CHAPTER XXI.

CONVERGENCY AND DIVERGENCY OF SERIES.

276. An expression in which the successive terms are formed by some regular law is called a **series**; if the series terminate at some assigned term it is called a **finite series**; if the number of terms is unlimited, it is called an **infinite series**.

In the present chapter we shall usually denote a series by an expression of the form

$$u_1 + u_2 + u_3 + \ldots\ldots + u_n + \ldots\ldots$$

277. Suppose that we have a series consisting of n terms. The sum of the series will be a function of n; if n increases indefinitely, the sum either tends to become equal to a certain finite *limit*, or else it becomes infinitely great.

An infinite series is said to be **convergent** when the sum of the first n terms cannot numerically exceed some finite quantity however great n may be.

An infinite series is said to be **divergent** when the sum of the first n terms can be made numerically greater than any finite quantity by taking n sufficiently great.

278. If we can find the sum of the first n terms of a given series, we may ascertain whether it is convergent or divergent by examining whether the series remains finite, or becomes infinite, when n is made indefinitely great.

For example, the sum of the first n terms of the series

$$1 + x + x^2 + x^3 + \ldots \text{ is } \frac{1 - x^n}{1 - x}.$$

If x is numerically less than 1, the sum approaches to the finite limit $\frac{1}{1-x}$, and the series is therefore convergent.

If x is numerically greater than 1, the sum of the first n terms is $\frac{x^n - 1}{x - 1}$, and by taking n sufficiently great, this can be made greater than any finite quantity; thus the series is divergent.

If $x = 1$, the sum of the first n terms is n, and therefore the series is divergent.

If $x = -1$, the series becomes

$$1 - 1 + 1 - 1 + 1 - 1 + \ldots\ldots$$

The sum of an even number of terms is 0, while the sum of an odd number of terms is 1; and thus the sum oscillates between the values 0 and 1. This series belongs to a class which may be called *oscillating* or *periodic convergent series.*

279. There are many cases in which we have no method of finding the sum of the first n terms of a series. We proceed therefore to investigate rules by which we can test the convergency or divergency of a given series without effecting its summation.

280. *An infinite series in which the terms are alternately positive and negative is convergent if each term is numerically less than the preceding term.*

Let the series be denoted by

$$u_1 - u_2 + u_3 - u_4 + u_5 - u_6 + \ldots\ldots$$

where

$$u_1 > u_2 > u_3 > u_4 > u_5 \ldots\ldots$$

The given series may be written in each of the following forms:

$$(u_1 - u_2) + (u_3 - u_4) + (u_5 - u_6) + \ldots\ldots\ldots\ldots\ldots \quad (1),$$

$$u_1 - (u_2 - u_3) - (u_4 - u_5) - (u_6 - u_7) - \ldots\ldots\ldots\ldots \quad (2).$$

From (1) we see that the sum of any number of terms is a positive quantity; and from (2) that the sum of any number of terms is less than u_1; hence the series is convergent.

281. For example, the series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \dots\dots$$

is convergent. By putting $x = 1$ in Art. 223, we see that its sum is $\log_e 2$.

Again, in the series

$$\frac{2}{1} - \frac{3}{2} + \frac{4}{3} - \frac{5}{4} + \frac{6}{5} - \frac{7}{6} + \dots\dots,$$

each term is numerically less than the preceding term, and the series is therefore convergent. But the given series is the sum of

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \dots\dots, \quad \dots\dots\dots(1),$$

and

$$1 - 1 + 1 - 1 + 1 - 1 + \dots\dots, \quad \dots\dots\dots(2).$$

Now (1) is equal to $\log_e 2$, and (2) is equal to 0 or 1 according as the number of terms is even or odd. Hence the given series is convergent, and its sum continually approximates towards $\log_e 2$ if an even number of terms is taken, and towards $1 + \log_e 2$ if an odd number is taken.

282. *An infinite series in which all the terms are of the same sign is divergent if each term is greater than some finite quantity however small.*

For if each term is greater than some finite quantity a, the sum of the first n terms is greater than na; and this, by taking n sufficiently great, can be made to exceed any finite quantity.

283. Before proceeding to investigate further tests of convergency and divergency, we shall lay down two important principles, which may almost be regarded as axioms.

I. If a series is convergent it will remain convergent, and if divergent it will remain divergent, when we add or remove any *finite* number of its terms; for the sum of these terms is a finite quantity.

II. If a series in which all the terms are positive is convergent, then the series is convergent when some or all of the terms are negative; for the sum is clearly greatest when all the terms have the same sign.

We shall suppose that all the terms are positive, unless the contrary is stated.

284. *An infinite series is convergent if from and after some fixed term the ratio of each term to the preceding term is numerically less than some quantity which is itself numerically less than unity.*

Let the series beginning from the fixed term be denoted by

$$u_1 + u_2 + u_3 + u_4 + \dots\dots\dots;$$

and let $$\frac{u_2}{u_1} < r, \quad \frac{u_3}{u_2} < r, \quad \frac{u_4}{u_3} < r, \quad \dots\dots,$$

where $r < 1$.

Then $$u_1 + u_2 + u_3 + u_4 + \dots$$

$$= u_1\left(1 + \frac{u_2}{u_1} + \frac{u_3}{u_2}\cdot\frac{u_2}{u_1} + \frac{u_4}{u_3}\cdot\frac{u_3}{u_2}\cdot\frac{u_2}{u_1} + \dots\right)$$

$$< u_1(1 + r + r^2 + r^3 + \dots\dots);$$

that is, $< \dfrac{u_1}{1-r}$, since $r < 1$.

Hence the given series is convergent.

285. In the enunciation of the preceding article the student should notice the significance of the words "from and after a fixed term."

Consider the series

$$1 + 2x + 3x^2 + 4x^3 + \dots\dots + nx^{n-1} + \dots.$$

Here $$\frac{u_n}{u_{n-1}} = \frac{nx}{n-1} = \left(1 + \frac{1}{n-1}\right)x;$$

and by taking n sufficiently large we can make this ratio approximate to x as nearly as we please, and the ratio of each term to the preceding term will ultimately be x. Hence if $x < 1$ the series is convergent.

But the ratio $\dfrac{u_n}{u_{n-1}}$ will not be less than 1, until $\dfrac{nx}{n-1} < 1$; that is, until $n > \dfrac{1}{1-x}$.

Here we have a case of a convergent series in which the terms may increase up to a certain point and then begin to decrease. For example, if $x = \dfrac{99}{100}$, then $\dfrac{1}{1-x} = 100$, and the terms do not begin to decrease until after the 100^{th} term.

286. *An infinite series in which all the terms are of the same sign is divergent if from and after some fixed term the ratio of each term to the preceding term is greater than unity, or equal to unity.*

Let the fixed term be denoted by u_1. If the ratio is equal to unity, each of the succeeding terms is equal to u_1, and the sum of n terms is equal to nu_1; hence the series is divergent.

If the ratio is greater than unity, each of the terms after the fixed term is greater than u_1, and the sum of n terms is greater than nu_1; hence the series is divergent.

287. In the practical application of these tests, to avoid having to ascertain the particular term after which each term is greater or less than the preceding term, it is convenient to find the limit of $\frac{u_n}{u_{n-1}}$ when n is indefinitely increased; let this limit be denoted by λ.

If $\lambda < 1$, the series is convergent. [Art. 284.]

If $\lambda > 1$, the series is divergent. [Art. 286.]

If $\lambda = 1$, the series may be either convergent or divergent, and a further test will be required; for it may happen that $\frac{u_n}{u_{n-1}} < 1$ *but continually approaching to* 1 *as its limit when* n *is indefinitely increased.* In this case we cannot name any finite quantity r which is itself less than 1 and yet greater than λ. Hence the test of Art. 284 fails. If, however, $\frac{u_n}{u_{n-1}} > 1$ but continually approaching to 1 as its limit, the series is divergent by Art. 286.

We shall use "*Lim* $\frac{u_n}{u_{n-1}}$" as an abbreviation of the words "the limit of $\frac{u_n}{u_{n-1}}$ when n is infinite."

Example 1. Find whether the series whose n^{th} term is $\frac{(n+1)x^n}{n^2}$ is convergent or divergent.

Here $$\frac{u_n}{u_{n-1}} = \frac{(n+1)x^n}{n^2} \div \frac{nx^{n-1}}{(n-1)^2} = \frac{(n+1)(n-1)^2}{n^3} \cdot x,$$

$$\therefore \quad Lim \frac{u_n}{u_{n-1}} = x;$$

hence if $x<1$ the series is convergent;

if $x>1$ the series is divergent.

