7

Binomial Theorem

7.01 Introduction :

Ancient indian mathematicians were aware of expansion of expression $(a+b)^n$ Pingal gave a graphical form of expansion of coefficient three centuries B.C., which is called as Meruprastara . In 16th centuary Vamveli also found the coefficient of $(a+b)^n$, $n \le 7$. In the starting of 17th centuary Aatreya gave the information about the coefficient upto index 10.

French mathematician B. Pascal: gave to find the coefficients of binomial expansion in the form of triangle. This array of numbers known as Pascal's Triangle. Which is following as

Pascal's Triangle

Index	Coefficients
n = 1	1 1
n = 2	1 2 1
n = 3	1 3 3 1
n = 4	1 4 6 4 1
n = 5	1 5 10 10 5 1
n = 6	1 6 15 20 15 6 1
n = 7	1 7 21 35 35 21 7 1

In the Pascal's Triangle each row is bourided by 1 on both sides. Any entry, except the first and last, in a row is the sum of two entries in the preceding row, one on the immediate left and other on the immediate right.

7.02 Binomial expression :

An algebric expression containing two terms is known as binomial expression or only binomial. The terms may be positive or negative signs.

Example :

(i) $x+a$, where x is first term and a the second term

(ii) x^2-9 etc.

Binomial theorem :

The expansion of any index of a binomial expression is done in the form of a series using a formula that formula is called as binomial theorem.

Binomial coefficient :

The coefficient of various index of x in the expansion of binomial expression $(x+a)^n$ are called as Binomial coefficients.

7.03 Binomial theorem for positive index :

$$(x+a)^n = {}^nC_0\, x^n + {}^nC_1\, x^{n-1}a + {}^nC_2\, x^{n-2}a^2 + ... + {}^nC_r\, x^{n-r}a^r + ... + {}^nC_n\, a^n$$

Proof : The proof is obtained by applying principle of mathematical induction

$$(x+a)^1 = x+a = {}^1C_0\, x^1 + {}^1C_1\, x^{1-1}a \qquad (1)$$

$$(x+a)^2 = x^2 + 2ax + a^2$$

$$= {}^2C_0\, x^2 + {}^2C_1\, x^{2-1}a + {}^2C_2\, x^{2-2}a^2 \qquad (2)$$

It is clear from (1) and (2) that the theorem is true for $n = 1$ & 2. Let it be true for $n = m$

then $$(x+a)^m = {}^mC_0\, x^m + {}^mC_1\, x^{m-1}a + {}^mC_2\, x^{m-2}a^2 + ... + {}^mC_r\, x^{m-r}a^r + ... + {}^mC_m a^m \qquad (3)$$

multiplying both sides by $(x+a)$

$$(x+a)(x+a)^m = (x+a)\left[{}^mC_0\, x^m + {}^mC_1\, x^{m-1}a + {}^mC_2\, x^{m-2}a^2 + ... + {}^mC_r\, x^{m-r}a^r + ... + {}^mC_m\, a^m\right]$$

$$(x+a)^{m+1} = x\left[{}^mC_0\, x^m + {}^mC_1\, x^{m-1}a + {}^mC_2\, x^{m-2}a^2 + ... + {}^mC_r\, x^{m-r}a^r + ... + {}^mC_m\, a^m\right] +$$

$$a\left[{}^mC_0\, x^m + {}^mC_1\, x^{m-1}a + {}^mC_2\, x^{m-2}a^2 + ... + {}^mC_r\, x^{m-r}a^r + ... + {}^mC_m\, a^m\right]$$

$$= {}^mC_0\, x^{m+1} + \left({}^mC_1 + {}^mC_0\right)x^m a + \left({}^mC_2 + {}^mC_1\right)x^{m-1}a^2$$

$$+\left({}^mC_3 + {}^mC_2\right)x^{m-2}a^3 + ... + \left({}^mC_r + {}^mC_{r-1}\right)x^{m-r+1}a^r + ... + {}^mC_m\, a^{m+1}$$

$$\because \quad {}^mC_1 + {}^mC_0 = {}^{m+1}C_1,$$

$${}^mC_2 + {}^mC_1 = {}^{m+1}C_2,$$

$$..... \quad ..... \quad .....$$

$${}^mC_r + {}^mC_{r-1} = {}^{m+1}C_r$$

$${}^mC_m = {}^{m+1}C_{m+1} = {}^mC_0 = {}^{m+1}C_0 = 1$$

$$\therefore \quad (x+a)^{m+1} = {}^{m+1}C_0\, x^{m+1} + {}^{m+1}C_1\, x^m a + {}^{m+1}C_2\, x^{m-1}a^2 + ... +$$

$${}^{m+1}C_r\, x^{m-r+1}a^r + ... + {}^{m+1}C_{m+1}\, a^{m+1} \qquad (4)$$

Thus from (4) it is proved that the theorem is true for $n = m+1$ also. Therfore, by principle of mathematical induction, the theorem is true for every positive integer n.

$$(x+a)^n = {}^nC_0 x^n + {}^nC_1\, x^{n-1}a + {}^nC_2\, x^{n-2}a^2 + ... + {}^nC_r\, x^{n-r}a^r + ... + {}^nC_n\, a^n$$

7.04 Various important forms of binomial theorem :

$$(x+a)^n = x^n + {}^nC_1\, x^{n-1}a + {}^nC_2\, x^{n-2}a^2 + ... + {}^nC_r\, x^{n-r}a^r + ... + a^n \qquad (1)$$

substituting $(-a)$ in place of a in (1)

$$(x-a)^n = x^n - {}^nC_1\, x^{n-1}a + {}^nC_2\, x^{n-2}a^2 - ... + (-1)^r\, {}^nC_r\, x^{n-r}a^r + ... + (-1)^n\, a^n \qquad (2)$$

interchanging a and x in (1)

$$(a+x)^n = a^n + {}^nC_1 a^{n-1}x + {}^nC_2 a^{n-2}x^2 + ... + {}^nC_r\, a^{n-r}x^r + ... + x^n \qquad (3)$$

substituting $(-x)$ in place of x in (3)

$$(a-x)^n = a^n - {}^nC_1 a^{n-1}x + {}^nC_2 a^{n-2}x^2 + \dots + (-1)^r a^{n-r}x^r + \dots + (-1)^n x^n \quad (4)$$

putting $x = 1$ and $a = x$ in (1)

$$(1+x)^n = 1 + {}^nC_1\, x + {}^nC_2\, x^2 + \dots + {}^nC_r\, x^r + \dots + x^n \quad (5)$$

putting $x = 1$ and $a = -x$ in (1)

$$(1-x)^n = 1 - {}^nC_1\, x + {}^nC_2\, x^2 - \dots + (-1)^r\, {}^nC_r\, x^r + \dots + (-1)^n x^n \quad (6)$$

Illustrative Examples

Example 1 : Expand $(2x + 3y)^5$ using Binomial Theorem.

Solution : Here first term $= 2x$, second term $= 3y$, and $n = 5$

$$(2x+3y)^5 = (2x)^5 + {}^5C_1(2x)^4(3y) + {}^5C_2(2x)^3(3y)^2 + {}^5C_3(2x)^2(3y)^3 + {}^5C_4(2x)(3y)^4 + (3y)^5$$

$$= (2x)^5 + 5(2x)^4(3y) + 10(2x)^3(3y)^2 + 10(2x)^2(3y)^3 + 5(2x)(3y)^4 + (3y)^5$$

$$= 32x^5 + 240x^4y + 720x^3y^2 + 1080x^2y^3 + 810xy^4 + 243y^5.$$

Example 2 : Expand $\left(2x + \frac{1}{x}\right)^4$ using Binomial Theorem.

Solution : $$\left(2x+\frac{1}{x}\right)^4 = (2x)^4 + {}^4C_1(2x)^3\left(\frac{1}{x}\right) + {}^4C_2(2x)^2\left(\frac{1}{x}\right)^2 + {}^4C_3(2x)\left(\frac{1}{x}\right)^3 + \left(\frac{1}{x}\right)^4$$

$$= (2x)^4 + 4(2x)^3\left(\frac{1}{x}\right) + 6(2x)^2\left(\frac{1}{x}\right)^2 + 4(2x)\left(\frac{1}{x}\right)^3 + \left(\frac{1}{x}\right)^4$$

$$= 16x^4 + 32x^2 + 24 + \frac{8}{x^2} + \frac{1}{x^4}.$$

Example 3 : Evaluate $(1 + \sqrt{5})^5 + (1 - \sqrt{5})^5$.

