6. Method of Induction and Binomial Theorem

- **Principle of Mathematical Induction**
  - There are some mathematical statements or results that are formulated in terms of *n*, where *n* is a positive integer. To prove such statements, the well-suited principle that is used, based on the specific technique, is known as the principle of mathematical induction.
  - To prove a given statement in terms of *n*, firstly, we assume the statement as P (*n*).

Thereafter, we examine the correctness of the statement for $n = 1$, i.e., P (1) is true.
Then, assuming that the statement is true for $n = k$, where *k* is a positive integer, we prove that the statement is true for $n = k + 1$, i.e., truth of P (*k*) implies the truth of P (*k* + 1). Then, we say P (*n*) is true for all natural numbers *n*.

Example: For all $n \in \mathbb{N}$, prove that

$$\frac{4}{3} + \left(\frac{4}{3}\right)^2 + \ldots\ldots + \left(\frac{4}{3}\right)^n = 4\left[\left(\frac{4}{3}\right)^n - 1\right]$$

Solution:
Let the given statement be P (*n*), i.e.,

$$P(n): \frac{4}{3} + \left(\frac{4}{3}\right)^2 + \ldots\ldots + \left(\frac{4}{3}\right)^n = 4\left[\left(\frac{4}{3}\right)^n - 1\right]$$

For $n = 1$, $P(n): \frac{4}{3} = 4\left[\frac{4}{3} - 1\right] = 4 \times \frac{1}{3} = \frac{4}{3}$, which is true.
Now, assume that P(*x*) is true for some positive integer *k*. This means

$$\frac{4}{3} + \left(\frac{4}{3}\right)^2 + \ldots\ldots + \left(\frac{4}{3}\right)^k = 4\left[\left(\frac{4}{3}\right)^k - 1\right] \qquad -(1)$$

We shall now prove that P (*k* + 1) is also true.

Now, we have

$$\left[\frac{4}{3}+\left(\frac{4}{3}\right)^2+\ldots\ldots+\left(\frac{4}{3}\right)^k\right]+\left(\frac{4}{3}\right)^{k+1}$$

$$=4\left[\left(\frac{4}{3}\right)^k-1\right]+\left(\frac{4}{3}\right)^{k+1}$$

$$=4\left(\frac{4}{3}\right)^k-4+\left(\frac{4}{3}\right)^k\times\frac{4}{3}$$

$$=\left(\frac{4}{3}\right)^k\times\left[4+\frac{4}{3}\right]-4$$

$$=\left(\frac{4}{3}\right)^k\times\frac{16}{3}-4$$

$$=\left(\frac{4}{3}\right)^k\times\frac{4}{3}\times4-4$$

$$=\left(\frac{4}{3}\right)^{k+1}\times4-4=4\left[\left(\frac{4}{3}\right)^{k+1}-1\right]$$

Thus, P (k + 1) is true whenever P (k) is true. Hence, from the principle of mathematical induction, the statement P (n) is true for all natural numbers n.

- The coefficients of the expansions of a binomial are arranged in an array. This array is called Pascal's triangle. It can be written as

Index	Coefficient(s)
0	${}^0C_0$ $(=1)$
1	${}^1C_0$ $(=1)$ ${}^1C_1$ $(=1)$
2	${}^2C_0$ $(=1)$ ${}^2C_1$ $(=2)$ ${}^2C_2$ $(=1)$
3	${}^3C_0$ $(=1)$ ${}^3C_1$ $(=3)$ ${}^3C_2$ $(=3)$ ${}^3C_3$ $(=1)$
4	${}^4C_0$ $(=1)$ ${}^4C_1$ $(=4)$ ${}^4C_2$ $(=6)$ ${}^4C_3$ $(=4)$ ${}^4C_4$ $(=1)$
5	

- **General Term:** The $(r + 1)^{th}$ term (denoted by T_{r+1}) is known as the general term of the expansion $(a + b)^n$ and it is given by $T_{r+1} = {}^nC_r\, a^{n-r} b^r$

Example 1: In the expansion of $(5x - 7y)^9$, find the general term?

Solution: $T_{r+1} = {}^9C_r\,(5x)^{9-r}\,(-7y)^r = (-1)^r\,{}^9C_r\,(5x)^{9-r}\,(7y)^r$

- **Middle term in the expansion of $(a + b)^n$:**
  - If n is even, then the number of terms in the expansion will be $n + 1$. Since n is even, $n + 1$ is odd. Therefore, the middle term is $\left(\frac{n}{2} + 1\right)^{th}$ term.
  - If n is odd, then $n + 1$ is even. So, there will be two middle terms in the expansion. They are $\left(\frac{n+1}{2}\right)^{th}$ term and $\left(\frac{n+1}{2} + 1\right)^{th}$ term.
- In the expansion of $\left(x + \frac{1}{x}\right)^{2n}$, where $x \neq 0$, the middle term is $\left(\frac{2n}{2} + 1\right)^{th}$, i.e., $(n + 1)^{th}$ term [since $2n$ is even].

