac{3g}{2l}\left(1+\frac{\kappa l}{m g}\right)\theta$$

Hence,
$$\omega_0 = \sqrt{\frac{3g}{2l}\left(1+\frac{\kappa l}{mg}\right)}.$$

4.45 (a) Let us locate the system when the threads are deviated through an angle $\alpha' < \alpha$, during the oscillations of the system (Fig.). From the conservation of mechanical energy of the system :

$$\frac{1}{2}\frac{mL^2}{12}\dot{\theta}^2 + mgl(1-\cos\alpha') = \text{constant} \qquad (1)$$



Where L is the length of the rod, θ is the angular deviation of the rod from its equilibrium position i.e. $\theta = 0$.

Differentiating Eqn. (1) w.r.t. time

$$\frac{1}{2}\frac{mL^2}{12}2\,\dot{\theta}\,\ddot{\theta} + mgl\sin\alpha'\,\dot{\alpha}' = 0$$

So, $$\frac{L^2}{12}\dot{\theta}\,\ddot{\theta} + gl\,\alpha'\,\dot{\alpha}' = 0 \text{ (for small } \alpha',\ \sin\alpha' = \alpha') \qquad (2)$$

But from the Fig.

$$\frac{L}{2}\theta = l\alpha' \text{ or } \alpha' = \frac{L}{2l}\theta$$

So, $$\dot{\alpha}' = \frac{L}{2l}\dot{\theta}$$

Putting these values of α' and $\frac{d\alpha'}{dt}$ in Eqn. (2) we get

$$\frac{d^2\theta}{dt^2} = -\frac{3g}{l}\theta$$

Thus the sought time period

$$T = \frac{2\pi}{\omega_0} = 2\pi\sqrt{\frac{l}{3g}}$$

(b) The sought oscillation energy

$$E = U_{extreme} = mgl(1-\cos\alpha) = mgl\,2\sin^2\frac{\alpha}{2}$$

$$= mgl\,2\frac{\alpha^2}{4} = \frac{mgl\alpha^2}{2} \text{ (because for small angle } \sin\theta = \theta \text{)}$$

4.46 The K.E. of the disc is $\frac{1}{2}I\dot{\varphi}^2 = \frac{1}{2}\left(\frac{mR^2}{2}\right)\dot{\varphi}^2 = \frac{1}{4}mR^2\dot{\varphi}^2$

The torsional potential energy is $\frac{1}{2}k\varphi^2$. Thus the total energy is :

$$\frac{1}{4}mR^2\dot{\varphi}^2 + \frac{1}{2}k\varphi^2 = \frac{1}{4}mR^2\dot{\varphi}_0^2 + \frac{1}{2}k\varphi_0^2$$

By definition of the amplitude φ_m, $\dot{\varphi} = 0$ when $\varphi = \varphi_m$. Thus total energy is

$$\frac{1}{2}k\varphi_m^2 = \frac{1}{4}mR^2\dot{\varphi}_0^2 + \frac{1}{2}k\varphi_0^2$$

or
$$\varphi_m = \varphi_0\sqrt{1 + \frac{mR^2}{2k}\frac{\dot{\varphi}_0^2}{\varphi_0^2}}$$

4.47 Moment of inertia of the rod equals $\frac{ml^2}{3}$ about its one end and perpendicular to its length

Thus rotational kinetic energy of the rod $= \frac{1}{2}\left(\frac{ml^2}{3}\right)\dot{\theta}^2 = \frac{ml^2}{6}\dot{\theta}^2$

when the rod is displaced by an angle θ its C.G. goes up by a distance $\frac{l}{2}(1-\cos\theta) = \frac{l\theta^2}{4}$ for small θ.

Thus the P.E. becomes : $mg\frac{l\theta^2}{4}$

As the mechanical energy of oscillation of the rod is conserved.

$$\frac{1}{2}\left(\frac{ml^2}{3}\right)\dot{\theta}^2 + \frac{1}{2}\left(\frac{mgl}{2}\right)\theta^2 = \text{Constant}$$

on differentiating w.r.t. time and for the simplifies we get : $\ddot{\theta} = -\frac{3g}{2l}\theta$ for small θ.

we see that the angular frequency ω is

$$= \sqrt{3g/2l}$$

we write the general solution of the angular oscillation as :

$$\theta = A\cos\omega t + B\sin\omega t$$

But $\theta = \theta_0$ at $t = 0$, so $A = \theta_0$

and $\dot{\theta} = \dot{\theta}_0$ at $t = 0$, so

$$B = \dot{\theta}_0/\omega$$

Thus
$$\theta = \theta_0\cos\omega t + \frac{\dot{\theta}_0}{\omega}\sin\omega t$$

Thus the K.E. of the rod

$$T = \frac{ml^2}{6}\dot{\theta}^2 = [-\omega\theta_0\sin\omega t + \dot{\theta}_0\cos\omega t]^2$$

$$= \frac{ml^2}{6}[\dot{\theta}_0^2\cos^2\omega t + \omega^2\theta_0^2\sin^2\omega t - 2\omega\theta_0\dot{\theta}_0\sin\omega t\cos\omega t]$$

On averaging over one time period the last term vanishes and $\langle\sin^2\omega t\rangle = \langle\cos^2\omega t\rangle = 1/2$. Thus

$$\langle T\rangle = \frac{1}{12}ml^2\dot{\theta}_0^2 + \frac{1}{8}mgl^2\theta_0^2 \quad (\text{where } \omega^2 = 3g/2l)$$

4.48 Let l = distance between the C.G. (C) of the pendulum and its point of suspension O Originally the pendulum is in inverted position and its C.G. is above O. When it falls to the normal (stable) position of equilibrium its C.G. has fallen by a distance $2\,l$. In the equilibrium position the total energy is equal to K.E. $= \frac{1}{2} I \omega^2$ and we have from energy conservation :

$$\frac{1}{2} I \omega^2 = m g 2 l \quad \text{or} \quad I = \frac{4 m g l}{\omega^2}$$

Angular frequency of oscillation for a physical pendulum is given by $\omega_0^2 = m g l / I$

Thus $$T = 2\pi \sqrt{\frac{I}{m g l}} = 2\pi \sqrt{\frac{4 m g l/\omega^2}{m g l}} = \frac{4\pi}{3}$$

4.49 Let, moment of inertia of the pendulum, about the axis, concerned is I, then writing $N_z = I \beta_z$, for the pendulum,

$$-m g x \sin a\, \theta = I \dot{\theta} \quad \text{or,} \quad \dot{\theta} = -\frac{m g x}{I}\, \theta \qquad \text{(For small } \theta \text{)}$$

which is the required equation for S.H.M. So, the frequency of oscillation,

$$\omega_1 = \sqrt{\frac{M g x}{I}} \quad \text{or,} \quad x = \frac{I}{M g} \sqrt{\omega_1^2} \qquad (1)$$

Now, when the mass m is attached to the pendulum, at a distance l below the oscillating axis,

$$-M g x \sin\theta' - m g l \sin\theta' = (I + m l^2) \frac{d^2 \theta'}{d t^2}$$

or, $$-\frac{g(M x + m l)}{(I + m l^2)}\, \theta' = \frac{d^2 \theta}{d t^2}, \text{ (For small } \theta \text{)}$$

which is again the equation of S.H.M., So, the new frequency,

$$\omega_2 = \sqrt{\frac{g(M x + m l)}{(I + m l^2)}} \qquad (2)$$

Solving Eqns. (1) and (2),

$$\omega_2 = \sqrt{\frac{g((I/g)\,\omega_1^2 + m l)}{(I + m l^2)}}$$

or, $$\omega_2^2 = \frac{I \omega_1^2 + m g l}{I + m l^2}$$

or, $$I(\omega_2^2 - \omega_1^2) = m g l - m \omega_2^2 l^2$$

and hence, $$I = m l^2 (\omega_2^2 - g/l) \big/ (\omega_1^2 - \omega_2^2) = 0{\cdot}8\ \text{g} \cdot \text{m}^2$$

4.50 When the two pendulums are joined rigidly and set to oscillate, each exert torques on the other, these torques are equal and opposite. We write the law of motion for the two pendulums as

$$I_1 \ddot{\theta} = -\omega_1^2 I_1 \theta + G$$

$$I_2 \ddot{\theta} = -\omega_2^2 I_2 \theta - G$$

where $\pm G$ is the torque of mutual interactions. We have written the restoring forces on each pendulum in the absence of the other as $-\omega_1^2 I_1 \theta$ and $-\omega_2^2 I_2 \theta$ respectively. Then

$$\ddot{\theta} = -\frac{I_1 \omega_1^2 + I_2 \omega_2^2}{I_1 + I_2} \theta = -\omega^2 \theta$$

Hence $$\omega = \sqrt{\frac{I_1 \omega_1^2 + I_2 \omega_2^2}{I_1 + I_2}}$$

4.51 Let us locate the rod when it is at small angular position θ relative to its equilibrium position. If a be the sought distance, then from the conservation of mechanical energy of oscillation

$$m g a (1 - \cos \theta) + \frac{1}{2} I_{OO'} (\dot{\theta})^2 = \text{constant}$$

Differentiating w.r.t. time we get :

$$m g a \sin \theta \dot{\theta} + \frac{1}{2} I_{OO'} 2 \dot{\theta} \ddot{\theta} = 0$$

But $I_{OO'} = \frac{m l^2}{12} + m a^2$ and for small θ, $\sin \theta = \theta$, we get

$$\ddot{\theta} = -\left(\frac{g a}{\frac{l^2}{12} + a^2} \right) \theta$$

Hence the time period of one full osscillation becomes

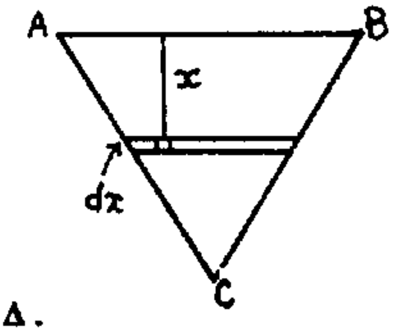
$$T = 2\pi \sqrt{\frac{\frac{l^2}{12} + a^2}{a g}} \quad \text{or} \quad T^2 = \frac{4\pi^2}{g} \left(\frac{l^2}{12 a} + a \right)$$

For T_{min}, obviously $\frac{d}{da} \left(\frac{l^2}{12 a} + a \right) = 0$

So, $$-\frac{l^2}{12 a^2} + 1 = 0 \quad \text{or} \quad a = \frac{l}{2\sqrt{3}}$$

Hence $$T_{min} = 2\pi \sqrt{\frac{l}{g\sqrt{3}}}$$

4.52 Consider the moment of inertia of the triangular plate about AB.



$$I = \iint x^2\, dm = \iint x^2 \rho\, dx\, dy$$

$$= \int_0^h x^2 \rho\, dx \frac{h-x}{h} \cdot \frac{2h}{\sqrt{3}} = \int_0^h x^2 \frac{2\rho}{\sqrt{3}} (h-x)\, dx$$

$$= \frac{2\rho}{\sqrt{3}} \left(\frac{h^4}{3} - \frac{h^4}{4} \right) = \frac{\rho h^4}{6\sqrt{3}} = \frac{m h^2}{6}$$

On using the area of the triangle $\Delta ABC = \frac{h^2}{\sqrt{3}}$ and $m = \rho \Delta$.