Solution : $$(1+\sqrt{5})^5 = 1 + {}^5C_1(\sqrt{5}) + {}^5C_2(\sqrt{5})^2 + {}^5C_3(\sqrt{5})^3 + {}^5C_4(\sqrt{5})^4 + (\sqrt{5})^5$$

$$= 1 + 5\sqrt{5} + 50 + 50\sqrt{5} + 125 + 25\sqrt{5} \quad (1)$$

and $$(1-\sqrt{5})^5 = 1 - {}^5C_1(\sqrt{5}) + {}^5C_2(\sqrt{5})^2 - {}^5C_3(\sqrt{5})^3 + {}^5C_4(\sqrt{5})^4 - (\sqrt{5})^5$$

$$= 1 - 5\sqrt{5} + 50 - 50\sqrt{5} + 125 - 25\sqrt{5} \quad (2)$$

adding (1) and (2)

$$(1+\sqrt{5})^5 + (1-\sqrt{5})^5 = 2[1 + 50 + 125]$$

$$= 2[176] = 352$$

Example 4 : Evaluate $(10\cdot1)^5$ using Binomial Theorem.

Solution : $(10\cdot1)^5 = (10 + \cdot1)^5$

$$= (10)^5 + {}^5C_1(10)^4(\cdot1) + {}^5C_2(10)^3(\cdot1)^2 + {}^5C_3(10)^2(\cdot1)^3 + {}^5C_4(10)(\cdot1)^4 + (\cdot1)^5$$

$$= 100000 + 5000 + 100 + 1 + \cdot 005 + \cdot 00001 = 105101 \cdot 00501$$

Example 5 : In the expansion of $(x+a)^n$ if A is sum of odd terms and B the sum of even terms then prove that :

(i) $\left(x^2 - a^2\right)^n = A^2 - B^2$,

(ii) $(x+a)^{2n} - (x-a)^{2n} = 4AB$

Solution : (i) $(x+a)^n = x^n + {}^nC_1 x^{n-1} a + {}^nC_2 x^{n-2} a^2 + {}^nC_3 x^{n-3} a^3 + {}^nC_4 x^{n-4} a^4 + \ldots\ldots + a^n$

$$= \left[x^n + {}^nC_2 x^{n-2} a^2 + {}^nC_4 x^{n-4} a^4 + \ldots\ldots\right] + \left[{}^nC_1 x^{n-1} a + {}^nC_3 x^{n-3} a^3 + \ldots\ldots\right]$$

= [sum of odd terms] + [sum of even terms]

$$= A + B \qquad (1)$$

similarly $\quad (x-a)^n = A - B \qquad (2)$

multiplying (1) and (2),

$$(x+a)^n (x-a)^n = (A+B)(A-B)$$

or $\quad [(x+a)(x-a)]^n = A^2 - B^2 \quad$ or $\quad \left(x^2 - a^2\right)^n = A^2 - B^2$.

(ii) Again : $\quad (x+a)^{2n} - (x-a)^{2n} = \left[(x+a)^n\right]^2 - \left[(x-a)^n\right]^2$

$$= \left[(x+a)^n + (x-a)^n\right]\left[(x+a)^n - (x-a)^n\right]$$

$$= \left[(A+B) + (A-B)\right]\left[(A+B) - (A-B)\right] = [2A][2B] = 4AB$$

Exercise 7.1

Expand the following (1 to 5) of each expression :

1. $(2-x)^3$ 2. $\left(\frac{2}{x} - \frac{x}{2}\right)^5$ 3. $\left(\frac{x}{3} + \frac{1}{x}\right)^6$ 4. $(3x+2y)^4$ 5. $\left(\sqrt{\frac{x}{a}} - \sqrt{\frac{9}{x}}\right)^6$

Expand the following (6 to 9) using Binomial Theorem :

6. $(96)^3$ 7. $(101)^4$ 8. $(99)^5$ 9. $(1 \cdot 1)^6$

10. Using Binomial Theorem, indicate which number is larger $(1 \cdot 1)^{10000}$ or 1000.

11. Find $(a+b)^4 - (a-b)^4$. Using its value, evaluate $\left(\sqrt{3}+\sqrt{2}\right)^4 - \left(\sqrt{3}-\sqrt{2}\right)^4$.

7.05 General term in binomial Expansion :

In the expansion $(r+1)^{\text{th}}$ term is called as the general term and represented by T_{r+1}

i.e. $T_{r+1} = {}^nC_r \, x^{n-r} a^r$

Putting $r = 0, 1, 2, 3, \ldots\ldots$ we can find the first, second, third and fourth term .... etc. which can be written as $T_1, T_2, T_3, T_4, \ldots\ldots$

Similarly $T_1 = {}^nC_0\, x^n$

$T_2 = {}^nC_1\, x^{n-1}a$

$T_3 = {}^nC_2\, x^{n-2}a^2$

$T_4 = {}^nC_3\, x^{n-3}a^3$

....

Some important knowledge about binomial expansion when index is positive integers.

(i) The total number of terms in the expansion of $(x+a)^n$ is one more then the index i.e. if the index is n then the expansion will have $(n+1)$ terms.

(ii) Powers of the first quantity 'x' go on decreasing by 1 whereas the powers of the second quantity 'a' is increased by 1, in the successive terms. The sum of powers of x and a equal to power of binomial.

7.06 Middle term in the expansion of $(x+a)^n$

1. If the index n is even then expansion will have odd terms, therefore it will have one middle terms i.e.

middle term $= \left(\frac{n}{2}+1\right) = \left(\frac{n+2}{2}\right) = \left[\frac{(n+1)+1}{2}\right]^{\text{th}}$ term

$$= \left[\frac{Number\ of\ terms \text{ in } \exp ansion + 1}{2}\right]^{\text{th}} \text{term}$$

2. If the index n is odd then expansion will have even terms, therefore it will have two middle terms i.e.

middle term $= \left(\frac{n+1}{2}\right)^{\text{th}}$ term or $\left(\frac{n+1}{2}+1\right) = \left(\frac{n+3}{2}\right)^{\text{th}}$ term

7.07 Coefficient of special power x^m in binomial expansion

If x^m comes in the T_{r+1} term of the expansion $\left(ax^p \pm \frac{b}{x^q}\right)^n$

then $T_{r+1} = {}^nC_r\left(ax^p\right)^{n-r}\left(\pm\frac{b}{x^q}\right)^r = {}^nC_r\, a^{n-r}(\pm b)^r\, x^{np-r(p+q)}$

$\therefore$ Find term using calculating the value of r using $np - r(p+q) = m$; r is always a positive integer

Thus the coefficient of x^m is ${}^nC_r\, a^{n-r}(\pm b)^r$

If the term independent of x is to be found out then

$$np - r(p+q) = 0 \qquad \therefore \qquad r = \frac{np}{p+q}.$$

7.08 Number of terms in the expansion of $(a+b+c)^n$

$$\because \quad (a+b+c)^n = \left[(a+b)+c\right]^n$$

$$= (a+b)^n + {}^nC_1(a+b)^{n-1}c + {}^nC_2(a+b)^{n-2}c^2 + \ldots\ldots + c^n$$

$\therefore$ number of terms in the expansion $(a+b+c)^n$

$= (n+1)$ term $+ n$ term $+ (n-1)$ term $+ ......+ 1$ term

$= \dfrac{(n+1)(n+2)}{2}$ terms. [By Sum of A.P. series]

Illustrative Examples

Example 6 : Find the middle term in the Expansion of $\left(\dfrac{a}{2}-\dfrac{b}{3}\right)^8$

Solution : The number of terms in the expansion $\left(\dfrac{a}{2}-\dfrac{b}{3}\right)^8$ is $= 8+1 = 9$ (odd)

$\therefore$ the middle term $=\left(\dfrac{n+2}{2}\right)=\left(\dfrac{8+2}{2}\right)=5^{\text{th}}$ term

$$= {}^8C_4\left(\frac{a}{2}\right)^4\left(-\frac{b}{3}\right)^4 = \frac{8.7.6.5}{1.2.3.4}\left(\frac{a}{2}\right)^4\left(\frac{b}{3}\right)^4 = \frac{70a^4b^4}{16\cdot 81} = \frac{35a^4b^4}{648}.$$

Example 7 : Prove that in the expansion of $(1+x)^{2n}$ the middle term $\dfrac{1.3.5......(2n-1)}{n!}2^n x^n$

Solution : $\because$ $2n$ is even number then the middle term is $\left(\dfrac{2n+2}{2}\right)$,

i.e. $(n+1)^{\text{th}}$ term

$(n+1)$ term $= T_{n+1} = {}^{2n}C_n\, x^n$

$$= \frac{(2n)!}{n!\,n!}x^n = \frac{\left[(2n)(2n-1)(2n-2).....6.5.4.3.2.1\right]x^n}{n!\,n!}$$

$$= \frac{\left[(2n)(2n-2).....6.4.2\right]\left[(2n-1)(2n-3).....5.3.1\right]}{n!\,n!}x^n$$

$$= \frac{(2)^n\left[n(n-1)(n-2).....\ 3.2.1\right]\left[(2n-1)(2n-3).....\ 5.3.1\right]}{n!\,n!}x^n$$

$$= \frac{(2)^n\, n!\left[(2n-1)(2n-3)....\ 5.3.1\right]}{n!\,n!}x^n$$

$$= \frac{(2)^n\left[1.3.5.\ .....(2n-3)(2n-1)\right]}{n!}x^n = \frac{1.3.5.\ ......(2n-1)}{n!}2^n\ x^n.$$

