## Chapter 6

## Applications of Derivatives

### Exercise - 6.5

Q1. Find the maximum & minimum values, if any, of the following functions.

i) $f(x) = (2x-1)^2 + 3$ (ii) $9x^2 + 12x + 2$

iii) $f(x) = -(x-1)^2 + 10$ (iv) $g(x) = x^3 + 1$

Sol: (i) $f(x) = (2x-1)^2 + 3$

As $(2x-1)^2 \geq 0 \ \forall x \in R$

$(2x-1)^2 + 3 \geq 3$

$\Rightarrow f(x) \geq 3$, Hence $f(x)$ has minimum value $= 3$, it does not have any maximum value.

ii) $f(x) = 9x^2 + 12x + 2$

$= (3x)^2 + 2(3x) \times 2 + 2$

$= [(3x)^2 + 2(3x).2 + 4] - 2$

$f(x) = (3x+2)^2 - 2$

As $(3x+2)^2 \geq 0 \ \forall x \in R$

$\therefore (3x^2+2)^2 - 2 \geq -2$

$\therefore f(x) \geq -2$, Hence $f(x)$ has minimum value $= -2$, it has no maximum value.

iii) $f(x) = -(x-1)^2 + 10$

As $(x+1)^2 \geq 0 \ \forall x \in R$

$-(x-1)^2 \leq 0$

$-(x-1)^2 + 10 \leq 10$

$\Rightarrow f(x) \leq 10$, Hence $f(x)$ has ~~ma~~ maximum value $= 10$. it has no minimum value.

iv) $g(x) = x^3 + 1$

Now $x^3 \to \infty$ as $x \to \infty$

and $x^3 \to -\infty$ as $x \to -\infty$

So, $g(x)$ has ~~neigh~~ neither maximum value, nor minimum value.

# Chapter-6

## Application of Derivatives

### Exercise 6.5

Q1 Find the maximum & minimum values, if any, of the following:-

i) $f(x) = |x+2| - 1$

Sol:- Given $f(x) = |x+2| - 1$

Now as $|x+2| \geq 0 \ \forall \ x \in R$.

So, $|x+2| \geq 0$

Adding $(-1)$ to both sides.

$|x+2| + (-1) \geq 0 + (-1)$

$\Rightarrow |x+2| - 1 \geq -1$

$\Rightarrow f(x) \geq -1$., Hence $f(x)$ has minimum value $= -1$,

Clearly $f(x) \to x$ as $x$ increases, so it has not maximum value.

ii) $g(x) = -|x+1| + 3$

Sol:- $g(x) = -|x+1| + 3$

as $|x+1| \geq 0 \ \forall \ x \in R$

$\Rightarrow -|x+1| \leq 0$ (sign of inequality reverse as we multiply by -ive value)

or $-|x+1| + 3 \leq 3$ (adding 3 both sides)

$\Rightarrow g(x) \leq 3$, Hence $g(x)$ has maximum value $= 3$

also $g(x) \to -\infty$ as $x$ increases, $\therefore$ it has no minimum value.

iii) $h(x) = \sin(2x) + 5$

Sol:- we know $|\sin 2x| \leq 1 \ \forall \ x \in R$

$\Rightarrow -1 \leq \sin 2x \leq 1 \ \forall \ x \in R$

adding 5 to each term

$-1 + 5 \leq \sin(2x) + 5 \leq 1 + 5$

$\Rightarrow 4 \leq \sin 2x + 5 \leq 6$

$\Rightarrow 4 \leq \cancel{f}h(x) \leq 6$ Hence $h(x)$ has max. value $= 6$, min value $= 4$.

iv) $f(x) = |\sin 4x + 3|$

Sol:- As $-1 \leq \sin 4x \leq 1 \ \forall \ x \in R$

adding 3 to each term

$-1 + 3 \leq \sin 4x + 3 \leq 1 + 3$

$2 \leq \sin 4x + 3 \leq 4$

$2 \leq f(x) \leq 4$

Hence $f(x)$ has max. value $= 4$, min. value $= 2$.

v) $h(x) = x+1$ , $x \in (-1, 1)$

Sol:- $x \in (-1,1)$.

$\Rightarrow \quad -1 < x < 1$

Adding 1 to all terms

$-1+1 < x+1 < 1+1$

$0 < x+1 < 2$

$0 < h(x) < 2$ , Hence $h(x)$ has neither max. value nor min. value

Q:3 Find local maxima and local minima, if any, of the following functions. Also find local maximum and local minimum values.

i) $f(x) = x^2$

ii) $g(x) = x^3 - 3x$

iii) $h(x) = \sin x + \cos x$ , $0 < x < \pi/2$

iv) $f(x) = \sin x - \cos x$ , $0 < x < 2\pi$

v) $f(x) = x^3 - 6x^2 + 9x + 15$

vi) $g(x) = \frac{x}{2} + \frac{2}{x}$ , $x > 0$

vii) $g(x) = \frac{1}{x^2+2}$

viii) $f(x) = x\sqrt{1-x}$ , $0 < x < 1$

Sol:- i) $f(x) = x^2$

$f'(x) = 2x$, $f''(x) = 2$

Now $f'(x) = 0 \Rightarrow 2x = 0$

$\Rightarrow x = 0$ (turning point)

As $f''(x) = 2$ (is positive)

So $f$ has local minima at $x=0$ and local minimum value of $f$ at $x=0$ is $f(0) = 0^2 = 0$.

ii) $g(x) = x^3 - 3x$

$g'(x) = 3x^2 - 3$ , $g''(x) = 6x$

$g'(x) = 0 \Rightarrow 3x^2 - 3 = 0$

$3(x^2 - 1) = 0$

$x^2 - 1 = 0$

$x = +1$ or $-1$

Two Cases:-

| at $x = -1$ | at $x = 1$ |
|---|---|
| $g''(x) = 6(-1)$ | $g''(x) = 6(1)$ |
| $= -6$ | $= 6$ (+ive value) |
| -ive value | local minima |
| local maxima | |

$\therefore f$ has local maximum value at $x = -1$ i.e $f(-1) = (-1)^3 - 3(-1)$

$= -1 + 3 = 2$ Ans

$f$ has local minimum value at $x = +1$, i.e $f(1) = (1)^3 - 3(1)$

$= 1 - 3 = -2$ Ans

iii) $h(x) = \sin x + \cos x$ ; $0 < x < \pi/2$

Sol: $h'(x) = \cos x - \sin x$, $h''(x) = -\sin x - \cos x$

$h'(x) = 0 \Rightarrow \cos x - \sin x = 0$

$\Rightarrow \cos x = \sin x$

$\Rightarrow \dfrac{\sin x}{\cos x} = 1$

$\tan x = 1$

$\Rightarrow x = \pi/4$ [as $x$ lies in $(0, \pi/2)$]

Now at $x = \pi/4$ $h''(\pi/4) = -\sin\frac{\pi}{4} - \cos\frac{\pi}{4} = -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} = -\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right) = -$ive value.

$\therefore x = \frac{\pi}{4}$ is point of local maxima and

local maximum value $f\left(\frac{\pi}{4}\right) = \sin\frac{\pi}{4} + \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \frac{2}{\sqrt{2}} = \sqrt{2}$

iv) $f(x) = \sin x - \cos x$ , $0 < x < 2\pi$

Sol $f'(x) = \cos x + \sin x$ , $f''(x) = -\sin x + \cos x$

$f'(x) = 0 \Rightarrow \cos x + \sin x = 0$

$\sin x = -\cos x$

$\dfrac{\sin x}{\cos x} = -1$

$\tan x = -1$ (-ive)

$\therefore x$ lies in 2nd and 4th quadrant.

