CHAPTER XIX.

INEQUALITIES.

245. Any quantity a is said to be greater than another quantity b when $a-b$ is positive; thus 2 is greater than -3, because $2-(-3)$, or 5 is positive. Also b is said to be less than a when $b-a$ is negative; thus -5 is less than -2, because $-5-(-2)$, or -3 is negative.

In accordance with this definition, zero must be regarded as greater than any negative quantity.

In the present chapter we shall suppose (unless the contrary is directly stated) that the letters always denote real and positive quantities.

246. If $a > b$, then it is evident that

$$a+c > b+c;$$

$$a-c > b-c;$$

$$ac > bc;$$

$$\frac{a}{c} > \frac{b}{c};$$

that is, *an inequality will still hold after each side has been increased, diminished, multiplied, or divided by the same* positive *quantity.*

247. If $$a-c > b,$$
by adding c to each side,

$$a > b+c;$$

which shews that *in an inequality any term may be transposed from one side to the other if its sign be changed.*

If $a > b$, then evidently $b < a$;
that is, *if the sides of an inequality be transposed, the sign of inequality must be reversed.*

If $a > b$, then $a - b$ is positive, and $b - a$ is negative; that is, $-a-(-b)$ is negative, and therefore

$$-a < -b;$$

hence, *if the signs of all the terms of an inequality be changed, the sign of inequality must be reversed.*

Again, if $a > b$, then $-a < -b$, and therefore

$$-ac < -bc;$$

that is, *if the sides of an inequality be multiplied by the same negative quantity, the sign of inequality must be reversed.*

248. If $a_1 > b_1$, $a_2 > b_2$, $a_3 > b_3$, ... $a_m > b_m$, it is clear that

$$a_1 + a_2 + a_3 + \ldots + a_m > b_1 + b_2 + b_3 + \ldots + b_m;$$

and
$$a_1 a_2 a_3 \ldots a_m > b_1 b_2 b_3 \ldots b_m.$$

249. If $a > b$, and if p, q are positive integers, then $\sqrt[q]{a} > \sqrt[q]{b}$, or $a^{\frac{1}{q}} > b^{\frac{1}{q}}$; and therefore $a^{\frac{p}{q}} > b^{\frac{p}{q}}$; that is, $a^n > b^n$, where n is any positive quantity.

Further, $\dfrac{1}{a^n} < \dfrac{1}{b^n}$; that is $a^{-n} < b^{-n}$.

250. The square of every real quantity is positive, and therefore greater than zero. Thus $(a-b)^2$ is positive;

$$\therefore \quad a^2 - 2ab + b^2 > 0;$$

$$\therefore \quad a^2 + b^2 > 2ab.$$

Similarly
$$\frac{x+y}{2} > \sqrt{xy};$$

that is, *the arithmetic mean of two positive quantities is greater than their geometric mean.*

The inequality becomes an equality when the quantities are equal.

251. The results of the preceding article will be found very useful, especially in the case of inequalities in which the letters are involved symmetrically.

Example 1. If a, b, c denote positive quantities, prove that

$$a^2+b^2+c^2>bc+ca+ab;$$

and $$2(a^3+b^3+c^3)>bc(b+c)+ca(c+a)+ab(a+b).$$

For $$b^2+c^2>2bc \quad\text{.........}\quad (1);$$
$$c^2+a^2>2ca;$$
$$a^2+b^2>2ab;$$

whence by addition $$a^2+b^2+c^2>bc+ca+ab.$$

It may be noticed that this result is true for *any* real values of a, b, c.

Again, from (1) $$b^2-bc+c^2>bc \quad\text{.........}\quad (2);$$
$$\therefore\ b^3+c^3>bc(b+c) \quad\text{.........}\quad (3).$$

By writing down the two similar inequalities and adding, we obtain

$$2(a^3+b^3+c^3)>bc(b+c)+ca(c+a)+ab(a+b).$$

It should be observed that (3) is obtained from (2) by introducing the factor $b+c$, and that if this factor be negative the inequality (3) will no longer hold.

