06

Trigonometric Ratios

6.01. Introduction

In class IX, We have studied trigonometric ratios of acute angles. In this chapter we will find trigonometric ratios of specific angles of right angled tringle, 0^o, 30^o, 45^o, 60^o and 90^o.

6.02. Trigonometric Ratios of Angle 0^o

Let rotating ray AC, makes acute angle $\angle XAC = \theta$ with its initial position AX, from point A draw perpendicular CB on AX, which is so small. As line AC tends to AX then length of CB tends to zero. In this case line AC and A coincides and $\angle XAC = \theta = 0°$ and $AC = AB$, $\therefore$ $CB = 0$ (zero).

Thus, values of trigonometric ratios corresponding to 0^o will be as follows :

$$\sin 0^\circ = \frac{CB}{CA} = \frac{0}{CA} = 0$$

$$\cos 0^\circ = \frac{AB}{CA} = \frac{CA}{CA} = 1$$

$$\tan 0^\circ = \frac{CB}{AB} = \frac{0}{AB} = 0$$

$$\cot 0^\circ = \frac{AB}{CB} = \frac{AB}{0} = \infty$$

$$\sec 0^\circ = \frac{CA}{AB} = \frac{CA}{CA} = 1$$

$$\text{cosec}\, 0^\circ = \frac{CA}{CB} = \frac{CA}{0} = \infty$$

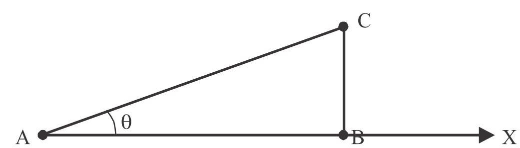

Fig. 6.01

6.03. Trigonometric Ratios of Angle 90^o

From ΔCBA, it is clear that as θ increases, length of CB decreases and point B gets very close to C and when θ become equal to 90^o, then points B and C will coincide. In this case $CB = 0$ and $CA = AB$.

$$\sin 90^\circ = \frac{AB}{CA} = \frac{AB}{AB} = 1$$

$$\cos 90^\circ = \frac{CB}{CA} = \frac{0}{AB} = 0$$

$$\tan 90^\circ = \frac{AB}{CB} = \frac{AB}{0} = \infty$$

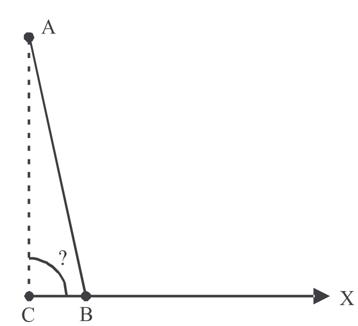

Fig. 6.02

$$\cot 90^\circ = \frac{CB}{AB} = \frac{0}{AB} = 0$$

$$\sec 90^\circ = \frac{CA}{CB} = \frac{CA}{0} = \infty$$

$$\operatorname{cosec} 90^\circ = \frac{CA}{AB} = \frac{AB}{AB} = 1$$

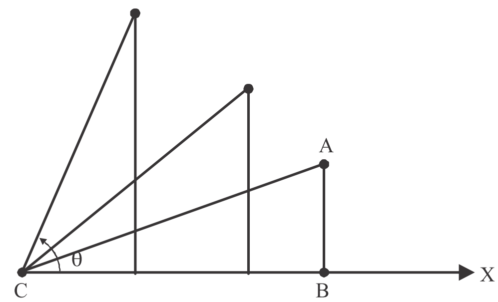

Fig. 6.03

6.04. Trigonometric Ratios of 30° and 60°

Construct an equilateral triangle ABC of each side 2 a. Each angle of an equilateral triangle is of 60°. AD is perpendicular from vertex A to side BC and point D is mid point of side BC.