$\therefore \tan x = -1 = -\tan \pi/4$

$x = \left(\pi - \frac{\pi}{4}\right)$ or $x = \left(2\pi - \frac{\pi}{4}\right)$

$x = \frac{3\pi}{4}$, $x = \frac{7}{4}\pi$

Two Cases :-

at $x = \frac{3\pi}{4}$

$f''(x) = -\sin\frac{3\pi}{4} + \cos 3\pi/4$

$= -\sin\frac{\pi}{4} - \cos\frac{\pi}{4}$

$= -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}$

$= -\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right) = -$ive

$f$ has local maxima at $x = \frac{3\pi}{4}$

at $x = \frac{7\pi}{4}$

$f''(x) = -\sin\frac{7\pi}{4} + \cos\frac{7\pi}{4}$

$= -\sin\left(2\pi - \frac{\pi}{4}\right) + \cos\left(2\pi - \frac{\pi}{4}\right)$

$= +\sin\left(\frac{\pi}{4}\right) + \cos\frac{\pi}{4}$

$= \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = +$ive.

$f$ has local minima at $x = \frac{7\pi}{4}$

Local maximum value $= f\left(\frac{3\pi}{4}\right) = \sin\frac{3\pi}{4} - \cos\frac{3\pi}{4} = \sin\left(\pi - \frac{\pi}{4}\right) - \cos\left(\pi - \frac{\pi}{4}\right)$

$= \sin\frac{\pi}{4} + \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \frac{2}{\sqrt{2}} = \sqrt{2}$

Local minimum value $= f\left(\frac{7\pi}{4}\right) = \sin\frac{7\pi}{4} - \cos\frac{7\pi}{4} = \sin\left(2\pi - \frac{\pi}{4}\right) - \cos\left(2\pi - \frac{\pi}{4}\right)$

$= -\sin\frac{\pi}{4} - \cos\frac{\pi}{4} = -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} = \frac{-2}{\sqrt{2}} = -\sqrt{2}$

(v) $f(x) = x^3 - 6x^2 + 9x + 15$

$f'(x) = 3x^2 - 12x + 9$ , $f''(x) = 6x - 12$

$f'(x) = 0 \Rightarrow 3x^2 - 12x + 9 = 0$

or $x^2 - 4x + 3 = 0$

$x^2 - 3x - x + 3 = 0$

$x(x-3) - 1(x-3) = 0$

$(x-3)(x-1) = 0$

$x = 3, 1.$

Two Cases.

at $x = 1$

$f''(x) = 6(1) - 12$

$= 6 - 12 = -6$

$-$ive

f has local maxima

at $x = 3$

$f''(3) = 6(3) - 12$

$= 18 - 12 = 6$

$+$ive

f has local minima

$\therefore$ Local maxi. value of $f = f(1) = (1)^3 - 6(1)^2 + 9(1) + 15$

$= 1 - 6 + 9 + 15 = 19$

Local min. value of $f = f(3) = (3)^3 - 6(3)^2 + 9(3) + 15$

$= 27 - 54 + 27 + 15 = 15$

vi) $g(x) = \frac{x}{2} + \frac{2}{x}$ ; $x > 0$

$g'(x) = \frac{1}{2} - \frac{2}{x^2}$ , $g''(x) = 0 - 2\left(-\frac{2}{x^3}\right) = \frac{4}{x^3}$

$g'(x) = 0 \Rightarrow \frac{1}{2} - \frac{2}{x^2} = 0$

$\Rightarrow \frac{2}{x^2} = \frac{1}{2}$

$x^2 = 4$

or $x = +2$ or $-2$

but $x > 0$, so we take only $x = 2$

$g''(2) = \frac{4}{(2)^3} = \frac{4}{8} = \frac{1}{2}$ (+ive)

$\therefore$ f has only local minima at $x = 2$

Local minimum value $= f(2)$

$= \frac{\cancel{2}}{\cancel{2}} + \frac{\cancel{2}}{\cancel{2}} = 1 + 1 = 2$

vii) $g(x) = \frac{1}{x^2+2}$

$g'(x) = \frac{d}{dx}(x^2+2)^{-1} = \frac{-1}{(x^2+2)^2} \times 2x = -\frac{2x}{(x^2+2)^2}$

$g''(x) = \frac{(x^2+2)^2(-2) - (2x)(2(x^2+2).2x)}{(x^2+2)^4} = \frac{-2(2-3x^2)}{(x^2+2)^3} = \frac{-2(2-3x^2)}{(x^2+3)^3}$

$g'(x) = 0 \Rightarrow \dfrac{-2x}{(x^2+2)^2} = 0$

$\Rightarrow \dfrac{2x}{(x^2+2)^2} = 0 \Rightarrow 2x = 0$

or $x = 0$

Now at $x=0$,

$g''(x) = \dfrac{-2(2-3(0)^2)}{((0)^2+2)^3} = \dfrac{-2(2)}{(2)^3} = \dfrac{-4}{8} = \dfrac{-1}{2}$ (-ive).

$\therefore$ $g(x)$ has local maxima at $x=0$

Local maximum value $= g(0) = \dfrac{1}{(0)^2+2} = \dfrac{1}{2}$

viii) $f(x) = x\sqrt{1-x}$

$f'(x) = x \cdot \dfrac{1}{2\sqrt{1-x}} \times (-1) + \sqrt{1-x} \cdot 1 = \dfrac{-x}{2\sqrt{1-x}} + \dfrac{\sqrt{1-x}}{1} = \dfrac{-x+2(1-x)}{2\sqrt{1-x}} = \dfrac{-x+2-2x}{2\sqrt{1-x}}$

$f'(x) = \dfrac{2-3x}{2\sqrt{1-x}}$.

$f''(x) = \dfrac{1}{2}\left[\dfrac{(\sqrt{1-x})(-3) - (2-3x)\left(\frac{1}{2\sqrt{1-x}}(-1)\right)}{(1-x)}\right] = \dfrac{1}{2}\left[\dfrac{-\frac{3\sqrt{1-x}}{1} + \frac{(2-3x)}{2\sqrt{1-x}}}{(1-x)}\right]$

$= \dfrac{1}{2}\left[\dfrac{-6(1-x)+2-3x}{2(1-x)^{3/2}}\right] = \dfrac{1}{4}\left[\dfrac{-6+6x+2-3x}{(1-x)^{3/2}}\right] = \dfrac{1}{4}\left[\dfrac{3x-4}{(1-x)^{3/2}}\right]$

$f'(x) = 0 \Rightarrow \dfrac{2-3x}{2\sqrt{1-x}} = 0 \Rightarrow 2-3x = 0 \Rightarrow x = \dfrac{2}{3}$

at $x = 2/3$

$f''(x) = \dfrac{1}{4}\left[\dfrac{3(2/3)-4}{\left(1-\frac{2}{3}\right)^{3/2}}\right] = \dfrac{1}{4}\left[\dfrac{2-4}{\left(\frac{1}{3}\right)^{3/2}}\right] = \dfrac{-2}{4\left(\frac{1}{3}\right)^{2/3}} = \dfrac{-1}{2\left(\frac{1}{3}\right)^{2/3}}$ (-ive).