Example 2. If x may have any real value find which is the greater, x^3+1 or x^2+x.

$$x^3+1-(x^2+x)=x^3-x^2-(x-1)$$
$$=(x^2-1)(x-1)$$
$$=(x-1)^2(x+1).$$

Now $(x-1)^2$ is positive, hence

$$x^3+1 > \text{ or } < x^2+x$$

according as $x+1$ is positive or negative; that is, according as $x >$ or < -1.

If $x=-1$, the inequality becomes an equality.

252. Let a and b be two positive quantities, S their sum and P their product; then from the identity

$$4ab=(a+b)^2-(a-b)^2,$$

we have

$$4P=S^2-(a-b)^2, \text{ and } S^2=4P+(a-b)^2.$$

Hence, if S is given, P is greatest when $a=b$; and if P is given, S is least when

$$a=b;$$

that is, *if the sum of two positive quantities is given, their product is greatest when they are equal;* and *if the product of two positive quantities is given, their sum is least when they are equal.*

253. *To find the greatest value of a product the sum of whose factors is constant.*

Let there be n factors $a, b, c, \dots k$, and suppose that their sum is constant and equal to s.

Consider the product $abc \dots k$, and suppose that a and b are any two unequal factors. If we replace the two unequal factors a, b by the two equal factors $\frac{a+b}{2}$, $\frac{a+b}{2}$ the product is increased while the sum remains unaltered; hence *so long as the product contains two unequal factors it can be increased without altering the sum of the factors;* therefore the product is greatest when all the factors are equal. In this case the value of each of the n factors is $\frac{s}{n}$, and the greatest value of the product is $\left(\frac{s}{n}\right)^n$, or

$$\left(\frac{a+b+c+\dots+k}{n}\right)^n.$$

Cor. If $a, b, c, \dots k$ are *unequal*,

$$\left(\frac{a+b+c+\dots+k}{n}\right)^n > abc \dots k;$$

that is,

$$\frac{a+b+c+\dots+k}{n} > (abc \dots k)^{\frac{1}{n}}.$$

By an extension of the meaning of the terms *arithmetic mean* and *geometric mean* this result is usually quoted as follows:

the arithmetic mean of any number of positive quantities is greater than the geometric mean.

Example. Shew that $(1^r+2^r+3^r+\dots+n^r)^n > n^n (\lfloor n)^r$; where r is any real quantity.

Since $$\frac{1^r+2^r+3^r+\dots+n^r}{n} > (1^r \,.\, 2^r \,.\, 3^r \dots\dots n^r)^{\frac{1}{n}};$$

$$\therefore \left(\frac{1^r+2^r+3^r+\dots+n^r}{n}\right)^n > 1^r \,.\, 2^r \,.\, 3^r \dots\dots n^r, \text{ that is, } > (\lfloor n)^r;$$

whence we obtain the result required.

254. *To find the greatest value of* $a^m b^n c^p \dots$ *when* $a + b + c + \dots$ *is constant;* $m, n, p, \dots$ *being positive integers.*

Since $m, n, p, \dots$ are constants, the expression $a^m b^n c^p \dots$ will be greatest when $\left(\frac{a}{m}\right)^m \left(\frac{b}{n}\right)^n \left(\frac{c}{p}\right)^p \dots$ is greatest. But this last expression is the product of $m + n + p + \dots$ factors whose sum is $m\left(\frac{a}{m}\right) + n\left(\frac{b}{n}\right) + p\left(\frac{c}{p}\right) + \dots$, or $a + b + c + \dots$, and therefore constant. Hence $a^m b^n c^p \dots$ will be greatest when the factors

$$\frac{a}{m}, \ \frac{b}{n}, \ \frac{c}{p}, \dots$$

are all equal, that is, when

$$\frac{a}{m} = \frac{b}{n} = \frac{c}{p} = \dots = \frac{a + b + c + \dots}{m + n + p + \dots}.$$

Thus the greatest value is

$$m^m n^n p^p \dots \left(\frac{a + b + c + \ \dots}{m + n + p + \dots}\right)^{m+n+p+\dots}$$

Example. Find the greatest value of $(a+x)^3 (a-x)^4$ for any real value of x numerically less than a.