Example 8 : Find the coefficient of x^{-17} in the expansion of $\left(x^4-\dfrac{1}{x^3}\right)^{15}$

Solution : Let, x^{-17}, T_{r+1} then

$$T_{r+1} = {}^{15}C_r\left(x^4\right)^{15-r}\left(-\frac{1}{x^3}\right)^r = {}^{15}C_r\,(x)^{60-4r}\,\frac{(-1)^r}{(x)^{3r}} = {}^{15}C_r\,(-1)^r\,(x)^{60-7r} \qquad (1)$$

The index of x should be -17

$\therefore \quad 60 - 7r = -17$

$7r = 77 \Rightarrow r = 11$

putting the value of r in (1) we have the coefficient of

$$x^{-17} = {}^{15}C_{11}(-1)^{11} = -{}^{15}C_4 = -\frac{15.14.13.12}{1.2.3.4} = -1365.$$

Example 9 : Find the term independent of x in the expansion of $\left(2x^2 + \frac{1}{2x}\right)^9$

Solution : let the $(r+1)$th term is independent of x

$$T_{r+1} = {}^9C_r\left(2x^2\right)^{9-r}\left(\frac{1}{2x}\right)^r = {}^9C_r\,(2)^{9-r}\,(x)^{18-2r}\,\frac{1}{(2)^r\,(x)^r} = {}^9C_r\,(2)^{9-2r}\,(x)^{18-3r}$$

For independent term of x, the index of x is zero

$\therefore \quad 18 - 3r = 0 \qquad$ or $\quad r = 6$

$\therefore \quad$ Term independent of $x = T_7 = {}^9C_6(2)^{9-12} = \frac{9.8.7}{1.2.3}.\frac{1}{2^3} = \frac{21}{2}$

Example 10 : In the expansion of $(1+x)^{20}$ if the ratio of coefficients of n terms and $(n+1)$ terms are $1:2$ then find n.

Solution :

$$T_n = {}^{20}C_{n-1}\,x^{n-1} \qquad (1)$$

$$T_{n+1} = {}^{20}C_n\,x^n \qquad (2)$$

According to question,

$$\frac{T_n}{T_{n+1}} = \frac{{}^{20}C_{n-1}}{{}^{20}C_n} = \frac{1}{2}$$

or

$$\frac{\dfrac{20!}{(n-1)!\,(20-n+1)!}}{\dfrac{20!}{(n)!\,(20-n)!}} = \frac{1}{2}$$

or

$$\frac{(n)!\,(20-n)!}{(n-1)!\,(21-n)!} = \frac{1}{2}$$

or

$$\frac{n(n-1)!\,(20-n)!}{(n-1)!\,(21-n)(20-n)!} = \frac{1}{2}$$

or

$$\frac{n}{21-n} = \frac{1}{2}$$

Simplifying, $\quad n = 7$.

Exercise 7.2

1. Find the marked term in the following bionomial expansion.

(i) In 5 th term $\left(a + 2x^3\right)^{17}$ **(ii)** In 9 th term $\left(\frac{x}{y} - \frac{3y}{x^2}\right)^{12}$ **(iii)** In 6 th term $\left(\frac{2}{\sqrt{x}} - \frac{x^2}{2}\right)^9$

2. Find the coefficient of :

(i) x^{-7} in expansion of $\left(ax - \frac{1}{bx^2}\right)^8$ (ii) x^4 in expansion of $\left(x^4 + \frac{1}{x^3}\right)^{15}$

(iii) x^6 in expansion of $\left(a - bx^2\right)^{10}$

3. Find the term independent of x in the following expansion of :

(i) $\left(x - \frac{1}{x^2}\right)^{12}$ (ii) $\left(\sqrt{x} - \frac{3}{x^2}\right)^{10}$, $x > 0$

(iii) $\left(\sqrt{\frac{x}{3}} + \frac{3}{2x^2}\right)^{10}$ (iv) $\left(x^2 - \frac{1}{x^2}\right)^{10}$

4. Find the middle terms in the following expansion of :

(i) $\left(\frac{x}{2} + 2y\right)^6$ (ii) $\left(3a - \frac{a^3}{6}\right)^9$

(iii) $\left(x + \frac{1}{x}\right)^{2n}$ (iv) $\left(3x - \frac{2}{x^2}\right)^{15}$

5. Show that the coefficient of middle term in the expansion of $(1 + x)^n$ is $\frac{1.3.5.....(n-1)}{2.4.6......n} 2^n$

where n is a even positive integer. If n is odd then coefficient of both middle term will be $\frac{1.3.5.....n}{2.4.6......(n+1)} 2^n$

6. If the coefficient of x^7 and x^{-7} in the expansion of $\left(ax + \frac{1}{bx}\right)^{11}$ are equal then prove that : $ab - 1 = 0$.

7. If the coefficient of 5th, 6th and 7th term in the expansion of $(1 + y)^n$ are in A.P. then find the value of n.

8. The second, third and fourth terms in the binomial expansion $(x + a)^n$ are 240, 720 and 1080, respectively. Find x, a and n.

9. The coefficients of three consecutive terms in the expansion of $(1 + a)^n$ are in the ratio 1 : 7 : 42. Find the value of n.

10. Find a positive value of m for which the coefficient of x^2 in the expansion $(1 + x)^m$ is 6.

7.09 Properties of binomial coefficients :

In the expansion of $(1 + x)^n$. The coefficient of x for various powers are ${}^nC_0, {}^nC_1, {}^nC_2, ..., {}^nC_n$. They are also represented by $C_0, C_1, C_2, ... C_n$ i.e.

Hence, $(1 + x)^n = C_0 + C_1x + C_2x^2 + C_3x^3 + ... + C_nx^n$ [from section 7.04]

(i) Putting $x = 1$ we have

$$(1+1)^n = C_0 + C_1 + C_2 + C_3 + ... + C_n \quad \Rightarrow \quad 2^n = C_0 + C_1 + C_2 + C_3 + ... + C_n \quad (1)$$

(ii) Putting $x = -1$ we have

$$(1-1)^n = C_0 - C_1 + C_2 - C_3 + ... + (-1)^n C_n \quad \Rightarrow \quad 0 = C_0 - C_1 + C_2 - C_3 + ... + (-1)^n C_n \quad (2)$$

adding (1) and (2)

$$2\left(C_0+C_2+C_4+\ldots\right)=2^n \quad \Rightarrow \quad C_0+C_2+C_4+\ldots=2^{n-1} \qquad (3)$$

again subracting (2) from (1)

$$2\left(C_1+C_3+C_5+\ldots\right)=2^n \quad \Rightarrow \quad C_1+C_3+C_5+\ldots=2^{n-1} \qquad (4)$$

therefore, from (3) and (4)

$$C_0+C_2+C_4+\ldots=C_1+C_3+C_5+\ldots=2^{n-1} \qquad (5)$$

Illustrative Examples

Example 11 : Evaluate ${}^{20}C_1+{}^{20}C_2+{}^{20}C_3+\ldots+{}^{20}C_{20}$

Solution : Since ${}^{n}C_0+{}^{n}C_1+{}^{n}C_2+\ldots+{}^{n}C_n=2^n$ (1)

putting $n=20$ in (1)

$${}^{20}C_0+{}^{20}C_1+{}^{20}C_2+\ldots+{}^{20}C_{20}=2^{20}$$

or $$1+{}^{20}C_1+{}^{20}C_2+\ldots+{}^{20}C_{20}=2^{20} \qquad \left[\because {}^{20}C_0=1\right]$$

or $${}^{20}C_1+{}^{20}C_2+{}^{20}C_3+\ldots+{}^{20}C_{20}=2^{20}-1$$

Example 12 : If $C_0, C_1, C_2, \ldots., C_n$ are bionmial coefficients then find value of

$$C_0+2.C_1+3.C_2+\ldots..+(n+1)\cdot C_n$$

Solution : Given expression is

$$=C_0+2.C_1+3.C_2+\ldots..+(n+1)C_n$$

$$=\left(C_0+C_1+C_2+\ldots..+C_n\right)+\left(C_1+2.C_2+3.C_3+\ldots..n.C_n\right)$$

$$=2^n+\left[{}^{n}C_1+2.{}^{n}C_2+3.{}^{n}C_3+n.{}^{n}C_n\right]=2^n+\left[n+2.\frac{n(n-1)}{1.2}+\ldots..+n.1\right]$$

$$=2^n+\left[n+n(n-1)+\frac{n(n-1)(n-2)}{1.2}+\ldots..+n\right]=2^n+n\left[1+(n-1)+\frac{(n-1)(n-2)}{1.2}+\ldots..+1\right]$$

$$=2^n+n\left[1+{}^{(n-1)}C_1+{}^{(n-1)}C_2+\ldots..+{}^{(n-1)}C_{n-1}\right]=2^n+n.2^{n-1}=2^{n-1}(n+2).$$

Example 13 : Prove that $C_1+2.C_2+3.C_3+\ldots..+n.C_n=n.2^{n-1}$.