Thus K.E. $= \frac{1}{2} \frac{m h^2}{6} \dot{\theta}^2$

P.E. $= m g \frac{h}{3} (1 - \cos\theta) = \frac{1}{2} m g h \frac{\theta^2}{3}$

Here θ is the angle that the instantaneous plane of the plate makes with the equilibrium position which is vertical. (The plate rotates as a rigid body)

Thus $$E = \frac{1}{2} \frac{m h^2}{6} \dot{\theta}^2 + \frac{1}{2} \frac{m g h}{3} \theta^2$$

Hence $$\omega^2 = \frac{2g}{h} = \frac{m g h}{3} \Big/ \frac{m h^2}{6}$$

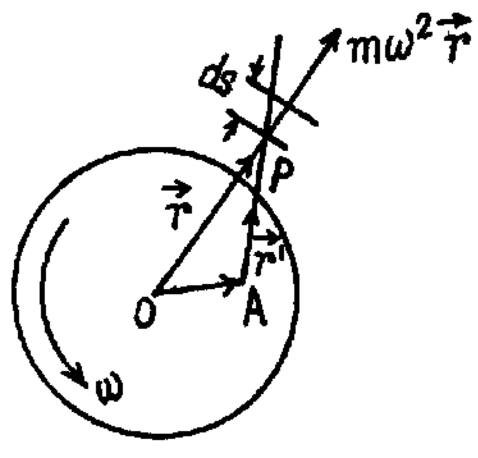
So $$T = 2\pi \sqrt{\frac{h}{2g}} = \pi \sqrt{\frac{2h}{g}}\,. \quad \text{and} \quad l_{reduced} = h/2.$$

4.53 Let us go to the rotating frame, in which the disc is stationary. In this frame the rod is subjected to coriolis and centrifugal forces, $\mathbf{F}_{cor}$ and $\mathbf{F}_{cf}$, where

$\mathbf{F}_{cor} = \int 2\, dm\, (\mathbf{v}' \times \vec{\omega}_0)$ and $\mathbf{F}_{cf} = \int dm\, \omega_0^2 \mathbf{r}$,

where $\mathbf{r}$ is the position of an elemental mass of the rod (Fig.) with respect to point O (disc's centre) and

$$\mathbf{v}' = \frac{d\mathbf{r}'}{dt}$$



As $$\mathbf{r} = \mathrm{OP} = \mathrm{OA} + \mathrm{AP}$$

So, $$\frac{d\mathbf{r}}{dt} = \frac{d(\mathrm{AP})}{dt} = \mathbf{v}' \qquad \text{(as OA is constant)}$$

As the rod is vibrating transversely, so $\mathbf{v}'$ is directed perpendicular to the length of the rod. Hence $2\, dm\, (\mathbf{v}' \times \vec{\omega})$ for each elemental mass of the rod is directed along PA. Therefore the net torque of coriolis about A becomes zero. The not torque of centrifulgal force about point A :

Now, $$\vec{\tau}_{cf(A)} = \int \mathrm{AP} \times dm\, \omega_0^2 \mathbf{r} = \int \mathrm{AP} \times \left(\frac{m}{l} \right) ds\, \omega_0^2 (\mathrm{OA} + \mathrm{AP})$$

$$= \int \mathbf{AP} \times \left(\frac{m}{l} ds \right) \omega_0^2 \, \mathbf{OA} = \int \frac{m}{l} ds \, \omega_0^2 \, s \, a \sin\theta \, (-\mathbf{k})$$

$$= \frac{m}{l} \omega_0^2 a \sin\theta \, (-\mathbf{k}) \int_0^l s \, ds = m \, \omega_0^2 \, a \frac{l}{2} \sin\theta \, (-\mathbf{k})$$

So, $$\tau_{cf(Z)} = \vec{\tau}_{cf(A)} \cdot \mathbf{k} = - m \, \omega_0^2 \, a \frac{l}{2} \sin\theta$$

According to the equation of rotational dynamics : $\tau_{A(Z)} = I_A \, \alpha_Z$

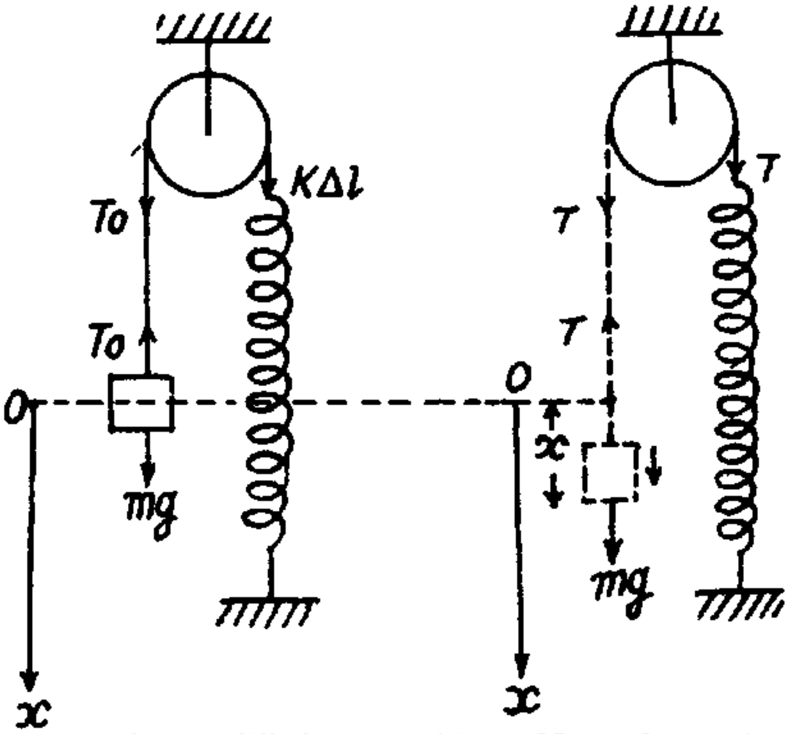
or, $$- m \, \omega_0^2 \, a \frac{l}{2} \sin\theta = \frac{m \, l^2}{3} \ddot{\theta}$$

or, $$\ddot{\theta} = - \frac{3}{2} \frac{\omega_0^2 \, a}{l} \sin\theta$$

Thus, for small θ, $$\ddot{\theta} = - \frac{3}{2} \frac{\omega^2 a}{2 l} \theta$$

This implies that the frequency ω_0 of oscillation is $\omega_0 = \sqrt{\frac{3 \, \omega^2 a}{2 l}}$

4.54 The physical system consists with a pulley and the block. Choosing an intertial frame, let us direct the x-axis as shown in the figure.



Initially the system is in equilibrium position. Now from the condition of translation equilibrium for the block

$$T_0 = m g \qquad (1)$$

Similarly for the rotational equilibrium of the pulley

$$\kappa \, \Delta/R = T_0 R$$

or. $$T_0 = \kappa \, \Delta \, l \qquad (2)$$

from Eqns. (1) and (2) $$\Delta l = \frac{m g}{\kappa} \qquad (3)$$

Now let us disturb the equilibrium of the system no matter in which way to analyse its motion. At an arbitrary position shown in the figure, from Newton's second law of motion for the block

$$F_x = m w_x$$

$$m g - T = m w = m \ddot{x} \qquad (4)$$

Similarly for the pulley

$$N_z = I \beta_z$$

$$T R - \kappa (\Delta l + x) R = I \ddot{\theta} \qquad (5)$$

But $$w = \beta R \quad \text{or,} \quad \ddot{x} = R \ddot{\theta} \qquad (6)$$

from (5) and (6) $$T R - \kappa (\Delta l + x) R = \frac{I}{R} \ddot{x} \qquad (7)$$

Solving (4) and (7) using the initial condition of the problem

$$-\kappa R x = \left(m R + \frac{I}{R} \right) \ddot{x}$$

or, $$\ddot{x} = -\left(\frac{\kappa}{m + \frac{I}{R^2}} \right) x$$

Hence the sought time period, $T = \dfrac{2\pi}{\omega_0} = 2\pi \sqrt{\dfrac{m + I/R^2}{\kappa}}$

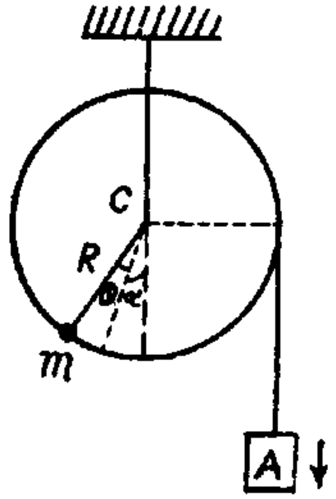
Note : we may solve this problem by using the conservation of mechanical energy also

4.55 At the equilbrium position, $N_{oz} = 0$ (Net torque about 0)

So, $$m_A g R - m g R \sin \alpha = 0 \quad \text{or} \quad m_A = m \sin \alpha \qquad (1)$$

From the equation of rotational dynamics of a solid body about the stationary axis (say z-axis) of rotation i.e. from $N_z = I \beta_z$

when the pulley is rotated by the small angular displacement θ in clockwise sense relative to the equilibrium position (Fig.), we get :



$$m_A g R - m g R \sin(\alpha + \theta)$$

$$= \left[\frac{M R^2}{2} + m R^2 + m_A R^2 \right] \ddot{\theta}$$

Using Eqn. (1)

$$m g \sin \alpha - m g (\sin \alpha \cos \theta + \cos \alpha \sin \theta)$$

$$= \left\{ \frac{M R + 2 m (1 + \sin \alpha) R}{2} \right\} \ddot{\theta}$$

But for small θ, we may write $\cos\theta \approx 1$ and $\sin\theta \approx \theta$

