CHAPTER XVII.

EXPONENTIAL AND LOGARITHMIC SERIES.

219. In Chap. XVI. it was stated that the logarithms in common use were not found directly, but that logarithms are first found to another base, and then transformed to base 10.

In the present chapter we shall prove certain formulæ known as the **Exponential and Logarithmic Series**, and give a brief explanation of the way in which they are used in constructing a table of logarithms.

220. *To expand* a^x *in ascending powers of* x.

By the Binomial Theorem, if $n>1$,

$$\left(1+\frac{1}{n}\right)^{nx}$$

$$=1+nx\cdot\frac{1}{n}+\frac{nx(nx-1)}{\underline{|2}}\cdot\frac{1}{n^2}+\frac{nx(nx-1)(nx-2)}{\underline{|3}}\cdot\frac{1}{n^3}+\ldots\ldots$$

$$=1+x+\frac{x\left(x-\frac{1}{n}\right)}{\underline{|2}}+\frac{x\left(x-\frac{1}{n}\right)\left(x-\frac{2}{n}\right)}{\underline{|3}}+\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

By putting $x=1$, we obtain

$$\left(1+\frac{1}{n}\right)^n=1+1+\frac{1-\frac{1}{n}}{\underline{|2}}+\frac{\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)}{\underline{|3}}+\ldots\ldots(2).$$

But $$\left(1+\frac{1}{n}\right)^{nx}=\left\{\left(1+\frac{1}{n}\right)^n\right\}^x;$$

hence the series (1) is the x^{th} power of the series (2); that is,

$$1 + x + \frac{x\left(x - \frac{1}{n}\right)}{\lfloor\underline{2}} + \frac{x\left(x - \frac{1}{n}\right)\left(x - \frac{2}{n}\right)}{\lfloor\underline{3}} + \ldots\ldots$$

$$= \left\{1 + 1 + \frac{1 - \frac{1}{n}}{\lfloor\underline{2}} + \frac{\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)}{\lfloor\underline{3}} + \ldots\right\}^x;$$

and this is true however great n may be. If therefore n be indefinitely increased we have

$$1 + x + \frac{x^2}{\lfloor\underline{2}} + \frac{x^3}{\lfloor\underline{3}} + \frac{x^4}{\lfloor\underline{4}} + \ldots\ldots = \left(1 + 1 + \frac{1}{\lfloor\underline{2}} + \frac{1}{\lfloor\underline{3}} + \frac{1}{\lfloor\underline{4}} + \ldots\ldots\right)^x.$$

The series $$1 + 1 + \frac{1}{\lfloor\underline{2}} + \frac{1}{\lfloor\underline{3}} + \frac{1}{\lfloor\underline{4}} + \ldots\ldots$$

is usually denoted by e; hence

$$e^x = 1 + x + \frac{x^2}{\lfloor\underline{2}} + \frac{x^3}{\lfloor\underline{3}} + \frac{x^4}{\lfloor\underline{4}} + \ldots\ldots.$$

Write cx for x, then

$$e^{cx} = 1 + cx + \frac{c^2x^2}{\lfloor\underline{2}} + \frac{c^3x^3}{\lfloor\underline{3}} + \ldots\ldots.$$

Now let $e^c = a$, so that $c = \log_e a$; by substituting for c we obtain

$$a^x = 1 + x\log_e a + \frac{x^2(\log_e a)^2}{\lfloor\underline{2}} + \frac{x^3(\log_e a)^3}{\lfloor\underline{3}} + \ldots\ldots.$$

This is the *Exponential Theorem.*

Cor. When n is infinite, the *limit* of $\left(1 + \frac{1}{n}\right)^n = e.$

[See Art. 266.]

Also as in the preceding investigation, it may be shewn that when n is indefinitely increased,

$$\left(1 + \frac{x}{n}\right)^n = 1 + x + \frac{x^2}{\lfloor\underline{2}} + \frac{x^3}{\lfloor\underline{3}} + \frac{x^4}{\lfloor\underline{4}} + \ldots\ldots;$$

that is, when n is infinite, the limit of $\left(1+\frac{x}{n}\right)^n = e^x$.