Solution : L.H.S. $C_1+2.C_2+3.C_3+\ldots..+n.C_n$

$$={}^{n}C_1+2.{}^{n}C_2+3.{}^{n}C_3+\ldots..+n.{}^{n}C_n$$

$$=n+2.\frac{n(n-1)}{2!}+3.\frac{n(n-1)(n-2)}{3!}+\ldots.+n.1$$

$$=n\left[1+(n-1)+\frac{(n-1)(n-2)}{2!}+\ldots..+1\right]$$

$$= n\left[{}^{n-1}C_0 + {}^{n-1}C_1 + {}^{n-1}C_2 + \ldots\ldots + {}^{n-1}C_{n-1}\right] = n.2^{n-1} = \text{R.H.S.}$$

Example 14 : In the expansion of $(1+x)^n$ if the bionomial coefficients are $C_0, C_1, C_2, \ldots\ldots, C_n$ then prove that.

$$C_0 + \frac{C_1}{2} + \frac{C_2}{3} + \ldots\ldots + \frac{C_n}{n+1} = \frac{2^{n+1}-1}{n+1}$$

Solution : L.H.S.

$$= C_0 + \frac{C_1}{2} + \frac{C_2}{3} + \ldots\ldots + \frac{C_n}{n+1}$$

$$= {}^nC_0 + \frac{{}^nC_1}{2} + \frac{{}^nC_2}{3} + \ldots\ldots + \frac{{}^nC_n}{n+1}$$

$$= 1 + \frac{n}{2} + \frac{n(n-1)}{3\ .\ 2!} + \ldots\ldots + \frac{1}{n+1}$$

$$= \frac{1}{n+1}\left[(n+1) + \frac{(n+1)n}{2} + \frac{(n+1)n(n-1)}{3!} + \frac{(n+1)n(n-1)(n-2)}{4!} + \ldots\ldots + 1\right]$$

$$= \frac{1}{n+1}\left[{}^{(n+1)}C_1 + {}^{(n+1)}C_2 + {}^{(n+1)}C_3 + {}^{(n+1)}C_4 + \ldots\ldots + {}^{n+1}C_{n+1}\right]$$

$$= \frac{1}{n+1}\left[{}^{(n+1)}C_0 + {}^{(n+1)}C_1 + {}^{(n+1)}C_2 + {}^{(n+1)}C_3 + \ldots\ldots + {}^{n+1}C_{n+1} - {}^{(n+1)}C_0\right]$$

[adding & substracting ${}^{(n+1)}C_0$]

$$= \frac{1}{n+1}\left[(2)^{n+1} - {}^{(n+1)}C_0\right] = \frac{2^{n+1}-1}{n+1} = \text{R.H.S.}$$

Example 15 : In the expansion of $(1+x)^n$ if the bionomial coefficients are $C_0, C_1, C_2, \ldots., C_n$ then prove that.

$$C_0^2 + C_1^2 + C_2^2 + \ldots\ldots + C_n^2 = \frac{(2n)!}{(n!)^2}$$

Solution : $(1+x)^n = C_0 + C_1x + C_2x^2 + \ldots\ldots + C_nx^n$ (1)

and $(x+1)^n = C_0x^n + C_1x^{n-1} + C_2x^{n-2} + \ldots\ldots + C_n$ (2)

multiplying (1) and (2)

$$(1+x)^{2n} = \left(C_0 + C_1x + C_2x^2 + \ldots\ldots + C_nx^n\right)\left(C_0x^n + C_1x^{n-1} + C_2x^{n-2} + \ldots\ldots + C_n\right)$$

comparing the coefficients of x^n in both sides, we have

$${}^{2n}C_n = C_0^2 + C_1^2 + C_2^2 + \ldots\ldots + C_n^2 \qquad \therefore \qquad C_0^2 + C_1^2 + C_2^2 + \ldots\ldots + C_n^2 = \frac{2n!}{n!\ n!} = \frac{2n!}{(n!)^2}$$

Exercise 7.3

1. In the expansion of $(1+x)^n$ if the bionomial coefficients are $C_0, C_1, C_2, \ldots., C_n$ then find the value of,

(i) ${}^8C_1 + {}^8C_2 + {}^8C_3 + \ldots + {}^8C_8$ (ii) ${}^8C_1 + {}^8C_3 + {}^8C_5 + {}^8C_7$

In the expansion of $(1+x)^n$ if the bionomial coefficients are $C_0, C_1, C_2, \ldots, C_n$ then Prove that

2. $C_0 + 3.C_1 + 5.C_2 + \ldots + (2n+1).C_n = (n+1)2^n$

3. $C_0C_2 + C_1C_3 + C_2C_4 + \ldots + C_{n-2}C_n = \dfrac{(2n)!}{(n-2)!\,(n+2)!}$

4. $C_0 + 2C_1 + 4C_2 + 6C_3 + \ldots + 2n\,C_n = 1 + n\,2^n$

5. $\left(1+\dfrac{C_1}{C_0}\right)\left(1+\dfrac{C_2}{C_1}\right)\left(1+\dfrac{C_3}{C_2}\right)\ldots\left(1+\dfrac{C_n}{C_{n-1}}\right) = \dfrac{(n+1)^n}{n!}$

6. In the expansion of $\left(1 + x - 2x^2\right)^6$ is given by $1 + a_1x + a_2x^2 + a_3x^3 + \ldots + a_{12}x^{12}$, then prove that : $a_2 + a_4 + a_6 + \ldots + a_{12} = 31$

7.10 Binomial theorem for rational index :

In a bionomial term if the index is fractional or negative, expansion is only possible if the first term of binomial is 1 and the second term is less then one. i.e. we always expand the Binomial as $(1+x)^n$ where x is always less than 1 i.e. $-1 < x < 1$. In this condition the formula for the Binomial is

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots + \frac{n(n-1)\ldots(n-r+1)}{r!}x^r + \ldots$$

Here the terms are infinite. i.e. the terms in expansion are infinite, this is called as binomial series.

In the expansion of $(x+a)^n$ if x is less then a then $(x+a)^n = a^n\left(1+\dfrac{x}{a}\right)^n$ and if the value of a is less then x then we expand by converting into the form $(x+a)^n = x^n\left(1+\dfrac{a}{x}\right)^n$

General Term for the Expansion :

$$T_{r+1} = \frac{n(n-1)(n-2)\ldots(n-r+1)}{r!}x^r$$

Note : When n is fractional or negative then ${}^nC_r$ is meaningless thus coefficients of different terms should not be written like this ${}^nC_1, {}^nC_2, \ldots$ instead it should belike the one given below.

7.11 Some important expansions :

$$(1-x)^n = 1 - nx + \frac{n(n-1)}{2!}x^2 + \ldots + (-1)^r\frac{n(n-1)(n-2)\ldots(n-r+1)}{r!}x^r + \ldots \quad (1)$$

$$(1+x)^{-n} = 1 - nx + \frac{n(n+1)}{2!}x^2 + \ldots + (-1)^r\frac{n(n+1)(n+2)\ldots(n+r-1)}{r!}x^r + \ldots \quad (2)$$

$$(1-x)^{-n} = 1 + nx + \frac{n(n+1)}{2!}x^2 + \ldots + \frac{n(n+1)(n+2)\ldots(n+r-1)}{r!}x^r + \ldots \quad (3)$$

In (2) and (3) putting $n = 1, 2, 3$

$$(1-x)^{-1}=1+x+x^2+x^3+.....+x^r+.....$$

$$(1+x)^{-1}=1-x+x^2-x^3+.....+(-1)^r x^r+.....$$

$$(1+x)^{-2}=1-2x+3x^2-4x^3+.....+(-1)^r (r+1)x^r+.....$$

$$(1-x)^{-2}=1+2x+3x^2+4x^3+.....+(r+1)x^r+.....$$

$$(1+x)^{-3}=1-3x+6x^2-10x^3+.....+(-1)^r \frac{(r+1)(r+2)}{2!}x^r+.....$$

$$(1-x)^{-3}=1+3x+6x^2+10x^3+.....+\frac{(r+1)(r+2)}{2!}x^r+.....$$

Illustrative Examples

Example 16 : Expand $(1+x)^{3/2}$ up to first four terms.