The given expression is greatest when $\left(\frac{a+x}{3}\right)^3 \left(\frac{a-x}{4}\right)^4$ is greatest; but the sum of the factors of this expression is $3\left(\frac{a+x}{3}\right) + 4\left(\frac{a-x}{4}\right)$, or $2a$; hence $(a+x)^3 (a-x)^4$ is greatest when $\frac{a+x}{3} = \frac{a-x}{4}$, or $x = -\frac{a}{7}$.

Thus the greatest value is $\frac{6^3 \cdot 8^4}{7^7} a^7$.

255. The determination of *maximum* and *minimum* values may often be more simply effected by the solution of a quadratic equation than by the foregoing methods. Instances of this have already occurred in Chap. IX.; we add a further illustration.

Example. Divide an odd integer into two integral parts whose product is a maximum.

Denote the integer by $2n+1$; the two parts by x and $2n+1-x$; and the product by y; then $(2n+1)x - x^2 = y$; whence

$$2x = (2n+1) \pm \sqrt{(2n+1)^2 - 4y};$$

but the quantity under the radical must be positive, and therefore y cannot be greater than $\frac{1}{4}(2n+1)^2$, or $n^2+n+\frac{1}{4}$; and since y is integral its greatest value must be n^2+n; in which case $x=n+1$, or n; thus the two parts are n and $n+1$.

256. Sometimes we may use the following method.

Example. Find the minimum value of $\dfrac{(a+x)(b+x)}{c+x}$.

Put $c+x=y$; then

$$\text{the expression} = \frac{(a-c+y)(b-c+y)}{y}$$

$$= \frac{(a-c)(b-c)}{y} + y + a - c + b - c$$

$$= \left(\frac{\sqrt{(a-c)(b-c)}}{\sqrt{y}} - \sqrt{y}\right)^2 + a - c + b - c + 2\sqrt{(a-c)(b-c)}.$$

Hence the expression is a minimum when the square term is zero; that is when $y=\sqrt{(a-c)(b-c)}$.

Thus the minimum value is

$$a-c+b-c+2\sqrt{(a-c)(b-c)};$$

and the corresponding value of x is $\sqrt{(a-c)(b-c)}-c$.

EXAMPLES. XIX. a.

1. Prove that $(ab+xy)(ax+by)>4abxy$.

2. Prove that $(b+c)(c+a)(a+b)>8abc$.

3. Shew that the sum of any real positive quantity and its reciprocal is never less than 2.

4. If $a^2+b^2=1$, and $x^2+y^2=1$, shew that $ax+by<1$.

5. If $a^2+b^2+c^2=1$, and $x^2+y^2+z^2=1$, shew that
$$ax+by+cz<1.$$

6. If $a>b$, shew that $a^ab^b>a^bb^a$, and $\log\frac{b}{a}<\log\frac{1+b}{1+a}$.

7. Shew that $(x^2y+y^2z+z^2x)(xy^2+yz^2+zx^2)>9x^2y^2z^2$.

8. Find which is the greater $3ab^2$ or a^3+2b^3.

9. Prove that $a^3b+ab^3<a^4+b^4$.

10. Prove that $6abc<bc(b+c)+ca(c+a)+ab(a+b)$.

11. Shew that $b^2c^2+c^2a^2+a^2b^2>abc(a+b+c)$.

12. Which is the greater x^3 or x^2+x+2 for positive values of x?

13. Shew that $x^3+13a^2x > 5ax^2+9a^3$, if $x > a$.

14. Find the greatest value of x in order that $7x^2+11$ may be greater than x^3+17x.

15. Find the minimum value of $x^2-12x+40$, and the maximum value of $24x-8-9x^2$.

16. Shew that $(\underline{|n})^2 > n^n$, and $2.4.6...2n < (n+1)^n$.

17. Shew that $(x+y+z)^3 > 27xyz$.

18. Shew that $n^n > 1.3.5...(2n-1)$.

19. If n be a positive integer greater than 2, shew that

$$2^n > 1+n\sqrt{2^{n-1}}.$$

20. Shew that $(\underline{|n})^3 < n^n\left(\dfrac{n+1}{2}\right)^{2n}$.

21. Shew that

(1) $(x+y+z)^3 > 27(y+z-x)(z+x-y)(x+y-z)$.