$\therefore$ $f$ has local maxima. at $x = 2/3$

Max. value of $f = f\left(\frac{2}{3}\right) = \frac{2}{3}\sqrt{1-\frac{2}{3}} = \frac{2}{3}\sqrt{\frac{1}{3}} = \frac{2}{3}\sqrt{\frac{3}{9}} = \frac{2\sqrt{3}}{9}$ Ans

Q.4 Prove that the following functions do not have maxima or minima.

i) $f(x) = e^x$  ii) $g(x) = \log x$  iii) $h(x) = x^3 + x^2 + x + 1$

Sol:- i) $f(x) = e^x$

Now $f'(x) = e^x$

$f'(x) = 0 \Rightarrow e^x = 0$

but this gives no real value of $x$ for which $e^x = 0$

Therefore $f(x)$ has no maxima or minima.

ii) $g(x) = \log(x)$

Sol. $g'(x) = \frac{1}{x}$,

Now $g'(x) = 0 \Rightarrow \frac{1}{x} = 0$ or $x = \frac{1}{0}$.

But $x = \frac{1}{0}$ not possible (not exists).

$\therefore g(x)$ does not have maxima or minima.

iii) $h(x) = x^3 + x^2 + x + 1$

Sol:- $h'(x) = 3x^2 + 2x + 1$

$h'(x) = 0 \Rightarrow 3x^2 + 2x + 1.$

$$x = \frac{-2 \pm \sqrt{(2)^2 - 4(3)1}}{2 \times 3}$$

$$x = \frac{-2 \pm \sqrt{4-12}}{6} = \frac{-2 \pm \sqrt{-8}}{2}$$

Clearly $x$ is not real.

$\therefore h(x)$ has neither maxima nor minima.

Q5 Find the absolute maximum and absolute minimum value of the following.

i) $f(x) = x^3$, $x \in [-2, 2]$

ii) $f(x) = \sin x + \cos x$, $x \in [0, \pi]$

iii) $f(x) = 4x - \frac{1}{2}x^2$, $x \in [-2, \frac{9}{2}]$

iv) $f(x) = (x-1)^2 + 3$ $x \in [-3, 1]$

Sol:- i) $f(x) = x^3$

$f'(x) = 3x^2$

$f'(x) = 0 \Rightarrow 3x^2 = 0$

or $x = 0$

$\therefore$ at $x = 0$ $f(0) = (0)^3 = 0$

at $x = -2$ $f(-2) = (-2)^3 = -8$

at $x = +2$ $f(+2) = (2)^3 = 8$

$\therefore$ Absolute max. value of $f = 8$

Absolute min. value of $f = -8$

ii) $f(x) = \sin x + \cos x$ ; $x \in [0, \pi]$

Sol: $f'(x) = \cos x - \sin x$

$f'(x) = 0 \Rightarrow \cos x - \sin x = 0$

or $\sin x = \cos x$

$\Rightarrow \frac{\sin x}{\cos x} = 1$

$\Rightarrow \tan x = 1. \Rightarrow x = \pi/4.$

$f(0) = \sin(0) + \cos(0) = 0 + 1 = 1$

$f(\frac{\pi}{4}) = \sin\frac{\pi}{4} + \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \frac{2}{\sqrt{2}} = \sqrt{2}$

$f(\pi) = \sin\pi + \cos\pi = 0 - 1 = -1$

Absolute max. value of $f = \sqrt{2}$

Absolute min. value of $f = -1$

iii) $f(x) = 4x - \frac{1}{2}x^2$, $x \in [-2, \frac{9}{2}]$

$f'(x) = 4 - \frac{\cancel{2}x}{\cancel{2}} = 4 - x$

$f'(x) = 0 \Rightarrow 4 - x = 0$

$x = 4$

at $x = -2$, $f(-2) = 4(-2) - \frac{1}{2}(-2)^2$

$= -8 - \frac{1}{2} \times 4 = -8 - 2 = -10$

$f(4) = 4(4) - \frac{1}{2}(4)^2 = 16 - \frac{\cancel{16}^{8}}{\cancel{2}} = 8$

$f(\frac{9}{2}) = 4(\frac{9}{2}) - \frac{1}{2}(\frac{9}{2})^2 = 18 - \frac{81}{8} = \frac{63}{8}$

Absolute max. value of $f = 8$

Absolute min. value of $f = -10$

iv) $f(x) = (x-1)^2 + 3$, $x \in [-3, 1]$

Sol: $f'(x) = 2(x-1)$.

$f'(x) = 0 \Rightarrow 2x - 2 = 0$

or $x = 1$

at $x = -3$ $f(-3) = (-3-1)^2 + 3 = 16 + 3 = 19$

at $x = 1$, $f(1) = (1-1)^2 + 3 = 0 + 3 = 3$

Absolute max. value of $f = 19$

Absolute min. value of $f = 3$.

Q.6. Find the maximum profit a company can make, if its profit function is given by $p(x) = 41 + 72x - 18x^2$

Sol: $p(x) = 41 + 72x - 18x^2$

$p'(x) = 0 + 72 - 36x$

$p'(x) = 0 \Rightarrow +72 - 36x = 0$

$36x = 72 \Rightarrow x = 2$

$p''(x) = -36$ (-ive).

$\therefore$ Max. profit

$= p(2) = 41 + 72(2) - 18(2)^2$

$= 41 + 144 - 72 = 113$ Ans

Q7 Find both max. and min. value of $3x^4-8x^3+12x^2-48x+25$ on $[0,3]$.

Sol: Let $f(x)=3x^4-8x^3+12x^2-48x+25$

$f'(x)=12x^3-24x^2+24x-48$

$f''(x)=36x^2-48x+24$

$f'(x)=0 \Rightarrow 12x^3-24x^2+24x-48=0$

$12x(x^2-2x+2)-48=0$

$12(x^3-2x^2+2x-4)=0$

$x^3-2x^2+2x-4=0$

$x^2(x-2)+2(x-2)=0$

$(x-2)(x^2+2)=0$

$x=2, -\sqrt{-2}, +\sqrt{-2}$

As $-\sqrt{-2}, +\sqrt{-2}$ are imaginary

So rejected

At $x=2$

$f''(x)=36(2)^2-48(2)+24$

$=36(4)-96+24=144-96+24$

$=72$

Now at $x=0$

$f(0)=3(0)^4-8(0)^3+12(0)^2-48(0)+25=25$

at $x=2$

$f(2)=3(2)^4-8(2)^3+12(2)^2-48(2)+25$

$=48-64+48-96+25=-39$

at $x=3$

$f(3)=3(3)^4-8(3)^3+12(3)^2-48(3)+25$

$=243-216+108-144+25=16$

$\therefore$ Absolute max. value of $f=25$

Absolute min. value of $f=-39$.

Q8 At what point in interval $[0,2\pi]$, does $\sin 2x$ attains maximum value.