Solution : $(1+x)^{3/2}=1+\frac{3}{2}(x)+\frac{\frac{3}{2}\left(\frac{3}{2}-1\right)}{2!}x^2+\frac{\frac{3}{2}\left(\frac{3}{2}-1\right)\left(\frac{3}{2}-2\right)}{3!}x^3+...........$

$$=1+\frac{3x}{2}+\frac{3}{2}.\frac{1}{2}.\frac{1}{2}x^2+\frac{3}{2}.\frac{1}{2}.\left(-\frac{1}{2}\right).\frac{1}{6}x^3+.....$$

$$=1+\frac{3x}{2}+\frac{3x^2}{8}-\frac{x^3}{16}+.....$$

Example 17 : Expand $(2+3x)^{-4}$ up to four terms

Solution : $(2+3x)^{-4}=2^{-4}\left[1+\frac{3x}{2}\right]^{-4}$

$$=2^{-4}\left[1+(-4)\left(\frac{3x}{2}\right)+\frac{(-4)(-4-1)}{2!}\left(\frac{3x}{2}\right)^2+\frac{(-4)(-4-1)(-4-2)}{3!}\left(\frac{3x}{2}\right)^3+.....\right]$$

$$=\frac{1}{2^4}\left[1-4\left(\frac{3x}{2}\right)+\frac{4.5}{2!}\left(\frac{3x}{2}\right)^2-\frac{4.5.6}{3!}\left(\frac{3x}{2}\right)^3+.....\right]=\frac{1}{2^4}\left[1-6x+\frac{45}{2}x^2-\frac{135}{2}x^3+.....\right].$$

Example 18 : Find the general term in the expansion of $(1-2x)^{-1/2}$.

Solution : The general term in the expansion of $(1-x)^{-n}$

$$T_{r+1}=\frac{n(n+1)(n+2).....(n+r-1)}{r!}x^r$$

In the given binomial $n=1/2$ and $2x$ is there in place of x therefore the general term in the expansion of $(1-2x)^{-1/2}$ will be

$$T_{r+1}=\frac{\frac{1}{2}\left(\frac{1}{2}+1\right)\left(\frac{1}{2}+2\right).....\left(\frac{1}{2}+r-1\right)}{r!}(2x)^r$$

$$=\frac{\frac{1}{2}.\frac{3}{2}.\frac{5}{2}......\frac{(2r-1)}{2}}{r!}(2)^r x^r=\frac{1.3.5......(2r-1)}{(2)^r\ r!}(2)^r x^r=\frac{1.3.5......(2r-1)}{r!}x^r.$$

Example 19 : Find the coefficient of x^r in the expansion of $(1+x)^{5/2}$.

Solution : The general term in the expansion of $(1+x)^{5/2}$

$$(r+1)\text{ th term} = \frac{\frac{5}{2}\left(\frac{5}{2}-1\right)\left(\frac{5}{2}-2\right).....\left(\frac{5}{2}-r+1\right)}{r!}x^r$$

$$= \frac{\left(\frac{5}{2}\right)\left(\frac{3}{2}\right)\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)......\left(\frac{7-2r}{2}\right)}{r!}x^r$$

$$= \frac{(5)(3)(1)(-1)(-3).....(7-2r)}{(2)^r\ r!}x^r$$

$$= (-1)^{r-3}\ \frac{5.3.3.5.7......(2r-7)}{(2)^r\ r!}x^r$$

$\therefore$ coefficient of $x^r = (-1)^{r-3}\ \dfrac{5.3.3.5.7......(2r-7)}{(2)^r\ r!}$

Example 20 : Find the coefficient of x^3 in the expansion of $\dfrac{(1+3x)^2}{(1-2x)}$.

Solution : $\dfrac{(1+3x)^2}{(1-2x)} = (1+3x)^2(1-2x)^{-1} = \left(1+6x+9x^2\right)\left(1+2x+4x^2+8x^3+.....\right)$

$$= 1+(6+2)x+(9+12+4)x^2+(18+24+8)x^3+.....$$

$\therefore$ coefficient of $x^3 = (18+24+8) = 50$

Example 21 : If $|x|<1$ then find the coefficient of x^4 in the expansion of $\left(1+2x+3x^2+....\right)^{1/2}$.

Solution : Given expansion $\left(1+2x+3x^2+....\right)^{1/2} = \left[(1-x)^{-2}\right]^{1/2} = (1-x)^{-1}$

$\therefore$ $(r+1)$ th term of $(1-x)^{-1} = \dfrac{(-1)(-2)(-3)......(-1-r+1)}{r!}x^r = (-1)^r\ \dfrac{1.2.3......r}{r!}.x^r$

$\therefore$ coefficient of $x^4 = (-1)^4\ \dfrac{1.2.3.4}{4!} = 1$

Example 22 : If $y = 3x+6x^2+10x^3+.....$ then prove that

$$x = \frac{y}{3} - \frac{1.4}{3^2\ 2!}.y^2 + \frac{1.4.7}{3^3\ 3!}.y^3 - .....$$

Solution : Adding 1 to both the sides in the given expansion

$1+y = 1+3x+6x^2+10x^3+.....$ or $(1+y) = (1-x)^{-3}$

Therefore $(1-x) = (1+y)^{-1/3}$

$$= 1+\left(-\frac{1}{3}\right)y + \frac{\left(-\frac{1}{3}\right)\left(-\frac{1}{3}-1\right)}{2!}y^2 + \frac{\left(-\frac{1}{3}\right)\left(-\frac{1}{3}-1\right)\left(-\frac{1}{3}-2\right)}{3!}y^3+...$$

$$=1-\frac{y}{3}+\frac{1.4}{3^2.2!}y^2-\frac{1.4.7}{3^3.3!}y^3+.....$$

or $$-x=-\left[\frac{y}{3}-\frac{1.4}{3^2.2!}y^2+\frac{1.4.7}{3^3.3!}y^3-.....\right]$$

or $$x=\frac{y}{3}-\frac{1.4}{3^2.2!}y^2+\frac{1.4.7}{3^3.3!}y^3-.....$$

Example 23 : Prove that

$$\left(1-x+x^2-x^3+...\infty\right)\left(1+x+x^2+x^3+...\infty\right)=\left(1+x^2+x^4+...\infty\right)$$

Solution : $\because$ $(1+x)^{-1}=1-x+x^2-x^3+....\infty$

and $(1-x)^{-1}=1+x+x^2+x^3+....\infty$

$\therefore$ L.H.S. $=\left(1-x+x^2-x^3+...\infty\right)\left(1+x+x^2+x^3+....\infty\right)$

$$=(1+x)^{-1}(1-x)^{-1}=\left(1-x^2\right)^{-1}$$

$$=1+x^2+x^4+....\infty=\text{ R.H.S.}$$

Exercise 7.4

1. Expand the following expansion up to four terms :

   (i) $\left(1+x^2\right)^{-2}$ **(ii)** $\left(1-\frac{x}{2}\right)^{1/2}$ **(iii)** $\left(3-2x^2\right)^{-2/3}$ **(iv)** $\frac{1}{\sqrt{5+4x}}$

2. Find the required terms in following expansion :

   (i) $(1-3x)^{-1/3}$, - 4th term **(ii)** $(1+x)^{5/2}$, - 7h term

   (iii) $(1+2x)^{-1/2}$, - 8th term

3. Find the general term in the expansion of :

   (i) $\left(a^3-x^3\right)^{2/3}$ **(ii)** $(1-2x)^{-3/2}$ **(iii)** $(1-x)^{-p/q}$

4. If $x<3$ then find the coefficient of x^5 in the expansion of $(3-x)^{-8}$

5. Find the coefficient of x^6 in the expansion of $\left(a+2bx^2\right)^{-3}$

6. Find the coefficient of x^{10} in the expansion of $\frac{1+3x^2}{\left(1-x^2\right)^3}$

7. Find the coefficient of x^r in the expansion of $\left(1-2x+3x^2-4x^3+.....\right)^n$ If $x=\frac{1}{2}$ and $n=1$ then find the value of binomial.

8. Prove that $\left(1+x+x^2+x^3+.....\right)^2=1+2x+3x^2+.....$

9. Prove that $\left(1+x+x^2+x^3+...\right)\left(1+3x+6x^2+...\right)=\left(1+2x+3x^2+...\right)^2$

10. If $x=2y+3y^2+4y^3+.....$ then express y in terms of increasing indices of x.

7.12 Applications of binomial theorem :

Illustrative Examples

Example 24 : Find the value of $(1\cdot 003)^4$ upto three places of decimal.