(2) $xyz > (y+z-x)(z+x-y)(x+y-z)$.

22. Find the maximum value of $(7-x)^4(2+x)^5$ when x lies between 7 and -2.

23. Find the minimum value of $\dfrac{(5+x)(2+x)}{1+x}$.

*257. *To prove that if* a *and* b *are positive and unequal,* $\dfrac{a^m+b^m}{2} > \left(\dfrac{a+b}{2}\right)^m$, *except when* m *is a positive proper fraction.*

We have $a^m+b^m = \left(\dfrac{a+b}{2}+\dfrac{a-b}{2}\right)^m + \left(\dfrac{a+b}{2}-\dfrac{a-b}{2}\right)^m$; and since $\dfrac{a-b}{2}$ is less than $\dfrac{a+b}{2}$, we may expand each of these expressions in ascending powers of $\dfrac{a-b}{2}$. [Art. 184.]

$$\therefore \frac{a^m+b^m}{2} = \left(\frac{a+b}{2}\right)^m + \frac{m(m-1)}{1.2}\left(\frac{a+b}{2}\right)^{m-2}\left(\frac{a-b}{2}\right)^2$$
$$+ \frac{m(m-1)(m-2)(m-3)}{1.2.3.4}\left(\frac{a+b}{2}\right)^{m-4}\left(\frac{a-b}{2}\right)^4 + \ldots.$$

(1) If m is a positive integer, or any negative quantity, all the terms on the right are positive, and therefore

$$\frac{a^m + b^m}{2} > \left(\frac{a+b}{2}\right)^m.$$

(2) If m is positive and less than 1, all the terms on the right after the first are negative, and therefore

$$\frac{a^m + b^m}{2} < \left(\frac{a+b}{2}\right)^m.$$

(3) If $m > 1$ and positive, put $m = \frac{1}{n}$ where $n < 1$; then

$$\left(\frac{a^m + b^m}{2}\right)^{\frac{1}{m}} = \left(\frac{a^{\frac{1}{n}} + b^{\frac{1}{n}}}{2}\right)^n;$$

$$\therefore \left(\frac{a^m + b^m}{2}\right)^{\frac{1}{m}} > \frac{(a^{\frac{1}{n}})^n + (b^{\frac{1}{n}})^n}{2}, \text{ by (2)};$$

$$\therefore \left(\frac{a^m + b^m}{2}\right)^{\frac{1}{m}} > \frac{a+b}{2}.$$

$$\therefore \frac{a^m + b^m}{2} > \left(\frac{a+b}{2}\right)^m.$$

Hence the proposition is established. If $m = 0$, or 1, the inequality becomes an equality.

*258. *If there are* n *positive quantities* a, b, c,...k, *then*

$$\frac{a^m + b^m + c^m + \ldots + k^m}{n} > \left(\frac{a+b+c+\ldots+k}{n}\right)^m$$

unless m *is a positive proper fraction.*

Suppose m to have any value not lying between 0 and 1.

Consider the expression $a^m + b^m + c^m + \ldots + k^m$, and suppose that a and b are unequal; if we replace a and b by the two equal quantities $\frac{a+b}{2}$, $\frac{a+b}{2}$, the value of $a+b+c+\ldots+k$ remains unaltered, but the value of $a^m + b^m + c^m + \ldots + k^m$ is diminished, since

$$a^m + b^m > 2\left(\frac{a+b}{2}\right)^m.$$

Hence so long as any two of the quantities $a, b, c, \ldots k$ are unequal the expression $a^m + b^m + c^m + \ldots + k^m$ can be diminished without altering the value of $a + b + c + \ldots + k$; and therefore the value of $a^m + b^m + c^m + \ldots + k^m$ will be least when all the quantities $a, b, c, \ldots k$ are equal. In this case each of the quantities is equal to

$$\frac{a + b + c + \ldots + k}{n};$$

and the value of $a^m + b^m + c^m + \ldots + k^m$ then becomes

$$n\left(\frac{a + b + c + \ldots + k}{n}\right)^m.$$

Hence when $a, b, c, \ldots k$ are unequal,

$$\frac{a^m + b^m + c^m + \ldots + k^m}{n} > \left(\frac{a + b + c + \ldots + k}{n}\right)^m.$$

If m lies between 0 and 1 we may in a similar manner prove that the sign of inequality in the above result must be reversed.