If $x=1$, then $Lim\ \dfrac{u_n}{u_{n-1}}=1$, and a further test is required.

Example 2. Is the series

$$1^2+2^2x+3^2x^2+4^2x^3+\ldots\ldots$$

convergent or divergent?

Here $$Lim\ \frac{u_n}{u_{n-1}}=Lim\ \frac{n^2x^{n-1}}{(n-1)^2x^{n-2}}=x.$$

Hence if $x<1$ the series is convergent;

if $x>1$ the series is divergent.

If $x=1$ the series becomes $1^2+2^2+3^2+4^2+\ldots$, and is obviously divergent.

Example 3. In the series

$$a+(a+d)r+(a+2d)r^2+\ldots+(a+\overline{n-1}\,.\,d)\,r^{n-1}+\ldots,$$

$$Lim\ \frac{u_n}{u_{n-1}}=Lim\ \frac{a+(n-1)d}{a+(n-2)d}\cdot r=r;$$

thus if $r<1$ the series is convergent, and the sum is finite. [See Art. 60, Cor.]

288. *If there are two infinite series in each of which all the terms are positive, and if the ratio of the corresponding terms in the two series is always finite, the two series are both convergent, or both divergent.*

Let the two infinite series be denoted by

$$u_1+u_2+u_3+u_4+\ldots\ldots\ldots,$$

and $$v_1+v_2+v_3+v_4+\ldots\ldots\ldots.$$

The value of the fraction

$$\frac{u_1+u_2+u_3+\ldots\ldots\ldots+u_n}{v_1+v_2+v_3+\ldots\ldots\ldots+v_n}$$

lies between the greatest and least of the fractions

$$\frac{u_1}{v_1},\ \frac{u_2}{v_2},\ \ldots\ldots\ \frac{u_n}{v_n},$$ [Art. 14.]

and is therefore a *finite* quantity, L say;

$$\therefore\ u_1+u_2+u_3+\ldots+u_n=L(v_1+v_2+v_3+\ldots+v_n).$$

Hence if one series is finite in value, so is the other; if one series is infinite in value, so is the other; which proves the proposition.

289. The application of this principle is very important, for by means of it we can compare a given series with an *auxiliary series* whose convergency or divergency has been already established. The series discussed in the next article will frequently be found useful as an auxiliary series.

290. *The infinite series*

$$\frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \dots$$

is always divergent except when p *is positive and greater than* 1.

CASE I. Let $p > 1$.

The first term is 1; the next two terms together are less than $\frac{2}{2^p}$; the following four terms together are less than $\frac{4}{4^p}$; the following eight terms together are less than $\frac{8}{8^p}$; and so on. Hence the series is less than $1 + \frac{2}{2^p} + \frac{4}{4^p} + \frac{8}{8^p} + \dots$;

that is, less than a geometrical progression whose common ratio $\frac{2}{2^p}$ is less than 1, since $p > 1$; hence the series is convergent.

CASE II. Let $p = 1$.

The series now becomes $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \dots$

The third and fourth terms together are greater than $\frac{2}{4}$ or $\frac{1}{2}$; the following four terms together are greater than $\frac{4}{8}$ or $\frac{1}{2}$; the following eight terms together are greater than $\frac{8}{16}$ or $\frac{1}{2}$; and so on. Hence the series is greater than

$$1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \dots,$$

and is therefore divergent. [Art. 286.]

CASE III. Let $p < 1$, or negative.

Each term is now greater than the corresponding term in Case II., therefore the series is divergent.

Hence the series is always divergent except in the case when p is positive and greater than unity.

Example. Prove that the series

$$\frac{2}{1}+\frac{3}{4}+\frac{4}{9}+\ldots+\frac{n+1}{n^2}+\ldots$$

is divergent.

Compare the given series with $1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}+\ldots$.

Thus if u_n and v_n denote the n^{th} terms of the given series and the auxiliary series respectively, we have

$$\frac{u_n}{v_n}=\frac{n+1}{n^2}\div\frac{1}{n}=\frac{n+1}{n};$$

hence $Lim\,\frac{u_n}{v_n}=1$, and therefore the two series are both convergent or both divergent. But the auxiliary series is divergent, therefore also the given series is divergent.

This completes the solution of Example 1. Art. 287.

291. In the application of Art. 288 it is necessary that the limit of $\frac{u_n}{v_n}$ should be finite; this will be the case if we find our auxiliary series in the following way:

Take u_n, the n^{th} term of the given series and retain only the highest powers of n. Denote the result by v_n; then the limit of $\frac{u_n}{v_n}$ is finite by Art. 270, and v_n may be taken as the n^{th} term of the auxiliary series.

Example 1. Shew that the series whose n^{th} term is $\dfrac{\sqrt[3]{2n^2-1}}{\sqrt[4]{3n^3+2n+5}}$ is divergent.

As n increases, u_n approximates to the value

$$\frac{\sqrt[3]{2n^2}}{\sqrt[4]{3n^3}},\quad\text{or}\quad\frac{\sqrt[3]{2}}{\sqrt[4]{3}}\cdot\frac{1}{n^{\frac{1}{12}}}.$$

Hence, if $v_n=\dfrac{1}{n^{\frac{1}{12}}}$, we have $Lim\,\dfrac{u_n}{v_n}=\dfrac{\sqrt[3]{2}}{\sqrt[4]{3}}$, which is a finite quantity; therefore the series whose n^{th} term is $\dfrac{1}{n^{\frac{1}{12}}}$ may be taken as the auxiliary series. But this series is divergent [Art. 290]; therefore the given series is divergent.

Example 2. Find whether the series in which

$$u_n = \sqrt[3]{n^3+1} - n$$

is convergent or divergent.

Here
$$u_n = n\left(\sqrt[3]{1+\frac{1}{n^3}} - 1\right)$$

$$= n\left(1 + \frac{1}{3n^3} - \frac{1}{9n^6} + \dots - 1\right)$$

$$= \frac{1}{3n^2} - \frac{1}{9n^5} + \dots$$

If we take $v_n = \frac{1}{n^2}$, we have

$$\frac{u_n}{v_n} = \frac{1}{3} - \frac{1}{9n^3} + \dots$$

$$\therefore\ Lim\,\frac{u_n}{v_n} = \frac{1}{3}.$$

But the auxiliary series

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots \frac{1}{n^2} + \dots$$

is convergent, therefore the given series is convergent.

292. *To shew that the expansion of* $(1+x)^n$ *by the Binomial Theorem is convergent when* $x < 1$.

Let u_r, u_{r+1} represent the r^{th} and $(r+1)^{th}$ terms of the expansion; then

$$\frac{u_{r+1}}{u_r} = \frac{n-r+1}{r}x.$$

When $r > n+1$, this ratio is negative; that is, from this point the terms are alternately positive and negative when x is positive, and always of the same sign when x is negative. Now when r is infinite, $Lim\,\frac{u_{r+1}}{u_r} = x$ numerically; therefore since $x < 1$ the series is convergent if all the terms are of the same sign; and therefore *a fortiori* it is convergent when some of the terms are positive and some negative. [Art. 283.]

293. *To shew that the expansion of* a^x *in ascending powers of* x *is convergent for every value of* x.

Here $\frac{u_n}{u_{n-1}} = \frac{x\log_e a}{n-1}$; and therefore $Lim\,\frac{u_n}{u_{n-1}} < 1$ whatever be the value of x; hence the series is convergent.

294. *To shew that the expansion of* $\log(1+x)$ *in ascending powers of* x *is convergent when* x *is numerically less than* 1.

Here the numerical value of $\frac{u_n}{u_{n-1}} = \frac{n-1}{n}x$, which in the limit is equal to x; hence the series is convergent when x is less than 1.

If $x=1$, the series becomes $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots$, and is convergent. [Art. 280.]

If $x=-1$, the series becomes $-1-\frac{1}{2}-\frac{1}{3}-\frac{1}{4}-\ldots$, and is divergent. [Art. 290.] This shews that the logarithm of zero is infinite and negative, as is otherwise evident from the equation $e^{-\infty}=0$.

295. The results of the two following examples are important, and will be required in the course of the present chapter.

Example 1. Find the limit of $\frac{\log x}{x}$ when x is infinite.

Put $x=e^y$; then

$$\frac{\log x}{x} = \frac{y}{e^y} = \frac{y}{1+y+\frac{y^2}{\underline{|2}}+\frac{y^3}{\underline{|3}}+\ldots}$$

$$= \frac{1}{\frac{1}{y}+1+\frac{y}{\underline{|2}}+\frac{y^2}{\underline{|3}}+\ldots};$$

also when x is infinite y is infinite; hence the value of the fraction is zero.