$BD = DC = a$ and ? BAD ? 30°

∴ $\angle D = 90^\circ$ in ΔABC

$$AB^2 = AD^2 + BD^2$$

$$(2a)^2 = AD^2 + a^2$$

$$AD^2 = 4a^2 - a^2$$

$$AD = \sqrt{3}a$$

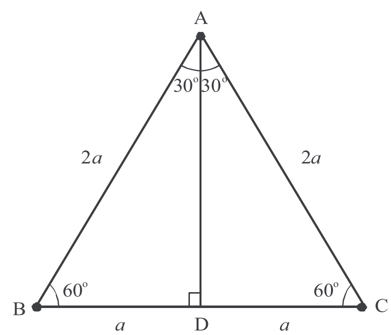

Fig. 6.04

Trigonometric Ratios of 30°

In right angles ADB , base $(AD) = \sqrt{3}a$, perpendicular $(BD) = a$, hypotenuse $(AB) = 2a$ and $= DAB = 30^\circ$

$$\sin 30^\circ = \frac{BD}{AB} = \frac{a}{2a} = \frac{1}{2}$$

$$\cos 30^\circ = \frac{AD}{AB} = \frac{\sqrt{3}a}{2a} = \frac{\sqrt{3}}{2}$$

$$\tan 30^\circ = \frac{BD}{AD} = \frac{a}{\sqrt{3}.a} = \frac{1}{\sqrt{3}}$$

$$\cot 30^\circ = \frac{AD}{BD} = \frac{\sqrt{3}.a}{a} = \frac{\sqrt{3}a}{a} = \sqrt{3}$$

$$\sec 30^\circ = \frac{AB}{AD} = \frac{2a}{\sqrt{3}a} = \frac{2}{\sqrt{3}}$$

$$\operatorname{cosec} 30^\circ = \frac{AB}{BD} = \frac{2a}{a} = 2$$

Trigonometric Ratios of 60°

In right angled ΔADB, base $(BD) = a$,

Perpendicular $(AD) = a.\sqrt{3}$, hypotenuse $(AB) = 2a$ and $=$ $ABD = 60^\circ$

$$\sin 60^\circ = \frac{AD}{AB} = \frac{a.\sqrt{3}}{2a} = \frac{\sqrt{3}}{2}$$

$$\cos 60^\circ = \frac{BD}{AB} = \frac{a}{2a} = \frac{1}{2}$$

$$\tan 60^\circ = \frac{AD}{BD} = \frac{a.\sqrt{3}}{a} = \sqrt{3}$$

$$\cot 60^\circ = \frac{BD}{AD} = \frac{a}{a\sqrt{3}} = \frac{1}{\sqrt{3}}$$

$$\sec 60^\circ = \frac{AB}{BD} = \frac{2a}{a} = 2$$

$$\cos ec60^\circ = \frac{AB}{AD} = \frac{2a}{a.\sqrt{3}} = \frac{2}{\sqrt{3}}$$

6.05. Trigonometric ratios of 45°

Construct a right tringle ABC in which

$\angle B = 90°$ and $\angle A = 45^\circ$

then in ΔABC,

$$\angle A + \angle B + \angle C = 180^\circ$$

$$45^\circ + 90^\circ + \angle C = 180^\circ$$

$$\angle C = 45^\circ$$

$\therefore$ $\angle A = \angle C$

$\therefore$ $AB = BC$

Let AB = BC = a

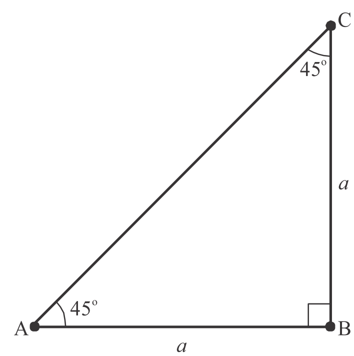

Fig. 6.05

In ΔABC, from Bodhayan theorem

$$AC^2 = AB^2 + BC^2 = a^2 + a^2 = 2a^2$$

$$AC = \sqrt{2a^2} = \sqrt{2}a$$

ΔABC में, $\angle A = 45^\circ$, base $(AB) = a$, perpendicular $(BC) = a$, hypotenuse $(AC) = \sqrt{2}a$