It is given by ${}^{2n}C_n x^n \left(\frac{1}{x}\right)^n = {}^{2n}C_n$ which is a constant.

This term is called the term independent of x or the constant term.

Note: In the expansion of $(a + b)^n$, r^{th} term from the end = $(n - r + 2)^{th}$ term from the beginning

Example 2: In the expansion of $\left(\frac{x^3}{4} - \frac{12}{x}\right)^4$, find the middle term and find the term which is independent of x.

Solution: As 4 is even, the middle term in the expansion of $\left(\frac{x^3}{4} - \frac{12}{x}\right)^4$ is the $\left(\frac{4}{2} + 1\right)^{th}$ term, i.e., 3^{rd} term, which is given by

$$T_3 = T_{2+1} = {}^4C_2 \left(\frac{x^3}{4}\right)^2 \left(\frac{-12}{x}\right)^2$$

$$= 6 \times \frac{x^6}{16} \times \frac{144}{x^2}$$

$$= 54x^4$$

Now, we will find the term in the expansion which is independent of x. Suppose $(r + 1)^{th}$ term is independent of x.

The $(r + 1)^{th}$ term in the expansion of $(a + b)^n$ is given by $T_{r+1} = {}^nC_r\, a^{n-r}\, b^r$

Hence, the $(r + 1)^{th}$ term in the expansion of $\left(\frac{x^3}{4} - \frac{12}{x}\right)^4$ is given by

Binomial Theorem for Any Index

For $n \in$ R and x<1,

- 1+xn=1+nx+nn-12!x2+nn-1n-23!x3+...+nn-1n-2...n-r+1r!xr+...
- 1-xn=1-nx+nn-12!x2-nn-1n-23!x3+...+-1rnn-1n-2...n-r+1r!xr+...
- 1+x-n=1-nx+nn+12!x2-nn+1n+23!x3+...+-1rnn+1n+2...n+r-1r!xr+...

Some expansions:

- **1+x-1=1-x+x2-x3+...**
- **1-x-1=1+x+x2+x3+...**
- 1+x-2=1-2x+3x2-4x3+...
- 1-x-2=1+2x+3x2+4x3+...

Properties of Binomial Coefficients

1. ${}^nC_r = {}^nC_{r-1}$
2. ${}^nC_r = {}^nC_s \Rightarrow r = s \text{ or } r + s = n$
3. ${}^nC_r + {}^nC_{r+1} = {}^{n+1}C_{r+1}$
4. $\dfrac{{}^nC_r}{{}^{n+1}C_{r+1}} = \dfrac{r+1}{n+1}$
5. $\dfrac{{}^nC_r}{{}^nC_{r+1}} = \dfrac{r+1}{n-r}$

Series of Binomial Coefficients

$$(1+x)^n = {}^nC_0 + {}^nC_1 x + {}^nC_2 x^2 + \ldots + {}^nC_n x^n \qquad \ldots (1)$$

1. Sum of the binomial coefficients in the expansion of $(1 + x)^n = 2^n$

2. $\sum_{r=0}^{n} (-1)^r \, {}^nC_r = 0$

3. In the expansion $(1 + x)^n$:

Sum of the binomial coefficients at odd position = Sum of the binomial coefficients at even position

$${}^nC_0 + {}^nC_2 + {}^nC_4 + \ldots = {}^nC_1 + {}^nC_3 + {}^nC_5 + \ldots = 2^{n-1}$$

4. ${}^nC_1 + 2\,{}^nC_2 + 3\,{}^nC_3 + \ldots\ldots + n\,{}^nC_n = n\,2^{n-1}$

5. ${}^nC_1 - 2\,{}^nC_2 + 3\,{}^nC_3 - \ldots + (-1)^{n-1} n\,{}^nC_n = 0$

6. ${}^nC_0\;{}^nC_r + {}^nC_1\;{}^nC_{r+1} + {}^nC_2\;{}^nC_{r+2} + \ldots\ldots + {}^nC_{n-r}\;{}^nC_n = {}^{2n}C_{n+r}$

7. ${}^nC_0 + \dfrac{{}^nC_1}{2} + \dfrac{{}^nC_2}{3} + \ldots + \dfrac{{}^nC_n}{n+1} = \dfrac{2^{n+1}-1}{n+1}$