Chapter 2: Mathematical Methods

EXERCISES [PAGE 29]

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Choose the correct option.

The resultant of two forces 10 N and 15 N acting along +x and - x-axes respectively, is

1. 25 N along + x-axis
2. 25 N along - x-axis
3. 5 N along + x-axis
4. **5 N along - x-axis**

SOLUTION

5 N along - x-axis

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Choose the correct option.
For two vectors to be equal, they should have the

1. same magnitude
2. same direction
3. **same magnitude and direction**
4. same magnitude but opposite direction

SOLUTION

same magnitude and direction

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Choose the correct option.
The magnitude of scalar product of two unit vectors perpendicular to each other is

1. **zero**
2. 1
3. -1
4. 2

SOLUTION

zero

Exercises | Q 1. (iv) | Page 29

Choose the correct option.

The magnitude of the vector product of two unit vectors making an angle of 60q with each other is

1

2

3/2

$\frac{\sqrt{3}}{2}$

SOLUTION

$\frac{\sqrt{3}}{2}$

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Choose the correct option.

If $\vec{A}, \vec{B}$ and $\vec{C}$ are three vectors, then which of the following is not correct?

$\vec{A} \cdot \left(\vec{B} + \vec{C}\right) = \vec{A} \cdot \vec{B} + \vec{A} \cdot \vec{C}$

$\vec{A} \cdot \vec{B} = \vec{B} \cdot \vec{A}$

$\vec{A} \times \vec{B} = \vec{B} \times \vec{A}$

$\vec{A} \times \left(\vec{B} \times \vec{C}\right) = \vec{A} \times \vec{B} + \vec{B} \times \vec{C}$

SOLUTION

$\vec{A} \times \vec{B} = \vec{B} \times \vec{A}$

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Answer the following question.

Show that $\vec{a} = \frac{\hat{i} - \hat{j}}{\sqrt{2}}$ is a unit vector.

SOLUTION

Let $\hat{a}$ be unit vector of s $\vec{a}$.

$$\therefore \hat{a} = \frac{\vec{a}}{\left|\vec{a}\right|}$$

Now, $\left|\vec{a}\right| = \sqrt{a_x^2 + a_y^2} = \sqrt{\left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{-1}{\sqrt{2}}\right)^2}$ = 1

$\therefore \hat{a} = \frac{\vec{a}}{1} \Rightarrow \vec{a}$ itself is a unit vector.

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Answer the following question.

If $\vec{v}_1 = 3\hat{i} + 4\hat{j} + \hat{k}$ and $\vec{v}_2 = \hat{i} - \hat{j} - \hat{k}$, determine the magnitude of $\vec{v}_1 + \vec{v}_2$.

SOLUTION

$$\vec{v}_1 + \vec{v}_2 = \left(3\hat{i} + 4\hat{j} + \hat{k}\right) + \left(\hat{i} - \hat{j} - \hat{k}\right)$$

$$= 3\hat{i} + \hat{i} + 4\hat{j} - \hat{j} + \hat{k} - \hat{k}$$

$$= 4\hat{i} + 3\hat{j}$$

$\therefore$ Magnitude of $\left(\vec{v}_1 + \vec{v}_2\right)$,

$\left|\vec{v}_1 + \vec{v}_2\right| = \sqrt{4^2 + 3^2} = \sqrt{25}$ = 5 units

Magnitude of $\vec{v}_1 + \vec{v}_2$ is **5 units**.

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Answer the following question.

For $\vec{v}_1 = 2\hat{i} - 3\hat{j}$ and $\vec{v}_2 = -6\hat{i} + 5\hat{j}$, determine the magnitude and direction of $\vec{v}_1 + \vec{v}_2$.

SOLUTION

$$\vec{v}_1 + \vec{v}_2 = \left(2\hat{i} - 3\hat{j}\right) + \left(-6\hat{i} + 5\hat{j}\right)$$

$$= \left(2\hat{i} - 6\hat{j}\right) + \left(-3\hat{i} + 5\hat{j}\right)$$

$$= -4\hat{i} + 2\hat{j}$$

$$\therefore \left|\vec{v}_1 + \vec{v}_2\right| = \sqrt{(-4)^2 + 2^2} = \sqrt{20} = \sqrt{4 \times 5} = 2\sqrt{5}$$

Comparing $\vec{v}_1 + \vec{v}_2$ with $\vec{R} = R_x\hat{i} + R_y\hat{j}$

$\Rightarrow R_x = -4 \text{ and } R_y = 2$

Taking θ to be angle made by R $\vec{R}$ with X – axis,

$$\therefore \theta = \tan^{-1}\left(\frac{R_y}{R_x}\right) = \tan^{-1}\left(\frac{2}{-4}\right)$$

$= \tan^{-1}\left(-\frac{1}{2}\right)$ with X-axis

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Answer the following question.