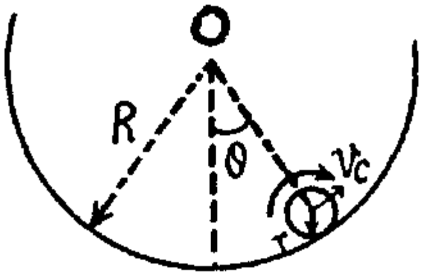
Thus we have

$$m g \sin\alpha - m g (\sin\alpha + \cos\alpha\,\theta) = \frac{\{MR + 2m(1+\sin\alpha)R\}}{2}\ddot{\theta}$$

Hence, $$\ddot{\theta} = -\frac{2 m g \cos\alpha}{[MR + 2m(1+\sin\alpha)R]}\theta$$

Hence the sought angular frequency $\omega_0 = \sqrt{\dfrac{2 m g \cos\alpha}{MR + 2mR(1+\sin\alpha)}}$

4.56 Let us locate solid cylinder when it is displaced from its stable equilibrium position by the small angle θ during its oscillations (Fig.). If v_c be the instantaneous speed of the C.M. (C) of the solid cylinder which is in pure rolling, then its angular velocity about its own centre C is



$$\omega = v_c/r \tag{1}$$

Since C moves in a circle of radius $(R-r)$, the speed of C at the same moment can be written as

$$v_c = \dot{\theta}(R-r) \tag{2}$$

Thus from Eqns (1) and (2)

$$\omega = \dot{\theta}\frac{(R-r)}{r} \tag{3}$$

As the mechanical energy of oscillation of the solid cylinder is conserved, i.e. $E = T + U =$ constant

So, $\frac{1}{2}m v_c^2 + \frac{1}{2}I_c\omega^2 + m g (R-r)(1-\cos\theta) =$ constant

(Where m is the mass of solid cylinder and I_c is the moment of inertia of the solid cylinder about an axis passing through its C.M. (C) and perpendicular to the plane of Fig. of solid cylinder)

or, $\frac{1}{2}m\omega^2 r^2 + \frac{1}{2}\frac{m r^2}{2}\omega^2 + m g (R-r)(1-\cos\theta) =$ constant (using Eqn (1) and $I_c = m r^2/2$)

$$\frac{3}{4}r^2(\dot{\theta})^2\frac{(R-r)^2}{r^2} + g(R-r)(1-\cos\theta) = \text{constant, (using Eqn. 3)}$$

Differentiating w.r.t. time

$$\frac{3}{4}(R-r)2\dot{\theta}\ddot{\theta} + g\sin\theta\,\dot{\theta} = 0$$

So, $$\ddot{\theta} = -\frac{2g}{3(R-r)}\theta, \text{ (because for small } \theta, \sin\theta \approx \theta)$$

Thus $$\omega_0 = \sqrt{\frac{2g}{3(R-r)}}$$

Hence the sought time period

$$T = \frac{2\pi}{\omega_0} = 2\pi\sqrt{\frac{3(R-r)}{2g}}$$

4.57 Let κ_1 and κ_2 be the spring constant of left and right sides springs. As the rolling of the solid cylinder is pure its lowest point becomes the instanteneous centre of rotation. If θ be the small angular displacement of its upper most point relative to its equilibrium position, the deformation of each spring becomes $(2R\theta)$. Since the mechanical energy of oscillation of the solid cylinder is conserved, $E = T + U =$ constant

i.e. $$\frac{1}{2}I_p(\dot{\theta})^2 + \frac{1}{2}\kappa_1(2R\theta)^2 + \frac{1}{2}\kappa_2(2R\theta)^2 = \text{constant}$$

Differentianting w.r.t. time

$$\frac{1}{2}I_p 2\dot{\theta}\ddot{\theta} + \frac{1}{2}(\kappa_1+\kappa_2)4R^2 2\theta\dot{\theta} = 0$$

or, $$\left(\frac{mR^2}{2} + mR^2\right)\ddot{\theta} + 4R^2\kappa\theta = 0$$

$$\left(\text{Because } I_P = I_C + mR^2 = \frac{mR^2}{2} + mR^2\right)$$

Hence $$\ddot{\theta} = -\frac{8}{3}\frac{\kappa}{m}\theta$$

Thus $\omega_0 = \frac{8\kappa}{3m}$ and sought time period

$$T = \frac{2\pi}{\omega_0} = 2\pi\sqrt{\frac{3m}{8\kappa}} = \pi\sqrt{\frac{3m}{2\kappa}}$$

4.58 In the C.M. frame (which is rigidly attached with the centre of mass of the two cubes) the cubes oscillates. We know that the kinetic energy of two body system equals $\frac{1}{2}\mu v_{rel}^2$, where μ is the reduced mass and v_{rel} is the modulus of velocity of any one body particle relative to other. From the conservation of mechanical energy of oscillation :

$$\frac{1}{2}\kappa x^2 + \frac{1}{2}\mu\left\{\frac{d}{dt}(l_0 + x)\right\}^2 = \text{constant}$$

Here l_0 is the natural length of the spring.

Differenting the above equation w.r.t time, we get :

$$\frac{1}{2}\kappa 2x\dot{x} + \frac{1}{2}\mu 2\dot{x}\ddot{x} = 0 \left[\text{becomes } \frac{d(l_0+x)}{dt} = \dot{x}\right]$$

Thus $\ddot{x} = -\frac{\kappa}{\mu}x \quad \left(\text{where } \mu = \frac{m_1 m_2}{m_1 + m_2}\right)$

Hence the natural frequency of oscillation : $\omega_0 = \sqrt{\frac{\kappa}{\mu}}$ where $\mu = \frac{m_1 m_2}{m_1 + m_2}$

4.59 Suppose the balls 1 & 2 are displaced by x_1, x_2 from their initial position. Then the energy is : $E = \frac{1}{2} m_1 \dot{x}_1^2 + m_2 \dot{x}^2 + \frac{1}{2} k (x_1 - x_2)^2 = \frac{1}{2} m_1 v_1^2$

Also total momentum is : $m_1 \dot{x}_1 + m_2 \dot{x}_2 = m_1 v_1$

Define $$X = \frac{m_1 x_1 + m_1 x_2}{m_1 + m_2}, \; x = x_1 - x_2$$

Then $$x_1 = X + \frac{m_2}{m_1 + m_2} x, \quad x_2 = X - \frac{m_1}{m_1 + m_2} x$$

$$E = \frac{1}{2} (m_1 + m_2) \dot{X}^2 + \frac{1}{2} \frac{m_1 m_2}{m_1 + m_2} \dot{x}^2 + \frac{1}{2} k x^2$$

Hence $$\dot{X} = \frac{m_1 v_1}{m_1 + m_2}$$

So $$\frac{1}{2} \frac{m_1 m_2}{m_1 + m_2} \dot{x}^2 + \frac{1}{2} k x^2 = \frac{1}{2} m_1 v_1^2 - \frac{1}{2} \frac{m_1^2 v_1^2}{m_1 + m_2} = \frac{1}{2} \frac{m_1 m_2}{m_1 + m_2} v_1^2$$

(a) From the above equation

We see $\omega = \sqrt{\frac{k}{\mu}} = \sqrt{\frac{3 \times 24}{2}} = 6 \text{ s}^{-1}$, when $\mu = \frac{m_1 m_2}{m_1 + m_2} = \frac{2}{3}$ kg.

(b) The energy of oscillation is

$$\frac{1}{2} \frac{m_1 m_2}{m_2 + m_2} v_1^2 = \frac{1}{2} \frac{2}{3} \times (0.12)^2 = 48 \times 10^{-4} = 4.8 \text{ mJ}$$

We have $x = a \sin (\omega t + \alpha)$

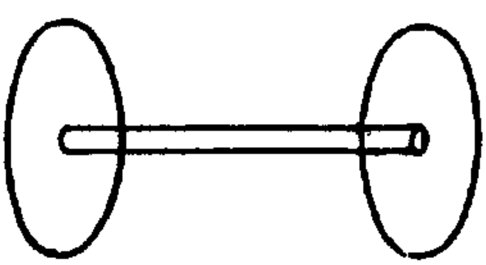
Initially $x = 0$ at $t = 0$ so $\alpha = 0$

Then $x = a \sin \omega t$. Also $\dot{x} = v_1$ at $t = 0$.

So $\omega a = v_1$ and hence $a = \frac{v_1}{\omega} = \frac{12}{6} = 2$ cm.

4.60 Suppose the disc 1 rotates by angle θ_1 and the disc 2 by angle θ_2 in the opposite sense. Then total torsion of the rod $= \theta_1 + \theta_2$

and torsional P.E. $= \frac{1}{2} \kappa (\theta_1 + \theta_2)^2$



The K.E. of the system (neglecting the moment of inertia of the rod) is

$$\frac{1}{2} I_1 \dot{\theta}_1^2 + \frac{1}{2} I_2 \dot{\theta}_2^2$$

So total energy of the rod

$$E = \frac{1}{2} I_1 \dot{\theta}_1^2 + \frac{1}{2} I_2 \dot{\theta}_2^2 + \frac{1}{2} \kappa (\theta_1 + \theta_2)^2$$

We can put the total angular momentum of the rod equal to zero since the frequency associated with the rigid rotation of the whole system must be zero (and is known).

Thus $$I_1\theta_1 = I_2\dot{\theta}_2 \quad \text{or} \quad \frac{\dot{\theta}_1}{1/I_1} = \frac{\dot{\theta}_2}{1/I_2} = \frac{\dot{\theta}_1+\dot{\theta}_2}{1/I_1+1/I_2}$$

So $$\dot{\theta}_1 = \frac{I_2}{I_1+I_2}(\dot{\theta}_1+\dot{\theta}_2) \quad \text{and} \quad \dot{\theta}_2 = \frac{I_1}{I_1+I_2}(\dot{\theta}_1+\dot{\theta}_2)$$

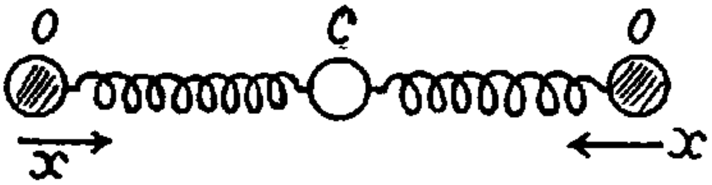
and $$E = \frac{1}{2}\frac{I_1 I_2}{I_1+I_2}(\dot{\theta}_1+\dot{\theta}_2)^2 + \frac{1}{2}\kappa(\theta_1+\theta_2)^2$$

The angular oscillation, frequency corresponding to this is

$$\omega^2 = \kappa \Big/ \frac{I_1 I_2}{I_1+I_2} = \kappa/I' \quad \text{and} \quad T = 2\pi\sqrt{\frac{I'}{\kappa}}, \text{ where } I' = \frac{I_1 I_2}{I_1+I_2}$$

4.61 In the first mode the carbon atom remains fixed and the oxygen atoms move in equal & opposite steps. Then total energy is

(1)

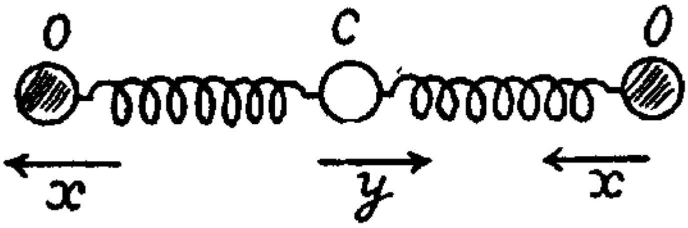


$$\frac{1}{2}2m_0\dot{x}^2 + \frac{1}{2}2\kappa x^2$$

where x is the displacement of one of the 0 atom (say left one). Thus

$$\omega_1^2 = \kappa/m_0.$$

(2)



In this mode the oxygen atoms move in equal steps in the same direction but the carbon atom moves in such a way as to keep the centre of mass fixed.