Example 2. Shew that when n is infinite the limit of $nx^n=0$, when $x<1$.

Let $x=\frac{1}{y}$, so that $y>1$;

also let $y^n=z$, so that $n\log y=\log z$; then

$$nx^n = \frac{n}{y^n} = \frac{1}{z}\cdot\frac{\log z}{\log y} = \frac{1}{\log y}\cdot\frac{\log z}{z}.$$

Now when n is infinite z is infinite, and $\frac{\log z}{z}=0$; also $\log y$ is finite; therefore $\qquad Lim\, nx^n=0.$

296. It is sometimes necessary to determine whether the product of an infinite number of factors is finite or not.

Suppose the product to consist of n factors and to be denoted by

$$u_1u_2u_3 \ldots\ldots u_n;$$

then if as n increases indefinitely $u_n<1$, the product will ultimately be zero, and if $u_n>1$ the product will be infinite; hence in order that the product may be finite, u_n must tend to the limit 1.

Writing $1+v_n$ for u_n, the product becomes

$$(1+v_1)(1+v_2)(1+v_3)\ldots\ldots(1+v_n).$$

Denote the product by P and take logarithms; then

$$\log P = \log(1+v_1)+\log(1+v_2)+\ldots+\log(1+v_n)\ \ldots\ldots(1),$$

and in order that the product may be finite this series must be convergent.

Choose as an auxiliary series

$$v_1+v_2+v_3+\ldots\ldots+v_n\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

Now $$Lim\,\frac{\log(1+v_n)}{v_n} = Lim\left(\frac{v_n-\frac{1}{2}v_n^2+\ldots}{v_n}\right)=1,$$

since the limit of v_n is 0 when the limit of u_n is 1.

Hence if (2) is convergent, (1) is convergent, and the given product finite.

Example. Shew that the limit, when n is infinite, of

$$\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{3}{4}\cdot\frac{5}{4}\cdot\frac{5}{6}\cdot\frac{7}{6}\ldots\ldots\frac{2n-1}{2n}\cdot\frac{2n+1}{2n}$$

is finite.

The product consists of $2n$ factors; denoting the successive pairs by $u_1, u_2, u_3,\ldots$ and the product by P, we have

$$P=u_1u_2u_3\ldots\ldots u_n,$$

where $$u_n=\frac{2n-1}{2n}\cdot\frac{2n+1}{2n}=1-\frac{1}{4n^2};$$

but $$\log P=\log u_1+\log u_2+\log u_3+\ldots+\log u_n\ \ldots\ldots\ldots(1),$$

and we have to shew that this series is finite.

Now $$\log u_n=\log\left(1-\frac{1}{4n^2}\right)=-\frac{1}{4n^2}-\frac{1}{32n^4}-\ldots;$$

therefore as in Ex. 2, Art. 291 the series is convergent, and the given product is finite.

297. In mathematical investigations infinite series occur so frequently that the necessity of determining their convergency or divergency is very important; and unless we take care that the series we use are convergent, we may be led to absurd conclusions. [See Art. 183.]

For example, if we expand $(1-x)^{-2}$ by the Binomial Theorem, we find

$$(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \ldots\ldots$$

But if we obtain the sum of n terms of this series as explained in Art. 60, it appears that

$$1 + 2x + 3x^2 + \ \ldots \ + nx^{n-1} = \frac{1-x^n}{(1-x)^2} - \frac{nx^n}{1-x};$$

whence

$$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + \ \ldots + nx^{n-1} + \frac{x^n}{(1-x)^2} + \frac{nx^n}{1-x}.$$

By making n infinite, we see that $\dfrac{1}{(1-x)^2}$ can only be regarded as the true equivalent of the infinite series

$$1 + 2x + 3x^2 + 4x^3 + \ldots\ldots$$

when $\dfrac{x^n}{(1-x)^2} + \dfrac{nx^n}{1-x}$ vanishes.

If n is infinite, this quantity becomes infinite when $x = 1$, or $x > 1$, and diminishes indefinitely when $x < 1$, [Art. 295], so that it is only when $x < 1$ that we can assert that

$$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + \ldots\ldots \text{to inf.};$$

and we should be led to erroneous conclusions if we were to use the expansion of $(1-x)^{-2}$ by the Binomial Theorem as if it were true for all values of x. In other words, we can introduce the infinite series $1 + 2x + 3x^2 + \ldots$ into our reasoning without error if the series is convergent, but we cannot do so when the series is divergent.

The difficulties of divergent series have compelled a distinction to be made between a series and its *algebraical equivalent.* For example, if we divide 1 by $(1-x)^2$, we can always obtain as many terms as we please of the series

$$1 + 2x + 3x^2 + 4x^3 + \ \ldots\ldots$$

whatever x may be, and so in a certain sense $\dfrac{1}{(1-x)^2}$ may be called its *algebraical equivalent;* yet, as we have seen, the equivalence does not really exist except when the series is convergent. It is therefore more appropriate to speak of $\frac{1}{(1-x)^2}$ as the *generating function* of the series

$$1 + 2x + 3x^2 + \ldots\ldots$$

being that function which when developed by ordinary algebraical rules will give the series in question.

The use of the term *generating function* will be more fully explained in the chapter on Recurring Series.

EXAMPLES. XXI. a.

Find whether the following series are convergent or divergent:

1. $\frac{1}{x} - \frac{1}{x+a} + \frac{1}{x+2a} - \frac{1}{x+3a} + \ldots\ldots$,

x and a being positive quantities.

2. $\frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \frac{1}{4.5} + \ldots\ldots$

3. $\frac{1}{xy} - \frac{1}{(x+1)(y+1)} + \frac{1}{(x+2)(y+2)} - \frac{1}{(x+3)(y+3)} + \ldots\ldots$,

x and y being positive quantities.

4. $\frac{x}{1.2} + \frac{x^2}{2.3} + \frac{x^3}{3.4} + \frac{x^4}{4.5} + \ldots\ldots$

5. $\frac{x}{1.2} + \frac{x^2}{3.4} + \frac{x^3}{5.6} + \frac{x^4}{7.8} + \ldots\ldots$

6. $1 + \frac{2^2}{\lfloor 2} + \frac{3^2}{\lfloor 3} + \frac{4^2}{\lfloor 4} + \ldots\ldots$

7. $\sqrt{\frac{1}{2}} + \sqrt{\frac{2}{3}} + \sqrt{\frac{3}{4}} + \sqrt{\frac{4}{5}} + \ldots\ldots$

8. $1 + 3x + 5x^2 + 7x^3 + 9x^4 + \ldots\ldots$

9. $\frac{2}{1^p} + \frac{3}{2^p} + \frac{4}{3^p} + \frac{5}{4^p} + \ldots\ldots$

10. $1 + \frac{x}{2} + \frac{x^2}{5} + \frac{x^3}{10} + \ldots + \frac{x^n}{n^2+1} + \ldots\ldots$

11. $x + \frac{3}{5}x^2 + \frac{8}{10}x^3 + \frac{15}{17}x^4 + \ldots + \frac{n^2-1}{n^2+1}x^n + \ldots\ldots$

12. $1+\frac{2}{5}x+\frac{6}{9}x^2+\frac{14}{17}x^3+\ldots+\frac{2^n-2}{2^n+1}x^{n-1}+\ldots\ldots$

13. $\frac{1}{1^p}+\frac{1}{3^p}+\frac{1}{5^p}+\frac{1}{7^p}+\ldots\ldots$

14. $2x+\frac{3x^2}{8}+\frac{4x^3}{27}+\ldots+\frac{(n+1)x^n}{n^3}+\ldots\ldots$

15. $\left(\frac{2^2}{1^2}-\frac{2}{1}\right)^{-1}+\left(\frac{3^3}{2^3}-\frac{3}{2}\right)^{-2}+\left(\frac{4^4}{3^4}-\frac{4}{3}\right)^{-3}+\ldots\ldots$

16. $1+\frac{1}{2^2}+\frac{2^2}{3^3}+\frac{3^3}{4^4}+\frac{4^4}{5^5}+\ldots\ldots$

17. Test the series whose general terms are

(1) $\sqrt{n^2+1}-n$. (2) $\sqrt{n^4+1}-\sqrt{n^4-1}$.

18. Test the series

(1) $\frac{1}{x}+\frac{1}{x+1}+\frac{1}{x+2}+\frac{1}{x+3}+\ldots\ldots,$

(2) $\frac{1}{x}+\frac{1}{x-1}+\frac{1}{x+1}+\frac{1}{x-2}+\frac{1}{x+2}+\ldots\ldots$

x being a positive fraction.