By putting $\frac{x}{n} = -\frac{1}{m}$, we have

$$\left(1-\frac{x}{n}\right)^n = \left(1+\frac{1}{m}\right)^{-mx} = \left\{\left(1+\frac{1}{m}\right)^m\right\}^{-x}.$$

Now m is infinite when n is infinite;

thus the limit of $\left(1-\frac{x}{n}\right)^n = e^{-x}$.

Hence the limit of $\left(1-\frac{1}{n}\right)^n = e^{-1}$.

221. In the preceding article no restriction is placed upon the value of x; also since $\frac{1}{n}$ is less than unity, the expansions we have used give results arithmetically intelligible. [Art. 183.]

But there is another point in the foregoing proof which deserves notice. We have assumed that when n is infinite

the limit of $\dfrac{x\left(x-\frac{1}{n}\right)\left(x-\frac{2}{n}\right)\ldots\left(x-\frac{r-1}{n}\right)}{\lfloor r}$ is $\dfrac{x^r}{\lfloor r}$

for all values of r.

Let us denote the value of

$$\frac{x\left(x-\frac{1}{n}\right)\left(x-\frac{2}{n}\right)\ldots\left(x-\frac{r-1}{n}\right)}{\lfloor r} \text{ by } u_r.$$

Then $\dfrac{u_r}{u_{r-1}} = \dfrac{1}{r}\left(x-\dfrac{r-1}{n}\right) = \dfrac{x}{r} - \dfrac{1}{n} + \dfrac{1}{nr}$.

Since n is infinite, we have

$$\frac{u_r}{u_{r-1}} = \frac{x}{r}; \text{ that is, } u_r = \frac{x}{r}u_{r-1}.$$

It is clear that the limit of u_2 is $\frac{x^2}{\lfloor 2}$; hence the limit of u_3 is $\frac{x^3}{\lfloor 3}$; that of u_4 is $\frac{x^4}{\lfloor 4}$; and generally that of u_r is $\frac{x^r}{\lfloor r}$.

222. The series

$$1 + 1 + \frac{1}{\lfloor 2} + \frac{1}{\lfloor 3} + \frac{1}{\lfloor 4} + \ldots\ldots,$$

which we have denoted by e, is very important as it is the base to which logarithms are first calculated. Logarithms to this base are known as the Napierian system, so named after Napier their inventor. They are also called *natural* logarithms from the fact that they are the first logarithms which naturally come into consideration in algebraical investigations.

When logarithms are used in theoretical work it is to be remembered that the base e is always understood, just as in arithmetical work the base 10 is invariably employed.

From the series the approximate value of e can be determined to any required degree of accuracy; to 10 places of decimals it is found to be 2·7182818284.

Example 1. Find the sum of the infinite series

$$1 + \frac{1}{\lfloor 2} + \frac{1}{\lfloor 4} + \frac{1}{\lfloor 6} + \ldots\ldots$$

We have $$e = 1 + 1 + \frac{1}{\lfloor 2} + \frac{1}{\lfloor 3} + \frac{1}{\lfloor 4} + \ldots\ldots;$$

and by putting $x = -1$ in the series for e^x,

$$e^{-1} = 1 - 1 + \frac{1}{\lfloor 2} - \frac{1}{\lfloor 3} + \frac{1}{\lfloor 4} - \ldots\ldots$$

$$\therefore\ e + e^{-1} = 2\left(1 + \frac{1}{\lfloor 2} + \frac{1}{\lfloor 4} + \frac{1}{\lfloor 6} + \ldots\ldots\right);$$

hence the sum of the series is $\frac{1}{2}(e + e^{-1})$.

Example 2. Find the coefficient of x^r in the expansion of $\dfrac{1 - ax - x^2}{e^x}$.

$$\frac{1 - ax - x^2}{e^x} = (1 - ax - x^2)\,e^{-x}$$

$$= (1 - ax - x^2)\left\{1 - x + \frac{x^2}{\lfloor 2} - \frac{x^3}{\lfloor 3} + \ldots + \frac{(-1)^r x^r}{\lfloor r} + \ldots\ldots\right\}.$$

The coefficient required $= \dfrac{(-1)^r}{\lfloor r} - \dfrac{(-1)^{r-1} a}{\lfloor r-1} - \dfrac{(-1)^{r-2}}{\lfloor r-2}$

$$= \frac{(-1)^r}{\lfloor r}\{1 + ar - r(r-1)\}.$$

223. *To expand* $\log_e(1+x)$ *in ascending powers of* x.