Thus $$2m_0 x + m_c y = 0 \quad \text{or, } y = -\frac{2m_0}{m_c}x$$

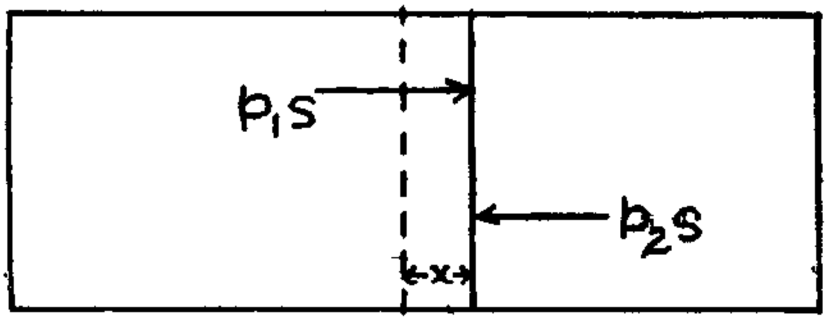
$$\text{KE.} = \frac{1}{2}2m_0\dot{x}^2 + \frac{1}{2}m_c\left(\frac{2m_0}{m_c}\dot{x}\right)^2 = \frac{1}{2}2m_0\dot{x}^2 + \frac{1}{2}2m_0\frac{2m_0}{m_c}\dot{x}^2 \qquad = \frac{1}{2}2m_0\left(1+\frac{2m_0}{m_c}\right)\dot{x}^2$$

$$\text{P.E.} = \frac{1}{2}k\left(1+\frac{2m_0}{m_c}\right)^2 x^2 \qquad + \frac{1}{2}\kappa\left(1+\frac{2m_0}{m_c}\right)^2 x^2 = \frac{1}{2}2\kappa\left(1+\frac{2m_0}{m_c}\right)^2 x^2$$

Thus $$\omega_2^2 = \frac{\kappa}{m_0}\left(1+\frac{2m_0}{m_c}\right) \quad \text{and} \quad \omega_2 = \omega_1\sqrt{1+\frac{2m_0}{m_c}}$$

Hence, $$\omega_2 = \omega_1\sqrt{1+\frac{32}{12}} = \omega_1\sqrt{\frac{11}{3}} = 1{\cdot}91\,\omega_1$$

4.62 Let, us displace the piston through small distance x, towards right, then from $F_x = m w_x$



or, $$(p_1 - p_1) S = -m \ddot{x} \tag{1}$$

But, the process is adiabatic, so from $P V^{\gamma} =$ const.

$$p_2 = \frac{p_0 V_0^{\gamma}}{(V_0 - S x)^{\gamma}} \quad \text{and} \quad p_1 = \frac{p_0 V_0^{\gamma}}{(V_0 + S x)^{\gamma}},$$

as the new volumes of the left and the right parts are now $(V_0 + S x)$ and $(V_0 - S x)$ respectively.

So, the Eqn (1) becomes.

$$\frac{p_0 V_0^{\gamma} S}{m} \left\{ \frac{1}{(V_0 - S x)^{\gamma}} - \frac{1}{(V_0 + S x)^{\gamma}} \right\} = -\ddot{x}$$

or, $$\frac{p_0 V_0^{\gamma} S}{m} \left\{ \frac{(V_0 + S x)^{\gamma} - (V_0 - S x)^{\gamma}}{(V_0^2 - S^2 x^2)^{\gamma}} \right\} = -\ddot{x}$$

or, $$\frac{p_0 V_0^{\gamma} S}{m} \left\{ \frac{\left(1 + \frac{\gamma S x}{V_0}\right) - \left(1 - \frac{\gamma S x}{V_0}\right)}{V_0^{\gamma} \left(1 - \frac{\gamma S^2 x^2}{V_0^2}\right)} \right\} = -\ddot{x}$$

Neglecting the term $\frac{\gamma S^2 x^2}{V_0^2}$ in the denominator, as it is very small, we get,

$$\ddot{x} = -\frac{2 p_0 S^2 \gamma x}{m V_0},$$

which is the equation for S.H.M. and hence the oscillating frequency.

$$\omega_0 = S \sqrt{\frac{2 p_0 \gamma}{m V_0}}$$

4.63 In the absence of the charge, the oscillation period of the ball

$$T = 2 \pi \sqrt{l/g}$$

when we impart the charge q to the ball, it will be influenced by the induced charges on the conducting plane. From the electric image method the electric force on the ball by the plane equals $\frac{q^2}{4 \pi \varepsilon_0 (2 h)^2}$ and is directed downward. Thus in this case the effective acceleration of the ball

$$g' = g + \frac{q^2}{16\pi\varepsilon_0 m h^2}$$

and the corresponding time period

$$T' = 2\pi\sqrt{\frac{l}{g'}} = 2\pi\sqrt{\frac{l}{g + \frac{q^2}{16\pi\varepsilon_0 m h^2}}}$$

From the conditon of the problem

$$T = \eta T'$$

So, $$T^2 = \eta^2 T'^2 \quad \text{or} \quad \frac{1}{g} = \eta^2\left(\frac{1}{g + \frac{q^2}{16\pi\varepsilon_0 m h^2}}\right)$$

Thus on solving

$$q = 4h\sqrt{\pi\varepsilon_0 m g(\eta^2 - 1)} = 2\,\mu C$$

4.64 In a magnetic field of induction B the couple on the magnet is $-MB\sin\theta = -MB\theta$ equating this to $I\ddot{\theta}$ we get

$$I\ddot{\theta} + MB\theta = 0$$

or $$\omega^2 = \frac{MB}{I} \quad \text{or} \quad T = 2\pi\sqrt{\frac{I}{MB}}.$$

Given $$T_2 = T_1/\eta$$

o. $$\sqrt{\frac{1}{B_2}} = \sqrt{\frac{1}{B_1}}\cdot\frac{1}{\eta} \quad \text{or} \quad \frac{1}{B_2} = \frac{1}{B_1}\cdot\frac{1}{\eta^2}$$

or $$B_2 = \eta^2 B_1$$

The induction of the field increased η^2 times.

4.65 We have in the circuit at a certain instant of time (t), from Faraday's law of electromagnetic induction :

$$L\frac{di}{dt} = Bl\frac{dx}{dt} \quad \text{or} \quad L\,di = Bl\,dx$$

As at $$t = 0,\ x = 0, \quad \text{so} \quad Li = Blx \quad \text{or} \quad i = \frac{Bl}{L}x \quad (1)$$

For the rod from the second law of motion $F_x = m\,w_x$

$$-ilB = m\ddot{x}$$

Using Eqn. (1), we get : $$\ddot{x} = -\left(\frac{l^2B^2}{mL}\right)x = -\omega_0^2 x \qquad (2)$$

where $$\omega_0 = lB\big/\sqrt{mL}$$

The solution of the above differential equation is of the form

$$x = a \sin(\omega_0 t + \alpha)$$

From the initial condition, at $t = 0$, $x = 0$, so $\alpha = 0$

Hence, $$x = a \sin \omega_0 t \qquad (3)$$

Differentiating w.r.t. time, $\dot{x} = a\,\omega_0 \cos \omega_0 t$

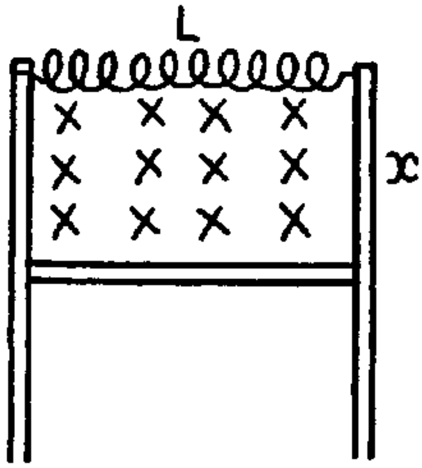
But from the initial condition of the problem at $t = 0$, $\dot{x} = v_0$

Thus $$v_0 = a\,\omega_0 \quad \text{or} \quad a = v_0/\omega_0 \qquad (4)$$

Putting the value of a from Eqn. (4) into Eqn. (3), we obtained

$$x = \frac{v_0}{\omega_0} \sin \omega_0 t \left(\text{where } \omega_0 = \frac{lB}{\sqrt{mL}} \right)$$

4.66 As the connector moves, an emf is set up in the circuit and a current flows, since the emf is $\xi = -B\,l\,\dot{x}$, we must have : $-B\,l\,\dot{x} + L\dfrac{dI}{dt} = 0$



so, $I = B\,l\,x/L$

provided x is measured from the initial position.

We then have

$$m\ddot{x} = -\frac{B\,l\,x}{L}.B.l + mg$$

for by Lenz's law the induced current will oppose downward sliding. Finally

$$\ddot{x} + \frac{(B\,l)^2}{mL} x = g$$

on putting $$\omega_0 = \frac{B\,l}{\sqrt{mL}}$$

$$\ddot{x} + \omega_0^2 x = g$$

A solution of this equation is $x = \dfrac{g}{\omega_0^2} + A\cos(\omega_0 t + \alpha)$

But $x = 0$ and $\dot{x} = 0$ at $t = 0$. This gives

$$x = \frac{g}{\omega_0^2}(1 - \cos \omega_0 t).$$

4.67 We are given $x = a_0 e^{-\beta t} \sin \omega t$

(a) The velocity of the point at $t = 0$ is obtained from

$$v_0 = (\dot{x})_{t=0} = \omega\,a_0$$

The term "oscillation amplitude at the moment $t = 0$" is meaningless. Probably the implication is the amplitude for $t << \dfrac{1}{\beta}$. Then $x = a_0 \sin \omega t$ and amplitude is a_0.