Sol. Let $f(x)=\sin 2x$

$f'(x)=2\cos 2x$

$f'(x)=0 \Rightarrow 2\cos 2x=0$

$\Rightarrow 2x=(2n+1)\frac{\pi}{2}$

$x=(2n+1)\frac{\pi}{4}$

$n=0, x=\pi/4$

$n=1, x=3\pi/4$

$n=2 \quad x=\frac{5\pi}{4}$

$n=3 \quad x=7\pi/4$

at $x=\frac{\pi}{4}$ $f(\frac{\pi}{4})=\sin\frac{2\pi}{4}=\sin\frac{\pi}{2}=1$

at $x=\frac{3\pi}{4}$, $f(\frac{3\pi}{4})=\sin 2\times\frac{3\pi}{4}=\sin\frac{3\pi}{2}=-1$

at $x=\frac{5\pi}{4}$, $f(\frac{5\pi}{4})=\sin 2\times\frac{5\pi}{4}=\sin\frac{5\pi}{2}$

$=\sin(2\pi+\frac{\pi}{2})=\sin\frac{\pi}{2}=1$

at $x=\frac{7\pi}{4}$, $f(\frac{7\pi}{4})=\sin 2(\frac{7\pi}{4})=\sin\frac{7\pi}{2}$

$=\sin(2\pi+\frac{3\pi}{2})=\sin\frac{3\pi}{2}=-1$

$\therefore f$ has absolute max. value at

$x=\frac{\pi}{4}$ & $\frac{5\pi}{4}$.

Q.9. What is the max. value of the function $\sin x + \cos x$ ?

Sol:- $f(x) = \sin x + \cos x$

$f'(x) = \cos x - \sin x$

$f'(x) = 0 \Rightarrow \cos x - \sin x = 0$

$\Rightarrow \cos x = \sin x.$

$\Rightarrow \frac{\sin x}{\cos x} = 1.$

$\Rightarrow \tan x = 1 = \tan \frac{\pi}{4}$

$\Rightarrow x = \left(n\pi + \frac{\pi}{4}\right)$

Now

$f\left(n\pi + \frac{\pi}{4}\right) = \sin\left(n\pi + \frac{\pi}{4}\right) + \cos\left(n\pi + \frac{\pi}{4}\right)$

$= (-1)^n \sin\frac{\pi}{4} + (-1)^n \cos\frac{\pi}{4}$

$= (-1)^n \frac{1}{\sqrt{2}} + (-1)^n \frac{1}{\sqrt{2}}$

$= 2(-1)^n \frac{1}{\sqrt{2}} = (-1)^n \sqrt{2}$

$\therefore$ if $n$ is odd $f\left(n\pi + \frac{\pi}{4}\right) = -\sqrt{2}$

if $n$ is even $f(n\pi + \pi/4) = +\sqrt{2}$

$\therefore$ Absolute max. value of $f = \sqrt{2}$

Absolute min. value of $f = -\sqrt{2}$.

Q.10 Find maximum value of $2x^3 - 24x + 107$ in $[1, 3]$ and in $[-3, -1]$

Sol: $f(x) = 2x^3 - 24x + 107$

$f'(x) = 6x^2 - 24$

$f'(x) = 0 \Rightarrow 6x^2 - 24 = 0$

$x^2 - 4 = 0$

$x = +2, -2$

Now for $[1, 3]$

$f(1) = 2(1)^3 - 24(1) + 107 = 2 - 24 + 107 = 85$

$f(2) = 2(2)^3 - 24(2) + 107 = 16 - 48 + 107 = 75$

$f(3) = 2(3)^3 - 24(3) + 107 = 54 - 72 + 107 = 89$

Max. value of $f = 89$ at $x = 3$.

for $[-1, -3]$.

$f(-1) = 2(-1)^3 - 24(-1) + 107 = -2 + 24 + 107 = 129$

$f(-2) = 2(-2)^3 - 24(-2) + 107 = -16 + 48 + 107 = 139$

$f(-3) = 2(-3)^3 - 24(-3) + 107 = -54 + 72 + 107 = 125$

$\therefore$ max. value of $f = 139$ at $x = -2$

Q.11 It is given $x = 1$, $x^4 - 62x^2 + ax - 9$ has maximum value on $[0, 2]$. find a.

Sol: $f(x) = x^4 - 62x^2 + ax + 9$

$f'(x) = 4x^3 - 124x + a$

As $f$ has max. at $x = 1$ $\therefore$

$f'(1) = 0 \Rightarrow 4(1)^3 - 124(1) + a = 0$

$4 - 124 + a = 0 \Rightarrow a = 124 - 4 = 120$ Ans

Q12 Find the maximum and minimum value of $x+\sin 2x$ on $[0,2\pi]$

Sol:- $f(x) = x+\sin 2x$

$f'(x) = 1+2\cos 2x$

$f'(x)=0 \Rightarrow 1+2\cos 2x = 0$

$2\cos 2x = -1$

$\cos 2x = -\frac{1}{2} = -\cos\frac{\pi}{3}$

$\cos 2x = \cos(\pi - \pi/3)$

$\cos 2x = \cos 2\frac{\pi}{3}$

$2x = 2n\pi \pm \frac{2\pi}{3}$, $n = 0, 1, 2, \ldots$

$\therefore$ ~~$n=0$, $2x$~~

or $x = n\pi \pm \frac{\pi}{3}$ — (*)

$n=0$, $x=\frac{\pi}{3}$ as we need only +ive values of $x$.

$n=1$, $x = \pi \pm \frac{\pi}{3} \Rightarrow x = \frac{2\pi}{3}$ & $\frac{4\pi}{3}$

$n=2$ $x = 2\pi \pm \frac{\pi}{3} \Rightarrow x = 2\pi - \frac{\pi}{3} = \frac{5\pi}{3}$

So $x = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$ (take $\pi = 3.14$)

$\therefore f(\frac{\pi}{3}) = \frac{\pi}{3} + \sin\frac{2\pi}{3} = \frac{\pi}{3} + \frac{\sqrt{3}}{2} = 1.05 + 0.87 = 1.92$ approx

$f(\frac{2\pi}{3}) = \frac{2\pi}{3} + \sin\frac{4\pi}{3} = \frac{2\pi}{3} - \frac{\sqrt{3}}{2} = 2.10 - 0.87 = 1.23$ approx

$f(\frac{4\pi}{3}) = \frac{4\pi}{3} + \sin\frac{8\pi}{3} = \frac{4\pi}{3} + \frac{\sqrt{3}}{2} = 4.2 + .087 = 5.07$ approx

$f(\frac{5\pi}{3}) = \frac{5\pi}{3} \sin\frac{10\pi}{3} = \frac{5\pi}{3} - \frac{\sqrt{3}}{2} = 5.25 - 0.87 = 4.38$ approx

$f(0) = 0 + \sin 2(0) = 0 + 0 = 0$

$f(2\pi) = 2\pi + \sin 2(2\pi)$

$= 2\pi + \sin 4\pi = 2\pi + 0$

$= 2\pi = 6.28$ approx.

$\therefore$ Absolute max. value of $f = 2\pi$

Absolute min value of $f = 0$

Q13 Find two numbers whose sum is 24 and product is as large as possible.

Sol: Let first number $= x$

Second number $= 24 - x$

Acc to question product of numbers is maximum.

Let Product $P = x(24-x) = 24x - x^2$

Now for $P$ to be max, $\frac{dP}{dx} = 0$, $\frac{d^2P}{dx^2} = -$ive

$\frac{dP}{dx} = 24 - 2x$, $\frac{d^2P}{dx^2} = -2$ (-ive)

$\frac{dP}{dx} = 0 \Rightarrow 24 - 2x = 0$

$\Rightarrow 24 = 2x$

$\Rightarrow x = \frac{24}{2} = 12$

$\therefore$ first number $= 12$

Second number $= 24 - 12 = 12$ Ans

**Q14** Find $x$ and $y$ such that $x+y=60$ and $xy^3$ is maximum.

Sol: $x+y=60 \Rightarrow x=60-y$

Let $P = xy^3 = (60-y)y^3 = 60y^3 - y^4$

For P to be maximum, $\frac{dP}{dx}=0$, $\frac{d^2P}{dx^2} = -ive$

$\frac{dP}{dx} = 180y^2 - 4y^3$

$\frac{dP}{dx}=0 \Rightarrow 180y^2 - 4y^3 = 0$

or $4y^2(45-y)=0$

$y=0$ or $y=45$

$\frac{d^2P}{dx^2} = 360y - 12y^2$

at $y=45$

$\frac{d^2P}{dx^2} = 360(45) - 12(45)^2$

$= 16200 - 24300 = -8100$ (-ive)

$\therefore$ P is max.