Solution : $(1\cdot 003)^4 = (1+\cdot 003)^4$

$$= 1+4(\cdot 003)+\frac{4\times 3}{2!}(\cdot 003)^2+\ldots\ldots = 1+\cdot 012+\ldots\ldots$$

$= 1\cdot 012$ (neglecting the higher powers)

Example 25 : If x is so small that its squares and higher powers are neglected then prove that

$$\frac{\sqrt{(1+2x)}+(16+3x)^{1/4}}{(1-x)^2} = 3+\frac{227}{32}x.$$

Solution : Given expression $= \dfrac{\sqrt{(1+2x)}+(16+3x)^{1/4}}{(1-x)^2}$

$$= \frac{(1+2x)^{1/2}+2\left(1+\frac{3x}{16}\right)^{1/4}}{(1-x)^2}$$

$$= \frac{\left[1+\frac{1}{2}(2x)+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)(2x)^2}{2!}+\ldots\right]+2\left[1+\frac{1}{4}\left(\frac{3x}{16}\right)+\frac{\frac{1}{4}\left(\frac{1}{4}-1\right)}{2!}\left(\frac{3x}{16}\right)^2+\ldots\right]}{(1-x)^2}$$

$$= \frac{\left[1+\frac{1}{2}(2x)\right]+2\left[1+\frac{1}{4}\left(\frac{3x}{16}\right)\right]}{(1-x)^2}$$ [neglecting x^2 and higher powers of x]

$$= \left[(1+x)+2\left(1+\frac{3x}{64}\right)\right](1-x)^{-2}$$

$$= \left(3+\frac{35}{32}x\right)(1+2x+\ldots)$$

$$= 3+\frac{35}{32}x+6x$$ (neglecting x^2 and higher powers of x)

$$= 3+\frac{227}{32}x$$

Example 26 : Find the value of $(126)^{1/3}$ upto 5 places of decimals.

Solution : $(126)^{1/3} = (126)^{1/3} = (125+1)^{1/3} = (5^3+1)^{1/3}$

$$= 5\left(1+\frac{1}{5^3}\right)^{1/3} = 5\left[1+\frac{1}{3}\left(\frac{1}{5^3}\right)+\frac{\frac{1}{3}\left(\frac{1}{3}-1\right)}{2!}\left(\frac{1}{5^3}\right)^2+\ldots\ldots\right]$$

$$=5\left[1+\frac{1}{3}.\frac{1}{5^3}+\frac{\frac{1}{3}\left(-\frac{2}{3}\right)}{2!}.\frac{1}{5^6}+.....\right]=5\left[1+\frac{1}{3}.\frac{1}{5^3}-\frac{1}{9}.\frac{1}{5^6}+.....\right]$$

$$=5+\frac{1}{3\times 5^2}-\frac{1}{9\times 5^5}+.....=5+\cdot 01333-\cdot 000035+.....=5+\cdot 01330=5\cdot 01330$$

Example 27 : If p and q are almost equal then prove that

$$\frac{(n+1)p+(n-1)q}{(n-1)p+(n+1)q}=\left(\frac{p}{q}\right)^{1/n}$$

Solution : Since p and q are almost equal then let $p=q+h$, where h is very small quantity whose square and higher powers can be neglected.

L.H.S. $$=\frac{(n+1)p+(n-1)q}{(n-1)p+(n+1)q}=\frac{(n+1)(q+h)+(n-1)q}{(n-1)(q+h)+(n+1)q}$$

$$=\frac{nq+q+nh+h+nq-q}{nq-q+nh-h+nq+q}=\frac{2nq+nh+h}{2nq+nh-h}$$

$$=\frac{2nq+(n+1)h}{2nq+(n-1)h}=\frac{(2nq)\left[1+\frac{(n+1)h}{2nq}\right]}{(2nq)\left[1+\frac{(n-1)h}{2nq}\right]}$$

$$=\left[1+\frac{(n+1)h}{2nq}\right]\left[1+\frac{(n-1)h}{2nq}\right]^{-1}$$

$$=\left[1+\frac{(n+1)h}{2nq}\right]\left[1-\frac{(n-1)h}{2nq}\right]$$

[neglecting higher powers of h]

$$=1+\frac{(n+1)h}{2nq}-\frac{(n-1)h}{2nq}$$

$$=1+\frac{h}{2nq}(n+1-n+1)=1+\frac{h}{2nq}(2)=1+\frac{h}{nq}.$$

R.H.S. $$=\left(\frac{p}{q}\right)^{1/n}=\left(\frac{q+h}{q}\right)^{1/n}=\left(1+\frac{h}{q}\right)^{1/n}$$

$$=1+\frac{h}{nq}=\text{L.H.S.}$$ [neglecting higher powers of h]

Exercise 7.5

1. If y is very less as compared to x then prove that $\frac{(x-y)^n}{(x+y)^n}=1-\frac{2ny}{x}$, where y^2 and higher powers are

neglected.

2. Find the value of the following expressions, x is so small that its square and higher powers are neglected.

(i) $\dfrac{(9+2x)^{1/2}(3+4x)}{(1+x)^{1/5}}$ **(ii)** $\dfrac{\sqrt{1-2x}+(1+3x)^{4/3}}{3+x+\sqrt{4-x}}$ **(iii)** $\dfrac{\left(1+\frac{3}{4}x\right)^{-4}\sqrt{16-3x}}{(8+x)^{2/3}}$

3. Evaluate :

(i) $\sqrt{30}$ upto 4 places of decimal. **(ii)** $(1\cdot 03)^{1/3}$ upto 4 places of decimal.

(iii) $\dfrac{1}{(8\cdot 16)^{1/3}}$ upto 4 places of decimal. **(iv)** cuberoot of 126 upto 5 places of decimal.

4. If x is a pproximately equal to 1, then prove that

(i) $\dfrac{mx^m - nx^n}{m-n} = x^{m+n}$ **(ii)** $\dfrac{ax^b - bx^a}{x^b - x^a} = \dfrac{1}{1-x}$

5. If p and q are almost equal then prove that :

$$\frac{q+2p}{p+2q} = \left(\frac{p}{q}\right)^{1/3}$$

7.13 Sum of series by binomial theorem :

To find the sum of a binomial series, its terms are compered with the corresponding terms of the below given series

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots$$

Firstly the terms are arranged as first term being 1 and second term as x, where $|x| < 1$ and the series should be in the ascending powers of x .

Now comparing the second and third terms with standard binomial expansion and get two equation is n and x and putting the values in standard binomial to find the sum.

Illustrative Examples

Example 28 : Find the value of series $1+\dfrac{1}{4}+\dfrac{1.3}{4.8}+\dfrac{1.3.5}{4.8.12}+\ldots$

Solution : Given series $1+\dfrac{1}{4}+\dfrac{1.3}{4.8}+\dfrac{1.3.5}{4.8.12}+\ldots$

Standard series $(1+x)^n = 1 + nx + \dfrac{n(n-1)}{2!}x^2 + \dfrac{n(n-1)(n-2)}{3!}x^3 + \ldots$

comparing the terms of series

$$nx = 1/4 \quad (1)$$

$$\frac{n(n-1)}{2!}x^2 = \frac{1.3}{4.8} \quad (2)$$

equation (2) by the square of equation (1)

$$\frac{n(n-1)x^2}{2!\ n^2x^2} = \frac{1.3}{4.8}(4)^2 \quad \Rightarrow \quad \frac{(n-1)}{2!\,.\,n} = \frac{1.3.4.4}{4.8} \Rightarrow \frac{(n-1)}{n} = \frac{3}{1}$$

$\Rightarrow 3n = n-1 \Rightarrow n = -1/2$

by substituting the value of n in equation (1)

$$\left(-\frac{1}{2}\right)x = \frac{1}{4} \Rightarrow x = -\frac{1}{2}$$

$\therefore$ sum of the terms $=(1+x)^n=\left[1-\tfrac{1}{2}\right]^{-1/2}=\left[\tfrac{1}{2}\right]^{-1/2}=[2]^{1/2}=\sqrt{2}$

Example 29 : Find the value of $1+\frac{2}{3}.\frac{1}{2}+\frac{2.5}{3.6}\frac{1}{(2)^2}+\frac{2.5.8}{3.6.9}.\frac{1}{(2)^3}+...$

Solution : Series $1+\frac{2}{3}.\frac{1}{2}+\frac{2.5}{3.6}\frac{1}{(2)^2}+\frac{2.5.8}{3.6.9}.\frac{1}{(2)^3}+...$

Standard series $(1+x)^n=1+nx+\frac{n(n-1)}{2!}x^2+...$

On comparing the terms of series

$$nx = \frac{2}{3}.\frac{1}{2}, \qquad (1)$$

$$\frac{n(n-1)}{2!}x^2 = \frac{2.5}{3.6}.\frac{1}{(2)^2} \qquad (2)$$

Dividing (2) by the square of equation (1)

$$\frac{n(n-1)}{2!}.\frac{x^2}{n^2x^2} = \frac{2.5}{3.6}.\frac{1}{(2)^2}.\frac{3^2.2^2}{2^2.1} \quad \Rightarrow \quad \frac{n(n-1)}{2!\ .\ n^2} = \frac{2.5.3.2.3.2}{3.6.2.2.2.2}$$

or $\frac{(n-1)}{n} = \frac{5}{2} \quad \Rightarrow \quad 2n-2 = 5n \quad \Rightarrow \quad n=-2/3$

by putting the value of n in equation (1) we have $-2/3x = 2/3.1/2 \Rightarrow x=-1/2$