(b) $\dot{x} = (-\beta\,a_0 \sin \omega t + \omega\,a_0 \cos \omega t)\,e^{-\beta t} = 0$

when the displacement is an extremum. Then

$$\tan \omega t = \frac{\omega}{\beta}$$

or

$$\omega t = \tan^{-1} \frac{\omega}{\beta} + n\pi, \; n = 0, 1, 2, \dots$$

4.68 Given $\varphi = \varphi_0 e^{-\beta t} \cos \omega t$

we have $\dot{\varphi} = -\beta\varphi - \omega\varphi_0 e^{-\beta t} \sin \omega t$

$$\ddot{\varphi} = -\beta\dot{\varphi} + \beta\omega\varphi_0 e^{-\beta t} \sin \omega t - \omega^2 \varphi_0 e^{-\beta t} \cos \omega t$$

$$= \beta^2 \varphi + 2\beta\omega\varphi_0 e^{-\beta t} \sin \omega t - \omega^2 \varphi$$

so

(a) $(\dot{\varphi})_0 = -\beta\varphi_0, \; (\ddot{\varphi})_0 = (\beta^2 - \omega^2)\varphi_0$

(b) $\dot{\varphi} = -\varphi_0 e^{-\beta t}(\beta \cos \omega t + \omega \sin \omega t)$ becomes maximum (or minimum) when

$$\ddot{\varphi} = \varphi_0(\beta^2 - \omega^2) e^{-\beta t} \cos \omega t + 2\beta\omega\varphi_0 e^{-\beta t} \sin \omega t = 0$$

or

$$\tan \omega t = \frac{\omega^2 - \beta^2}{2\beta\omega}$$

and

$$t_n = \frac{1}{\omega}\left[\tan^{-1} \frac{\omega^2 - \beta^2}{2\beta\omega} + n\pi\right], \; n = 0, 1, 2, \dots$$

4.69 We write $x = a_0 e^{-\beta t} \cos(\omega t + \alpha)$.

(a) $x(0) = 0 \Rightarrow \alpha = \pm \frac{\pi}{2} \Rightarrow x = \mp a_0 e^{-\beta t} \sin \omega t$

$$\dot{x}(0) = (\dot{x})_{t=0} = \mp \omega a_0$$

Since a_0 is $+ve$, we must choose the upper sign if $\dot{x}(0) < 0$ and the lower sign if $\dot{x}(0) > 0$. Thus

$$a_0 = \frac{|\dot{x}(0)|}{\omega} \text{ and } \alpha = \begin{cases} +\frac{\pi}{2} & \text{if } \dot{x}(0) < 0 \\ -\frac{\pi}{2} & \text{if } \dot{x}(0) > 0 \end{cases}$$

(b) we write $x = \text{Re}\, A\, e^{-\beta t + i\omega t}$, $A = a_0 e^{i\alpha}$

Then

$$\dot{x} = v_x = \text{Re}\,(-\beta + i\omega) A e^{-\beta t + i\omega t}$$

From $v_x(0) = 0$ we get $Re\,(-\beta + i\omega) A = 0$

This implies $A = \pm i(\beta + i\omega) B$ where B is real and positive. Also

$$x_0 = \text{Re}\, A = \mp \omega B$$

Thus

$$B = \frac{|x_0|}{\omega} \text{ with } + \text{ sign in } A \text{ if } x_0 < 0$$

$-$ sign in A if $x_0 > 0$

So $$A = \pm i \frac{\beta + i\omega}{\omega} |x_0| = \left(\mp 1 + \pm \frac{i\beta}{\omega}\right)|x_0|$$

Finally $$a_0 = \sqrt{1 + \left(\frac{\beta}{\omega}\right)^2}\, |x_0|$$

$$\tan\alpha = \frac{-\beta}{\omega},\ \alpha = \tan^{-1}\left(\frac{-\beta}{\omega}\right)$$

α is in the 4[th] quadrant $\left(-\frac{\pi}{2} < \alpha < 0\right)$ if $x_0 > 0$ and α is in the 2[nd] quadrant $\left(\frac{\pi}{2} < \alpha < \pi\right)$ if $x_0 < 0$.

4.70 $x = a_0 e^{-\beta t} \cos(\omega t + \alpha)$

Then $$(\dot{x})_{t=0} = -\beta a_0 \cos\alpha - \omega a_0 \sin\alpha = 0$$

or $$\tan\alpha = -\frac{\beta}{\omega}.$$

Also $$(x)_{t=0} = a_0 \cos\alpha = \frac{a_0}{\eta}$$

$$\sec^2\alpha = \eta^2,\ \tan\alpha = -\sqrt{\eta^2 - 1}$$

Thus $$\beta = \omega\sqrt{\eta^2 - 1}$$

(We have taken the amplitude at $t = 0$ to be a_0).

4.71 We write $x = a_0 e^{-\beta t} \cos(\omega t + \alpha)$

$$= Re\, A e^{-\beta t + i\omega t},\ A = a_0 e^{i\alpha}$$

$$\dot{x} = Re\, A(-\beta + i\omega) e^{-\beta t + i\omega t}$$

Velocity amplitude as a function of time is defined in the following manner. Put $t = t_0 + \tau$, then

$$x = Re\, A e^{-\beta(t_0 + \tau)}\, e^{i\omega(t_0 + \tau)}$$

$$= Re\, A e^{-\beta t_0}\, e^{i\omega t_0 + i\omega\tau} = Re A\, e^{-\beta t_0} e^{i\omega t}$$

for $\tau << \frac{1}{\beta}$. This means that the displacement amplitude around the time t_0 is $a_0 e^{-\beta t_0}$ and we can say that the displacement amplitude at time t is $a_0 e^{-\beta t}$. Similarly for the velocity amplitude.

Clearly

(a) Velocity amplitude at time $t = a_0 \sqrt{\beta^2 + \omega^2}\, e^{-\beta t}$

Since $$A(-\beta + i\omega) = a_0 e^{i\alpha}(-\beta + i\omega)$$

$$= a_0 \sqrt{\beta^2 + \omega^2}\, e^{i\gamma}$$

where γ is anotner constant.

(b) $x(0) = 0 \Rightarrow Re\, A = 0$ or $A = \pm i a_0$
where a_0 is real and positive.

Also
$$v_x(0) = \dot{x}_0 = Re \pm i a_0 (-\beta + i\omega)$$
$$= \mp \omega a_0$$

Thus $a_0 = \dfrac{|\dot{x}_0|}{\omega}$ and we take $-(+)$ sign if $\dot{x}_0$ is negative (positive). Finally the velocity amplitude is obtained as

$$\frac{|\dot{x}_0|}{\omega} \sqrt{\beta^2 + \omega^2}\, e^{-\beta t}.$$

4.72 The first oscillation decays faster in time. But if one takes the natural time scale, the period T for each oscillation, the second oscillation attenuates faster during that period.

4.73 By definition of the logarithemic decrement $\left(\lambda = \beta \dfrac{2\pi}{\omega}\right)$ we get for the original decrement λ_0

$$\lambda_0 = \beta \frac{2\pi}{\sqrt{\omega_0^2 - \beta^2}} \quad \text{and finally } \lambda = \frac{2\pi n \beta}{\sqrt{\omega_0^2 - n^2 \beta^2}}$$

Now
$$\frac{\beta}{\sqrt{\omega_0^2 - \beta^2}} = \frac{\lambda_0}{2\pi} \quad \text{or} \quad \frac{\beta}{\omega_0} = \frac{\lambda_0/2\pi}{\sqrt{1 + \left(\dfrac{\lambda_0}{2\pi}\right)^2}}$$

so
$$\frac{\lambda/2\pi}{\sqrt{1 + \left(\dfrac{\lambda}{2\pi}\right)^2}} = \frac{n \dfrac{\lambda_0}{2\pi}}{\sqrt{1 + \left(\dfrac{\lambda_0}{2\pi}\right)^2}}$$

Hence
$$\frac{\lambda}{2\pi} = \frac{n \lambda_0 / 2\pi}{\sqrt{1 - (n^2 - 1)\left(\dfrac{\lambda_0}{2\pi}\right)^2}}$$

For critical damping
$$\omega_0 = n_c \beta$$

$$\frac{1}{n_c} = \frac{\beta}{\omega_0} = \frac{\lambda_0/2\pi}{\sqrt{1 + \left(\dfrac{\lambda_0}{2\pi}\right)^2}} \quad \text{or} \quad n_c = \sqrt{1 + \left(\frac{2\pi}{\lambda_0}\right)^2}$$

4.74 The Eqn of the dead weight is

$$m\ddot{x} + 2\beta m\dot{x} + m\omega_0^2 x = mg$$

so $$\Delta x = \frac{g}{\omega_0^2} \quad \text{or} \quad \omega_0^2 = \frac{g}{\Delta x}.$$

Now $$\lambda = \frac{2\pi\beta}{\omega} = \frac{2\pi\beta}{\sqrt{\omega_0^2 - \beta^2}} \quad \text{or} \quad \frac{\omega_0}{\sqrt{\omega_0^2 - \beta^2}} = \sqrt{1 + \left(\frac{\lambda}{2\pi}\right)^2}$$

Thus $$T = \frac{2\pi}{\sqrt{\omega_0^2 - \beta^2}} = \frac{2\pi}{\omega_0}\sqrt{1 + \left(\frac{\lambda}{2\pi}\right)^2}$$

$$= 2\pi\sqrt{\frac{\Delta x}{g}}\sqrt{1 + \left(\frac{\lambda}{2\pi}\right)^2} = \sqrt{\frac{\Delta x}{g}(4\pi^2 + \lambda^2)} = 0.70 \text{ sec.}$$

4.75 The displacement amplitude decrease η times every n oscillations. Thus

$$\frac{1}{\eta} = e^{-\beta \cdot \frac{2\pi}{\omega} \cdot n}$$

or $$\frac{2\pi n\beta}{\omega} = \ln\eta \quad \text{or} \quad \frac{\beta}{\omega} = \frac{\ln\eta}{2\pi n}.$$