From Art. 220,

$$a^y = 1 + y\log_e a + \frac{y^2(\log_e a)^2}{\lfloor 2} + \frac{y^3(\log_e a)^3}{\lfloor 3} + \dots\dots$$

In this series write $1+x$ for a; thus

$$(1+x)^y$$
$$= 1 + y\log_e(1+x) + \frac{y^2}{\lfloor 2}\{\log_e(1+x)\}^2 + \frac{y^3}{\lfloor 3}\{\log_e(1+x)\}^3 + \dots (1).$$

Also by the Binomial Theorem, when $x < 1$ we have

$$(1+x)^y = 1 + yx + \frac{y(y-1)}{\lfloor 2}x^2 + \frac{y(y-1)(y-2)}{\lfloor 3}x^3 + \dots\dots (2).$$

Now in (2) the coefficient of y is

$$x + \frac{(-1)}{1 \cdot 2}x^2 + \frac{(-1)(-2)}{1 \cdot 2 \cdot 3}x^3 + \frac{(-1)(-2)(-3)}{1 \cdot 2 \cdot 3 \cdot 4}x^4 + \dots\dots;$$

that is, $$x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots\dots.$$

Equate this to the coefficient of y in (1); thus we have

$$\log_e(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots\dots.$$

This is known as the *Logarithmic Series.*

Example. If $x<1$, expand $\{\log_e(1+x)\}^2$ in ascending powers of x.

By equating the coefficients of y^2 in the series (1) and (2), we see that the required expansion is double the coefficient of y^2 in

$$\frac{y(y-1)}{1 \cdot 2}x^2 + \frac{y(y-1)(y-2)}{1 \cdot 2 \cdot 3}x^3 + \frac{y(y-1)(y-2)(y-3)}{1 \cdot 2 \cdot 3 \cdot 4}x^4 + \dots\dots;$$

that is, double the coefficient of y in

$$\frac{y-1}{1 \cdot 2}x^2 + \frac{(y-1)(y-2)}{1 \cdot 2 \cdot 3}x^3 + \frac{(y-1)(y-2)(y-3)}{1 \cdot 2 \cdot 3 \cdot 4}x + \dots\dots.$$

Thus $\{\log_e(1+x)\}^2 = 2\left\{\frac{1}{2}x^2 - \frac{1}{3}\left(1+\frac{1}{2}\right)x^3 + \frac{1}{4}\left(1+\frac{1}{2}+\frac{1}{3}\right)x^4 - \dots\dots\right\}.$

224. Except when x is very small the series for $\log_e(1+x)$ is of little use for numerical calculations. We can, however, deduce from it other series by the aid of which Tables of Logarithms may be constructed.

By writing $\frac{1}{n}$ for x we obtain $\log_e \frac{n+1}{n}$; hence

$$\log_e(n+1) - \log_e n = \frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3} - \dots\dots\dots\dots (1).$$

By writing $-\frac{1}{n}$ for x we obtain $\log_e \frac{n-1}{n}$; hence, by changing signs on both sides of the equation,

$$\log_e n - \log_e(n-1) = \frac{1}{n} + \frac{1}{2n^2} + \frac{1}{3n^3} + \dots\dots\dots\dots (2).$$

From (1) and (2) by addition,

$$\log_e(n+1) - \log_e(n-1) = 2\left(\frac{1}{n} + \frac{1}{3n^3} + \frac{1}{5n^5} + \dots\right) \dots\dots (3).$$

From this formula by putting $n=3$ we obtain $\log_e 4 - \log_e 2$, that is $\log_e 2$; and by effecting the calculation we find that the value of $\log_e 2 = \cdot 69314718\dots$; whence $\log_e 8$ is known.

Again by putting $n=9$ we obtain $\log_e 10 - \log_e 8$; whence we find $\log_e 10 = 2\cdot 30258509\dots$.