19. Shew that the series

$$1+\frac{2^p}{\lfloor 2}+\frac{3^p}{\lfloor 3}+\frac{4^p}{\lfloor 4}+\ldots\ldots$$

is convergent for all values of p.

20. Shew that the infinite series

$$u_1+u_2+u_3+u_4+\ldots\ldots$$

is convergent or divergent according as $Lim\,\sqrt[n]{u_n}$ is <1, or >1.

21. Shew that the product

$$\frac{2}{1}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{4}{5}\cdot\frac{6}{5}\ldots\ldots\frac{2n-2}{2n-3}\cdot\frac{2n-2}{2n-1}\cdot\frac{2n}{2n-1}$$

is finite when n is infinite.

22. Shew that when $x=1$, no term in the expansion of $(1+x)^n$ is infinite, except when n is negative and numerically greater than unity.

*298. The tests of convergency and divergency we have given in Arts. 287, 291 are usually sufficient. The theorem proved in the next article enables us by means of the auxiliary series

$$\frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \dots + \frac{1}{n^p} + \dots$$

to deduce additional tests which will sometimes be found convenient.

*299. *If* u_n, v_n *are the general terms of two infinite series in which all the terms are positive, then the* u-*series will be convergent when the* v-*series is convergent if after some particular term* $\frac{u_n}{u_{n-1}} < \frac{v_n}{v_{n-1}}$; *and the* u-*series will be divergent when the* v-*series is divergent if* $\frac{u_n}{u_{n-1}} > \frac{v_n}{v_{n-1}}$.

Let us suppose that u_1 and v_1 are the particular terms.

CASE I. Let $\frac{u_2}{u_1} < \frac{v_2}{v_1}$, $\frac{u_3}{u_2} < \frac{v_3}{v_2}$, ...; then

$$u_1 + u_2 + u_3 + \dots\dots$$

$$= u_1 \left(1 + \frac{u_2}{u_1} + \frac{u_3}{u_2} \cdot \frac{u_2}{u_1} + \dots\right)$$

$$< u_1 \left(1 + \frac{v_2}{v_1} + \frac{v_3}{v_2} \cdot \frac{v_2}{v_1} + \dots\right);$$

that is, $$< \frac{u_1}{v_1}(v_1 + v_2 + v_3 + \dots).$$

Hence, if the v-series is convergent the u-series is also convergent.

CASE II. Let $\frac{u_2}{u_1} > \frac{v_2}{v_1}$, $\frac{u_3}{u_2} > \frac{v_3}{v_2}$ ......; then

$$u_1 + u_2 + u_3 + \dots\dots$$

$$= u_1 \left(1 + \frac{u_2}{u_1} + \frac{u_3}{u_2} \cdot \frac{u_2}{u_1} + \dots\right)$$

$$> u_1 \left(1 + \frac{v_2}{v_1} + \frac{v_3}{v_2} \cdot \frac{v_2}{v_1} + \dots\right);$$

that is, $$> \frac{u_1}{v_1}(v_1 + v_2 + v_3 + \dots).$$

Hence, if the v-series is divergent the u-series is also divergent.

*300. We have seen in Art. 287 that a series is convergent or divergent according as the limit of the ratio of the n^{th} term to the *preceding* term is less than 1, or greater than 1. In the remainder of the chapter we shall find it more convenient to use this test in the equivalent form:

A series is convergent or divergent according as the limit of the ratio of the n^{th} term to the *succeeding* term is greater than 1, or less than 1; that is, according as $Lim \frac{u_n}{u_{n+1}} > 1$, or < 1.

Similarly the theorem of the preceding article may be enunciated:

The u-series will be convergent when the v-series is convergent provided that $Lim \frac{u_n}{u_{n+1}} > Lim \frac{v_n}{v_{n+1}}$; and the u-series will be divergent when the v-series is divergent provided that

$$Lim \frac{u_n}{u_{n+1}} < Lim \frac{v_n}{v_{n+1}}.$$

*301. *The series whose general term is* u_n *is convergent or divergent according as* $Lim \left\{ n \left(\frac{u_n}{u_{n+1}} - 1 \right) \right\} > 1$, or < 1.

Let us compare the given series with the auxiliary series whose general term v_n is $\frac{1}{n^p}$.

When $p > 1$ the auxiliary series is convergent, and in this case the given series is convergent if

$$\frac{u_n}{u_{n+1}} > \frac{(n+1)^p}{n^p}, \text{ or } \left(1 + \frac{1}{n}\right)^p;$$

that is, if $$\frac{u_n}{u_{n+1}} > 1 + \frac{p}{n} + \frac{p(p-1)}{2n^2} + \dots;$$

or $$n\left(\frac{u_n}{u_{n+1}} - 1\right) > p + \frac{p(p-1)}{2n} + \dots;$$

that is, if $$Lim \left\{ n \left(\frac{u_n}{u_{n+1}} - 1 \right) \right\} > p.$$

But the auxiliary series is convergent if p is greater than 1 by a finite quantity however small; hence the first part of the proposition is established.

When $p<1$ the auxiliary series is divergent, and by proceeding as before we may prove the second part of the proposition.

Example. Find whether the series

$$\frac{x}{1}+\frac{1}{2}\cdot\frac{x^3}{3}+\frac{1\,.\,3}{2\,.\,4}\cdot\frac{x^5}{5}+\frac{1\,.\,3\,.\,5}{2\,.\,4\,.\,6}\cdot\frac{x^7}{7}+\dots$$

is convergent or divergent.

Here $Lim\,\dfrac{u_n}{u_{n+1}}=\dfrac{1}{x^2}$; hence if $x<1$ the series is convergent, and if $x>1$ the series is divergent.

If $x=1$, $Lim\,\dfrac{u_n}{u_{n+1}}=1$. In this case

$$u_n=\frac{1\,.\,3\,.\,5\,\dots\dots\,(2n-3)}{2\,.\,4\,.\,6\,\dots\dots\,(2n-2)}\cdot\frac{1}{2n-1},$$

and
$$\frac{u_n}{u_{n+1}}=\frac{2n\,(2n+1)}{(2n-1)\,(2n-1)};$$

$$\therefore\quad n\left(\frac{u_n}{u_{n+1}}-1\right)=\frac{n\,(6n-1)}{(2n-1)^2};$$

$$\therefore\quad Lim\left\{n\left(\frac{u_n}{u_{n+1}}-1\right)\right\}=\frac{3}{2};$$

hence when $x=1$ the series is convergent.

*302. *The series whose general term is* u_n *is convergent or divergent, according as* $Lim\left(n\log\dfrac{u_n}{u_{n+1}}\right)>1$, *or* <1.

Let us compare the given series with the series whose general term is $\dfrac{1}{n^p}$.

When $p>1$ the auxiliary series is convergent, and in this case the given series is convergent if

$$\frac{u_n}{u_{n+1}}>\left(1+\frac{1}{n}\right)^p; \qquad \text{[Art. 300.]}$$

that is, if
$$\log\frac{u_n}{u_{n+1}}>p\log\left(1+\frac{1}{n}\right);$$

or if
$$\log\frac{u_n}{u_{n+1}}>\frac{p}{n}-\frac{p}{2n^2}+\dots;$$

that is, if $$Lim\left(n\log\frac{u_n}{u_{n+1}}\right)>p.$$

Hence the first part of the proposition is established.

When $p<1$ we proceed in a similar manner; in this case the auxiliary series is divergent.

Example. Find whether the series

$$x+\frac{2^2x^2}{\lfloor 2}+\frac{3^3x^3}{\lfloor 3}+\frac{4^4x^4}{\lfloor 4}+\frac{5^5x^5}{\lfloor 5}+\dots$$

is convergent or divergent.

Here $$\frac{u_n}{u_{n+1}}=\frac{n^nx^n}{\lfloor n}\div\frac{(n+1)^{n+1}x^{n+1}}{\lfloor n+1}=\frac{n^n}{(n+1)^nx}=\frac{1}{\left(1+\frac{1}{n}\right)^nx};$$

$$\therefore\ Lim\frac{u_n}{u_{n+1}}=\frac{1}{ex}.\qquad\text{[Art. 220 Cor.].}$$

Hence if $x<\frac{1}{e}$ the series is convergent, if $x>\frac{1}{e}$ the series is divergent.