So $$Q = \frac{\omega}{2\beta} = \frac{\pi n}{\ln\eta} = 499.$$

4.76 From $x = a_0 e^{-\beta t}\cos(\omega t + \alpha)$, we get using

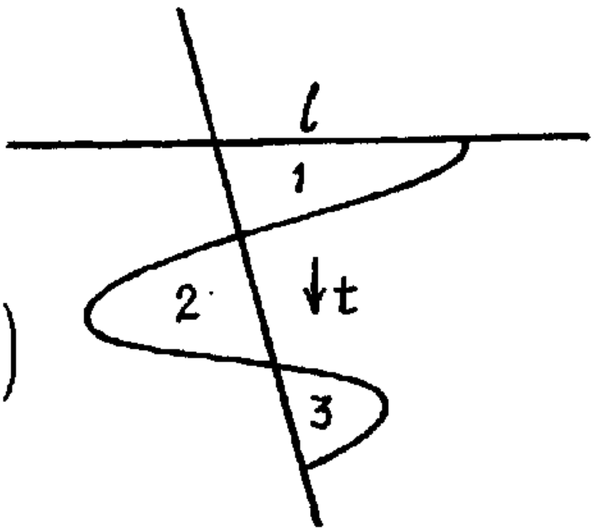
$$(x)_{t=0} = l = a_0\cos\alpha$$

$$0 = (\dot{x})_{t=0} = -\beta a_0 \cos\alpha - \omega a_0 \sin\alpha$$

Then $$\tan\alpha = -\frac{\beta}{\omega} \quad \text{or} \quad \cos\alpha = \frac{\omega}{\sqrt{\omega^2 + \beta^2}}$$

and $$x = \frac{l\sqrt{\omega^2 + \beta^2}}{\omega} e^{-\beta t}\cos\left(\omega t - \tan^{-1}\frac{\beta}{\omega}\right)$$

$$x = 0 \text{ at } t = \frac{1}{\omega}\left(n\pi + \frac{\pi}{2} + \tan^{-1}\frac{\beta}{\omega}\right)$$



Total distance travelled in the first lap $= l$

To get the maximum displacement in the second lap we note that

$$\dot{x} = \left[-\beta\cos\left(\omega t - \tan^{-1}\frac{\beta}{\omega}\right) - \omega\sin\left(\omega t - \tan^{-1}\frac{\beta}{\omega}\right)\right]$$

$$x\,\frac{l\sqrt{\omega^2 + \beta^2}}{\omega} e^{-\beta t} = 0$$

when $$\omega t = \pi, 2\pi, 3\pi, \dots \text{ etc}.$$

Thus $\qquad \ddot{x}_{max} = -a_0 e^{-\pi\beta/\omega} \cos\alpha = -l e^{-\pi\beta/\omega}$ for $t = \pi/\omega$

so, distance traversed in the 2^{nd} lap $= 2 l e^{-\pi\beta/\omega}$

Continuing total distance traversed $= l + 2 l e^{-\pi\beta/\omega} + 2 l e^{-2\pi\beta/\omega} + \ldots\ldots$

$$= l + \frac{2 l e^{-\pi\beta/\omega}}{1 - e^{-\beta\pi/\omega}} = l + \frac{2 l}{e^{\beta\pi/\omega} - 1}$$

$$= l \frac{e^{\beta\pi/\omega} + 1}{e^{\beta\pi/\omega} - 1} = l \frac{1 + e^{\lambda/2}}{e^{\lambda/2} - 1}$$

where $\lambda = \dfrac{2\pi\beta}{\omega}$ is the logarithemic decrement. Substitution gives 2 metres.

4.77 For an undamped oscillator the mechanical energy $E = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} m \omega_0^2 x^2$ is conserved. For a damped oscillator.

$$x = a_0 e^{-\beta t} \cos(\omega t + \alpha), \ \omega = \sqrt{\omega_0^2 - \beta^2}$$

and $$E(t) = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} m \omega_0^2 x^2$$

$$= \frac{1}{2} m a_0^2 e^{-2\beta t} \left[\beta^2 \cos^2(\omega t + \alpha) + 2\beta\omega \cos(\omega t + \alpha) \times \sin(\omega t + \alpha) + \omega^2 \sin^2(\omega t + \alpha)\right]$$

$$+ \frac{1}{2} m a_0^2 \omega_0^2 e^{-2\beta t} \cos^2(\omega t + \alpha)$$

$$= \frac{1}{2} m a_0^2 \omega_0^2 e^{-2\beta t} + \frac{1}{2} m a_0^2 \beta^2 e^{-2\beta t} \cos(2\omega t + 2\alpha) + \frac{1}{2} m a_0^2 \beta\omega e^{-2\beta t} \sin(2\omega t + 2\alpha)$$

If $\beta << \omega$, then the average of the last two terms over many oscillations about the time t will vanish and

$$\langle E(t) \rangle = \frac{1}{2} m a_0^2 \omega_0^2 e^{-2\beta t}$$

and this is the relevant mechanical energy.

In time τ this decreases by a factor $\dfrac{1}{\eta}$ so

$$e^{-2\beta\tau} = \frac{1}{\eta} \quad \text{or} \quad \tau = \frac{\ln\eta}{2\beta}.$$

$$\beta = \frac{\ln\eta}{2\tau}$$

and $$\lambda = \frac{2\pi\beta}{\sqrt{\omega_0^2 - \beta^2}} = \frac{2\pi}{\sqrt{\left(\frac{\omega_0}{\beta}\right)^2 - 1}} = \frac{2\pi}{\sqrt{\frac{4 g \tau^2}{l \ln^2\eta} - 1}} \quad \text{since } \omega_0^2 = \frac{g}{l}.$$

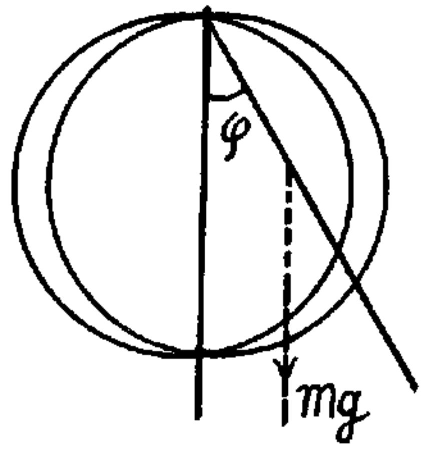
and $$Q = \frac{\pi}{\lambda} = \frac{1}{2}\sqrt{\frac{4 g \tau^2}{l \ln^2\eta} - 1} \approx 130.$$

4.78 The restoring couple is

$$\Gamma = -mgR\sin\varphi \simeq -mgR\varphi$$

The moment of inertia is

$$I = \frac{3mR^2}{2}$$



Thus for undamped oscillations

$$\frac{3mR^2}{2}\ddot{\varphi} + mgR\varphi = 0$$

so, $\omega_0^2 = \dfrac{2g}{3R}$

Also

$$\lambda = \frac{2\pi\beta}{\omega} = \frac{2\pi\beta}{\sqrt{\omega_0^2 - \beta^2}}$$

Hence

$$\frac{\beta}{\sqrt{\omega_0^2 - \beta^2}} = \frac{\lambda}{2\pi} \quad \text{or} \quad \frac{\omega_0}{\sqrt{\omega_0 - \beta^2}} = \sqrt{1 + \left(\frac{\lambda}{2\pi}\right)^2}$$

Hence finally the period T of small oscillation comes to

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{\omega_0} \times \frac{\omega_0}{\sqrt{\omega_0^2 - \beta^2}}\, 2\pi \sqrt{\frac{3R}{2g}\left(1 + \left(\frac{\lambda}{2\pi}\right)^2\right)}$$

$$= \sqrt{\frac{3R}{2g}(4\pi^2 + \lambda^2)} = 0.90 \text{ sec.}$$

4.79 Let us calculate the moment G_1 of all the resistive forces on the disc. When the disc rotates an element $(r\,dr\,d\theta)$ with coordinates (r, θ) has a velocity $r\dot{\varphi}$, where φ is the instantaneous angle of rotation from the equilibrium position and r is measured from the centre. Then

$$G_1 = \int_0^{2\pi} d\theta \int_0^R dr \cdot r \cdot (F_1 \times r)$$

$$= \int_0^R \eta\, r\dot{\varphi} r^2\, d\gamma \times 2\pi = \frac{\eta\pi R^4}{2}\dot{\varphi}$$

Also moment of inertia $= \dfrac{mR^2}{2}$

Thus

$$\frac{mR^2}{2}\ddot{\varphi} + \frac{\pi\eta R^4}{2}\dot{\varphi} + \alpha\varphi = 0$$

or

$$\ddot{\varphi} + 2\frac{\pi\eta R^2}{2m}\dot{\varphi} + \frac{2\alpha}{mR^2}\varphi = 0$$

Hence

$$\omega_0^2 = \frac{2\alpha}{mR^2} \quad \text{and} \quad \beta = \frac{\pi\eta R^2}{2m}$$

and angular frequency $\omega = \sqrt{\left(\frac{2\alpha}{mR^2}\right) - \left(\frac{\pi\eta R^2}{2m}\right)^2}$

Note :- normally by frequency we mean $\frac{\omega}{2\pi}$.

4.80 From the law of viscosity, force per unit area $= \eta \frac{dv}{dx}$

so when the disc executes torsional oscillations the resistive couple on it is

$$= \int_0^R \eta \cdot 2\pi r. \frac{r\varphi}{h} \cdot r \cdot dr \times 2 = \frac{\eta \pi R^4}{h} \dot{\varphi}$$

(factor 2 for the two sides of the disc; see the figure in the book)

where φ is torsion. The equation of motion is

$$I\ddot{\varphi} + \frac{\eta \pi R^4}{h} \dot{\varphi} + c\varphi = 0$$

Comparing with $\ddot{\varphi} + 2\beta\dot{\varphi} + \omega_0^2 \varphi = 0$ we get

$$\beta = \eta \pi R^4 / 2hI$$

Now the logarithmic decrement λ is given by $\lambda = \beta T$, $T =$ time period

Thus $\eta = 2\lambda h I / \pi R^4 T$

4.81 If $\varphi =$ angle of deviation of the frame from its normal position, then an e.m.f.

$$\varepsilon = B a^2 \dot{\varphi}$$

is induced in the frame in the displaced position and a current $\frac{\varepsilon}{R} = \frac{B a^2 \dot{\varphi}}{R}$ flows in it. A couple

$$\frac{B a^2 \dot{\varphi}}{R} \cdot B \cdot a \cdot a = \frac{B^2 a^4}{R} \dot{\varphi}$$

then acts on the frame in addition to any elastic restoring couple $c\varphi$. We write the equation of the frame as

$$I\ddot{\varphi} + \frac{B^2 a^4}{R} \dot{\varphi} + c\varphi = 0$$

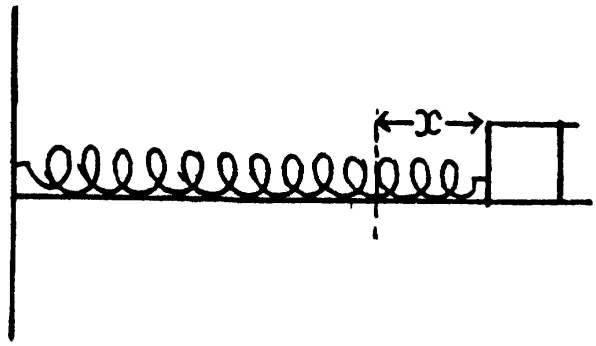
Thus $\beta = \frac{B^2 a^4}{2IR}$ where β is defined in the book.