To convert Napierian logarithms into logarithms to base 10 we multiply by $\frac{1}{\log_e 10}$, which is the *modulus* [Art. 216] of the common system, and its value is $\frac{1}{2\cdot 30258509\dots}$, or $\cdot 43429448\dots$; we shall denote this modulus by μ.

In the *Proceedings of the Royal Society of London*, Vol. XXVII. page 88, Professor J. C. Adams has given the values of e, μ, $\log_e 2$, $\log_e 3$, $\log_e 5$ to more than 260 places of decimals.

225. If we multiply the above series throughout by μ, we obtain formulæ adapted to the calculation of *common logarithms.*

Thus from (1), $\mu \log_e(n+1) - \mu \log_e n = \frac{\mu}{n} - \frac{\mu}{2n^2} + \frac{\mu}{3n^3} - \dots$;

that is,

$$\log_{10}(n+1) - \log_{10} n = \frac{\mu}{n} - \frac{\mu}{2n^2} + \frac{\mu}{3n^3} - \dots\dots\dots\dots \quad (1).$$

Similarly from (2),

$$\log_{10} n - \log_{10}(n-1) = \frac{\mu}{n} + \frac{\mu}{2n^2} + \frac{\mu}{3n^3} + \dots\dots\dots\dots \quad (2).$$

From either of the above results we see that if the logarithm of one of two consecutive numbers be known, the logarithm of the other may be found, and thus a table of logarithms can be constructed.

It should be remarked that the above formulæ are only needed to calculate the logarithms of *prime* numbers, for the logarithm of a *composite* number may be obtained by adding together the logarithms of its component factors.

In order to calculate the logarithm of any one of the smaller prime numbers, we do not usually substitute the number in either of the formulæ (1) or (2), but we endeavour to find some value of n by which division may be easily performed, and such that either $n+1$ or $n-1$ contains the given number as a factor. We then find $\log(n+1)$ or $\log(n-1)$ and deduce the logarithm of the given number.

Example. Calculate log 2 and log 3, given $\mu = \cdot 43429448$.

By putting $n=10$ in (2), we have the value of $\log 10 - \log 9$; thus

$$1 - 2\log 3 = \cdot 043429448 + \cdot 002171472 + \cdot 000144765 + \cdot 000010857$$
$$+ \cdot 000000868 + \cdot 000000072 + \cdot 000000006;$$

$$1 - 2\log 3 = \cdot 045757488,$$

$$\log 3 = \cdot 477121256.$$

Putting $n=80$ in (1), we obtain $\log 81 - \log 80$; thus

$$4\log 3 - 3\log 2 - 1 = \cdot 005428681 - \cdot 000033929 + \cdot 000000283 - \cdot 000000003;$$

$$3\log 2 = \cdot 908485024 - \cdot 005395032,$$

$$\log 2 = \cdot 301029997.$$

In the next article we shall give another series for $\log_e(n+1) - \log_e n$ which is often useful in the construction of Logarithmic Tables. For further information on the subject the reader is referred to Mr Glaisher's article on *Logarithms* in the *Encyclopædia Britannica.*

226. In Art. 223 we have proved that

$$\log_e(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots;$$

changing x into $-x$, we have

$$\log_e(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \dots.$$

By subtraction,

$$\log_e \frac{1+x}{1-x} = 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \dots\right).$$

Put $\dfrac{1+x}{1-x} = \dfrac{n+1}{n}$, so that $x = \dfrac{1}{2n+1}$; we thus obtain

$$\log_e(n+1) - \log_e n = 2\left\{\frac{1}{2n+1} + \frac{1}{3(2n+1)^3} + \frac{1}{5(2n+1)^5} + \dots\right\}.$$

NOTE. This series converges very rapidly, but in practice is not always so convenient as the series in Art. 224.