$$\sin 45^\circ = \frac{BC}{AC} = \frac{a}{\sqrt{2}a} = \frac{1}{\sqrt{2}}$$

$$\cos 45^\circ = \frac{AB}{AC} = \frac{a}{\sqrt{2}a} = \frac{1}{\sqrt{2}}$$

$$\tan 45^\circ = \frac{BC}{AB} = \frac{a}{a} = 1$$

$$\cot 45^\circ = \frac{AB}{BC} = \frac{a}{a} = 1$$

$$\sec 45^\circ = \frac{AC}{AB} = \frac{\sqrt{2}a}{a} = \sqrt{2}$$

$$\cos ec 45^\circ = \frac{AC}{BC} = \frac{\sqrt{2}a}{a} = \sqrt{2}$$

Table of Trigonometric Ratios of Specific Angles

Angle (θ) → (Degree/Radian) Trigonometric Ratio ↓	$0^\circ / 0$	$30^\circ / \pi/6$	$45^\circ / \pi/4$	$60^\circ / \pi/3$	$90^\circ / \pi/2$
sin	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
cos	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
tan	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	∞
cot	∞	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$	0
sec	1	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2	∞
cosec	∞	2	$\sqrt{2}$	$\frac{2}{\sqrt{3}}$	1

Example 1. Find the value of $\tan^2 60^\circ + 3\cos^2 30^\circ$

Solution : $\tan^2 60^\circ + 3\cos^2 30^\circ$ (Putting the values of trignometric ratios)

$$= \left(\sqrt{3}\right)^2 + 3\left(\frac{\sqrt{3}}{2}\right)^2 = 3 + 3 \times \frac{3}{4}$$

$$= 3 + \frac{9}{4} = \frac{12+9}{4} = \frac{21}{4}$$

Example 2. Find the value of $\sin 60^\circ \cos 30^\circ + \sin 30^\circ \cos 60^\circ$

Solution : $\sin 60^\circ \cos 30^\circ + \sin 30^\circ \cos 60^\circ$

$$= \frac{\sqrt{3}}{2} . \frac{\sqrt{3}}{2} + \frac{1}{2} . \frac{1}{2} = \frac{3}{4} + \frac{1}{4} = \frac{3+1}{4} = \frac{4}{4} = 1$$

Example 3. Prove that $4\sin 30^\circ \sin^2 60^\circ + 3\cos 60^\circ \tan 45^\circ = 2\sec^2 60^\circ - \operatorname{cos}ec^2 90^\circ$

Solution : $(L.H.S.) = 4\sin 30^\circ \sin^2 60^\circ + 3\cos 60^\circ \tan 45^\circ$

$$= 4.\frac{1}{2}.\left(\frac{\sqrt{3}}{2}\right)^2 + 3\times\frac{1}{2}.1 = \frac{3}{2}+\frac{3}{2} = 3$$

$(R.H.S.) = 2\sec^2 60^\circ - \cos ec^2 90^\circ$

$$= 2.\left(\sqrt{2}\right)^2 - (1)^2 = 2\times 2 - 1 = 4 - 1 = 3$$

$\therefore L.H.S. = R.H.S.$

Example 4. Find the value of $\dfrac{\cos 45^\circ}{\sec 30^\circ + \cos ec 30^\circ}$

Solution : $\dfrac{\cos 45^\circ}{\sec 30^\circ + \cos ec 30^\circ}$

$$= \frac{\frac{1}{\sqrt{2}}}{\frac{2}{\sqrt{3}}+2} = \frac{\frac{1}{\sqrt{2}}}{\frac{2+2\sqrt{3}}{\sqrt{3}}} = \frac{1}{\sqrt{2}}.\frac{\sqrt{3}}{2+2\sqrt{3}} = \frac{\sqrt{3}}{2\sqrt{2}\left(\sqrt{3}+1\right)}$$

$$= \frac{\sqrt{6}}{4}\left[\frac{\sqrt{3}-1}{\left(\sqrt{3}+1\right)\left(\sqrt{3}-1\right)}\right] = \frac{\sqrt{6}\left(\sqrt{3}-1\right)}{4(3-1)} = \frac{\sqrt{6}\left(\sqrt{3}-1\right)}{8}$$