The proposition may be stated verbally as follows:

The arithmetic mean of the mth *powers of* n *positive quantities is greater than the* mth *power of their arithmetic mean in all cases except when* m *lies between* 0 *and* 1.

*259. *If* a *and* b *are positive integers, and* a > b, *and if* x *be a positive quantity,*

$$\left(1 + \frac{x}{a}\right)^a > \left(1 + \frac{x}{b}\right)^b.$$

For

$$\left(1 + \frac{x}{a}\right)^a = 1 + x + \left(1 - \frac{1}{a}\right)\frac{x^2}{\lfloor 2} + \left(1 - \frac{1}{a}\right)\left(1 - \frac{2}{a}\right)\frac{x^3}{\lfloor 3} + \ldots (1),$$

the series consisting of $a + 1$ terms; and

$$\left(1 + \frac{x}{b}\right)^b = 1 + x + \left(1 - \frac{1}{b}\right)\frac{x^2}{\lfloor 2} + \left(1 - \frac{1}{b}\right)\left(1 - \frac{2}{b}\right)\frac{x^3}{\lfloor 3} + \ldots (2),$$

the series consisting of $b + 1$ terms.

After the second term, each term of (1) is greater than the corresponding term of (2); moreover the number of terms in (1) is greater than the number of terms in (2); hence the proposition is established.

*260. *To prove that* $\sqrt[x]{\frac{1+x}{1-x}} > \sqrt[y]{\frac{1+y}{1-y}}$,

if x *and* y *are proper fractions and positive, and* x > y.

For $$\sqrt[x]{\frac{1+x}{1-x}} > \text{ or } < \sqrt[y]{\frac{1+y}{1-y}},$$

according as $$\frac{1}{x}\log\frac{1+x}{1-x} > \text{ or } < \frac{1}{y}\log\frac{1+y}{1-y}.$$

But $$\frac{1}{x}\log\frac{1+x}{1-x} = 2\left(1+\frac{x^2}{3}+\frac{x^4}{5}+\dots\right), \text{ [Art. 226]};$$

and $$\frac{1}{y}\log\frac{1+y}{1-y} = 2\left(1+\frac{y^2}{3}+\frac{y^4}{5}+\dots\right).$$

$$\therefore \quad \frac{1}{x}\log\frac{1+x}{1-x} > \frac{1}{y}\log\frac{1+y}{1-y},$$

and thus the proposition is proved.

*261. *To prove that* $(1+x)^{1+x}(1-x)^{1-x} > 1$, *if* $x<1$, *and to deduce that* $$a^a b^b > \left(\frac{a+b}{2}\right)^{a+b}.$$

Denote $(1+x)^{1+x}(1-x)^{1-x}$ by P; then

$$\log P = (1+x)\log(1+x) + (1-x)\log(1-x)$$

$$= x\{\log(1+x) - \log(1-x)\} + \log(1+x) + \log(1-x)$$

$$= 2x\left(x+\frac{x^3}{3}+\frac{x^5}{5}+\dots\right) - 2\left(\frac{x^2}{2}+\frac{x^4}{4}+\frac{x^6}{6}+\dots\right)$$

$$= 2\left(\frac{x^2}{1\,.\,2}+\frac{x^4}{3\,.\,4}+\frac{x^6}{5\,.\,6}+\dots\right).$$