227. The following examples illustrate the subject of the chapter.

Example 1. If α, β are the roots of the equation $ax^2+bx+c=0$, shew that $\quad \log(a-bx+cx^2) = \log a + (\alpha+\beta)x - \dfrac{\alpha^2+\beta^2}{2}x^2 + \dfrac{\alpha^3+\beta^3}{3}x^3 - \dots$

Since $\alpha+\beta = -\dfrac{b}{a}$, $\alpha\beta = \dfrac{c}{a}$, we have

$$a - bx + cx^2 = a\{1 + (\alpha+\beta)x + \alpha\beta x^2\}$$
$$= a(1+\alpha x)(1+\beta x).$$

$$\therefore \log(a-bx+cx^2) = \log a + \log(1+\alpha x) + \log(1+\beta x)$$

$$= \log a + \alpha x - \frac{\alpha^2 x^2}{2} + \frac{\alpha^3 x^3}{3} - \dots + \beta x - \frac{\beta^2 x^2}{2} + \frac{\beta^3 x^3}{3} - \dots$$

$$= \log a + (\alpha+\beta)x - \frac{\alpha^2+\beta^2}{2}x^2 + \frac{\alpha^3+\beta^3}{3}x^3 - \dots$$

Example 2. Prove that the coefficient of x^n in the expansion of $\log(1+x+x^2)$ is $-\dfrac{2}{n}$ or $\dfrac{1}{n}$ according as n is or is not a multiple of 3.

$$\log(1+x+x^2) = \log\frac{1-x^3}{1-x} = \log(1-x^3) - \log(1-x)$$

$$= -x^3 - \frac{x^6}{2} - \frac{x^9}{3} - \dots - \frac{x^{3r}}{r} - \dots + \left(x + \frac{x^2}{2} + \frac{x^3}{3} + \dots + \frac{x^r}{r} + \dots\right).$$

If n is a multiple of 3, denote it by $3r$; then the coefficient of x^n is $-\frac{1}{r}$ from the first series, together with $\frac{1}{3r}$ from the second series; that is, the coefficient is $-\frac{3}{n}+\frac{1}{n}$, or $-\frac{2}{n}$.

If n is not a multiple of 3, x^n does not occur in the first series, therefore the required coefficient is $\frac{1}{n}$.

228. *To prove that* e *is incommensurable.*

For if not, let $e=\frac{m}{n}$, where m and n are positive integers;

then
$$\frac{m}{n}=1+1+\frac{1}{\underline{|2}}+\frac{1}{\underline{|3}}+\ldots+\frac{1}{\underline{|n}}+\frac{1}{\underline{|n+1}}+\ldots.$$

multiply both sides by $\underline{|n}$;

$$\therefore\ m\underline{|n-1}=\text{integer}+\frac{1}{n+1}+\frac{1}{(n+1)(n+2)}+\frac{1}{(n+1)(n+2)(n+3)}+\ldots.$$

But
$$\frac{1}{n+1}+\frac{1}{(n+1)(n+2)}+\frac{1}{(n+1)(n+2)(n+3)}+\ldots$$
is a proper fraction, for it is greater than $\frac{1}{n+1}$ and less than the geometrical progression
$$\frac{1}{n+1}+\frac{1}{(n+1)^2}+\frac{1}{(n+1)^3}+\ldots;$$
that is, less than $\frac{1}{n}$; hence an integer is equal to an integer plus a fraction, which is absurd; therefore e is incommensurable.

EXAMPLES. XVII.

1. Find the value of
$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\ldots.$$

2. Find the value of
$$\frac{1}{2}-\frac{1}{2\cdot 2^2}+\frac{1}{3\cdot 2^3}-\frac{1}{4\cdot 2^4}+\frac{1}{5\cdot 2^5}-\ldots.$$

3. Shew that

$$\log_e(n+a)-\log_e(n-a)=2\left(\frac{a}{n}+\frac{a^3}{3n^3}+\frac{a^5}{5n^5}+\dots\right).$$

4. If $$y=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\dots,$$

shew that $$x=y+\frac{y^2}{\lfloor 2}+\frac{y^3}{\lfloor 3}+\dots.$$

5. Shew that

$$\frac{a-b}{a}+\frac{1}{2}\left(\frac{a-b}{a}\right)^2+\frac{1}{3}\left(\frac{a-b}{a}\right)^3+\dots=\log_e a-\log_e b.$$