Example 5. Prove that $3\tan^2 30^\circ - \frac{4}{3}\sin^2 60^\circ - \frac{1}{2}\cos ec^2 45^\circ + \frac{4}{3}\sin^2 90^\circ = \frac{1}{3}$

Solution : $(L.H.S.) = 3\tan^2 30^\circ - \frac{4}{3}\sin^2 60^\circ - \frac{1}{2}\cos ec^2 45^\circ + \frac{4}{3}\sin^2 90^\circ$

$$= 3\left(\frac{1}{\sqrt{3}}\right)^2 - \frac{4}{3}\left(\frac{\sqrt{3}}{2}\right)^2 - \frac{1}{2}\left(\sqrt{2}\right)^2 + \frac{4}{3}(1)^2 = 3.\left(\frac{1}{3}\right) - \frac{4}{3}.\left(\frac{3}{4}\right) - \frac{1}{2}.(2) + \frac{4}{3}$$

$$= 1 - 1 - 1 + \frac{4}{3} = \frac{1}{3} \quad (R.H.S.)$$

Example 6. If $\tan 3x = \sin 45^\circ \cos 45^\circ + \sin 30^\circ$, then find the value of x. $(x < 90^\circ)$

Solution : Given, $\tan 3x = \sin 45^\circ \cos 45^\circ + \sin 30^\circ$

$$\tan 3x = \frac{1}{\sqrt{2}}.\frac{1}{\sqrt{2}} + \frac{1}{2} = \frac{1}{2} + \frac{1}{2}$$

or $\tan 3x = 1 \Rightarrow \tan 3x = \tan 45^\circ$ or $\tan 3x = \tan 225^\circ$.

or $3x = 45^\circ$ or $x = 15^\circ$ and $3x = 225^\circ$ or $x = 75^\circ$

Example 7. If $\sin(A+B)=1$ and $\cos(A-B)=\frac{\sqrt{3}}{2}$ here $0^\circ<(A+B)\le 90^\circ$, $A>B$ Find the value of A and B.

Solution : Given $\sin(A+B)=1$

or $\sin(A+B)=\sin 90^\circ$

or $A+B=90^\circ \quad \ldots (1)$

and $\cos(A-B)=\frac{\sqrt{3}}{2}$

or $\cos(A-B)=\cos 30^\circ$

or $A-B=30^\circ \quad \ldots (2)$

On adding equations (1) and (2), we get

$$(A+B)+(A-B)=90+30^\circ$$

$$2A=120^\circ \qquad \text{or} \qquad A=60^\circ$$

Putting value of A in equation (1), we get

$$60^\circ+B=90^\circ$$

$$B=30^\circ$$

$\therefore \quad A=60^\circ, B=30^\circ$

Example 8. Find the value of $\frac{\sin 30^\circ+\tan 45^\circ-\cos ec 60^\circ}{\sec 30^\circ+\cos 60^\circ+\cot 45^\circ}$

Solution : $\frac{\sin 30^\circ+\tan 45^\circ-\cos ec 60^\circ}{\sec 30^\circ+\cos 60^\circ+\cot 45^\circ}$

$$=\frac{\frac{1}{2}+1-\frac{2}{\sqrt{3}}}{\frac{2}{\sqrt{3}}+\frac{1}{2}+1}=\frac{\frac{3}{2}-\frac{2}{\sqrt{3}}}{\frac{2}{\sqrt{3}}+\frac{3}{2}}$$

$$=\frac{\frac{3\sqrt{3}-4}{2\sqrt{3}}}{\frac{4+3\sqrt{3}}{2\sqrt{3}}}=\frac{3\sqrt{3}-4}{2\sqrt{3}}\times\frac{2\sqrt{3}}{4+3\sqrt{3}}=\left(\frac{3\sqrt{3}-4}{4+3\sqrt{3}}\right)\times\left(\frac{4-3\sqrt{3}}{4-3\sqrt{3}}\right)$$