$\therefore$ $y=45$, $x = 60-45 = 15$

**Q15** Find two numbers $x$ and $y$ such that sum is 35 and product $x^2y^5$ is maximum.

Sol: Given $x+y=35 \Rightarrow y = 35-x$.

Let Product $P = x^2y^5$

$P = x^2(35-x)^5$

$\frac{dP}{dx} = x^2 \cdot 5(35-x)^4(-1) + (35-x)^5(2x)$

$\frac{dP}{dx} = 0 \Rightarrow -5x^2(35-x)^4 + (35-x)^5 2x = 0$

or $(35-x)^4 x[2(35-x) - 5x] = 0$

$x(35-x)^4[70-7x]=0$

$\Rightarrow x=0,\ 35-x=0$ or $70-7x=0$

$\underline{x=0, x=35}$, or $x=10$

rejected

at $x=10$

$\frac{d^2P}{dx^2} = x(35-x)^4(-7) + (35-x)^4(70-7x) + x(70-7x)4(35-x)^3(-1)$

$= (35-x)^3[-7x(35-x) + (35-x)(70-7x) - 4x(70-7x)]$

$= (35-10)^3[-70(25) + 25 \times 0 - 4(0)]$

$= (25)^3[-1750] = -ive$

$\therefore$ P is max.

$\therefore$ Hence $x=10$, $y = 35-10 = 25$ Ans

**Q16.** Find two positive numbers whose sum is 16 and sum of cubes is min.

Sol: Let first No. $= x$, second no $= 16-x$.

Sum of Cubes $S = x^3 + (16-x)^3$

$\frac{dS}{dx} = 3x^2 + 3(16-x)^2(-1)$

$\frac{dS}{dx} = 0 \Rightarrow 3x^2 - 3(16-x)^2 = 0$

$3x^2 - 3(256 + x^2 - 32x) = 0$

$\cancel{3x^2} - 768 - \cancel{3x^2} + 96x = 0$

$96x = 768 \Rightarrow x = \frac{768}{96} = 8$

Now $\frac{d^2S}{dx^2} = 6x - 6(16-x)(-1)$

$= \cancel{6x} + 96 - \cancel{6x} = 96$ (+ive)

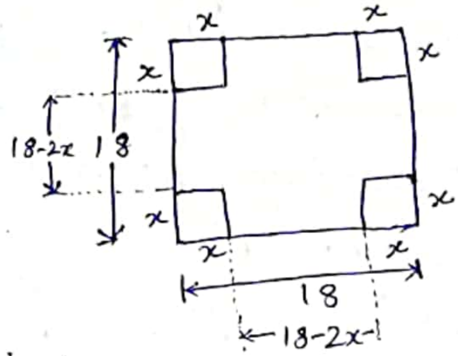
$\therefore$ S is min.

Hence $x=8$

$y = 16-8 = 8$ Ans

Q17 A square piece of tin of side 18cm is to be made into box without top. by cutting a square from each Corner and folding up the flaps. what should be the side of square to cut off so that volume of the box is maximum.

Sol: Let $x$ be the side of square to be cutt from each corner




As from the figure:-

Side of box $= 18-2x$ cm.

Let V be the volume of box.

$V = L \times B \times H = (18-2x)(18-2x)(x)$

$V = x(18-2x)^2$

Now for V to be max. $\frac{dV}{dx} = 0$, $\frac{d^2V}{dx^2} < 0$ (-ive)

$\frac{dV}{dx} = (18-2x)^2 + x \times 2(18-2x)(-2)$

$\frac{dV}{dx} = 0 \Rightarrow (18-2x)^2 - 4x(18-2x) = 0$

$(18-2x)[(18-2x)-4x] = 0$

$(18-2x)(18-6x) = 0$

$18-2x = 0$ | $18-6x=0$

$2x = 18$ | $6x = 18$

$x = 9$ | $x = 3$

$\frac{d^2V}{dx^2} = (18-2x)(-6) + (18-6x)(-2)$

$= -108 + 12x - 36 + 12x$

$= 24x - 144$.

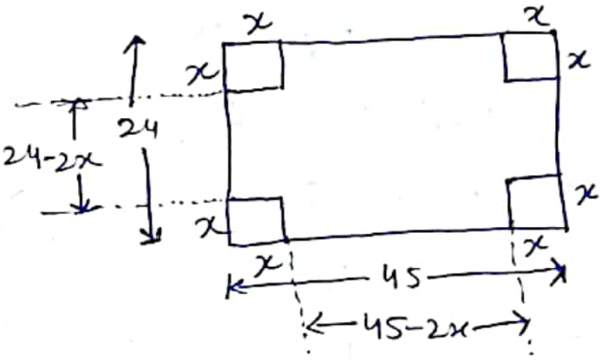
at $x=3$ $\frac{d^2V}{dx^2} = 24(3) - 144$

$= 72 - 144 = -72$ (-ive)

$\therefore$ V is max when $x = 3$

Hence side of square to be cutt off $= 3$cm Ans

Q18 A rectangular sheet of tin 45cm by 24cm is to be made into box without top, by cutting off square from each Corner and folding up the flaps. What should be the side of square to cutt off so that volume of box is maximum




Sol:- Let x cm. be the side of square to be cutt off.

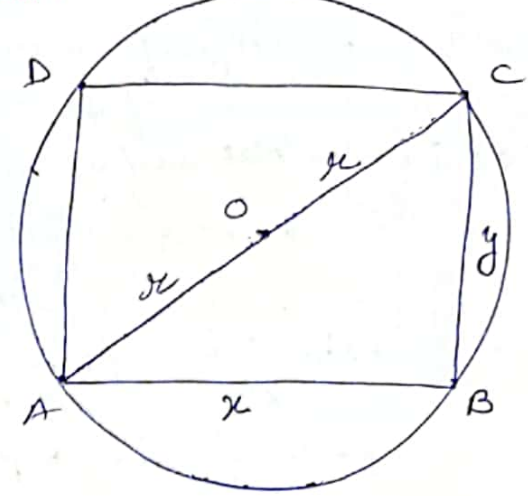
Now volume of box $V = L \times B \times h$

$= (45-2x)(24-2x)(x)$. (from figure)

$\therefore$ For V to be max - $\frac{dV}{dx} = 0$, $\frac{d^2V}{dx^2} < 0$

Do it yourself As in previous question no 17.

Q.19. Show that of all the rectangles inscribed in a given fixed circle, the square has the maximum area.