Find a vector which is parallel to $\vec{v} = \hat{i} - 2\hat{j}$ and has a magnitude 10.

SOLUTION

When two vectors are parallel, one vector is scalar multiple of another,

i.e., if $\vec{v}$ and $\vec{w}$ are parallel then, $\vec{w} = n\vec{v}$ where, n is scalar.

This means, $\vec{w} = nv_x\hat{i} + nv_y\hat{j}$

$= n\hat{i} - 2n\hat{j}$ .....$(\because v_x = l, v_y = 2)$

$$\therefore \left|\vec{w}\right| = \sqrt{(n)^2 + (-2n)^2} = \sqrt{5}n$$

Given: $\left|\overrightarrow{w}\right| = 10$

$\therefore$ n = $\frac{10}{\sqrt{5}} = 2\sqrt{5}$

$\therefore \overrightarrow{w} = 2\sqrt{5}\hat{i} - 2\left(2\sqrt{5}\right)\hat{j}$

$= 2\sqrt{5}\hat{i} - 4\sqrt{5}\hat{j}$

$= \frac{2\sqrt{5} \times \sqrt{5}}{\sqrt{5}}\hat{i} - \frac{4\sqrt{5} \times \sqrt{5}}{\sqrt{5}}\hat{j}$

$\therefore \overrightarrow{w} = \frac{10}{5}\hat{i} - \frac{20}{\sqrt{5}}\hat{j}$

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Answer the following question.

Show that vectors $\overrightarrow{a} = 2\hat{i} + 5\hat{j} - 6\hat{k}$ and $\overrightarrow{b} = \hat{i} + \frac{5}{2}\hat{j} - 3\hat{k}$ are parallel.

SOLUTION

Let the angle between the two vectors be θ.

$\therefore$ cos θ = $\frac{\overrightarrow{a} \cdot \overrightarrow{b}}{\left|\overrightarrow{a}\right|\left|\overrightarrow{b}\right|}$

$$= \frac{\left(2\hat{i} + 5\hat{j} - 6\hat{k}\right) \cdot \left(\hat{i} + \frac{5}{2}\hat{j} - 3\hat{k}\right)}{\sqrt{2^2 + 5^2 + (-6)^2} \times \sqrt{1^2 + \left(\frac{5}{2}\right)^2 + (-3)^2}}$$

$$= \frac{2 + \frac{25}{2} + 18}{\sqrt{65} \times \sqrt{65/4}}$$

$$= \frac{65/2}{65/2} = 1$$

$\Rightarrow \theta = \cos^{-1}(1) = 0°$

$\Rightarrow$ Two vectors are parallel.

Alternate method:

$$\vec{a} = 2\hat{i} + 5\hat{j} - 6\hat{k} = 2\left(\hat{i} + \frac{5}{2}\hat{j} - 3\hat{k}\right) = 2\vec{b}$$

Since $\vec{a}$ is a scalar multiple of $\vec{b}$, the vectors are parallel.

Exercises | Q 3. (i) | Page 29

Solve the following problem.

Determine $\vec{a} \times \vec{b}$, given $\vec{a} = 2\hat{i} + 3\hat{j}$ and $\vec{b} = 3\hat{i} + 5\hat{j}$.

SOLUTION

Using determinant for vectors in two dimensions,

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} \\ a_x & a_y \\ b_x & b_y \end{vmatrix} = \begin{vmatrix} \hat{i} & \hat{j} \\ 2 & 3 \\ 3 & 5 \end{vmatrix}$$

$$= [(2 \times 5) - (3 \times 3)]\hat{k} = (10 - 9)\hat{k} = \hat{k}$$

$\vec{a} \times \vec{b}$ gives $\hat{k}$

Exercises | Q 3. (ii) | Page 29

Solve the following problem.

Show that vectors $\vec{a} = 2\hat{i} + 3\hat{j} + 6\hat{k}$, $\vec{b} = 3\hat{i} - 6\hat{j} + 2\hat{k}$ and $\vec{c} = 6\hat{i} + 2\hat{j} - 3\hat{k}$ are mutually perpendicular.