[Multiplying Numerator and Denominator by $(4-3\sqrt{3})$

$$= \frac{-(4-3\sqrt{3})(4-3\sqrt{3})}{(4)^2-(3\sqrt{3})^2} = \frac{-(4-3\sqrt{3})^2}{16-27}$$

$$= \frac{-(16+27-24\sqrt{3})}{-11} = \frac{43-24\sqrt{3}}{11}$$

$$\therefore \qquad \frac{\sin 30^\circ + \tan 45^\circ - \operatorname{cosec} 60^\circ}{\sec 30^\circ + \cos 60^\circ + \cot 45^\circ} = \frac{43-24\sqrt{3}}{11}$$

Exercise 6.1

Find the value of the following :

1. $2\sin 45^\circ \cos 45^\circ$
2. $\cos 45^\circ \cos 60^\circ - \sin 45^\circ \sin 60^\circ$
3. $\sin^2 30^\circ + 2\cos^2 45^\circ + 3\tan^2 60^\circ$
4. $3\sin 60^\circ - 4\sin^3 60^\circ$
5. $\dfrac{5\cos^2 60^\circ + 4\sin^2 30^\circ + \tan^2 45^\circ}{\sin^2 30^\circ + \cos^2 45^\circ}$
6. $4\cot^2 45^\circ - \sec^2 60^\circ + \sin^2 60^\circ + \cos^2 90^\circ$
7. $\dfrac{4}{\cot^2 30^\circ} + \dfrac{1}{\sin^2 30^\circ} - \cos^2 45^\circ$
8. $\dfrac{\tan^2 60^\circ + 4\sin^2 45^\circ + \sin^2 90^\circ}{3\sec^2 30^\circ + \cos ec^2 60^\circ - \cot^2 30^\circ}$
9. $\dfrac{\sin 30^\circ - \sin 90^\circ + 2\cos 0^\circ}{\tan 30^\circ \tan 60^\circ}$
10. $\dfrac{2\tan 30^\circ}{1-\tan^2 30^\circ}$
11. Find the value of x in the following :

    (i) $\cos x = \cos 60^\circ \cos 30^\circ + \sin 60^\circ \sin 30^\circ$

    (ii) $\sin 2x = \sin 60^\circ \cos 30^\circ - \cos 60^\circ \sin 30^\circ$

    (iii) $\sqrt{3}\tan 2x = \sin 30^\circ + \sin 45^\circ \cos 45^\circ + 2\sin 90^\circ$

Prove that :

12. $\dfrac{\cos 30^\circ + \sin 60^\circ}{1+\cos 60^\circ + \sin 30^\circ} = \dfrac{\sqrt{3}}{2}$

13. $4\cot^2 45^\circ - \sec^2 60^\circ - \sin^2 30^\circ = -\dfrac{1}{4}$

14. $4\sin 30^\circ \sin^2 60^\circ + 3\cos 60^\circ \tan 45^\circ = 2\sec^2 45^\circ - \operatorname{cosec}^2 90^\circ$

15. $\operatorname{cosec}^2 45^\circ \sec^2 30^\circ \sin^3 90^\circ \cos 60^\circ \ ? \ \dfrac{4}{3}$

16. $\dfrac{\sin 60^\circ + \sin 30^\circ}{\sin 60^\circ - \sin 30^\circ} = \dfrac{\tan 60^\circ + \tan 45^\circ}{\tan 60^\circ - \tan 45^\circ}$

17. $2\left(\cos^2 45^\circ + \tan^2 60^\circ\right) - 6\left(\sin^2 45^\circ - \tan^2 30^\circ\right) = 6$

18. $\left(\sec^2 30^\circ + \operatorname{cosec}^2 45^\circ\right)\left(2\cos 60^\circ + \sin 90^\circ + \tan 45^\circ\right) = 10$

19. $\left(1 - \sin 45^\circ + \sin 30^\circ\right)\left(1 + \cos 45^\circ + \cos 60^\circ\right) = \dfrac{7}{4}$

20. $\cos^2 0^\circ - 2\cot^2 30^\circ + 3\operatorname{cosec}^2 90^\circ = 2\left(\sec^2 45^\circ - \tan^2 60^\circ\right)$