Amplitude of oscillation die out according to $e^{-\beta t}$ so time required for the oscillations to decrease to $\frac{1}{e}$ of its value is

$$\frac{1}{\beta} = \frac{2IR}{B^2 a^4}$$

4.82 We shall denote the stiffness constant by κ. Suppose the spring is stretched by x_0 . The bar in then subject to two horizontal forces (1) restoring force $-\kappa x$ and (2) friction kmg opposing motion. If

$$x_0 > \frac{kmg}{\kappa} = \Delta$$



the bar will come back.
(If $x_0 \le \Delta$, the bar will stay put.)
The equation of the bar when it is moving to the left is

$$m\ddot{x} = -\kappa x + kmg$$

This equation has the solution

$$x = \Delta + (x_0 - \Delta) \cos\sqrt{\frac{k}{m}}\, t$$

where we have used $x = x_0, \dot{x} = 0$ at $t = 0$. This solution is only valid till the bar comes to rest. This happens at

$$t_1 = \pi \Big/ \sqrt{\frac{k}{m}}$$

and at that time $x = x_1 = 2\Delta - x_0$. if $x_0 > 2\Delta$ the tendency of the rod will now be to move to the right .(if $\Delta < x_0 < 2\Delta$ the rod will stay put now) Now the equation for rightward motion becomes

$$m\ddot{x} = -\kappa x - kmg$$

(the friction force has reversed).

We notice that the rod will move to the right only if

$$\kappa (x_0 - 2\Delta) > kmg \quad \text{i.e.} \quad x_0 > 3\Delta$$

In this case the solution is

$$x = -\Delta + (x_0 - 3\Delta) \cos\sqrt{\frac{k}{m}}\, t$$

Since $\quad x = 2\Delta - x_0$ and $\dot{x} = 0$ at $t = t_1 = \pi \Big/ \sqrt{\frac{k}{m}}$.

The rod will next come to rest at

$$t = t_2 = 2\pi \Big/ \sqrt{\frac{k}{m}}$$

and at that instant $x = x_2 = x_0 - 4\Delta$. However the rod will stay put unless $x_0 > 5\Delta$. Thus

(a) time period of one full oscillation $= 2\pi \Big/ \sqrt{\frac{k}{m}}$.

(b) There is no oscillation if $0 < x_0 < \Delta$
One half oscillation if $\quad \Delta < x_0 < 3\Delta$

2 half oscillation if $3\,\Delta < x_0 < 5\,\Delta$ etc.

We can say that the number of full oscillations is one half of the integer n

where $$n = \left[\frac{x_0 - \Delta}{2\,\Delta}\right]$$

where $[\,x\,]$ = smallest non-negative integer greater than x.

4.83 The equation of motion of the ball is

$$m(\ddot{x} + \omega_0^2 x) = F_0 \cos \omega t$$

This equation has the solution

$$x = A\cos(\omega_0 t + \alpha) + B\cos\omega t$$

where A and α are arbitrary and B is obtained by substitution in the above equation

$$B = \frac{F_0/m}{\omega_0^2 - \omega^2}$$

The conditions $x = 0$, $\dot{x} = 0$ at $t = 0$ give

$$A\cos\alpha + \frac{F_0/m}{\omega_0^2 - \omega^2} = 0 \quad \text{and} \quad -\omega_0 A \sin\alpha = 0$$

This gives $\alpha = 0$, $$A = -\frac{F_0/m}{\omega_0^2 - \omega^2} = \frac{F_0/m}{\omega^2 - \omega_0^2}$$

Finally, $$x = \frac{F_0/m}{\omega^2 - \omega_0^2}(\cos\omega_0 t - \cos\omega t)$$

4.84 We have to look for solutions of the equation

$$m\ddot{x} + kx = F, \; 0 < t_1 < \tau,$$

$$m\ddot{x} + kx = 0, \; t > \tau$$

subject to $x(0) = \dot{x}(0) = 0$ where F is constant.

The solution of this equation will be sought in the form

$$x = \frac{F}{k} + A\cos(\omega_0 t + \alpha), \; 0 \le t \le \tau$$

$$x = B\cos(\omega_0(t - \tau) + \beta), \; t > \tau$$

A and α will be determined from the boundary condition at $t = 0$.

$$0 = \frac{F}{k} + A\cos\alpha$$

$$0 = -\omega_0 A\sin\alpha$$

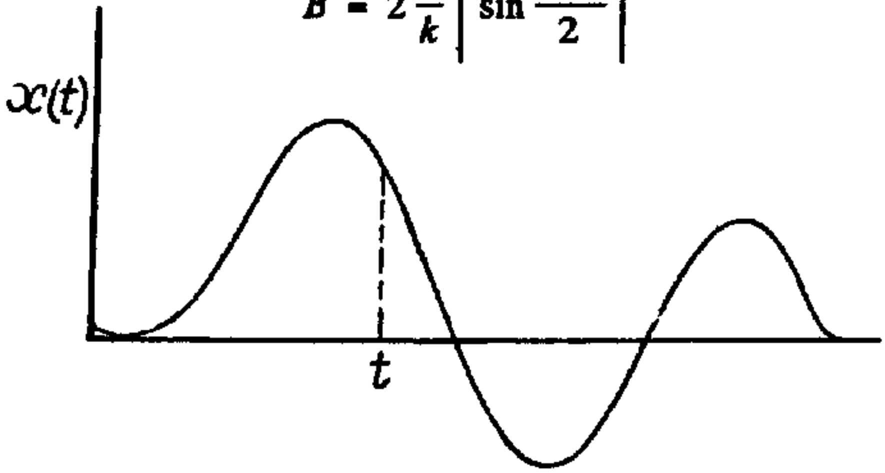
Thus $\alpha = 0$ and $A = -\dfrac{F}{k}$ and $x = \dfrac{F}{k}(1 - \cos\omega_0 t)\;\; 0 \le t < \tau.$

B and β will be determined by the continuity of x and $\dot{x}$ at $t = \tau$. Thus

$$\frac{F}{k}(1 - \cos\omega_0\tau) = B\cos\beta \quad \text{and} \quad \not{\omega_0}\,\frac{F}{k}\sin\omega_0\tau = -\not{\omega_0}\,B\sin\beta$$

Thus $$B^2 = \left(\frac{F}{k}\right)^2 (2 - 2\cos\omega_0\tau)$$

or $$B = 2\frac{F}{k}\left|\sin\frac{\omega_0\tau}{2}\right|$$



4.85 For the spring $mg = \kappa\,\Delta l$
where κ is its stifness coefficient. Thus

$$\omega_0^2 = \frac{\kappa}{m} = \frac{g}{\Delta l},$$

The equation of motion of the ball is

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = \frac{F_0}{m}\cos\omega t$$

Here $$\lambda = \frac{2\pi\beta}{\sqrt{\omega_0^2 - \beta^2}} \text{ or } \frac{\beta}{\omega} = \frac{\lambda/2\pi}{\sqrt{1 + (\lambda/2\pi)^2}}$$

To find the solution of the above equation we look for the solution of the auxiliary equation

$$\ddot{z} + 2\beta\dot{z} + \omega_0^2 z = \frac{F_0}{m}e^{i\omega t}$$

Clearly we can take $\text{Re}\, z = x$. Now we look for a particular integral for z of the form

$$z = A\,e^{i\omega t}$$

Thus, substitution gives A and we get

$$z = \frac{(F_0/m)\,e^{i\omega t}}{\omega_0^2 - \omega^2 + 2i\beta\omega}$$

so taking the real part

$$x = \frac{(F_0/m)\left[(\omega_0^2 - \omega^2)\cos\omega t + 2\beta\omega\sin\omega t\right]}{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2}$$

$$= \frac{F_0}{m}\frac{\cos(\omega t - \varphi)}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2}}, \quad \varphi = \tan^{-1}\frac{2\beta\omega}{\omega_0^2 - \omega^2}$$

The amplitude of this oscillation is maximum when the denominator is minimum.
Tthis happens when

$\omega^4 - 2\,\omega_0^2\,\omega^2 + 4\,\beta^2\,\omega^2 + \omega_0^4 = (\,\omega^2 - \omega_0^2 + 2\,\beta^2\,) + 4\,\beta^2\,\omega_0^2 - 4\,\beta^4$ is minimum. i.e for $\omega^2 = \omega_0^2 - 2\,\beta^2$

Thus
$$\omega_{res}^2 = \omega_0^2\left(1 - \frac{2\,\beta^2}{\omega_0^2}\right)$$

$$= \frac{g}{\Delta l}\left[1 - \frac{2\left(\frac{\lambda}{2\,\pi}\right)^2}{1 + \left(\frac{\lambda}{2\,\pi}\right)^2}\right] = \frac{g}{\Delta l}\,\frac{1 - \left(\frac{\lambda}{2\,\pi}\right)^2}{1 + \left(\frac{\lambda}{2\,\pi}\right)^2}$$

and
$$a_{res} = \frac{F_0/m}{\sqrt{4\,\beta^2\,\omega_0^2 - 4\,\beta^4}} = \frac{F_0/m}{2\,\beta\sqrt{\omega_0^2 - \beta^2}} = \frac{F_0/m}{2\,\beta^2}\cdot\frac{\lambda}{2\,\pi}$$

$$= \frac{F_0}{2\,m\,\omega_0^2}\cdot\frac{1 + \left(\frac{\lambda}{2\,\pi}\right)^2}{\lambda/2\,\pi} = \frac{F_0\,\Delta l\,\lambda}{4\,\pi\,m\,g}\left(1 + \frac{4\,\pi^2}{\lambda^2}\right)$$

4.86 Since $a = \dfrac{F_0/m}{\sqrt{(\,\omega^2 - \omega_0^2 + 2\,\beta^2\,)^2 + 4\,\beta^2\,(\,\omega_0^2 - \beta^2\,)}}$

we must have
$$\omega_1^2 - \omega_0^2 + 2\,\beta^2 = -\,(\,\omega_2^2 - \omega_0^2 + 2\,\beta^2\,)$$

or
$$\omega_0^2 - 2\,\beta^2 = \frac{\omega_1^2 + \omega_2^2}{2} = \omega_{res}^2$$

4.87
$$x = \frac{F_0}{m}\,\frac{(\,\omega_0^2 - \omega^2\,)\cos\omega\,t + 2\,\beta\,\omega\sin\omega\,t}{\sqrt{(\,\omega^2 - \omega_0^2\,)^2 + 4\,\beta^2\,\omega^2}}$$