If $x=\frac{1}{e}$, then $$\frac{u_n}{u_{n+1}}=\frac{e}{\left(1+\frac{1}{n}\right)^n};$$

$$\therefore\ \log\frac{u_n}{u_{n+1}}=\log e-n\log\left(1+\frac{1}{n}\right)$$

$$=1-n\left(\frac{1}{n}-\frac{1}{2n^2}+\frac{1}{3n^3}-\dots\right)$$

$$=\frac{1}{2n}-\frac{1}{3n^2}+\dots;$$

$$\therefore\ n\log\frac{u_n}{u_{n+1}}=\frac{1}{2}-\frac{1}{3n}+\dots;$$

$$\therefore\ Lim\left(n\log\frac{u_n}{u_{n+1}}\right)=\frac{1}{2};$$

hence when $x=\frac{1}{e}$ the series is divergent.

*303. If $Lim\frac{u_n}{u_{n+1}}=1$, and also $Lim\left\{n\left(\frac{u_n}{u_{n+1}}-1\right)\right\}=1$, the tests given in Arts. 300, 301 are not applicable.

To discover a further test we shall make use of the auxiliary series whose general term is $\frac{1}{n(\log n)^p}$. In order to establish the convergency or divergency of this series we need the theorem proved in the next article.

*304. *If* $\phi(\mathrm{n})$ *is positive for all positive integral values of* n *and continually diminishes as* n *increases, and if* a *be any positive integer, then the two infinite series*

$$\phi(1) + \phi(2) + \phi(3) + \ldots + \phi(\mathrm{n}) + \ldots,$$

and $$\mathrm{a}\phi(\mathrm{a}) + \mathrm{a}^2\phi(\mathrm{a}^2) + \mathrm{a}^3\phi(\mathrm{a}^3) + \ldots + \mathrm{a}^{\mathrm{n}}\phi(\mathrm{a}^{\mathrm{n}}) + \ldots,$$

are both convergent, or both divergent.

In the first series let us consider the terms

$$\phi(a^k + 1),\ \phi(a^k + 2),\ \phi(a^k + 3), \ldots\ldots \phi(a^{k+1}) \quad \ldots\ldots\ldots(1)$$

beginning with the term which follows $\phi(a^k)$.

The number of these terms is $a^{k+1} - a^k$, or $a^k(a-1)$, and each of them is greater than $\phi(a^{k+1})$; hence their sum is greater than $a^k(a-1)\phi(a^{k+1})$; that is, greater than $\dfrac{a-1}{a} \times a^{k+1}\phi(a^{k+1})$.

By giving to k in succession the values 0, 1, 2, 3,... we have

$$\phi(2) + \phi(3) + \phi(4) + \ldots\ldots + \phi(a) > \frac{a-1}{a} \times a\phi(a);$$

$$\phi(a+1) + \phi(a+2) + \phi(a+3) + \ldots\ldots + \phi(a^2) > \frac{a-1}{a} \times a^2\phi(a^2);$$

....

therefore, by addition, $$S_1 - \phi(1) > \frac{a-1}{a} S_2,$$

where S_1, S_2 denote the sums of the first and second series respectively; therefore if the second series is divergent so also is the first.

Again, each term of (1) is less than $\phi(a^k)$, and therefore the sum of the series is less than $(a-1) \times a^k\phi(a^k)$.

By giving to k in succession the values 0, 1, 2, 3... we have

$$\phi(2) + \phi(3) + \phi(4) + \ldots\ldots + \phi(a) < (a-1) \times \phi(1);$$

$$\phi(a+1) + \phi(a+2) + \phi(a+3) + \ldots\ldots + \phi(a^2) < (a-1) \times a\phi(a);$$

....

therefore, by addition

$$S_1 - \phi(1) < (a-1)\{S_2 + \phi(1)\};$$

hence if the second series is convergent so also is the first.

Note. To obtain the general term of the second series we take $\phi(n)$ the general term of the first series, write a^n instead of n and multiply by a^n.

*305. *The series whose general term is* $\dfrac{1}{n(\log n)^p}$ *is convergent if* $p > 1$, *and divergent if* $p = 1$, *or* $p < 1$.

By the preceding article the series will be convergent or divergent for the same values of p as the series whose general term is

$$a^n \times \frac{1}{a^n(\log a^n)^p}, \text{ or } \frac{1}{(n \log a)^p}, \text{ or } \frac{1}{(\log a)^p} \times \frac{1}{n^p}.$$

The constant factor $\dfrac{1}{(\log a)^p}$ is common to every term; therefore the given series will be convergent or divergent for the same values of p as the series whose general term is $\dfrac{1}{n^p}$. Hence the required result follows. [Art. 290.]

*306. *The series whose general term is* u_n *is convergent or divergent according as Lim* $\left[\left\{n\left(\dfrac{u_n}{u_{n+1}} - 1\right) - 1\right\}\log n\right] > 1$, or < 1.

Let us compare the given series with the series whose general term is $\dfrac{1}{n(\log n)^p}$.

When $p > 1$ the auxiliary series is convergent, and in this case the given series is convergent by Art. 299, if

$$\frac{u_n}{u_{n+1}} > \frac{(n+1)\{\log(n+1)\}^p}{n(\log n)^p} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

Now when n is very large,

$$\log(n+1) = \log n + \log\left(1 + \frac{1}{n}\right) = \log n + \frac{1}{n}, \text{ nearly;}$$

Hence the condition (1) becomes

$$\frac{u_n}{u_{n+1}} > \left(1 + \frac{1}{n}\right)\left(1 + \frac{1}{n \log n}\right)^p;$$

that is, $$\frac{u_n}{u_{n+1}} > \left(1 + \frac{1}{n}\right)\left(1 + \frac{p}{n \log n}\right);$$

that is, $$\frac{u_n}{u_{n+1}} > 1 + \frac{1}{n} + \frac{p}{n \log n};$$

or $$n\left(\frac{u_n}{u_{n+1}}-1\right)>1+\frac{p}{\log n};$$

or $$\left\{n\left(\frac{u_n}{u_{n+1}}-1\right)-1\right\}\log n>p.$$

Hence the first part of the proposition is established. The second part may be proved in the manner indicated in Art. 301.

Example. Is the series

$$1+\frac{2^2}{3^2}+\frac{2^2\,.\,4^2}{3^2\,.\,5^2}+\frac{2^2\,.\,4^2\,.\,6^2}{3^2\,.\,5^2\,.\,7^2}+\ldots\ldots$$

convergent or divergent?

Here $$\frac{u_n}{u_{n+1}}=\frac{(2n+1)^2}{(2n)^2}=1+\frac{1}{n}+\frac{1}{4n^2}\;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\;(1).$$

$\therefore\ Lim\ \dfrac{u_n}{u_{n+1}}=1$, and we proceed to the next test.

From (1), $$n\left(\frac{u_n}{u_{n+1}}-1\right)=1+\frac{1}{4n}\;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\;(2).$$

$\therefore\ Lim\ \left\{n\left(\dfrac{u_n}{u_{n+1}}-1\right)\right\}=1$, and we pass to the next test.

From (2), $$\left\{n\left(\frac{u_n}{u_{n+1}}-1\right)-1\right\}\log n=\frac{\log n}{4n};$$

$$\therefore\ Lim\left[\left\{n\left(\frac{u_n}{u_{n+1}}-1\right)-1\right\}\log n\right]=0,$$

since $Lim\ \dfrac{\log n}{n}=0$ [Art. 295]; hence the given series is divergent.

*307. We have shewn in Art. 183 that the use of divergent series in mathematical reasoning may lead to erroneous results. But even when the infinite series are convergent it is necessary to exercise caution in using them.

For instance, the series

$$1-x+\frac{x^2}{\sqrt[4]{2}}-\frac{x^3}{\sqrt[4]{3}}+\frac{x^4}{\sqrt[4]{4}}-\frac{x^5}{\sqrt[4]{5}}+\ldots$$

is convergent when $x=1$. [Art. 280.] But if we multiply the series by itself, the coefficient of x^{2n} in the product is

$$\frac{1}{\sqrt[4]{2n}}+\frac{1}{\sqrt[4]{2n-1}}+\frac{1}{\sqrt[4]{2}\,.\sqrt[4]{2n-2}}+\ldots+\frac{1}{\sqrt[4]{r}.\ \sqrt[4]{2n-r}}+\ldots+\frac{1}{\sqrt[4]{2n}}.$$

Denote this by a_{2n}; then since

$$\frac{1}{\sqrt[4]{r}\cdot\sqrt[4]{2n-r}} > \frac{1}{(\sqrt[4]{n})^2}, \text{ or } > \frac{1}{\sqrt{n}},$$

$a_{2n} > \dfrac{2n+1}{\sqrt{n}}$, and is therefore infinite when n is infinite.