SOLUTION

As dot product of two perpendicular vectors is zero. Taking dot product of $\vec{a}$ and $\vec{b}$

$\vec{a} \cdot \vec{b} = \left(2\hat{i} + 3\hat{j} + 6\hat{k}\right) \cdot \left(3\hat{i} - 6\hat{j} + 2\hat{k}\right)$

$= \left(2\hat{i} + 3\hat{i}\right) + \left(3\hat{j} \times -6\hat{j}\right) + \left(6\hat{k} \times 2\hat{k}\right)$ ...$\left(\because \hat{i} \cdot \hat{j} = \hat{j} \cdot \hat{k} = \hat{k} \cdot \hat{i} = 0\right)$

= 6 - 18 + 12 .....$\left(\because \hat{i} \cdot \hat{j} = \hat{j} \cdot \hat{k} = \hat{k} \cdot \hat{i} = 1\right)$

= 0

Similarly, $\vec{b} \cdot \vec{c} = \left(3\hat{i} - 6\hat{j} + 2\hat{k}\right) \cdot \left(6\hat{i} + 2\hat{j} - 3\hat{k}\right)$

$= \left(3\hat{i} \times 6\hat{i}\right) + \left(-6\hat{j} \times 2\hat{j}\right) + \left(2\hat{k} \times -3\hat{k}\right)$ ..... $\left(\because \hat{i} \cdot \hat{j} = \hat{j} \cdot \hat{k} = \hat{k} \cdot \hat{i} = 0\right)$

= 18 - 12 - 6 ...$\left(\because \hat{i} \cdot \hat{i} = \hat{j} \cdot \hat{i} = \hat{k} \cdot \hat{i} = 1\right)$

= 0

Combining two results, we can say that given three vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ are mutually perpendicular to each other.

Exercises | Q 3. (iii) | Page 29

Solve the following problem.

Determine the vector product of $\vec{v}_1 = 2\hat{i} + 3\hat{j} - \hat{k}$ and $\vec{v}_2 = \hat{i} + 2\hat{j} - 3\hat{k}$

SOLUTION

As $\vec{A} \times \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$

Using determinant to find vector product,

$\therefore \vec{v}_1 \times \vec{v}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & -1 \\ 1 & 2 & -3 \end{vmatrix}$

$= [(3 \times -3) - (-1 \times 2)]\hat{i} + [(-1 \times 1) - (2 \times -3)]\hat{j} + [(2 \times 2) - (3 \times 1)]\hat{k}$

$= [-9 + 2]\hat{i} + [-1 + 6]\hat{j} + [4 - 3]\hat{k}$

$= -7\hat{i} + 5\hat{j} + \hat{k}$

Exercises | Q 3. (iv) | Page 29

Solve the following problem.

Given $\overrightarrow{v}_1 = 5\hat{i} + 2\hat{j}$ and $\overrightarrow{v}_2 = a\hat{i} - 6\hat{j}$ are perpendicular to each other, determine the value of a.

SOLUTION

As $\overrightarrow{v}_1$ and $\overrightarrow{v}_2$ are perpendicular to each other, θ = 90°

$$\overrightarrow{v}_1 \cdot \overrightarrow{v}_2 = 0$$

$\therefore \left(5\hat{i} + 2\hat{j}\right) \cdot \left(a\hat{i} - 6\hat{j}\right) = 0$

$\therefore \left(5\hat{i} \cdot a\hat{i}\right) + \left(2\hat{j} \cdot 6\hat{j}\right) = 0 \; .... \left(\because \hat{i} \cdot \hat{j} = \hat{j} \cdot \hat{i} = 0\right)$

$\therefore 5a + (-12) = 0 \quad ...\left(\because \hat{i} \cdot \hat{i} = \hat{j} \cdot \hat{j} = 1\right)$

$\therefore 5a = 12$

$\therefore a = \frac{12}{5}$

Exercises | Q 3. (v)(i) | Page 29

Solve the following problem.