21. If $x = 30^\circ$ then prove that

(i) $\sin 3x = 3\sin x - 4\sin^3 x$

(ii) $\tan 2x = \dfrac{2\tan x}{1-\tan^2 x}$

(iii) $\sin x = \sqrt{\dfrac{1-\cos 2x}{2}}$

(iv) $\cos 3x = 4\cos^3 x - 3\cos x$

22. If $A = 60^\circ$ and $B = 30^\circ$ then prove that :

$$\cot(A-B) = \frac{\cot A \cot B + 1}{\cot B - \cot A}$$

Miscellaneous Exercise 6

Multiple Choice Questions (from 1 to 5)

1. The value of $\tan^2 60^\circ$ is:

(a) 3 (b) $\dfrac{1}{3}$ (c) 1 (d) ∞

2. The value of $2\sin^2 60^\circ \cos 60^\circ$ will be:

(a) $\dfrac{4}{3}$ (b) $\dfrac{5}{2}$ (c) $\dfrac{3}{4}$ (d) $\dfrac{1}{3}$

3. If $\operatorname{cosec}\theta = \frac{2}{\sqrt{3}}$ then value of θ is:

(a) $\frac{\pi}{4}$ (b) $\frac{\pi}{3}$ (c) $\frac{\pi}{2}$ (d) $\frac{\pi}{6}$

4. The value of $\cos^2 45^\circ$ will be :

(a) $\frac{1}{\sqrt{2}}$ (b) $\frac{\sqrt{3}}{2}$ (c) $\frac{1}{2}$ (d) $\frac{1}{\sqrt{3}}$

5. If $\theta = 45^\circ$ then, value of $\frac{1-\cos 2\theta}{\sin 2\theta}$ is :

(a) 0 (b) 1 (c) 2 (d) ∞

Prove that :

6. $\cos 60^\circ = 2\cos^2 30^\circ - 1$

7. $\sin 60^\circ = \frac{2\tan 30^\circ}{1+\tan^2 30^\circ}$

8. $\cos 60^\circ = \frac{1-\tan^2 30^\circ}{1+\tan^2 30^\circ}$

9. $(\sin 45^\circ + \cos 45^\circ)^2 = 2$

10. $4\tan 30^\circ \sin 45^\circ \sin 60^\circ \sin 90^\circ = \sqrt{2}$

11. Find the value of $\sin^2 60^\circ \cot^2 60^\circ$.

12. Find the value of $4\cos^3 30^\circ - 3\cos 30^\circ$

13. If $\cot\theta = \frac{1}{\sqrt{3}}$ then prove that $\frac{1-\cos^2\theta}{2-\sin^2\theta} = \frac{3}{5}$

14. Prove that $3\left(\tan^2 30^\circ + \cot^2 30^\circ\right) - 8\left(\sin^2 45^\circ + \cos^2 30^\circ\right) = 0$

15. Prove that $4\left(\sin^4 30^\circ + \cos 60^\circ\right) - 3\left(\cos^2 45^\circ - \sin^2 90^\circ\right) = \frac{15}{4}$

16. Prove that $\frac{\cos 30^\circ + \sin 60^\circ}{1+\cos 60^\circ + \sin 30^\circ} = \frac{\sqrt{3}}{2}$

17. Prove that $2\left(\cos^2 45^\circ + \tan^2 60^\circ\right) - 6\left(\sin^2 45^\circ - \tan^2 30^\circ\right) = 6$

Answer Sheet

Exercise 6.1

(1) 1 **(2)** $\frac{1-\sqrt{3}}{2\sqrt{2}}$ **(3)** $10\frac{1}{4}$ **(4)** 0 **(5)** $\frac{67}{12}$ **(6)** $\frac{3}{4}$

(7) $\frac{13}{6}$ **(8)** $\frac{18}{7}$ **(9)** $\frac{3}{2}$ **(10)** $\sqrt{3}$ **(11)** (i) 30^o (ii) 15^o (iii) 30^o

Miscellaneous Exercise 6

1. (a) 2. (c) 3. (b) 4. (c) 5. (b) 11. $\frac{1}{4}$ 12. 0