If $x=1$, the product becomes

$$a_0 - a_1 + a_2 - a_3 + \ldots + a_{2n} - a_{2n+1} + a_{2n+2} - \ldots,$$

and since the terms a_{2n}, a_{2n+1}, a_{2n+2} ... are infinite, the series has no arithmetical meaning.

This leads us to enquire under what conditions the product of two infinite convergent series is also convergent.

*308. Let us denote the two infinite series

$$a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots + a_{2n}x^{2n} + \ldots,$$

$$b_0 + b_1x + b_2x^2 + b_3x^3 + \ldots + b_{2n}x^{2n} + \ldots$$

by A and B respectively.

If we multiply these series together we obtain a result of the form

$$a_0b_0 + (a_1b_0 + a_0b_1)\,x + (a_2b_0 + a_1b_1 + a_0b_2)\,x^2 + \ldots$$

Suppose this series to be *continued to infinity* and let us denote it by C; then we have to examine under what conditions C may be regarded as the true arithmetical equivalent of the product AB.

First suppose that all the terms in A and B are positive.

Let A_{2n}, B_{2n}, C_{2n} denote the series formed by taking the first $2n+1$ terms of A, B, C respectively.

If we multiply together the two series A_{2n}, B_{2n}, the coefficient of each power of x in their product is equal to the coefficient of the like power of x in C as far as the term x^{2n}; but in $A_{2n}B_{2n}$ there are terms containing powers of x higher than x^{2n}, whilst x^{2n} is the highest power of x in C_{2n}; hence

$$A_{2n}B_{2n} > C_{2n}.$$

If we form the product A_nB_n the last term is $a_nb_nx^{2n}$; but C_{2n} includes all the terms in the product and some other terms besides; hence

$$C_{2n} > A_nB_n.$$

Thus C_{2n} is intermediate in value between $A_n B_n$ and $A_{2n} B_{2n}$, whatever be the value of n.

Let A and B be *convergent* series; put

$$A_n = A - X, \quad B_n = B - Y,$$

where X and Y are the remainders after n terms of the series have been taken; then when n is infinite X and Y are both indefinitely small.

$$\therefore \quad A_n B_n = (A - X)(B - Y) = AB - BX - AY + XY;$$

therefore the limit of $A_n B_n$ is AB, since A and B are both finite.

Similarly, the limit of $A_{2n} B_{2n}$ is AB.

Therefore C which is the limit of C_{2n} must be equal to AB since it lies between the limits of $A_n B_n$ and $A_{2n} B_{2n}$.

Next suppose the terms in A and B are not all of the same sign.

In this case the inequalities $A_{2n} B_{2n} > C_{2n} > A_n B_n$ are not necessarily true, and we cannot reason as in the former case.

Let us denote the aggregates of the positive terms in the two series by P, P' respectively, and the aggregates of the negative terms by N, N'; so that

$$A = P - N, \quad B = P' - N'.$$

Then if each of the expressions P, P', N, N' represents a convergent series, the equation

$$AB = PP' - NP' - PN' + NN',$$

has a meaning perfectly intelligible, for each of the expressions PP', NP', PN', NN' is a convergent series, by the former part of the proposition; and thus the product of the two series A and B is a convergent series.

Hence *the product of two series will be convergent provided that the sum of all the terms of the same sign in each is a convergent series.*

But if each of the expressions P, N, P', N' represents a divergent series (as in the preceding article, where also $P' = P$ and $N' = N$), then all the expressions PP', NP', PN', NN' are divergent series. When this is the case, a careful investigation is necessary in each particular example in order to ascertain whether the product is convergent or not.

*EXAMPLES. XXI. b.

Find whether the following series are convergent or divergent:

1. $1+\frac{1}{2}\cdot\frac{x^2}{4}+\frac{1.3.5}{2.4.6}\cdot\frac{x^4}{8}+\frac{1.3.5.7.9}{2.4.6.8.10}\cdot\frac{x^6}{12}+\ldots\ldots$

2. $1+\frac{3}{7}x+\frac{3.6}{7.10}x^2+\frac{3.6.9}{7.10.13}x^3+\frac{3.6.9.12}{7.10.13.16}x^4+\ldots\ldots$

3. $x^2+\frac{2^2}{3.4}x^4+\frac{2^2.4^2}{3.4.5.6}x^6+\frac{2^2.4^2.6^2}{3.4.5.6.7.8}x^8+\ldots\ldots$

4. $1+\frac{2x}{\lfloor 2}+\frac{3^2x^2}{\lfloor 3}+\frac{4^3x^3}{\lfloor 4}+\frac{5^4x^4}{\lfloor 5}+\ldots\ldots$

5. $1+\frac{1}{2}x+\frac{\lfloor 2}{3^2}x^2+\frac{\lfloor 3}{4^3}x^3+\frac{\lfloor 4}{5^4}x^4+\ldots\ldots$

6. $\frac{1^2}{2^2}+\frac{1^2.3^2}{2^2.4^2}x+\frac{1^2.3^2.5^2}{2^2.4^2.6^2}x^2+\ldots\ldots$

7. $1+\frac{a(1-a)}{1^2}+\frac{(1+a)a(1-a)(2-a)}{1^2.2^2}$

$$+\frac{(2+a)(1+a)a(1-a)(2-a)(3-a)}{1^2.2^2.3^2}+\ldots\ldots$$

a being a proper fraction.

8. $\frac{a+x}{1}+\frac{(a+2x)^2}{\lfloor 2}+\frac{(a+3x)^3}{\lfloor 3}+\ldots\ldots$

9. $1+\frac{a.\beta}{1.\gamma}x+\frac{a(a+1)\beta(\beta+1)}{1.2.\gamma(\gamma+1)}x^2$

$$+\frac{a(a+1)(a+2)\beta(\beta+1)(\beta+2)}{1.2.3.\gamma(\gamma+1)(\gamma+2)}x^3+\ldots\ldots$$

10. $x^2(\log 2)^q+x^3(\log 3)^q+x^4(\log 4)^q+\ldots\ldots$

11. $1+a+\frac{a(a+1)}{1.2}+\frac{a(a+1)(a+2)}{1.2.3}+\ldots\ldots$

12. If $\frac{u_n}{u_{n+1}}=\frac{n^k+An^{k-1}+Bn^{k-2}+Cn^{k-3}+\ldots}{n^k+an^{k-1}+bn^{k-2}+cn^{k-3}+\ldots}$, where k is a positive integer, shew that the series $u_1+u_2+u_3+\ldots\ldots$ is convergent if $A-a-1$ is positive, and divergent if $A-a-1$ is negative or zero.

CHAPTER XXII.

UNDETERMINED COEFFICIENTS.

309. In Art. 230 of the *Elementary Algebra*, it was proved that if any rational integral function of x vanishes when $x = a$, it is divisible by $x - a$. [See also Art. 514. Cor.]

Let $p_0x^n + p_1x^{n-1} + p_2x^{n-2} + \ldots\ldots + p_n$ be a rational integral function of x of n dimensions, which vanishes when x is equal to each of the unequal quantities

$$a_1,\ a_2,\ a_3,\ \ldots\ldots\ a_n.$$

Denote the function by $f(x)$; then since $f(x)$ is divisible by $x - a_1$, we have

$$f(x) = (x - a_1)(p_0x^{n-1} + \ldots\ldots),$$

the quotient being of $n - 1$ dimensions.

Similarly, since $f(x)$ is divisible by $x - a_2$, we have

$$p_0x^{n-1} + \ldots\ldots = (x - a_2)(p_0x^{n-2} + \ldots\ldots),$$

the quotient being of $n - 2$ dimensions; and

$$p_0x^{n-2} + \ldots\ldots = (x - a_3)(p_0x^{n-3} + \ldots\ldots).$$

....

Proceeding in this way, we shall finally obtain after n divisions

$$f(x) = p_0(x - a_1)(x - a_2)(x - a_3)\ldots\ldots(x - a_n).$$

310. *If a rational integral function of* n *dimensions vanishes for more than* n *values of the variable, the coefficient of each power of the variable must be zero.*

Let the function be denoted by $f(x)$, where

$$f(x) = p_0x^n + p_1x^{n-1} + p_2x^{n-2} + \ldots\ldots + p_n;$$

and suppose that $f(x)$ vanishes when x is equal to each of the unequal values a_1, a_2, a_3 ...... a_n; then

$$f(x) = p_0 (x - a_1)(x - a_2)(x - a_3) \ldots\ldots (x - a_n).$$

Let c be another value of x which makes $f(x)$ vanish; then since $f(c) = 0$, we have

$$p_0 (c - a_1)(c - a_2)(c - a_3) \ldots\ldots (c - a_n) = 0;$$

and therefore $p_0 = 0$, since, by hypothesis, none of the other factors is equal to zero. Hence $f(x)$ reduces to

$$p_1 x^{n-1} + p_2 x^{n-2} + p_3 x^{n-3} + \ldots\ldots + p_n.$$

By hypothesis this expression vanishes for more than n values of x, and therefore $p_1 = 0$.