Obtain a derivative of the following function: x sin x

SOLUTION

Using, $\frac{d}{dx}[f_1(x) \times f_2(x)] = f_1(x)\frac{df_2(x)}{dx} + \frac{df_1(x)}{dx}f_2(x)$

For $f_1(x) = x$ and $f_2(x) = \sin x$

$$\frac{d}{dx}(x \sin x) = x\frac{d(\sin x)}{dx} + \frac{d(x)}{dx}\sin x$$

= x cos x + 1 × sin x

= sin x + x cos x

Exercises | Q 3. (v)(ii) | Page 29

Solve the following problem.

Obtain derivative of the following function: $x^4 + \cos x$

SOLUTION

Using $\frac{d}{dx}[f_1(x) + f_2(x)] = \frac{df_1(x)}{dx} + \frac{df_2(x)}{dx}$

For $f_1(x) = x^4$ and $f_2(x) = \cos x$

$$\frac{d}{dx}\left(x^4 + \cos x\right) = \frac{d\left(x^4\right)}{dx} + \frac{d(\cos x)}{dx}$$

$= 4x^3 - \sin x$

Exercises | Q 3. (v)(iii) | Page 29

Solve the following problem.

Obtain derivative of the following function: $\frac{x}{\sin x}$

SOLUTION

Using $\frac{d}{dx}\left[\frac{f_1(x)}{f_2(x)}\right] = \frac{1}{f_2(x)}\frac{df_1(x)}{dx} - \frac{f_1(x)}{f_2^2(x)}\frac{df_2(x)}{dx}$

For $f_1(x) = x$ and $f_2(x) = \sin x$

$$\therefore \frac{d}{dx}\left(\frac{x}{\sin x}\right) = \frac{1}{\sin x} \times \frac{d(x)}{dx} - \frac{x}{\sin^2 x} \times \frac{d(\sin x)}{dx}$$

$$= \frac{1}{\sin x} \times 1 - \frac{x}{\sin^2 x} \times \cos x \quad \ldots.. \left[\because \frac{d}{dx}(\sin x) = \cos x\right]$$

$$= \frac{1}{\sin x} - \frac{x \cos x}{\sin^2 x}$$

Exercises | Q 3. (vi) | Page 29

Solve the following problem.

Using the rule for differentiation for quotient of two functions, prove that $\frac{d}{dx}\left(\frac{\sin x}{\cos x}\right) = \sec^2 x$

SOLUTION

Using, $\frac{d}{dx}\left[\frac{f_1(x)}{f_2(x)}\right] = \frac{1}{f_2(x)}\frac{df_1(x)}{dx} - \frac{f_1(x)}{f_2^2(x)}\frac{df_1(x)}{dx}$

For $f_1(x) = \sin x$ and $f_2(x) = \cos x$

$$\frac{d}{dx}\left(\frac{\sin x}{\cos x}\right) = \frac{1}{\cos x} \times \frac{d(\sin x)}{dx} - \frac{\sin x}{\cos^2 x} \times \frac{d(\cos x)}{dx}$$

$$= \frac{1}{\cos x} \times \cos x - \frac{\sin x}{\cos^2 x} \times (-\sin x)$$

$$= 1 + \frac{\sin^2 x}{\cos^2 x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x}$$

$$\therefore \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right) = \frac{1}{\cos^2 x} \; .... \left[\sin^2 x + \cos^2 x = 1\right]$$

$= \sec^2 x \quad .... \left(\because \frac{1}{\cos x} = \sec x\right)$

Exercises | Q 3. (vii)(i) | Page 29

Solve the following problem.

Evaluate the following integral: $\int_0^{\frac{\pi}{2}} \sin x \, dx$

SOLUTION

Using $\int_a^b$ f(x) dx = F(x)$|_a^b$

$\therefore \int_0^{\frac{\pi}{2}}$ sin x dx = - cos x $\Big|_0^{\pi/2} = -\left[\cos\left(\frac{\pi}{2}\right) - \cos 0\right]$

Since, $\cos\left(\frac{\pi}{2}\right) = 0 \text{ and } \cos 0 = 1$

$\int_0^{\frac{\pi}{2}}$ sin x dx = - (0 - 1) = 1

Exercises | Q 3. (vii)(ii) | Page 29

Solve the following problem.

Evaluate the following integral: $\int_1^5$ x dx

SOLUTION

Using, $\int_a^b$ f(x) dx = F(x)$|_a^b$

$\int_1^5$ x dx = $\frac{x^2}{2}\Big|_1^5$

$= \frac{5^2}{2} - \frac{1^2}{2}$

$= \frac{25 - 1}{2}$

$= \frac{24}{2}$

= 12