Hence $\log P$ is positive, and therefore $P>1$;

that is, $$(1+x)^{1+x}(1-x)^{1-x} > 1.$$

In this result put $x=\frac{z}{u}$, where $u>z$; then

$$\left(1+\frac{z}{u}\right)^{1+\frac{z}{u}}\left(1-\frac{z}{u}\right)^{1-\frac{z}{u}}>1;$$

$$\therefore \quad \left(\frac{u+z}{u}\right)^{u+z}\left(\frac{u-z}{u}\right)^{u-z}>1^u, \text{ or } 1;$$

$$\therefore \quad (u+z)^{u+z}(u-z)^{u-z}>u^{2u}.$$

Now put $u+z=a$, $u-z=b$, so that $u=\frac{a+b}{2}$;

$$\therefore \quad a^a b^b>\left(\frac{a+b}{2}\right)^{a+b}.$$

*EXAMPLES. XIX. b.

1. Shew that $27(a^4+b^4+c^4)>(a+b+c)^4$.

2. Shew that $n(n+1)^3<8(1^3+2^3+3^3+\ldots+n^3)$.

3. Shew that the sum of the m^{th} powers of the first n even numbers is greater than $n(n+1)^m$, if $m>1$.

4. If α and β are positive quantities, and $\alpha>\beta$, shew that

$$\left(1+\frac{1}{\alpha}\right)^{\alpha}>\left(1+\frac{1}{\beta}\right)^{\beta}.$$

Hence shew that if $n>1$ the value of $\left(1+\frac{1}{n}\right)^n$ lies between 2 and 2·718...

5. If a, b, c are in descending order of magnitude, shew that

$$\left(\frac{a+c}{a-c}\right)^a<\left(\frac{b+c}{b-c}\right)^b.$$

6. Shew that $\left(\frac{a+b+c+\ldots+k}{n}\right)^{a+b+c+\ldots+k}<a^a b^b c^c\ldots k^k$.

7. Prove that $\frac{1}{m}\log(1+a^m)<\frac{1}{n}\log(1+a^n)$, if $m>n$.

8. If n is a positive integer and $x<1$, shew that

$$\frac{1-x^{n+1}}{n+1}<\frac{1-x^n}{n}.$$

9. If a, b, c are in H. P. and $n > 1$, shew that $a^n + c^n > 2b^n$.

10. Find the maximum value of $x^3(4a-x)^5$ if x is positive and less than $4a$; and the maximum value of $x^{\frac{1}{2}}(1-x)^{\frac{1}{3}}$ when x is a proper fraction.

11. If x is positive, shew that $\log(1+x) < x$ and $> \dfrac{x}{1+x}$.

12. If $x+y+z=1$, shew that the least value of $\dfrac{1}{x}+\dfrac{1}{y}+\dfrac{1}{z}$ is 9; and that $(1-x)(1-y)(1-z) > 8xyz$.

13. Shew that $(a+b+c+d)(a^3+b^3+c^3+d^3) > (a^2+b^2+c^2+d^2)^2$.

14. Shew that the expressions

$$a(a-b)(a-c)+b(b-c)(b-a)+c(c-a)(c-b)$$

and

$$a^2(a-b)(a-c)+b^2(b-c)(b-a)+c^2(c-a)(c-b)$$

are both positive.

15. Shew that $(x^m+y^m)^n < (x^n+y^n)^m$, if $m > n$.

16. Shew that $a^b b^a < \left(\dfrac{a+b}{2}\right)^{a+b}$.

17. If a, b, c denote the sides of a triangle, shew that

(1) $a^2(p-q)(p-r)+b^2(q-r)(q-p)+c^2(r-p)(r-q)$

cannot be negative; p, q, r being any real quantities;

(2) $a^2yz+b^2zx+c^2xy$ cannot be positive, if $x+y+z=0$.

18. Shew that $\underline{|1}\ \underline{|3}\ \underline{|5} \ldots\ldots\ldots \underline{|2n-1} > (\underline{|n})^n$

19. If a, b, c, d, ... are p positive integers, whose sum is equal to n, shew that the least value of

$$\underline{|a}\,\underline{|b}\,\underline{|c}\,\underline{|d} \ldots\ldots\ldots \text{ is } (\underline{|q})^{p-r}(\underline{|q+1})^r,$$

where q is the quotient and r the remainder when n is divided by p.