Sol: Draw a circle with centre 'O' and radius 'r'. As circle is fixed so r is constant:-

Let ABCD be the rectangle inscribed in circle with $AB = x$, $BC = y$

Now $AC = 2r$ (Diameter of circle)

By Pythagoras Theorem

$AB^2 + BC^2 = AC^2 \Rightarrow x^2 + y^2 = (2r)^2$

$x^2 + y^2 = 4r^2$ or $y^2 = 4r^2 - x^2$ or $y = \sqrt{4r^2 - x^2}$ — (*)

Now Area of Rec. ABCD $\Delta = AB \times BC = x \times y = x\sqrt{4r^2 - x^2}$

For area to be max, $\frac{d\Delta}{dx} = 0$, $\frac{d^2\Delta}{dx^2} < 0$

$$\frac{d\Delta}{dx} = \sqrt{4r^2 - x^2} \times 1 + x \cdot \frac{1}{\cancel{2}\sqrt{4r^2 - x^2}} \times -\cancel{2}x = \frac{\sqrt{4r^2 - x^2}}{1} - \frac{x^2}{\sqrt{4r^2 - x^2}}$$

$$= \frac{4r^2 - x^2 - x^2}{\sqrt{4r^2 - x^2}} = \frac{4r^2 - 2x^2}{\sqrt{4r^2 - x^2}}$$

$$\frac{d\Delta}{dx} = 0 \Rightarrow \frac{4r^2 - 2x^2}{\sqrt{4r^2 - x^2}} = 0$$

$\Rightarrow 4r^2 - 2x^2 = 0$

or $2x^2 = 4r^2$

$x^2 = 2r^2$

$x = \sqrt{2}r$

We reject negative value as x is length of rectangle

Now. $$\frac{d^2\Delta}{dx^2} = \frac{\sqrt{4r^2 - x^2}\cdot(-4x) - (4r^2 - 2x^2) \times \frac{-x}{\sqrt{4r^2 - x^2}}}{(4r^2 - x^2)}$$

$$= \frac{-4x(4r^2 - x^2) + 4r^2x - 2x^3}{(4r^2 - x^2)^{3/2}}$$

$$= \frac{-16xr^2 + 4x^3 + 4r^2x - 2x^3}{(4r^2 - x^2)^{3/2}} = \frac{2x^3 - 12xr^2}{(4r^2 - x^2)^{3/2}}$$

at $x = \sqrt{2}r$

$$\frac{d^2\Delta}{dx^2} = \frac{2(\sqrt{2}r)^3 - 12(\sqrt{2}r)r^2}{(4r^2 - 2r^2)^{3/2}} = \frac{4\sqrt{2}r^3 - 12\sqrt{2}r^3}{(2r^2)^{3/2}}$$

$$= \frac{-8\sqrt{2}r^3}{(2r^2)^{3/2}} \text{ (-ive)}$$

Hence $\Delta$ is max. when

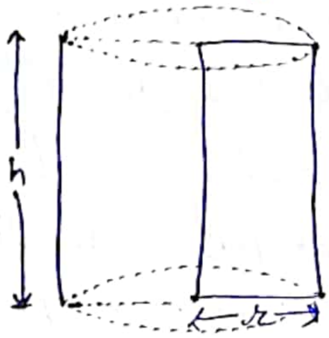
$x = \sqrt{2}r$

$\therefore y = \sqrt{4r^2 - 2r^2} = \sqrt{2r^2} = \sqrt{2}r$ i.e

$x = y \Rightarrow$ Rectangle is a square.

Q.20 Show that right circular cylinder of given surface area and maximum volume is such that its height is equal to diameter of base.

Sol. Let $h$ = height of cylinder.
$r$ = radius of its base




Let $S$ be the total surface area of cylinder, then

$$S = \pi r^2 + \pi r^2 + 2\pi r h = 2\pi r^2 + 2\pi r h$$

$$S = 2\pi(r^2 + rh)$$

or $r^2 + rh = \frac{S}{2\pi}$ suppose $\frac{S}{2\pi} = p$ (constant)

$$r^2 + rh = p \Rightarrow rh = p - r^2$$

$$h = \left(\frac{p - r^2}{r}\right) \quad \text{—} \circledast$$

Volume of Cylinder $= \pi r^2 h$.

$$V = \pi(r^2) \cdot \left(\frac{p - r^2}{r}\right) = \pi r(p - r^2)$$

For volume to be maximum.

$$\frac{dV}{dr} = 0, \quad \frac{d^2V}{dr^2} < 0$$

$$\frac{dv}{dr} = \pi[r(0 - 2r) + (p - r^2)1]$$

$$= \pi(-2r^2 + p - r^2]$$

$$= \pi(p - 3r^2).$$

$$\frac{dv}{dr} = 0 \Rightarrow \pi(p - 3r^2) = 0$$

$$\Rightarrow p - 3r^2 = 0$$

$$3r^2 = p$$

$$r^2 = p/3.$$

$$\Rightarrow r = \sqrt{\frac{p}{3}}$$

$$\frac{d^2v}{dr^2} = \pi(0 - 6r)$$

$$= -6\pi r \ (-\text{ive})$$

$\therefore$ volume is max.

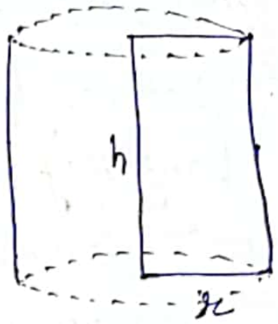
$$h = \frac{p - r^2}{r} = \frac{p - p/3}{\sqrt{p/3}} = \frac{2p/3}{\sqrt{p/3}}.$$

$$h = 2 \cdot \sqrt{p/3} = 2r$$

i.e. $h = 2r$

height = diameter of the base.

Q21 Of all the closed cylindrical cans, of given volume of 100 cubic cm, find the dimensions of can which has minimum surface area.




Sol:- Let $h$ = height of cylindrical can.
$r$ = radius of base.

If $S$ be the surface area and $V$ be the volume of can.

$S = 2\pi r^2 + 2\pi r h, \quad V = \pi r^2 h.$

Acc. to question $V = 100$

$\pi r^2 h = 100$

$h = \frac{100}{\pi r^2}$ — (*)

$S = 2\pi(r^2 + hr)$

$= 2\pi\left(r^2 + \frac{100}{\pi r^{\cancel{2}}} \cdot \cancel{r}\right)$

$S = 2\pi\left(r^2 + \frac{100}{\pi r}\right)$

for $S$ to be maximum, $\frac{dS}{dr} = 0$, $\frac{d^2S}{dr^2} < 0$

$\frac{dS}{dr} = 2\pi\left(2r - \frac{100}{\pi r^2}\right)$

$\frac{dS}{dr} = 0 \Rightarrow 2\pi\left(2r - \frac{100}{\pi r^2}\right) = 0$

$\Rightarrow \frac{2r}{1} - \frac{100}{\pi r^2} = 0$

$\frac{2\pi r^3 - 100}{\pi r^2} = 0$

$\Rightarrow 2\pi r^3 - 100 = 0$

$2\pi r^3 = 100$

$r^3 = \frac{\cancel{100}\,50}{\cancel{2}\pi}$

$r = \left(\frac{50}{\pi}\right)^{\frac{1}{3}}$

$\frac{d^2S}{dr^2} = 2\pi\left(2 - \frac{100}{\pi}\left(\frac{-2}{r^3}\right)\right)$

$= 2\pi\left(2 + \frac{200}{\pi r^3}\right)$

$= 2\pi\left(2 + \frac{\cancel{200}\,4}{\cancel{\pi}\left(\frac{\cancel{50}}{\cancel{\pi}}\right)}\right) = 2\pi(6)$

$\frac{d^2S}{dr^2} = 12\pi \; (+ve)$

$\therefore S$ is minimum

when $r = \left(\frac{50}{\pi}\right)^{\frac{1}{3}}$

$h = \frac{100}{\pi\left(\frac{50}{\pi}\right)^{2/3}} = \frac{100}{\cancel{\pi}\,\frac{50}{\cancel{\pi}}\left(\sqrt{\frac{50}{\pi}}\right)^{-\frac{1}{3}}} = 2\left(\frac{50}{\pi}\right)^{\frac{1}{3}}$
$= 2r$

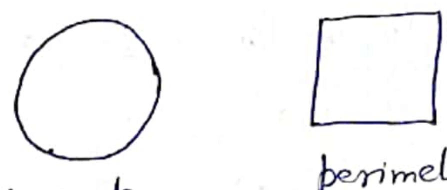
$h = 2r$. Ans.