$\therefore$ the sum of the series $=(1+x)^n=\left(1-\frac{1}{2}\right)^{-2/3}=\left(\frac{1}{2}\right)^{-2/3}=(2)^{2/3} \qquad (4)^{1/3}$

Example 30 : Find the value of $2+\frac{5}{2!\ .\ 3}+\frac{5.7}{3!\ .\ 3^2}+.....$

Solution : Series $1+1+\frac{5}{2!\ .\ 3}+\frac{5.7}{3!\ .\ 3^2}+.....$

Standard series $(1+x)^n = 1+nx+\frac{n(n-1)}{2!}x^2+.....$

On comparing the terms of series

$$nx = 1 \qquad (1)$$

$$\frac{n(n-1)}{2!}x^2 = \frac{5}{2!\times 3} \qquad (2)$$

Dividing (2) by the square of equation (1)

$$\frac{n(n-1)x^2}{2!\,n^2x^2} = \frac{5}{2!\times 3}$$

$$\Rightarrow \quad \frac{(n-1)}{n} = \frac{5}{3} \Rightarrow 5n = 3n - 3 \Rightarrow 2n = -3 \qquad \Rightarrow \quad n = -\frac{3}{2}$$

by putting the value of n in equation (1) we have

$$\frac{-3}{2}x = 1, \; x = -\frac{2}{3}$$

$\therefore$ the sum of the series

$$= (1+x)^n = \left(1 - \frac{2}{3}\right)^{-3/2} = \left(\frac{1}{3}\right)^{-3/2} = (3)^{3/2} = (27)^{1/2}$$

Example 31 : If $x = \frac{1}{3} + \frac{1.3}{3.6} + \frac{1.3.5}{3.6.9} + \frac{1.3.5.7}{3.6.9.12} + \ldots$, then prove that $x^2 + 2x - 2 = 0$

Solution : $1 + x = 1 + \frac{1}{3} + \frac{1.3}{3.6} + \frac{1.3.5}{3.6.9} + \ldots$

R.H.S. $= 1 + \frac{1}{3} + \frac{1.3}{3.6} + \frac{1.3.5}{3.6.9} + \ldots$

Standard series $\quad (1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \ldots$

comparing the terms of series

$$nx = \frac{1}{3} \qquad (1)$$

$$\frac{n(n-1)}{2!}x^2 = \frac{1.3}{3.6} \qquad (2)$$

Dividing (2) by the square of equation (1)

$$\frac{n(n-1)x^2}{2!\,.\,n^2x^2} = \frac{1.3}{3.6}.3^2$$

$$\Rightarrow \quad \frac{(n-1)}{2!\,.\,n} = \frac{1.3.3.3}{3.6} \Rightarrow \frac{(n-1)}{n} = \frac{3}{1} \Rightarrow 3n = n - 1$$

$$\Rightarrow \quad 2n = -1 \Rightarrow n = -1/2$$

by putting the value of n in equation (1)

$$-\frac{1}{2}.x = \frac{1}{3} \Rightarrow x = -\frac{2}{3}$$

therefore R.H.S. $= (1+x)^n = \left(1 - \frac{2}{3}\right)^{-1/2} = \left(\frac{1}{3}\right)^{-1/2} = (3)^{1/2} = \sqrt{3}$

$$\therefore \quad (x+1) = \sqrt{3} \Rightarrow (x+1)^2 = 3$$

$$\Rightarrow \quad x^2 + 2x + 1 = 3$$

$$\Rightarrow \quad x^2 + 2x - 2 = 0$$

Example 32 : Prove that : $x^n = 1 + n\left(1 - \frac{1}{x}\right) + \frac{n(n+1)}{2!}\left(1 - \frac{1}{x}\right)^2 + \ldots$

Solution : R.H.S. $=1+n\left(1-\frac{1}{x}\right)+\frac{n(n+1)}{2!}\left(1-\frac{1}{x}\right)^2+\ldots$

Standard series $(1+y)^m=1+my+\frac{m(m-1)}{2!}y^2+\ldots$

comparing the terms

$$my = n\left(1-\frac{1}{x}\right) \quad (1)$$

$$\frac{m(m-1)}{2!}y^2=\frac{n(n+1)}{2!}\left(1-\frac{1}{x}\right)^2 \quad (2)$$

Dividing (2) by the square of equation (1)

$$\frac{m(m-1)y^2}{2!\ m^2y^2}=\frac{n(n+1)}{2!}\frac{\left(1-\frac{1}{x}\right)^2}{n^2\left(1-\frac{1}{x}\right)^2}$$

$\Rightarrow$ $\frac{(m-1)}{m}=\frac{(n+1)}{n}\Rightarrow mn-n=mn+m$ $\Rightarrow$ $m=-n$

putting the value of n in equation (1)

$$-n.y=n\left(1-\frac{1}{x}\right)\Rightarrow y=-\left(1-\frac{1}{x}\right)$$

$\therefore$ sum of the series $=(1+y)^m=\left[1-\left(1-\frac{1}{x}\right)\right]^{-n}=\left[\frac{1}{x}\right]^{-n}=[x]^n=x^n=$ L.H.S

Exercise 7.6

Find the sum of infinite series

1. $1+\frac{2}{3}.\frac{1}{2}+\frac{2.5}{3.6}.\frac{1}{2^2}+\ldots$

2. $1+\frac{1}{3}.\frac{1}{4}+\frac{1.4}{3.6}.\frac{1}{4^2}+\ldots$

3. $1+\frac{1}{4}+\frac{1.4}{4.8}+\frac{1.4.7}{4.8.12}+\ldots$

4. $1+\frac{1}{10}+\frac{1.4}{10.20}+\frac{1.4.7}{10.20.30}+\ldots$

5. $1-\frac{1}{2}.\frac{1}{2}+\frac{1.3}{2.4}\left(\frac{1}{2}\right)^2-\frac{1.3.5}{2.4.6}\left(\frac{1}{2}\right)^3+\ldots$

Prove that (Q. 6-8)

6. $\sqrt{2}=1+\frac{1}{2^2}+\frac{1.3}{2!.\ 2^4}+\frac{1.3.5}{3!.\ 2^6}+\ldots$

7. $\sqrt{2}=\frac{7}{5}\left[1+\frac{1}{10^2}+\frac{1.3}{1.2}.\frac{1}{10^4}+\frac{1.3.5}{1.2.3}.\frac{1}{10^6}+\ldots\right]$

8. $\left(\frac{3}{2}\right)^{1/3}=1+\frac{1}{3^2}+\frac{1.4}{1.2}.\frac{1}{3^4}+\frac{1.4.7}{1.2.3}.\frac{1}{3^6}+\ldots$

9. If $y=\frac{1}{3}+\frac{1.3}{3.6}+\frac{1.3.5}{3.6.9}+\ldots\infty$, then prove that $y^2+2y-2=0$

10. Prove that $(1+x)^n=2^n\left[1-\frac{n(1-x)}{(1+x)}+\frac{n(n+1)}{2!}\left(\frac{1-x}{1+x}\right)^2-\ldots\right]$

Miscellaneous Exercise 7

1. The number of terms in the expansion of $\left(\frac{a}{x}+bx\right)^{12}$ is

(A) 11 (B) 13 (C) 10 (D) 14.

2. 7th terms in the expansion of $\left(\frac{1}{2}+a\right)^{8}$ is

(A) ${}^{8}C_7\left(\frac{1}{2}\right)(a)^7$ (B) ${}^{8}C_7\left(\frac{1}{2}\right)^7.a$ (C) ${}^{8}C_6\left(\frac{1}{2}\right)^2(a)^6$ (D) ${}^{8}C_6\left(\frac{1}{2}\right)^6(a)^2$.

3. Middle term in the expansion of $(a-x)^8$ is

(A) $56a^3x^5$ (B) $-56a^3x^5$ (C) $70a^4x^4$ (D) $-70a^4x^4$.

4. Constant term in the expansion of $\left(2x+\frac{1}{3x^2}\right)^9$ is

(A) fifth (B) fourth (C) sixth (D) seventh.