6. Find the Napierian logarithm of $\frac{1001}{999}$ correct to sixteen places of decimals.

7. Prove that $$e^{-1}=2\left(\frac{1}{\lfloor 3}+\frac{2}{\lfloor 5}+\frac{3}{\lfloor 7}+\dots\right).$$

8. Prove that

$$\log_e(1+x)^{1+x}(1-x)^{1-x}=2\left(\frac{x^2}{1.2}+\frac{x^4}{3.4}+\frac{x^6}{5.6}+\dots\right).$$

9. Find the value of

$$x^2-y^2+\frac{1}{\lfloor 2}(x^4-y^4)+\frac{1}{\lfloor 3}(x^6-y^6)+\dots.$$

10. Find the numerical values of the common logarithms of 7, 11 and 13; given $\mu=\cdot 43429448$, $\log 2=\cdot 30103000$.

11. Shew that if ax^2 and $\frac{a}{x^2}$ are each less unity

$$a\left(x^2+\frac{1}{x^2}\right)-\frac{a^2}{2}\left(x^4+\frac{1}{x^4}\right)+\frac{a^3}{3}\left(x^6+\frac{1}{x^6}\right)-\dots=\log_e\left(1+ax^2+a^2+\frac{a}{x^2}\right).$$

12. Prove that

$$\log_e(1+3x+2x^2)=3x-\frac{5x^2}{2}+\frac{9x^3}{3}-\frac{17x^4}{4}+\dots;$$

and find the general term of the series.

13. Prove that

$$\log_e\frac{1+3x}{1-2x}=5x-\frac{5x^2}{2}+\frac{35x^3}{3}-\frac{65x^4}{4}+\dots;$$

and find the general term of the series.

14. Expand $\frac{e^{5x}+e^x}{e^{3x}}$ in a series of ascending powers of x.

15. Express $\frac{1}{2}(e^{ix}+e^{-ix})$ in ascending powers of x, where $i=\sqrt{-1}$.

16. Shew that

$$\log_e(x+2h)=2\log_e(x+h)-\log_e x-\left\{\frac{h^2}{(x+h)^2}+\frac{h^4}{2(x+h)^4}+\frac{h^6}{3(x+h)^6}+\dots\right\}.$$

17. If α and β be the roots of $x^2-px+q=0$, shew that

$$\log_e(1+px+qx^2)=(\alpha+\beta)x-\frac{\alpha^2+\beta^2}{2}x^2+\frac{\alpha^3+\beta^3}{3}x^3-\dots.$$

18. If $x<1$, find the sum of the series

$$\frac{1}{2}x^2+\frac{2}{3}x^3+\frac{3}{4}x^4+\frac{4}{5}x^5+\dots.$$

19. Shew that

$$\log_e\left(1+\frac{1}{n}\right)^n=1-\frac{1}{2(n+1)}-\frac{1}{2.3(n+1)^2}-\frac{1}{3.4(n+1)^3}-\dots.$$

20. If $\log_e\frac{1}{1+x+x^2+x^3}$ be expanded in a series of ascending powers of x, shew that the coefficient of x^n is $-\frac{1}{n}$ if n be odd, or of the form $4m+2$, and $\frac{3}{n}$ if n be of the form $4m$.

21. Shew that

$$1+\frac{2^3}{\underline{|2}}+\frac{3^3}{\underline{|3}}+\frac{4^3}{\underline{|4}}+\dots=5e.$$

22. Prove that

$$2\log_e n-\log_e(n+1)-\log_e(n-1)=\frac{1}{n^2}+\frac{1}{2n^4}+\frac{1}{3n^6}+\dots.$$

23. Shew that $\frac{1}{n+1}+\frac{1}{2(n+1)^2}+\frac{1}{3(n+1)^3}+\dots$

$$=\frac{1}{n}-\frac{1}{2n^2}+\frac{1}{3n^3}-\dots.$$

24. If $\log_e\frac{9}{10}=-a$, $\log_e\frac{24}{25}=-b$, $\log_e\frac{81}{80}=c$, shew that

$$\log_e 2=7a-2b+3c,\ \log_e 3=11a-3b+5c,\ \log_e 5=16a-4b+7c;$$

and calculate $\log_e 2$, $\log_e 3$, $\log_e 5$ to 8 places of decimals.