Q22 A wire of length 28m is cut into two pieces. One piece is to be made into circle and other into square. What should be the length of two pieces so that combined area of two is minimum.

Sol: Let length of one piece $= x$ m

Length of second piece $= (28-x)$ m

Let circle is made from $x$ m. length and square is made from $(28-x)$ m. length.




Let $r =$ radius of circle.

$a =$ side of square

$2\pi r = x \Rightarrow r = \frac{x}{2\pi}$

$4a = 28-x \Rightarrow a = \left(\frac{28-x}{4}\right)$

If $A =$ Combined Area $=$ Ar of Circle $+$ Ar of Square

$$= \pi r^2 + a^2$$

$$A = \pi\left(\frac{x}{2\pi}\right)^2 + \left(\frac{28-x}{4}\right)^2$$

$$A = \frac{x^2}{4\pi} + \frac{1}{16}(28-x)^2$$

for A to be maximum, $\frac{dA}{dx} = 0$, $\frac{d^2A}{dx^2} < 0$

$$\frac{dA}{dx} = \frac{2x}{4\pi} + \frac{1}{16} \times 2(28-x)(-1)$$

$$\frac{dA}{dx} = \frac{2x}{4\pi} - \frac{1}{8}(28-x)$$

$$\frac{dA}{dx} = 0 \Rightarrow \frac{2x}{4\pi} - \frac{1}{8}(28-x) = 0$$

$$\frac{2x}{4\pi} = \frac{28-x}{8}$$

$$4x = (28-x)\pi$$

$$4x = 28\pi - \pi x$$

$$4x + \pi x = 28\pi$$

$$x(4+\pi) = 28\pi$$

$$x = \frac{28\pi}{4+\pi} \quad \text{---} \; (*)$$

$$\frac{d^2A}{dx^2} = \frac{2}{4\pi} - \frac{1}{8}(0-1)$$

$$= \left(\frac{1}{2\pi} + \frac{1}{8}\right) = +ive$$

so Area is minimum

Length of second piece $= 28 - x$

$$= \frac{28}{1} - \frac{28\pi}{4+\pi}$$

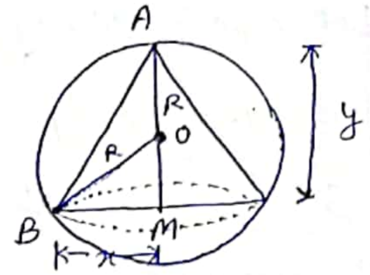
$$= \frac{28(4+\pi) - 28\pi}{4+\pi}$$

$$= \frac{112 + \cancel{28\pi} - \cancel{28\pi}}{4+\pi} = \frac{112}{4+\pi}$$

$\therefore$ length of pieces are

$$\frac{28\pi}{4+\pi}, \; \frac{112}{4+\pi}$$

Q.23. Prove that volume of largest cone that can inscribed in a sphere of radius R is $\frac{8}{27}$ of the volume of sphere.




Sol:- Let O be the centre and R be the radius of sphere. Let $BM = x$, $AM = y$, $OB = R$ (radius) $= AO$

In $\Delta OMB$. Now $OM = AM - AO = y - R$

$OM^2 + BM^2 = OB^2$

$(y-R)^2 + x^2 = R^2$

$y^2 + \cancel{R^2} - 2yR + x^2 = \cancel{R^2}$

$y^2 - 2Ry + x^2 = 0$

$x^2 = 2Ry - y^2$ —— (*)

Volume of cone $V = \frac{1}{3}\pi x^2 y$

$V = \frac{1}{3}\pi(2Ry - y^2)y$

$V = \frac{\pi}{3}(2Ry^2 - y^3)$.

For volume to be max; $\frac{dV}{dy} = 0$, $\frac{d^2V}{dy^2} < 0$

$\frac{dV}{dy} = \frac{\pi}{3}(4Ry - 3y^2)$

$\frac{dV}{dy} = 0 \Rightarrow \frac{\pi}{3}(4Ry - 3y^2) = 0$

$4Ry - 3y^2 = 0$

$y(4R - 3y) = 0$

$y = 0$ or $y = \frac{4}{3}R$

$\frac{d^2V}{dy^2} = \frac{\pi}{3}(4R - 6y)$

at $y = \frac{4}{3}R$

$\frac{d^2V}{dy^2} = \frac{\pi}{3}\left[4R - 6 \times \frac{4R}{3}\right]$

$= \frac{\pi}{3}[4R - 8R]$

$= -\frac{\pi 4R}{3} < 0$ (-ive)

$\therefore$ volume is maximum.

Now $y = \frac{4}{3}R$

from eq. (*)

$x^2 = 2R\left(\frac{4}{3}R\right) - \left(\frac{4}{3}R\right)^2$

$= \frac{8}{3}R^2 - \frac{16}{9}R^2 = \frac{24R^2 - 16R^2}{9} = \frac{8R^2}{9}$

$x^2 = \frac{8}{9}R^2$

$\therefore$ max. Volume $= \frac{1}{3}\pi x^2 y$

$= \frac{1}{3}\pi\left(\frac{8}{9}R^2\right) \cdot \frac{4}{3}R$

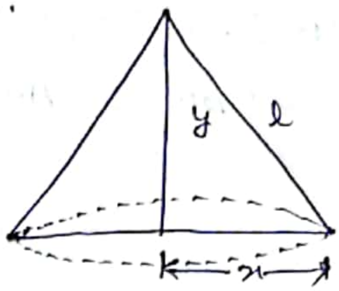
$= \frac{8}{27}\left[\frac{4}{3}\pi R^3\right]$

Max. Vol. $= \frac{8}{27}$ [volume of sphere]

Hence proved.

Q24 Show that right circular Cone of least Curved surface and given volume has an altitude equal to $\sqrt{2}$ times the radius of base.

Sol:- Let $x$ be radius of base & $y$ be height of cone.



Volume $V = \frac{1}{3}\pi x^2 y$. — ①

Let $S$ = Curved surface area of Cone.

$S = \pi \times x \times l$.

here $l^2 = x^2 + y^2$

$S = \pi x \sqrt{x^2+y^2}$

or $S^2 = \pi^2 x^2 (x^2+y^2)$

$V = \frac{1}{3}\pi x^2 y$

$x^2 y = \frac{3V}{\pi}$

or $x^2 = \left(\frac{3V}{\pi}\right)\frac{1}{y}$

or $x^2 = \frac{k}{y}$ where $k = \frac{3V}{\pi}$ (constant).