5. General term in the expansion of $(x+a)^n$ is

(A) ${}^{n}C_r\, x^{n-r}.a^r$ (B) ${}^{n}C_r\, x^r.a^r$ (C) ${}^{n}C_{n-r}\, x^{n-r}.a^r$ (D) ${}^{n}C_{n-r}\, x^r.a^{n-r}$

6. The term independent of x in the expansion of $\left(2x^2-\frac{1}{x}\right)^{12}$ is :

(A) 264 (B) –264 (C) 7920 (D) –7920

7. The coefficient of x^{-17} in the expansion of $\left(x^4-\frac{1}{x^3}\right)^{15}$ is :

(A) 1365 (B) –1365 (C) 3003 (D) –3003

8. If the coefficient of $(2r+4)$th and $(r-2)$th terms in the expansion of $(1+x)^{18}$ are equal then find the value of r :

(A) 5 (B) 6 (C) 7 (D) 8

9. If in the expansions of $(a+b)^n$ and $(a+b)^{n+3}$ ratio of second and third and fourth terms are equal then the value of n will be :

(A) 5 (B) 6 (C) 3 (D) 4

10. If the coefficient of 3rd and $(r+2)$th term in the expansion of $(1+x)^{2n}$ are equal then :

(A) $n=2r$ (B) $n=2r-1$ (C) $n=2r+1$ (D) $n=r+1$

11. Find the term independent of x in the expansion of $\left(2x^2-1/x\right)^{10}$.

12. Find the number of terms after simplifying $(x+a)^{200}+(x-a)^{200}$.

13. If $c_0, c_1, c_2 \ldots c_n$ are the coefficients of terms of $(1+x)^n$ then find the value of $c_0+c_2+c_4\ldots$

14. Find the value of ${}^{30}C_1 + {}^{30}C_2 + {}^{30}C_3 + ... + {}^{30}C_{30}$

15. Find the middle term in the expansion of $\left(\dfrac{a}{x} + \dfrac{x}{a}\right)^{10}$

16. Find the coefficients of x^5 in the expansion of $(1+2x)^6(1-x)^7$

17. If in the expansion of $(1+x)^{2n}$ the coefficients second, third and fourth terms are in A.P. then prove that $2n^2 - 9n + 7 = 0$

18. Using binomial theorem expand $\left(1 + \dfrac{x}{2} - \dfrac{2}{x}\right)^4, x \neq 0$

Important Points

1. If n is positive integer index, then number of terms in expansion of $(x+a)^n$ is $(n+1)$ and

$$(x+a)^n = {}^nC_0\, x^n + {}^nC_1\, x^{n-1}a + {}^nC_2\, x^{n-2}a^2 + ... + {}^nC_r\, x^{n-r}a^r + ... + {}^nC_n\, a^n.$$

2. General term $T_{r+1} = {}^nC_r\, x^{n-r}a^r$

3. If n is even then the middle term $= \left(\dfrac{n}{2} + 1\right) = \left(\dfrac{n+2}{2}\right)^{\text{th}}$ term

If n is odd then the middle term $\left(\dfrac{n+1}{2}\right)^{\text{th}}$ term and $\left(\dfrac{n+3}{2}\right)^{\text{th}}$ term

4. Binomial theorem for nay index

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + ...,$$

In this number of terms is infinite.

5. General term of series $(1+x)^n = T_{r+1} = \dfrac{n(n-1)(n-2).....(n-r+1)}{r!}x^r$

6. $(1+x)^{-n} = 1 - nx + \dfrac{n(n+1)}{2!}x^2 - \dfrac{n(n+1)(n+2)}{3!}x^3 + ...$

7. $(1-x)^{-n} = 1 + nx + \dfrac{n(n+1)}{2!}x^2 + \dfrac{n(n+1)(n+2)}{3!}x^3 + ...$

8. $(1+x)^{-1} = 1 - x + x^2 - ..... + (-1)^r x^r + ...$

9. $(1-x)^{-1} = 1 + x + x^2 + x^3 + ... + x^r + ...$

10. $(1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + ... + (-1)^r (r+1)x^r + ...$

11. $(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + ... + (r+1)x^r + ...$

12. $(1+x)^{-3} = 1 - 3x + 6x^2 - 10x^3 + ... + (-1)^r \dfrac{(r+1)(r+2)}{2}x^r + ...$

13. $(1-x)^{-3} = 1 + 3x + 6x^2 + 10x^3 + \ldots + \frac{(r+1)(r+2)}{2} x^r + \ldots$

Answers

Exercise 7.1

1. $8 - 12x + 6x^2 - x^3$ **2.** $\frac{32}{x^5} - \frac{40}{x^3} + \frac{20}{x} - 5x + \frac{5}{8}x^3 - \frac{x^5}{32}$. **3.** $\frac{x^6}{729} + \frac{2x^4}{81} + \frac{5x^2}{27} + \frac{20}{27} + \frac{5}{3x^2} + \frac{2}{x^4} + \frac{1}{x^6}$

4. $81x^4 + 216x^3y + 216x^2y^2 + 96xy^3 + 16y^4$. **5.** $\frac{x^3}{a^3} - \frac{6x^2}{a^2} + \frac{15x}{a} - 20 + 15\frac{9}{x} - 6\frac{a^2}{x^2} + \frac{a^3}{x^3}$

6. 884736 **7.** 104060401 **8.** 9509900499 **9.** 1.771516 **10.** $(1.1)^{10000}$ is greater

11. $8ab(a^2 + 6^2); 104\sqrt{6}$

Exercise 7.2

1. (i) ${}^{17}C_4 a^{13}\ 16x^{12}$ (ii) $3247695\frac{y^4}{x^{12}}$ (iii) $-63x^8$ **2.** (i) $-56\frac{a^3}{b^5}$ (ii) 6435 (iii) $-120a^7b^3$

3. (i) 495 (ii) 405 (iii) $\frac{5}{4}$ (iv) -252 **4.** (i) $20x^3y^3$ (ii) $\frac{189}{8}a^{17}, \frac{-21}{16}a^{19}$. (iii) $\frac{2n!}{(n!)2}$

(iv) $\frac{-6435\times 3^8\times 2^7}{x^6}, \frac{6435\times 3^7\times 2^8}{x^9}$ **7.** 7 or 14 **8.** $x = 2, a = 3, n = 5$

9. $n = 55$ **10.** $m = 4$

Exercise 7.3

1. (i) $(2)^8 - 1$ or 255 (ii) $(2)^7$ or 128

Exercise 7.4

1. **(i)** $1 - 2x^2 + 3x^4 - 4x^6$ **(ii)** $1 - \frac{x}{4} - \frac{x^2}{32} - \frac{x^3}{128}$

(iii) $\frac{1}{(3)^{2/3}}\left[1 + \frac{4}{9}x^2 + \frac{20}{81}x^4 + \frac{320}{2187}x^6\right]$ **(iv)** $\frac{1}{\sqrt{5}}\left[1 - \frac{2}{5}x + \frac{6}{25}x^2 - \frac{4}{25}x^3\right]$

2. **(i)** $\frac{14}{3}x^3$ **(ii)** $\frac{-5}{1024}x^6$ **(iii)** $\frac{-429}{16}x^7$

3. **(i)** $-\frac{2.1.4\ldots\ldots(3r-5)}{3^r\ r!}\cdot\frac{x^{3r}}{a^{3r-2}}$ **(ii)** $\frac{3.5.7\ldots\ldots(2r+1)}{r!}x^r$

(iii) $\frac{p(p+q)(p+2q)\ldots..\{p+(r-1)q\}}{r!}\left(\frac{x}{q}\right)^r$ **4.** $\frac{88}{(3)^{11}}$ **5.** $-80a^{-6}b^3$

6. 66 **7.** $(-1)^r\frac{2n(2n+1)(2n+2)\ldots..(2n+r-1)}{r!}, \frac{4}{9}$ **10.** $\frac{1}{2}x - \frac{3}{8}x^2 + \frac{5}{16}x^3 - \ldots..$

Exercise 7.5

2. **(i)** $9 + \frac{56}{5}x$ **(ii)** $\frac{2}{5} + \frac{27}{50}x$ **(iii)** $\left[1 - \frac{305}{96}x\right]$

3. **(i)** 5.4775 **(ii)** 1.0099 **(iii)** 0.4964 **(iv)** 5.01333

Exercise 7.6

1. $(4)^{1/3}$ **2.** $(4/3)^{1/3}$ **3.** $(4)^{1/3}$ **4.** $(10/7)^{1/3}$ **5.** $\sqrt{2/3}$

Miscellaneous Exercise 7

1. A **2.** C **3.** C **4.** B **5.** A **6.** C **7.** B

8. B **9.** A **10.** A **11.** –8064 **12.** 101 **13.** 2^{n-1} **14.** $2^{30}-1$

15. 252 **16.** 171 **18.** $\frac{16}{x}+\frac{8}{x^2}+\frac{32}{x^3}+\frac{16}{x^4}-4x+\frac{x^2}{2}+\frac{x^3}{2}+\frac{x^4}{16}-5$