In a similar manner we may shew that each of the coefficients p_2, p_3, ... p_n must be equal to zero.

This result may also be enunciated as follows:

If a rational integral function of n *dimensions vanishes for more than* n *values of the variable, it must vanish for every value of the variable.*

Cor. If the *function* $f(x)$ vanishes for more than n values of x, the *equation* $f(x) = 0$ has more than n roots.

Hence also, *if an equation of* n *dimensions has more than* n *roots it is an identity.*

Example. Prove that

$$\frac{(x-b)(x-c)}{(a-b)(a-c)} + \frac{(x-c)(x-a)}{(b-c)(b-a)} + \frac{(x-a)(x-b)}{(c-a)(c-b)} = 1.$$

This equation is of *two* dimensions, and it is evidently satisfied by each of the *three* values a, b, c; hence it is an identity.

311. *If two rational integral functions of* n *dimensions are equal for more than* n *values of the variable, they are equal for every value of the variable.*

Suppose that the two functions

$$p_0 x^n + p_1 x^{n-1} + p_2 x^{n-2} + \ldots\ldots + p_n,$$

$$q_0 x^n + q_1 x^{n-1} + q_2 x^{n-2} + \ldots\ldots + q_n,$$

are equal for more than n values of x; then the expression

$$(p_0 - q_0) x^n + (p_1 - q_1) x^{n-1} + (p_2 - q_2) x^{n-2} + \ldots\ldots + (p_n - q_n)$$

vanishes for more than n values of x; and therefore, by the preceding article,

$$p_0 - q_0 = 0, \; p_1 - q_1 = 0, \; p_2 - q_2 = 0, \ldots\ldots p_n - q_n = 0;$$

that is,

$$p_0 = q_0, \; p_1 = q_1, \; p_2 = q_2, \ldots\ldots p_n = q_n.$$

Hence the two expressions are *identical*, and therefore are equal for every value of the variable. Thus

if two rational integral functions are identically equal, we may equate the coefficients of the like powers of the variable.

This is the principle we assumed in the *Elementary Algebra*, Art. 227.

Cor. This proposition still holds if one of the functions is of lower dimensions than the other. For instance, if

$$p_0x^n + p_1x^{n-1} + p_2x^{n-2} + p_3x^{n-3} + \ldots\ldots + p_n$$
$$= q_2x^{n-2} + q_3x^{n-3} + \ldots\ldots + q_n,$$

we have only to suppose that in the above investigation $q_0 = 0$, $q_1 = 0$, and then we obtain

$$p_0 = 0, \; p_1 = 0, \; p_2 = q_2, \; p_3 = q_3, \ldots\ldots p_n = q_n.$$

312. The theorem of the preceding article is usually referred to as the *Principle of Undetermined Coefficients*. The application of this principle is illustrated in the following examples.

Example 1. Find the sum of the series

$$1 \,.\, 2 + 2 \,.\, 3 + 3 \,.\, 4 + \ldots\ldots\ldots\ldots + n(n+1).$$

Assume that

$$1 \,.\, 2 + 2 \,.\, 3 + 3 \,.\, 4 + \ldots + n(n+1) = A + Bn + Cn^2 + Dn^3 + En^4 + \ldots,$$

where A, B, C, D, E,... are quantities independent of n, whose values have to be determined.

Change n into $n+1$; then

$$1 \,.\, 2 + 2 \,.\, 3 + \ldots + n(n+1) + (n+1)(n+2)$$
$$= A + B(n+1) + C(n+1)^2 + D(n+1)^3 + E(n+1)^4 + \ldots.$$

By subtraction,

$$(n+1)(n+2) = B + C(2n+1) + D(3n^2+3n+1) + E(4n^3+6n^2+4n+1) + \ldots.$$

This equation being true for all integral values of n, the coefficients of the respective powers of n on each side must be equal; thus E and all succeeding coefficients must be equal to zero, and

$$3D = 1; \quad 3D + 2C = 3; \quad D + C + B = 2;$$

whence $$D = \frac{1}{3}, \quad C = 1, \quad B = \frac{2}{3}.$$

Hence the sum $$=A+\frac{2n}{3}+n^2+\frac{1}{3}n^3.$$

To find A, put $n=1$; the series then reduces to its first term, and

$$2=A+2, \text{ or } A=0.$$

Hence $$1.2+2.3+3.4+\ldots+n(n+1)=\frac{1}{3}n(n+1)(n+2).$$

NOTE. It will be seen from this example that when the n^{th} term is a rational integral function of n, it is sufficient to assume for the sum a function of n which is of one dimension higher than the n^{th} term of the series.

Example 2. Find the conditions that x^3+px^2+qx+r may be divisible by x^2+ax+b.

Assume $$x^3+px^2+qx+r=(x+k)(x^2+ax+b).$$

Equating the coefficients of the like powers of x, we have

$$k+a=p, \quad ak+b=q, \quad kb=r.$$

From the last equation $k=\frac{r}{b}$; hence by substitution we obtain

$$\frac{r}{b}+a=p, \text{ and } \frac{ar}{b}+b=q;$$

that is, $$r=b(p-a), \text{ and } ar=b(q-b);$$

which are the conditions required.

EXAMPLES. XXII. a.

Find by the method of Undetermined Coefficients the sum of

1. $1^2+3^2+5^2+7^2+\ldots$ to n terms.

2. $1.2.3+2.3.4+3.4.5+\ldots$ to n terms.

3. $1.2^2+2.3^2+3.4^2+4.5^2+\ldots$ to n terms.

4. $1^3+3^3+5^3+7^3+\ldots$ to n terms.

5. $1^4+2^4+3^4+4^4+\ldots$ to n terms.

6. Find the condition that $x^3-3px+2q$ may be divisible by a factor of the form $x^2+2ax+a^2$.

7. Find the conditions that ax^3+bx^2+cx+d may be a perfect cube.

8. Find the conditions that $a^2x^4+bx^3+cx^2+dx+f^2$ may be a perfect square.

9. Prove that $ax^2+2bxy+cy^2+2dx+2ey+f$ is a perfect square, if $b^2=ac$, $d^2=af$, $e^2=cf$.

10. If ax^3+bx^2+cx+d is divisible by x^2+h^2, prove that $ad=bc$.

11. If $x^5-5qx+4r$ is divisible by $(x-c)^2$, shew that $q^5=r^4$.

12. Prove the identities:

(1) $$\frac{a^2(x-b)(x-c)}{(a-b)(a-c)}+\frac{b^2(x-c)(x-a)}{(b-c)(b-a)}+\frac{c^2(x-a)(x-b)}{(c-a)(c-b)}=x^2.$$

(2) $$\frac{(x-b)(x-c)(x-d)}{(a-b)(a-c)(a-d)}+\frac{(x-c)(x-d)(x-a)}{(b-c)(b-d)(b-a)}$$
$$+\frac{(x-d)(x-a)(x-b)}{(c-d)(c-a)(c-b)}+\frac{(x-a)(x-b)(x-c)}{(d-a)(d-b)(d-c)}=1.$$

13. Find the condition that

$$ax^2+2hxy+by^2+2gx+2fy+c$$

may be the product of two factors of the form

$$px+qy+r,\quad p'x+q'y+r'.$$

14. If $\xi=lx+my+nz$, $\eta=nx+ly+mz$, $\zeta=mx+ny+lz$, and if the same equations are true for all values of x, y, z when ξ, η, ζ are interchanged with x, y, z respectively, shew that

$$l^2+2mn=1,\quad m^2+2ln=0,\quad n^2+2lm=0.$$

15. Shew that the sum of the products $n-r$ together of the n quantities $a, a^2, a^3, \dots a^n$ is

$$\frac{(a^{r+1}-1)(a^{r+2}-1)\dots(a^n-1)}{(a-1)(a^2-1)\dots(a^{n-r}-1)}\,a^{\frac{1}{2}(n-r)(n-r+1)}.$$

313. *If the infinite series* $a_0+a_1x+a_2x^2+a_3x^3+\dots\dots$ *is equal to zero for every finite value of* x *for which the series is convergent, then each coefficient must be equal to zero identically.*

Let the series be denoted by S, and let S_1 stand for the expression $a_1+a_2x+a_3x^2+\dots\dots$; then $S=a_0+xS_1$, and therefore, by hypothesis, $a_0+xS_1=0$ for all finite values of x. But since S is convergent, S_1 cannot exceed some finite limit; therefore by taking x small enough xS_1 may be made as small as we please. In this case the limit of S is a_0; but S is *always* zero, therefore a_0 must be equal to zero identically.