$S^2 = \pi^2 \frac{k}{y}\left(\frac{k}{y} + y^2\right) = \pi^2 k\left(\frac{k}{y^2} + y\right)$. — ⊗

For $S$ to be minimum. $\frac{d}{dy}(S^2) = 0$, $\frac{d^2}{dy^2}(S^2) > 0$

$\frac{d}{dy}(S^2) = \pi^2 k\left(-\frac{2k}{y^3} + 1\right)$

$\frac{d}{dy}(S^2) = 0 \Rightarrow \pi^2 k\left(-\frac{2k}{y^3} + 1\right)$

$\pi^2 k\left(-\frac{2k}{y^3} + 1\right) = 0$

$\Rightarrow -\frac{2k}{y^3} + 1 = 0$

$\Rightarrow \frac{2k}{y^3} = \frac{1}{1}$

$\Rightarrow y^3 = 2k$

$y = (2k)^{1/3}$

$\frac{d^2}{dy^2}(S^2) = \pi^2 k\left(-2k \times -\frac{3}{y^4}\right)$

$= \frac{6\pi^2 k^2}{y^4} = +$ive.

$\therefore S^2$ is minimum.

Now at $y = (2k)^{1/3}$

$x^2 = \frac{k}{y} = \frac{k}{(2k)^{1/3}}$

from — ①

$\frac{3V}{\pi} = x^2 y$

$x^2 y = k$

$x^2 = \frac{2k}{2(2k)^{1/3}} = \frac{(2k)}{2(2k)^{1/3}} = \frac{y^3}{2y} = \frac{y^2}{2}$

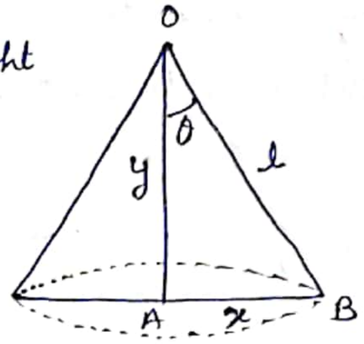
or $y^2 = 2x^2$

$y = \sqrt{2}x$

$\therefore S$ is minimum when

height $= \sqrt{2}$ (radius of base).

Q25 Show that the semi-vertical angle of the cone of maximum volume and of given slant height is $\tan^{-1}\sqrt{2}$.




Sol:- Let $x$ be radius of base of cone, $y$ be the height and '$l$' as the slant height of cone

Let $\theta$ = semi-vertical angle of cone

Now $x^2+y^2=l^2$

$x^2=l^2-y^2$

Volume of Cone $V=\frac{1}{3}\pi x^2 y=\frac{1}{3}\pi(l^2-y^2).y=\frac{1}{3}\pi(l^2y-y^3)$

Now for volume to be max. $\frac{dv}{dy}=0$, $\frac{d^2v}{dy^2}<0$

$\frac{dv}{dy}=\frac{1}{3}\pi(l^2-3y^2)$

$\frac{dv}{dy}=0 \Rightarrow \frac{1}{3}\pi(l^2-3y^2)=0$

$\Rightarrow l^2-3y^2=0$

$\Rightarrow l^2=3y^2$

$\Rightarrow l=\sqrt{3}y$ or $y=\frac{l}{\sqrt{3}}$.

also $\frac{d^2v}{dy^2}=\frac{1}{3}\pi(-6y)$

$=-2\pi y<0$

$\therefore$ Vol. of cone is max. at

$y=\frac{l}{\sqrt{3}}$.

Now.

$x^2=l^2-y^2$

$=l^2-\frac{l^2}{3}$

$=\frac{3l^2-l^2}{3}=\frac{2l^2}{3}$

$x=\sqrt{\frac{2}{3}}\,l$.

Now In $\Delta OAB$.

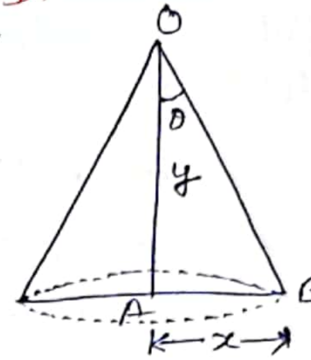
$\tan\theta=\frac{x}{y}=\frac{\frac{\sqrt{2}}{\cancel{\sqrt{3}}}\cancel{l}}{\frac{\cancel{l}}{\cancel{\sqrt{3}}}}$

$\tan\theta=\sqrt{2}$

or $\theta=\tan^{-1}\sqrt{2}$.

Hence semi-vertical angle $\theta=\tan^{-1}\sqrt{2}$.

Q 26 Show that semi-vertical angle of right circular Cone of given surface area and maximum volume is $\sin^{-1}(1/3)$.




Sol:- Let $x$ be the radius and $h$ be the height of right Circular Cone. Let $\theta$ be the semi-vertical angle.

Now Surface area $S = \pi r l + \pi x^2$

$$S = \pi \times x \cdot \sqrt{x^2+y^2} + \pi x^2$$

$$S = \pi\left[x\sqrt{x^2+y^2}+x^2\right)$$

or $x\sqrt{x^2+y^2}+x^2 = \frac{S}{\pi} = k$ (say) where $k$ is Constant.

$$x\sqrt{x^2+y^2}+x^2 = k$$

$$x\sqrt{x^2+y^2} = k - x^2$$

square both sides

$$x^2(x^2+y^2) = (k-x^2)^2$$

$$\cancel{x^4}+x^2y^2 = k^2+\cancel{x^4}-2kx^2$$

$$x^2y^2 = k^2-2kx^2$$

$$x^2y^2+2kx^2 = k^2$$

$$x^2(y^2+2k) = k^2$$

$$x^2 = \frac{k^2}{y^2+2k} \quad \text{—— } (*)$$

Now Volume $V = \frac{1}{3}\pi x^2 y$

$$V = \frac{1}{3}\pi\left(\frac{k^2}{y^2+2k}\right)\cdot y$$

$$V = \frac{k^2\pi}{3}\left[\frac{y}{y^2+2k}\right]$$

$$\frac{dv}{dy} = \frac{k^2\pi}{3}\left[\frac{(y^2+2k)1-y(2y)}{(y^2+2k)^2}\right]$$

$$= \frac{k^2\pi}{3}\left[\frac{2k-y^2}{(y^2+2k)^2}\right]$$

$$\frac{dv}{dy} = 0 \Rightarrow \frac{k^2\pi}{3}\left[\frac{2k-y^2}{(y^2+2k)^2}\right] = 0$$

$$\Rightarrow 2k-y^2 = 0$$

$$y^2 = 2k$$

$$y = \sqrt{2k}.$$

Now. [illegible] as $\frac{dv}{dy}$ gives $y = \sqrt{2k}$.

$\therefore y = \sqrt{2k}$ is the turning point

from eq. $(*)$

$$x^2 = \frac{k^2}{2k+2k} = \frac{k^2}{4k} = \frac{k}{4}$$

$$x = \sqrt{\frac{k}{4}} = \frac{\sqrt{k}}{2}$$

Now In $\Delta OAB$.

$$\sin\theta = \frac{AB}{OB} = \frac{x}{\sqrt{x^2+y^2}}$$

$$\sin\theta = \frac{\sqrt{k}/2}{\sqrt{k/4+2k}} = \frac{\sqrt{k}/2}{\sqrt{\frac{9k}{4}}}$$

$$\sin\theta = \frac{\sqrt{k}}{\sqrt{9k}} = \frac{1}{3}.$$

$$\Rightarrow \theta = \sin^{-1}\left(\frac{1}{3}\right).$$

Hence proved.