Then
$$\dot{x} = \frac{F_0\,\omega}{m}\,\frac{2\,\beta\,\omega\cos\omega\,t + (\,\omega^2 - \omega_0^2\,)\sin\omega\,t}{(\,\omega_0^2 - \omega^2\,)^2 + 4\,\beta^2\,\omega^2}$$

Thus the velocity amplitude is
$$V_0 = \frac{F_0\,\omega}{m\sqrt{(\,\omega_0^2 - \omega^2\,)^2 + 4\,\beta^2\,\omega^2}}$$

$$= \frac{F_0}{m\sqrt{\left(\frac{\omega_0^2}{\omega} - \omega\right)^2 + 4\,\beta^2}}$$

This is maximum when
$$\omega^2 = \omega_0^2 = \omega_{res}^2$$

and then
$$V_{0\,res} = \frac{F_0}{2\,m\,\beta}.$$

Now at half maximum $$\left(\frac{\omega_0^2}{\omega} - \omega\right)^2 = 12\,\beta^2$$

or $$\omega^2 \pm 2\sqrt{3}\,\beta\,\omega - \omega_0^2 = 0$$

$$\omega = \mp\,\beta\sqrt{3} + \sqrt{\omega_0^2 + 3\,\beta^2}$$

where we have rejected a solution with $-\,ve$ sign before there dical. Writing

$$\omega_1 = \sqrt{\omega_0^2 + 3\,\beta^2} + \beta\sqrt{3}\ ,\quad \omega_2 = \sqrt{\omega_0^2 + 3\,\beta^2} - \beta\sqrt{3}$$

we get (a) $\omega_{res} = \omega_0 = \sqrt{\omega_1\,\omega_2}$ (Velocity resonance frequency)

(b) $\beta = \dfrac{|\,\omega_1 - \omega_2\,|}{2\sqrt{3}}$ and damped oscillation frequency

$$\sqrt{\omega_0^2 - \beta^2} = \sqrt{\omega_1\,\omega_2 - \frac{(\,\omega_1 - \omega_2\,)^2}{12}}$$

4.88 In general for displacement amplitude

$$a = \frac{F_0}{m}\,\frac{1}{\sqrt{(\,\omega_0^2 - \omega^2\,)^2 + 4\,\beta^2\,\omega^2}}$$

$$= \frac{F_0}{m}\,\frac{1}{\sqrt{(\,\omega^2 - \omega_0^2 + 2\,\beta^2\,)^2 + 4\,\beta^2\,(\,\omega_0^2 - \beta^2\,)}}$$

Thus $$\eta = \frac{a_{res}}{a_{low}} = \frac{\omega_0^2}{\sqrt{4\,\beta^2\,(\,\omega_0^2 - \beta^2\,)}} = \frac{\omega_0^2}{2\,\beta\sqrt{\omega_0^2 - \beta^2}}$$

But $$\frac{\beta}{\omega_0} = \frac{\lambda/2\,\pi}{\sqrt{1 + (\,\lambda/2\,\pi\,)^2}}\ ,\quad \frac{\lambda}{2\,\pi} = \frac{\beta}{\sqrt{\omega_0^2 - \beta^2}}$$

Hence $$\eta = \frac{\omega_0^2}{2\,\beta^2}\cdot\frac{\lambda}{2\,\pi} = \frac{1}{2}\,\frac{1 + \left(\dfrac{\lambda}{2\,\pi}\right)^2}{\dfrac{\lambda}{2\,\pi}} = 2.90$$

4.89 The work done in one cycle is

$$A = \int F\,d\,x = \int_0^T F\,\mathrm{v}\,dt = \int_0^T F_0\cos\omega\,t\,(\,-\,\omega\,a\sin(\,\omega\,t - \varphi\,)\,)\,dt$$

$$= \int_0^T F_0\,\omega\,a\,(\,-\cos\omega\,t\sin\omega\,t\cos\varphi + \cos^2\omega\,t\sin\varphi\,)\,dt$$

$$= \frac{1}{2}F_0\,\omega\,a\,\frac{T}{2}\sin\varphi = \pi\,a\,F_0\sin\varphi$$

4.90 In the formula $x = a\cos(\omega t - \varphi)$

we have
$$a = \frac{F_0}{m}\frac{1}{\sqrt{(\omega_0^2-\omega^2)^2+4\beta^2\omega^2}}$$

$$\tan\varphi = \frac{2\beta\omega}{\omega_0^2-\omega^2}$$

Thus
$$\beta = \frac{(\omega_0^2-\omega^2)\tan\varphi}{2\omega}.$$

Hence
$$\omega_0 = \sqrt{K/m} = 20\,\text{s}^{-1}.$$

and (a) the quality factor

$$Q = \frac{\pi}{\beta T} = \frac{\sqrt{\omega_0^2-\beta^2}}{2\beta} = \frac{1}{2}\sqrt{\frac{4\omega^2\omega_0^2}{(\omega_0^2-\omega^2)^2\tan^2\varphi}-1} = 2.17$$

(b) work done is $A = \pi a F_0 \sin\varphi$

$$= \pi m a^2\sqrt{(\omega_0^2-\omega^2)^2+4\beta^2\omega^2}\sin\varphi = \pi m a^2\times 2\beta\omega$$

$$= \pi m a^2(\omega_0^2-\omega^2)\tan\varphi = 6\,\text{mJ}.$$

4.91 Here as usual $\tan\varphi = \dfrac{2\beta\omega}{\omega_0^2-\omega^2}$ where φ is the phase lag of the displacement

$$x = a\cos(\omega t-\varphi),\quad a = \frac{F_0}{m}\frac{1}{\sqrt{(\omega_0^2-\omega^2)^2+4\beta^2\omega^2}}$$

(a) Mean power developed by the force over one oscillation period

$$= \frac{\pi F_0 a\sin\varphi}{T} = \frac{1}{2}F_0 a\omega\sin\varphi$$

$$= \frac{F_0^2}{m}\frac{\beta\omega^2}{(\omega_0^2-\omega^2)^2+4\beta^2\omega^2} = \frac{F_0^2\beta}{m}\frac{1}{\left(\frac{\omega_0^2}{\omega}-\omega\right)^2+4\beta^2}$$

(b) Mean power $<P>$ is maximum when $\omega = \omega_0$ (for the denominator is then minimum). Also

$$<P>_{\max} = \frac{F_0^2}{4m\beta}$$

4.92 Given $\beta = \omega_0/\eta$. Then from the previous problem

$$<P> = \frac{F_0^2\omega_0}{\eta m}\cdot\frac{1}{\left(\frac{\omega_0^2}{\omega}-\omega\right)^2+4\frac{\omega_0^2}{\eta^2}}$$

At displacement resonance $\omega = \sqrt{\omega_0^2 - 2\beta^2}$

$$<P>_{res} = \frac{F_0^2\,\omega_0}{\eta\,m}\,\frac{1}{\dfrac{4\beta^4}{\omega_0^2 - 2\beta^2} + \dfrac{4\omega_0^2}{\eta^2}} = \frac{F_0^2\,\omega_0}{\eta\,m}\,\frac{1}{\dfrac{4\omega_0^4/\eta^4}{\omega_0^2\left(1 - \dfrac{2}{\eta^2}\right)} + 4\dfrac{\omega_0^2}{\eta^2}}$$

$$= \frac{F_0^2}{4\eta\,m\,\omega_0}\,\frac{\eta^2}{\dfrac{1}{\eta^2 - 2} + 1} = \frac{F_0^2\,\eta}{4\,m\,\omega_0}\,\frac{\eta^2 - 2}{\eta^2 - 1}$$

while $$<P>_{max} = \frac{F_0^2\,\eta}{4\,m\,\omega_0}.$$

Thus $$\frac{<P>_{max} - <P>_{res}}{<P>_{max}} = \frac{100}{\eta^2 - 1}\ \%$$

4.93 The equation of the disc is $\ddot{\varphi} + 2\beta\dot{\varphi} + \omega_0^2\varphi = \dfrac{N_m\cos\omega t}{I}$

Then as before $$\varphi = \varphi_m\cos(\omega t - \alpha)$$

where $$\varphi_m = \frac{N_m}{I\left[(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2\right]^{1/2}},\quad \tan\alpha = \frac{2\beta\omega}{\omega_0^2 - \omega^2}$$

(a) Work performed by frictional forces

$$= -\int N_r\,d\varphi \quad \text{where } N_r = -2I\beta\dot{\varphi} = -\int_0^T 2\beta I\dot{\varphi}^2\,dt = -2\pi\beta\omega I\varphi_m^2$$

$$= -\pi I\varphi_m^2\left[(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2\right]^{1/2}\sin\alpha = -\pi N_m\varphi_m\sin\alpha$$

(b) The quality factor

$$Q = \frac{\pi}{\lambda} = \frac{\pi}{\beta T} = \frac{\sqrt{\omega_0^2 - \beta^2}}{2\beta} = \frac{\omega\sqrt{\omega_0^2 - \beta^2}}{(\omega_0^2 - \omega^2)\tan\alpha} = \frac{1}{2\tan\alpha}\left\{\frac{4\omega^2\omega_0^2}{(\omega_0^2 - \omega^2)^2} - \frac{4\beta^2\omega^2}{(\omega_0^2 - \omega^2)^2}\right\}^{1/2}$$

$$= \frac{1}{2\tan\alpha}\left\{\frac{4\omega^2\omega_0^2 I^2\varphi_m^2}{N_m^2\cos^2\alpha} - \tan^2\alpha\right\}^{1/2} \quad \text{since} \quad \omega_0^2 = \omega^2 + \frac{N_m}{I\varphi_m}\cos\alpha$$

$$= \frac{1}{2\sin\alpha}\left\{\frac{4\omega^2\omega_0^2 I^2\varphi_m^2}{N_m^2} - \sin^2\alpha\right\}^{1/2}$$

$$= \frac{1}{2\sin\alpha}\left\{\frac{4\omega^2 I^2\varphi_m^2}{N_m^2}\left(\omega^2 + \frac{N_m\cos\alpha}{I\varphi_m}\right) + 1 - \cos^2\alpha\right\}^{1/2}$$

$$= \frac{1}{2\sin\alpha}\left\{\frac{4I^2\varphi_m^2}{N_m^2}\omega^4 + \frac{4I\varphi_m}{N_m}\omega^2\cos\alpha + \cos^2\alpha - 1\right\}^{1/2} = \frac{1}{2\sin\alpha}\left\{\left(\frac{2I\varphi_m\omega^2}{N_m} + \cos\alpha\right)^2 - 1\right\}^{1/2}$$