Removing the term a_0, we have $xS_1=0$ for all finite values of x; that is, $a_1+a_2x+a_3x^2+\dots\dots$ vanishes for all finite values of x.

Similarly, we may prove in succession that each of the coefficients a_1, a_2, a_3,...... is equal to zero identically.

314. *If two infinite series are equal to one another for every finite value of the variable for which both series are convergent, the coefficients of like powers of the variable in the two series are equal.*

Suppose that the two series are denoted by

$$a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots\ldots$$

and $$A_0 + A_1 x + A_2 x^2 + A_3 x^3 + \ldots\ldots;$$

then the expression

$$a_0 - A_0 + (a_1 - A_1)\, x + (a_2 - A_2)\, x^2 + (a_3 - A_3)\, x^3 + \ldots\ldots$$

vanishes for all values of x within the assigned limits; therefore by the last article

$$a_0 - A_0 = 0,\ \ a_1 - A_1 = 0,\ \ a_2 - A_2 = 0,\ \ a_3 - A_3 = 0, \ldots\ldots$$

that is, $$a_0 = A_0,\ \ a_1 = A_1,\ \ a_2 = A_2,\ \ a_3 = A_3, \ldots\ldots;$$

which proves the proposition.

Example 1. Expand $\dfrac{2+x^2}{1+x-x^2}$ in a series of ascending powers of x as far as the term involving x^5.

Let $$\frac{2+x^2}{1+x-x^2} = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots,$$

where $a_0, a_1, a_2, a_3, \ldots$ are constants whose values are to be determined; then

$$2 + x^2 = (1 + x - x^2)\,(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots).$$

In this equation we may equate the coefficients of like powers of x on each side. On the right-hand side the coefficient of x^n is $a_n + a_{n-1} - a_{n-2}$, and therefore, since x^2 is the highest power of x on the left, for all values of $n > 2$ we have

$$a_n + a_{n-1} - a_{n-2} = 0;$$

this will suffice to find the successive coefficients after the first three have been obtained. To determine these we have the equations

$$a_0 = 2,\ \ a_1 + a_0 = 0,\ \ a_2 + a_1 - a_0 = 1;$$

whence $$a_0 = 2,\ \ a_1 = -2,\ \ a_2 = 5.$$

Also $$a_3 + a_2 - a_1 = 0, \text{ whence } a_3 = -7;$$

$$a_4 + a_3 - a_2 = 0, \text{ whence } a_4 = 12;$$

and $$a_5 + a_4 - a_3 = 0, \text{ whence } a_5 = -19;$$

thus $$\frac{2+x^2}{1+x-x^2} = 2 - 2x + 5x^2 - 7x^3 + 12x^4 - 19x^5 + \ldots$$

Example 2. Prove that if n and r are positive integers

$$n^r - n(n-1)^r + \frac{n(n-1)}{\underline{|2}}(n-2)^r - \frac{n(n-1)(n-2)}{\underline{|3}}(n-3)^r + \ldots$$

is equal to 0 if r be less than n, and to $\underline{|n}$ if $r=n$.

We have $$(e^x-1)^n = \left(x + \frac{x^2}{\underline{|2}} + \frac{x^3}{\underline{|3}} + \frac{x^4}{\underline{|4}} + \ldots\ldots\right)^n$$

$$= x^n + \text{terms containing higher powers of } x \ldots (1).$$

Again, by the Binomial Theorem,

$$(e^x-1)^n = e^{nx} - ne^{(n-1)x} + \frac{n(n-1)}{1 \cdot 2}e^{(n-2)x} - \ldots\ldots, \ldots\ldots\ldots(2).$$

By expanding each of the terms e^{nx}, $e^{(n-1)x}$,... we find that the coefficient of x^r in (2) is

$$\frac{n^r}{\underline{|r}} - n \cdot \frac{(n-1)^r}{\underline{|r}} + \frac{n(n-1)}{\underline{|2}} \cdot \frac{(n-2)^r}{\underline{|r}} - \frac{n(n-1)(n-2)}{\underline{|3}} \cdot \frac{(n-3)^r}{\underline{|r}} + \ldots$$

and by equating the coefficients of x^r in (1) and (2) the result follows.

Example 3. If $$y = ax + bx^2 + cx^3 + \ldots\ldots,$$
express x in ascending powers of y as far as the term involving y^3.

Assume $$x = py + qy^2 + ry^3 + \ldots\ldots,$$
and substitute in the given series; thus

$$y = a(py + qy^2 + ry^3 + \ldots) + b(py + qy^2 + \ldots)^2 + c(py + qy^2 + \ldots)^3 + \ldots\ldots.$$

Equating coefficients of like powers of y, we have

$$ap = 1; \quad \text{whence } p = \frac{1}{a}.$$

$$aq + bp^2 = 0; \quad \text{whence } q = -\frac{b}{a^3}.$$

$$ar + 2bpq + cp^3 = 0; \quad \text{whence } r = \frac{2b^2}{a^5} - \frac{c}{a^4}.$$

Thus $$x = \frac{y}{a} - \frac{by^2}{a^3} + \frac{(2b^2 - ac)y^3}{a^5} + \ldots$$

This is an example of *Reversion of Series*.

Cor. If the series for y be given in the form

$$y = k + ax + bx^2 + cx^3 + \ldots$$

put $$y - k = z;$$

then $$z = ax + bx^2 + cx^3 + \ldots;$$

from which x may be expanded in ascending powers of z, that is of $y-k$.

EXAMPLES. XXII. b.

Expand the following expressions in ascending powers of x as far as x^3.

1. $\dfrac{1+2x}{1-x-x^2}$. 2. $\dfrac{1-8x}{1-x-6x^2}$. 3. $\dfrac{1+x}{2+x+x^2}$.

4. $\dfrac{3+x}{2-x-x^2}$. 5. $\dfrac{1}{1+ax-ax^2-x^3}$.

6. Find a and b so that the n^{th} term in the expansion of $\dfrac{a+bx}{(1-x)^2}$ may be $(3n-2)x^{n-1}$.

7. Find a, b, c so that the coefficient of x^n in the expansion of $\dfrac{a+bx+cx^2}{(1-x)^3}$ may be n^2+1.

8. If $y^2+2y=x(y+1)$, shew that one value of y is

$$\tfrac{1}{2}x+\tfrac{1}{8}x^2-\tfrac{1}{128}x^4+\ldots\ldots$$

9. If $cx^3+ax-y=0$, shew that one value of x is

$$\frac{y}{a}-\frac{cy^3}{a^4}+\frac{3c^2y^5}{a^7}-\frac{12c^3y^7}{a^{10}}\ldots\ldots$$

Hence shew that $x=\cdot00999999$ is an approximate solution of the equation $x^3+100x-1=0$. To how many places of decimals is the result correct?

10. In the expansion of $(1+x)(1+ax)(1+a^2x)(1+a^3x)\ldots\ldots$, the number of factors being infinite, and $a<1$, shew that the coefficient of x^r is

$$\frac{1}{(1-a)(1-a^2)(1-a^3)\ldots\ldots(1-a^r)}\,a^{\frac{1}{2}r(r-1)}.$$

11. When $a<1$, find the coefficient of x^n in the expansion of

$$\frac{1}{(1-ax)(1-a^2x)(1-a^3x)\ldots\ldots\text{to inf.}}$$

12. If n is a positive integer, shew that

(1) $n^{n+1}-n(n-1)^{n+1}+\dfrac{n(n-1)}{\underline{|2}}(n-2)^{n+1}-\ldots\ldots=\tfrac{1}{2}n\,\underline{|n+1}$;

(2) $n^n-(n+1)(n-1)^n+\dfrac{(n+1)n}{\underline{|2}}(n-2)^n-\ldots\ldots=1$;

the series in each case being extended to n terms; and

(3) $1^n-n2^n+\dfrac{n(n-1)}{1\,.\,2}3^n-\ldots\ldots=(-1)^n\,\underline{|n}$;

(4) $(n+p)^n-n(n+p-1)^n+\dfrac{n(n-1)}{\underline{|2}}(n+p-2)^n-\ldots\ldots=\underline{|n}$;

the series in the last two cases being extended to $n+1$ terms.